Eidgenössische Technische Hochschule Zürich
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Exercise 2: Two Dimensional Beam Elements

Timoshenko and Assumed Natural Strain Beam Elements (always 2-nodes)

Outlines

Part 1: Including Shear Deformation

- Stiffness Matrices 1

  from ***Bernoulli*** to ***Timoshenko***

- Shear Locking 1

Part 2: How to avoid Shear Locking

- Stiffness Matrices 2

  Assumed Natural Strain and ***Exact Timoshenko***

- Shear Locking 2

Including Shear Deformation

- from thin-walled structures (Bernoulli, Kirchhoff, Love)
  - *‘A straight line, normal to the mid surface (mid axis in the case of beams), remains **straight and normal** to the deformed mid surface (axis) throughout deformation’*
- to slightly thicker structures (Timoshenko, Mindlin and Reisser, Naghdi)
  - *‘A straight line, normal to the mid surface (mid axis in the case of beams), remains **straight** throughout deformation.’*
  - *‘Normal stresses in transverse direction are negligible.’*

Bernoulli – Shape Functions

A cubic ansatz for the trial function

$$\phi(\xi) = a_1 + a_2\xi + a_3\xi^2 + a_4\xi^3$$

with the given Dirichlet and Neumann boundary conditions

$$\phi_1 = \phi(0), \quad \phi_2 = \frac{\mathrm{d}\phi}{\mathrm{d}\xi}(0), \quad \phi_3 = \phi(1), \quad \phi_4 = \frac{\mathrm{d}\phi}{\mathrm{d}\xi}(1)$$

provides the following shape functions

$$N_1 = (1-\xi)^2(1+2\xi), \quad N_2 = \xi(1-\xi)^2,$$
$$N_3 = \xi^2(3-2\xi), \quad N_4 = \xi^2(\xi-1).$$

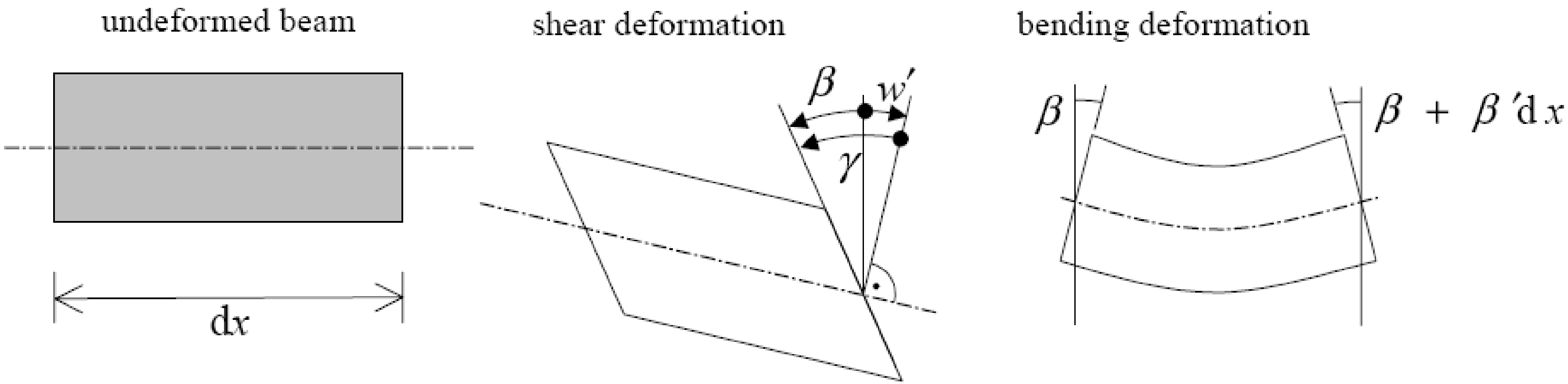
Bernoulli – Stiffness Matrix

Following a Bubnov-Galerkin concept including an integration over the square of the second derivatives of the shape functions results in the following stiffness matrix

$$K_{Bernoulli} = \frac{EI}{h^3}\begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix}$$

Timoshenko – Kinematics and Shape Functions

- Kinematics



- Linear Shape Functions (were used by most authors)

$$N_1 = \frac{1}{2}(1-\xi), \quad N_2 = \frac{1}{2}(1+\xi).$$

http://www.statik.bv.tu-muenchen.de/content/teaching/fem1/fem1.A1.pdf

Timoshenko – Bathe 1 (page 400)

Two-Dimensional Straight Beam Elements

Figure 5.19 shows the two-dimensional rectangular cross-section beam considered. Using the general expression of the principle of virtual work with the assumptions discussed above we have (see Exercise 5.32)

$$EI \int_0^L \left(\frac{d\beta}{dx}\right)\left(\frac{d\overline{\beta}}{dx}\right) dx + GAk \int_0^L \left(\frac{dw}{dx} - \beta\right)\left(\frac{d\overline{w}}{dx} - \overline{\beta}\right) dx = \int_0^L p\overline{w}\, dx + \int_0^L m\overline{\beta}\, dx \qquad (5.58)$$

where p and m are the transverse and moment loadings per unit length. Using now the interpolations

$$w = \sum_{i=1}^{q} h_i w_i; \qquad \beta = \sum_{i=1}^{q} h_i \theta_i \qquad (5.59)$$

where q is equal to the number of nodes used and the h_i are the one-dimensional interpolation functions listed in Fig. 5.3, we can directly employ the concepts of the isoparametric formulations discussed in Section 5.3 to establish all relevant element matrices. Let

$$\begin{aligned} w &= \mathbf{H}_w \hat{\mathbf{u}}; \qquad & \beta &= \mathbf{H}_\beta \hat{\mathbf{u}} \\ \frac{\partial w}{\partial x} &= \mathbf{B}_w \hat{\mathbf{u}}; \qquad & \frac{\partial \beta}{\partial x} &= \mathbf{B}_\beta \hat{\mathbf{u}} \end{aligned} \qquad (5.60)$$

Timoshenko – Bathe 2 (page 401)

where

$$\hat{\mathbf{u}}^T = [w_1 \; . \; . \; . \; w_q \quad \theta_1 \; . \; . \; . \; \theta_q]$$
$$\mathbf{H}_w = [h_1 \; . \; . \; . \; h_q \quad 0 \; . \; . \; . \; 0] \qquad (5.61)$$
$$\mathbf{H}_\beta = [0 \cdots 0 \quad h_1 \cdots h_q]$$

and

$$\mathbf{B}_w = J^{-1}\left[\frac{\partial h_1}{\partial r} \; \cdots \; \frac{\partial h_q}{\partial r} \quad 0 \; \cdots \; 0\right]$$
$$\mathbf{B}_\beta = J^{-1}\left[0 \; \cdots \; 0 \quad \frac{\partial h_1}{\partial r} \; \cdots \; \frac{\partial h_q}{\partial r}\right] \qquad (5.62)$$

where $J = \partial x / \partial r$; then we have for a single element,

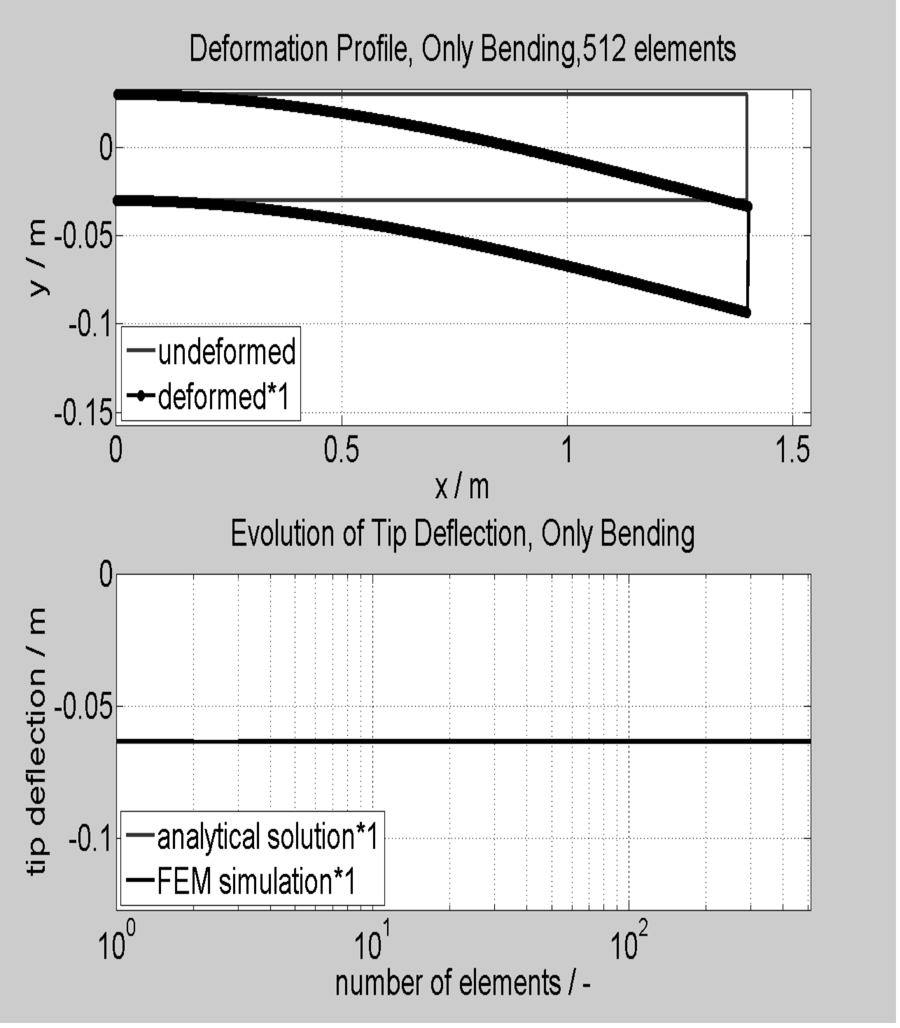
$$\mathbf{K} = EI \int_{-1}^{1} \mathbf{B}_\beta^T \mathbf{B}_\beta \det J \, dr + GAk \int_{-1}^{1} (\mathbf{B}_w - \mathbf{H}_\beta)^T (\mathbf{B}_w - \mathbf{H}_\beta) \det J \, dr$$
$$\mathbf{R} = \int_{-1}^{1} \mathbf{H}_w^T p \det J \, dr + \int_{1}^{-1} \mathbf{H}_\beta^T m \det J \, dr \qquad (5.63)$$

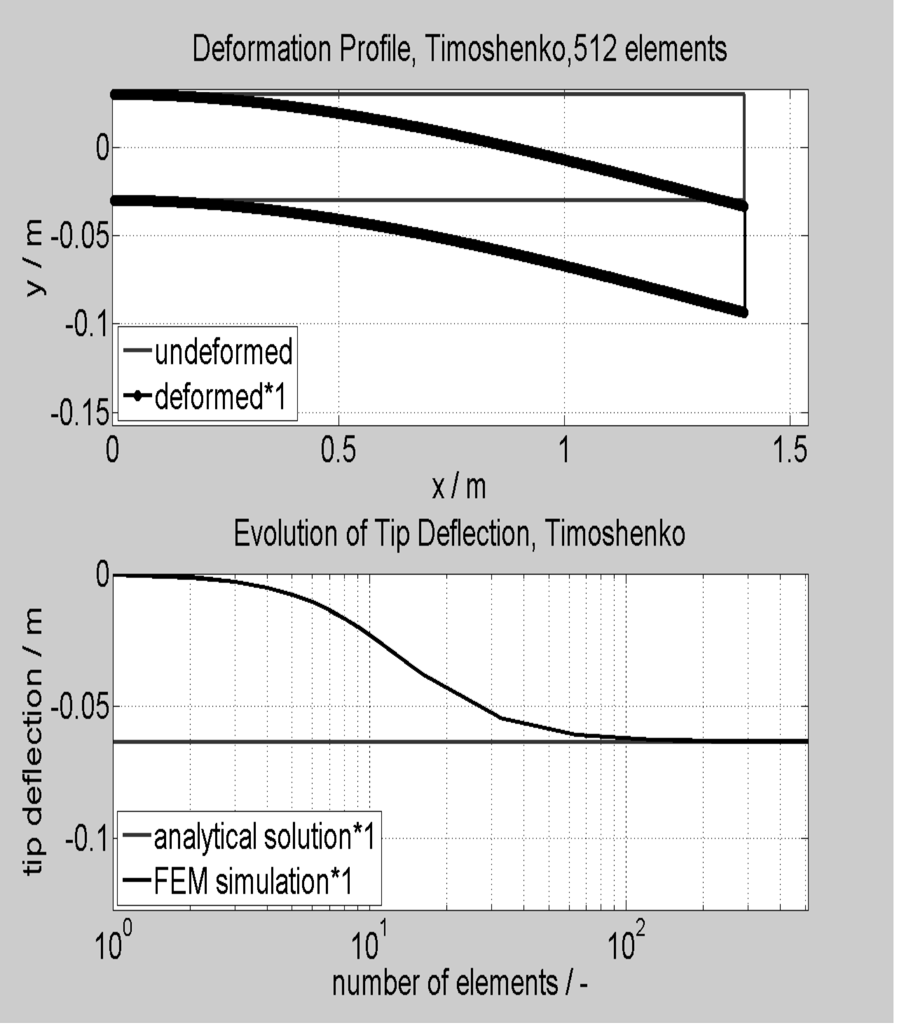
Timoshenko – Stiffness Matrix

Same procedure as for the Bernoulli beam element provides the following stiffness matrix

$$K_{Timoshenko} = kGA \begin{bmatrix} \frac{1}{h} & -\frac{1}{2} & -\frac{1}{h} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{h}{3}+\frac{EI}{kGAh} & \frac{1}{2} & \frac{h}{6}-\frac{EI}{kGAh} \\ -\frac{1}{h} & \frac{1}{2} & \frac{1}{h} & \frac{1}{2} \\ -\frac{1}{2} & \frac{h}{6}-\frac{EI}{kGAh} & \frac{1}{2} & \frac{h}{3}+\frac{EI}{kGAh} \end{bmatrix}$$

Shear Locking – Bernoulli vs. Timoshenko





How to avoid Shear Locking 1

- Selective Reduced Integration – Mixed Interpolation Approach – Assumed Natural Strain Method

The idea is to split the strain energy into individual parts and apply different integration rules to the corresponding contributions of the stiffness matrix.

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Mixed Interpolation Approach 1 – Ex. 4.30

$$\tilde{\Pi}^*_{\text{HR}} = \int_V \left(\frac{1}{2}\epsilon_{xx} E\epsilon_{xx} - \frac{1}{2}\gamma^{\text{AS}}_{xz} G\gamma^{\text{AS}}_{xz} + \gamma^{\text{AS}}_{xz} G\gamma_{xz} - \mathbf{u}^T\mathbf{f}^B\right) dV + \text{boundary terms}$$

where
$$\mathbf{u} = \begin{bmatrix} u \\ w \end{bmatrix}; \qquad \epsilon_{xx} = \frac{\partial u}{\partial x}; \qquad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

Now invoking $\delta\tilde{\Pi}^*_{\text{HR}} = 0$, we obtain corresponding to $\delta\mathbf{u}$, (not including boundary terms)

$$\int_V (\delta\epsilon_{xx} E\epsilon_{xx} + \delta\gamma_{xz} G\gamma^{\text{AS}}_{xz})\, dV = \int_V \delta\mathbf{u}^T\mathbf{f}^B\, dV \qquad \text{(b)}$$

and corresponding to $\delta\gamma^{\text{AS}}_{xz}$,

$$\int_V \delta\gamma^{\text{AS}}_{xz} G(\gamma_{xz} - \gamma^{\text{AS}}_{xz})\, dV = 0 \qquad \text{(c)}$$

Let
$$\hat{\mathbf{u}} = \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix}; \qquad \hat{\boldsymbol{\epsilon}} = [\gamma^{\text{AS}}]$$

Then we can write

$$\mathbf{u} = \mathbf{H}\hat{\mathbf{u}}; \qquad \epsilon_{xx} = \mathbf{B}_b\hat{\mathbf{u}}$$
$$\gamma_{xz} = \mathbf{B}_s\hat{\mathbf{u}}; \qquad \gamma^{\text{AS}}_{xz} = \mathbf{B}^{\text{AS}}_s\hat{\boldsymbol{\epsilon}}$$

Mixed Interpolation Approach 2 – Ex. 4.30

Substituting into (b) and (c), we obtain

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{u\epsilon} \\ \mathbf{K}_{u\epsilon}^T & \mathbf{K}_{\epsilon\epsilon} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{u}} \\ \hat{\boldsymbol{\epsilon}} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_B \\ \mathbf{0} \end{bmatrix} \qquad \text{(d)}$$

where
$$\mathbf{K}_{uu} = \int_V \mathbf{B}_b^T E \mathbf{B}_b \, dV; \qquad \mathbf{K}_{u\epsilon} = \int_V \mathbf{B}_s^T G \mathbf{B}_s^{AS} \, dV$$

$$\mathbf{K}_{\epsilon\epsilon} = -\int_V (\mathbf{B}_s^{AS})^T G \mathbf{B}_s^{AS} \, dV; \qquad \mathbf{R}_B = \int_V \mathbf{H}^T \mathbf{f}^B \, dV$$

We can now use static condensation on $\hat{\boldsymbol{\epsilon}}$ to obtain the final element stiffness matrix:

$$\mathbf{K} = \mathbf{K}_{uu} - \mathbf{K}_{u\epsilon} \mathbf{K}_{\epsilon\epsilon}^{-1} \mathbf{K}_{u\epsilon}^T$$

In our case, we have

$$\mathbf{H} = \begin{bmatrix} 0 & -\frac{z}{L}\left(\frac{L}{2} - x\right) & 0 & -\frac{z}{L}\left(\frac{L}{2} + x\right) \\ \frac{1}{L}\left(\frac{L}{2} - x\right) & 0 & \frac{1}{L}\left(\frac{L}{2} + x\right) & 0 \end{bmatrix}$$

$$\mathbf{B}_b = \begin{bmatrix} 0 & \frac{z}{L} & 0 & -\frac{z}{L} \end{bmatrix}$$

$$\mathbf{B}_s = \begin{bmatrix} -\frac{1}{L} & -\frac{1}{L}\left(\frac{L}{2} - x\right) & \frac{1}{L} & -\frac{1}{L}\left(\frac{L}{2} + x\right) \end{bmatrix}$$

$$\mathbf{B}_s^{AS} = [1]$$

Assumed Natural Strain (ANS) – Stiffness Matrix

Assuming a constant shear strain one obtains

$$K_{ANS} = kGA \begin{bmatrix} \frac{1}{h} & -\frac{1}{2} & -\frac{1}{h} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{h}{4} + \frac{EI}{kGAh} & \frac{1}{2} & \frac{h}{4} - \frac{EI}{kGAh} \\ -\frac{1}{h} & \frac{1}{2} & \frac{1}{h} & \frac{1}{2} \\ -\frac{1}{2} & \frac{h}{4} - \frac{EI}{kGAh} & \frac{1}{2} & \frac{h}{4} + \frac{EI}{kGAh} \end{bmatrix}$$

How to avoid Shear Locking 2

- Rigorously derive Shape Functions – Exact Timoshenko

Instead of assuming arbitrary shape functions, one can rigorously derive the shape functions, either via unit load method or directly from the differential equations.

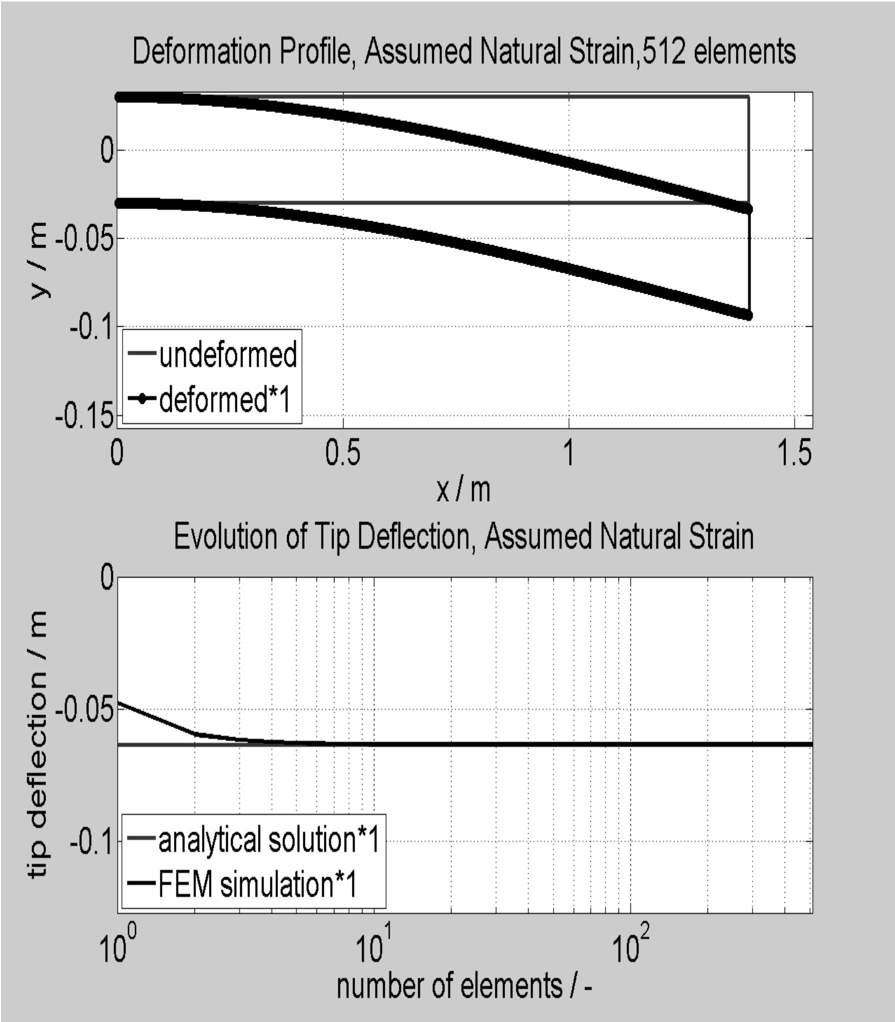
Eisenberger M., 1994, CNME 10 (9): 673-681

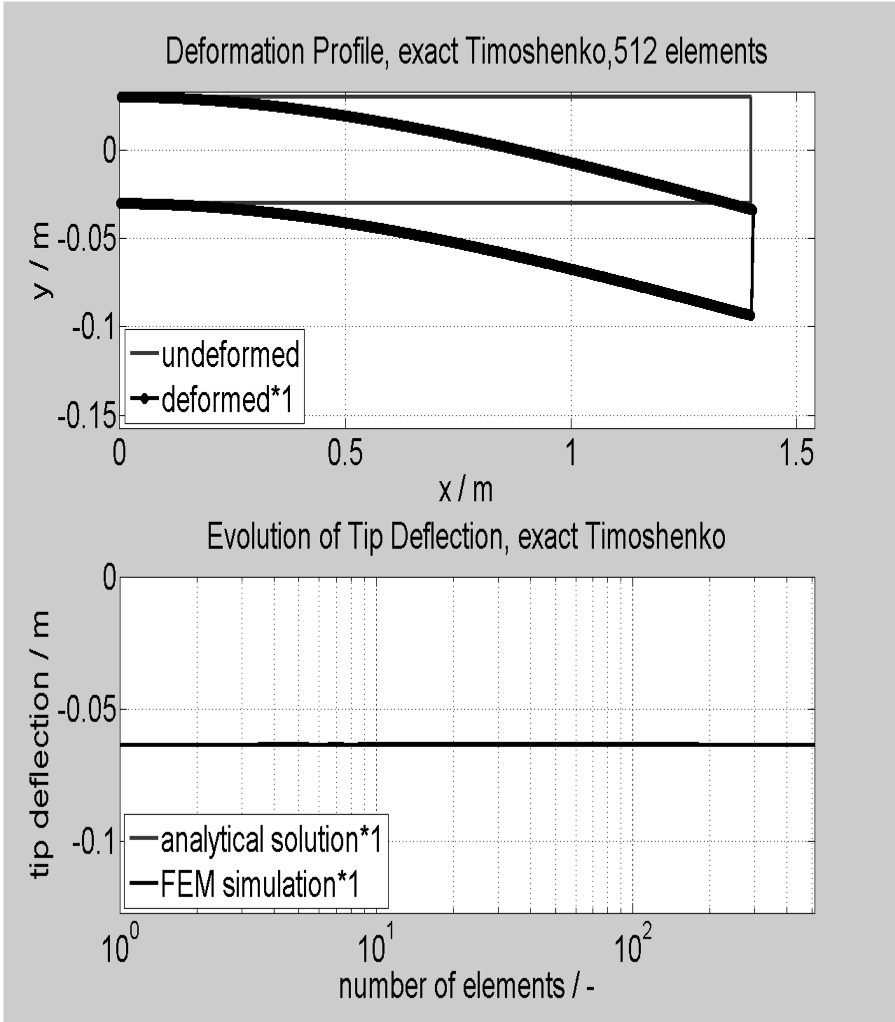
Exact Timoshenko – Stiffness Matrix

From the rigorously derived (exact) shape functions one obtains a stiffness matrix for a superconvergent exact Timoshenko beam element.

$$K_{\substack{exact\\Timoshenko}} = \frac{kGA}{\left(12+\frac{kGAh^2}{EI}\right)h}\begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2\left(1+3\frac{EI}{kGA}\right) & -6h & 2h^2\left(1-6\frac{EI}{kGA}\right) \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2\left(1-6\frac{EI}{kGA}\right) & -6h & 4h^2\left(1+3\frac{EI}{kGA}\right) \end{bmatrix}$$

Shear Locking – ANS vs. exact Timoshenko





What is the Reason for Shear Locking ?

The assumed shape functions, w and β, are not correct and therefore the relation

$$\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{u} \quad \text{with} \quad \mathbf{u} = \begin{bmatrix} w \\ \beta \end{bmatrix}$$

is not valid at each point of the element.

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