Finite Element Formulation for Beams

- Handout 2 -

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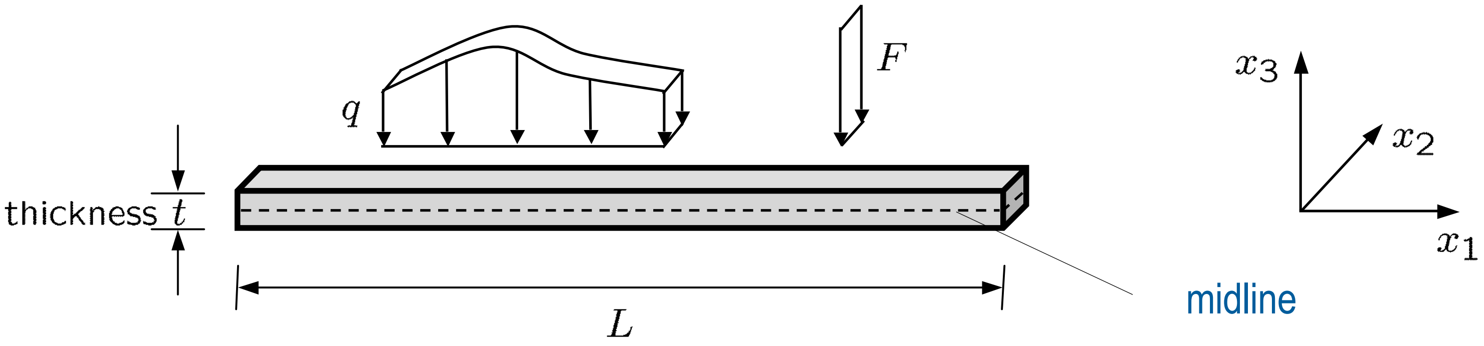
Completed Version

Review of Euler-Bernoulli Beam

- Physical beam model



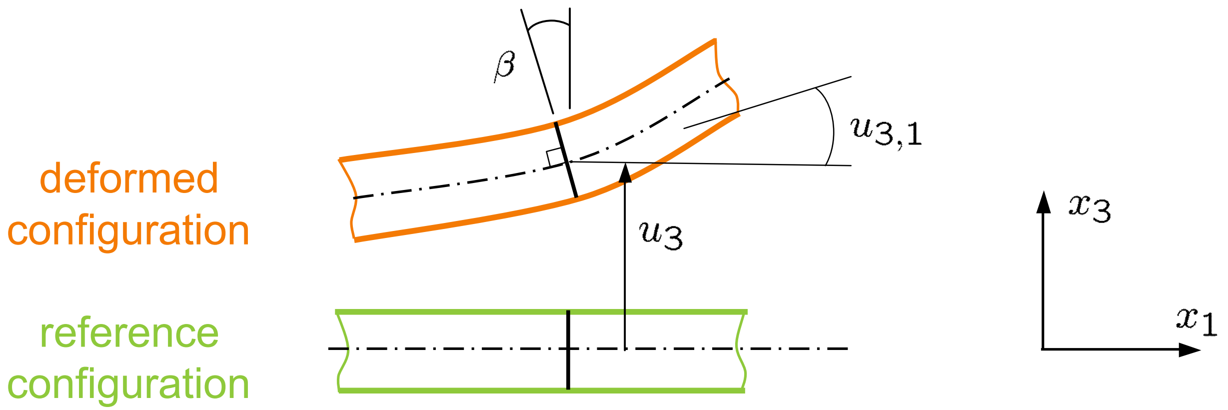
- Beam domain in three-dimensions

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_3 \in [-\tfrac{t}{2},\, \tfrac{t}{2}], x_2 \in [-\tfrac{1}{2},\, \tfrac{1}{2}], x_1 \in \Omega \subset \mathbb{R}\}$$

- Midline, also called the neutral axis, has the coordinate $x_3 = 0$
- Key assumptions: beam axis is in its unloaded configuration straight
- Loads are normal to the beam axis

Kinematics of Euler-Bernoulli Beam -1-

- Assumed displacements during loading



- Kinematic assumption: Material points on the normal to the midline remain on the normal during the deformation
  - Slope of midline: $\beta = \frac{\partial u_3}{\partial x_1} = u_{3,1}$
- The kinematic assumption determines the axial displacement of the material points across thickness

$$u_1 = -\beta x_3 \quad \text{with} \; -\frac{t}{2} \leq x_3 \leq \frac{t}{2}$$

  - Note this is valid only for small deflections, else $u_1 = \sin(-\beta) x_3$

Kinematics of Euler-Bernoulli Beam -2-

- Introducing the displacements into the strain equations of three-dimensional elasticity leads to
  - Axial strains

    $$\epsilon_{11} = u_{1,1} = -\beta_{,1} x_3 = -u_{3,11} x_3 = \kappa x_3 \qquad \text{(with curvature } \kappa = -u_{3,11}\text{)}$$

    - Axial strains vary linearly across thickness
  - All other strain components are zero
    - Shear strain in the $x_1 - x_3$ plane

      $$\epsilon_{13} = \frac{1}{2}\left(u_{1,3} - u_{3,1}\right) = \frac{1}{2}(-\beta + \beta) = 0$$

    - Through-the-thickness strain (no stretching of the midline normal during deformation)

      $$\epsilon_{33} = \frac{\partial u_3}{\partial x_3} = 0$$

    - No deformations in $x_1 - x_2$ and $x_2 - x_3$ planes so that the corresponding strains are zero

Weak Form of Euler-Bernoulli Beam

- The beam strains introduced into the internal virtual work expression of three-dimensional elasticity

$$\int_\Omega \int_{-t/2}^{t/2} \sigma_{ij}\epsilon_{ij}\, dx_3 dx_1 = \int_\Omega \int_{-t/2}^{t/2} \sigma_{11}\epsilon_{11}(v)\, dx_3 dx_1$$

$$= \int_\Omega \int_{-t/2}^{t/2} \sigma_{11} x_3 \kappa(v)\, dx_3 dx_1 = \int_\Omega m\kappa(v)\, dx_1$$

  - with the standard definition of bending moment: $m = \int_{-t/2}^{t/2} \sigma_{11} x_3\, dx_3$

- External virtual work

$$\int_\Omega qv\, dx_1$$

- Weak work of beam equation

$$\int_\Omega m\kappa(v)\, dx_1 = \int_\Omega qv\, dx_1 + \text{boundary terms}$$

  - Boundary terms only present if force/moment boundary conditions present

Stress-Strain Law

- The only non-zero stress component is given by Hooke's law

$$\sigma_{11} = E\epsilon_{11} = E\kappa x_3$$

  - This leads to the usual relationship between the moment and curvature

$$m = \int_{-t/2}^{t/2} \sigma_{11} x_3 \, dx_3 = \int_{-t/2}^{t/2} E\kappa x_3^2 \, dx_3 = EI\kappa$$

    - with the second moment of area $I = \int_{-t/2}^{t/2} x_3^2 \, dx_3$

- Weak form work as will be used for FE discretization

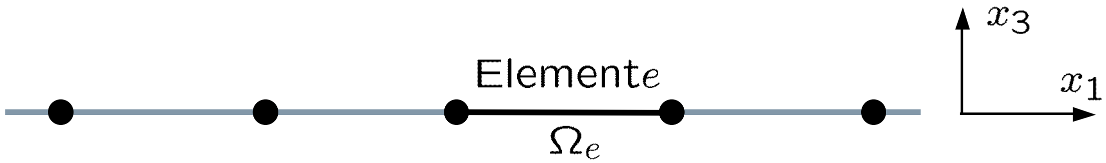
$$EI \int_\Omega \kappa(u_3)\kappa(v) \, dx_1 = \int_\Omega qv \, dx_1 + \text{boundary terms}$$

    - EI assumed to be constant

Finite Element Method

- Beam is represented as a (disjoint) collection of finite elements

$$\Omega := \bigcup_{\Omega_e \in \Omega} \Omega_e$$



- On <u>each element</u> displacements and the test function are interpolated using shape functions and the corresponding nodal values

$$u_3 = \sum_{K=1}^{NP} N^K u_3^K \Rightarrow \kappa(u_3) = -u_{3,11} = -\sum_{K=1}^{NP} N^K_{,11} u_3^K$$

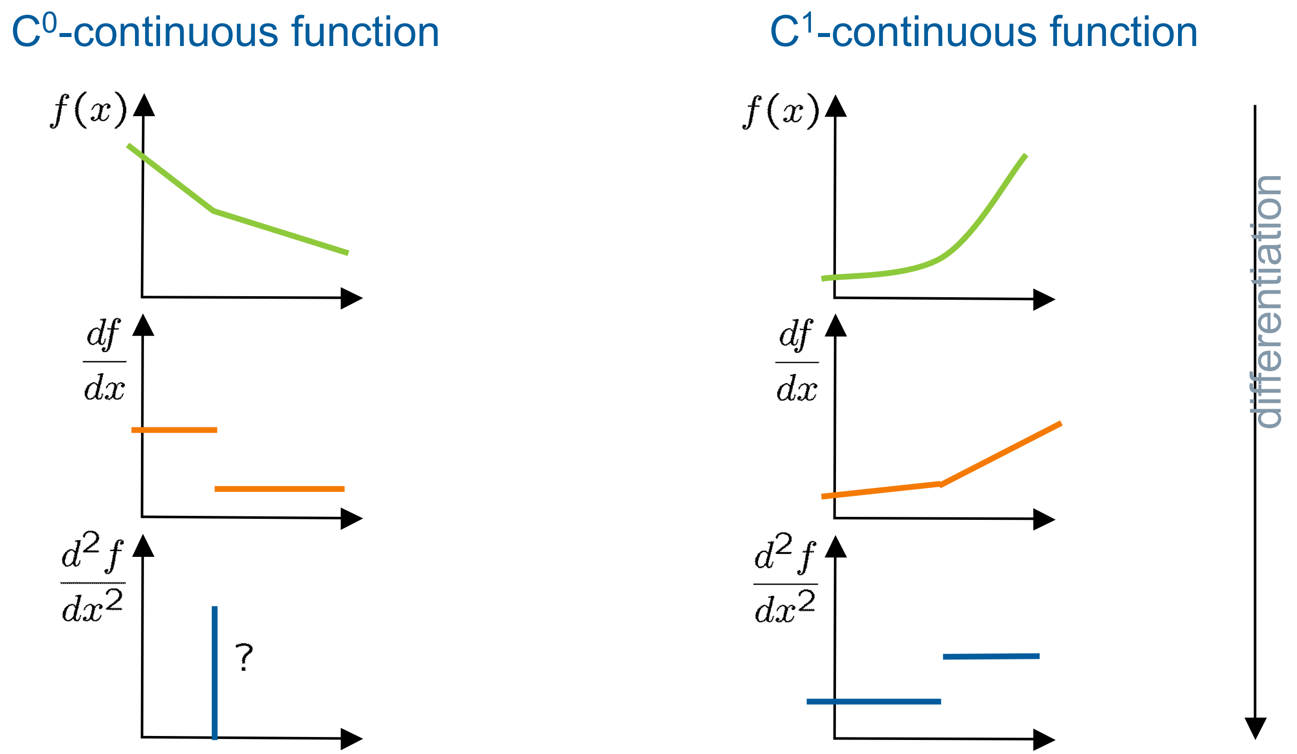
$$v = \sum_{K=1}^{NP} N^K v^K \Rightarrow \kappa(v) = -v_{,11} = -\sum_{K=1}^{NP} N^K_{,11} v^K$$

  - Number of nodes per element NP
  - Shape function of node K N^K
  - Nodal values of displacements $u_3^1, \dots, u_3^{NP}$
  - Nodal values of test functions $v^1, \dots, v^{NP}$

- To obtain the FE equations the preceding interpolation equations are introduced into the weak form
  - Note that the integrals in the weak form depend on the second order derivatives of u_3 and v

Aside: Smoothness of Functions

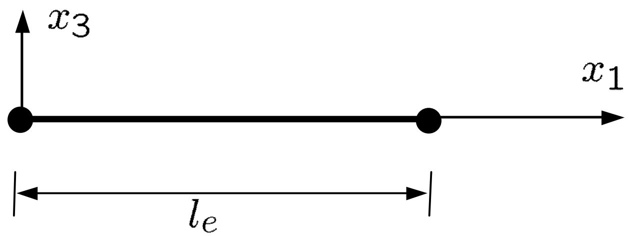
- A function f: $\Omega \rightarrow \Re$ is of class $C^k = C^k(\Omega)$ if its derivatives of order j, where $0 \leq j \leq k$, exist and are continuous functions
  - For example, a C^0 function is simply a continuous function
  - For example, a C^∞ function is a function with all the derivatives continuous



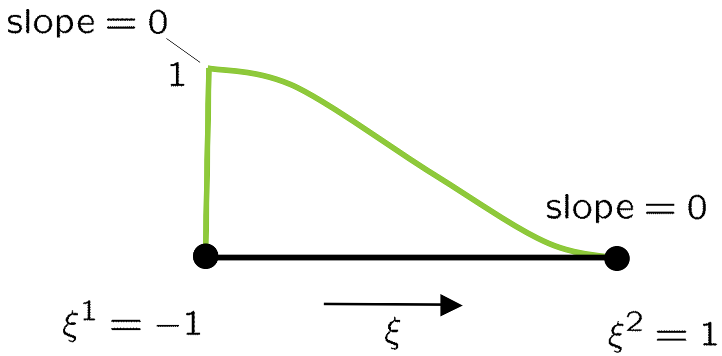
- The shape functions for the Euler-Bernoulli beam have to be C^1-continuous so that their second order derivatives in the weak form can be integrated

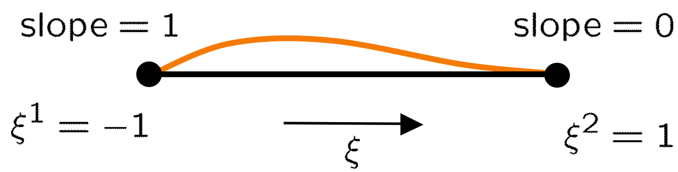
Hermite Interpolation -1-

- To achieve C[1]-smoothness Hermite shape functions can be used
  - Hermite shape functions for an element of length l_e



  - Shape functions of node 1



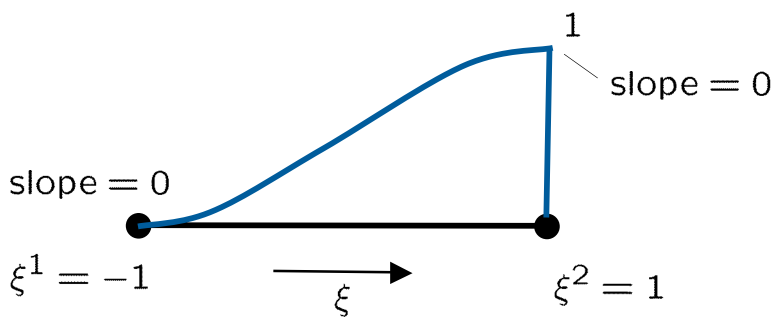


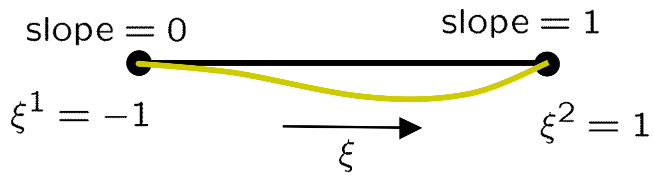
$$N^1(\xi) = \frac{1}{4}(1-\xi)^2(2+\xi) \qquad M^1(\xi) = \frac{l_e}{8}(1-\xi)^2(1+\xi)$$

  - with $\xi = \dfrac{2x_1}{l_e} - 1$

Hermite Interpolation -2-

- Shape functions of Node 2





$$N^2(\xi) = \frac{1}{4}(1+\xi)^2(2-\xi)$$

$$M^2(\xi) = \frac{l_e}{8}(1+\xi)^2(\xi-1)$$

- with $\xi = \frac{2x_1}{l_e} - 1$

Element Stiffness Matrix

- According to Hermite interpolation the degrees of freedom for each element are the displacements u_3 and slopes β at the two nodes
  - Interpolation of the displacements

$$u_3 = [N^1 \; M^1 \; N^2 \; M^2] \begin{bmatrix} u_3^1 \\ \beta^1 \\ u_3^2 \\ \beta^2 \end{bmatrix} \Rightarrow \kappa(u_3) = -\underbrace{[N^1_{,11} \; M^1_{,11} \; N^2_{,11} \; M^2_{,11}]}_{\text{"B"-matrix}} \underbrace{\begin{bmatrix} u_3^1 \\ \beta^1 \\ u_3^2 \\ \beta^2 \end{bmatrix}}_{w}$$

$$\Rightarrow \kappa(u_3) = -\sum_{K=1}^{\cdot} B^K w^K$$

  - Test functions are interpolated in the same way like displacements

$$\kappa(v) = -\sum_{L=1}^{4} B^L v^L$$

  - Introducing the displacement and test functions interpolations into weak form gives the element stiffness matris

$$EI \int_{\Omega_e} \kappa(u_3)\kappa(v)\, dx_1 = \sum_K \sum_L w^K v^L \underbrace{EI \int_{\Omega_e} B^K B^L \, dx_1}_{\boldsymbol{k}_e}$$

Element Load Vector

- Load vector computation analogous to the stiffness matrix derivation

$$\int_{\Omega_e} qv \, dx_1 = \sum_K v^K \underbrace{\int_{\Omega_e} qN^K dx_1}_{\boldsymbol{f}_e}$$

- The global stiffness matrix and the global load vector are obtained by assembling the individual element contributions
  - The assembly procedure is identical to usual finite elements

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}$$

    - Global stiffness matrix $\boldsymbol{K}$
    - Global load vector $\boldsymbol{F}$
    - All nodal displacements and rotations $\boldsymbol{w}$

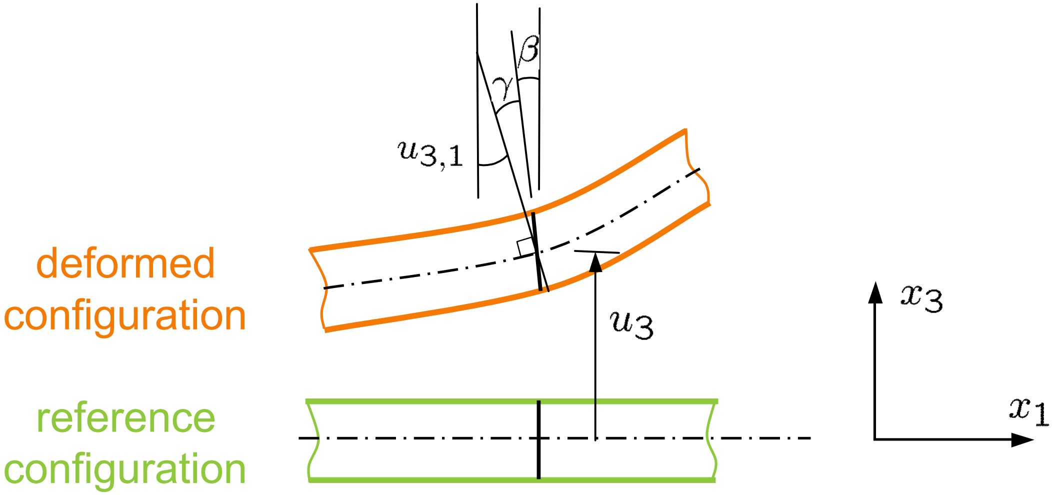
Stiffness Matrix of Euler-Bernoulli Beam

- Element stiffness matrix of an element with length l_e

$$
\boldsymbol{k}_e = EI \begin{bmatrix} \frac{12}{l_e^3} & \frac{6}{l_e^2} & -\frac{12}{l_e^3} & \frac{6}{l_e^2} \\ & \frac{4}{l_e} & -\frac{6}{l_e^2} & \frac{2}{l_e} \\ & & \frac{12}{l_e^3} & -\frac{6}{l_e^2} \\ \text{sym.} & & & \frac{4}{l_e} \end{bmatrix}
$$

Kinematics of Timoshenko Beam -1-

- Assumed displacements during loading



- Kinematic assumption: a plane section originally normal to the centroid remains plane, but in addition also shear deformations occur
  - Rotation angle of the normal: β
  - Angle of shearing: γ
  - Slope of midline: $u_{3,1} = \gamma + \beta$

- The kinematic assumption determines the axial displacement of the material points across thickness

$$u_1 = -\beta x_3 = (-u_{3,1} + \gamma) x_3$$

  - Note that this is only valid for small rotations, else $u_1 = \sin(-\beta) x_3$

Kinematics of Timoshenko Beam -2-

- Introducing the displacements into the strain equations of three-dimensional elasticity leads to
  - Axial strain

    $$\epsilon_{11} = -\beta_{,1} x_3 = \kappa x_3$$

    - Axial strain varies linearly across thickness
  - Shear strain

    $$\epsilon_{13} = \frac{1}{2}(-\beta + u_{3,1}) = \frac{1}{2}\gamma$$

    - Shear strain is constant across thickness
  - All the other strain components are zero

Weak Form of Timoshenko Beam

- The beam strains introduced into the internal virtual work expression of three-dimensional elasticity give

$$\int_\Omega \int_{-t/2}^{t/2} \left[\sigma_{11}\epsilon_{11}(v) + 2\sigma_{13}\epsilon_{13}(v)\right] \, dx_3 dx_1$$

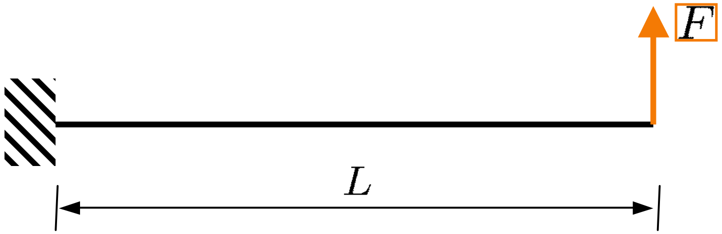
  - Hookes's law $\sigma_{11} = E\epsilon_{11}$ and $\sigma_{13} = G\gamma$
  - Introducing the expressions for strain and Hooke's law into the weak form gives

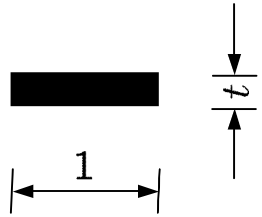
$$EI \int_\Omega \beta_{,1}\phi_{,1} \, dx + GAk \int_\Omega \left(u_{3,1} - \beta\right)\left(v_{3,1} - \phi\right) \, dx$$

    - virtual displacements and rotations: $v_3, \ \phi$
    - shear correction factor k necessary because across thickness shear stresses are parabolic according to elasticity theory but constant according to Timoshenko beam theory
    - shear correction factor for a rectangular cross section $k = \frac{5}{6}$
    - shear modulus $G = \frac{E}{2}$
- External virtual work similar to Euler-Bernoulli beam

Euler-Bernoulli vs. Timoshenko -1-

- Comparison of the displacements of a cantilever beam analytically computed with the Euler-Bernoulli and Timoshenko beam theories





- Bernoulli beam
  - Governing equation: $EIu_{3,1111} = 0$
  - Boundary conditions: $u_3(0) = 0 \quad u_{3,1}|_{x_1=0} = 0$

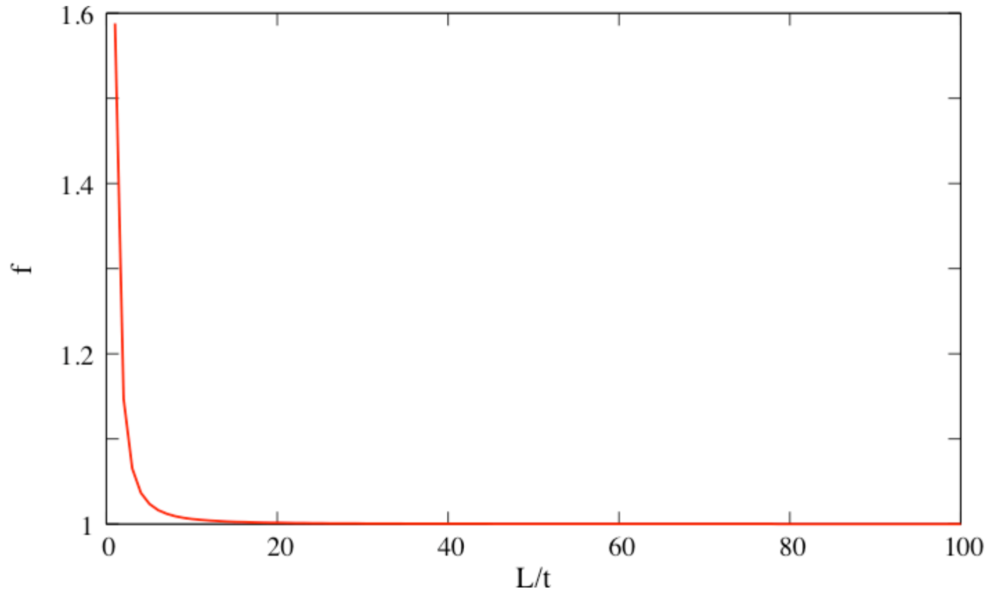
    $M(L) = -EIu_{3,11}|_{x_1=L} = 0 \quad Q(L) = -EIu_{3,111}|_{x_1=L} = F$

- Timoshenko beam
  - Governing equations: $EI\beta_{,11} = 0 \qquad GA\left(u_{3,11} + \beta_{,1}\right) = 0$
  - Boundary conditions: $u_3(0) = 0 \quad \beta(0) = 0$

    $GA(u_{3,1} + \beta)|_{x_1=L} = F \quad EI\beta_{,1}|_{x_1=L} = 0$

Euler-Bernoulli vs. Timoshenko -2-

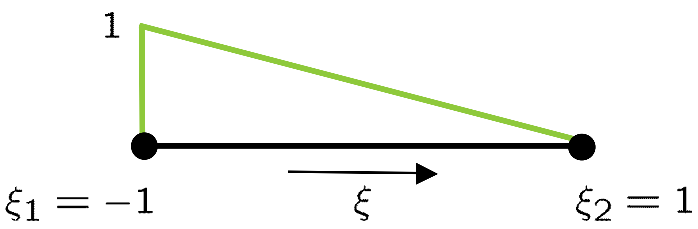
- Maximum tip deflection computed by integrating the differential equations
  - Bernoulli beam $$u_3^B(L) = \frac{4FL^3}{Et^3}$$
  - Timoshenko beam $$u_3^T(L) = \frac{4FL^3}{Et^3} + \frac{12FL}{5Et}$$
  - Ratio $$f = \frac{u_3^T(L)}{u_3^B(L)} = 1 + \frac{3}{5}\left(\frac{t}{L}\right)^2$$

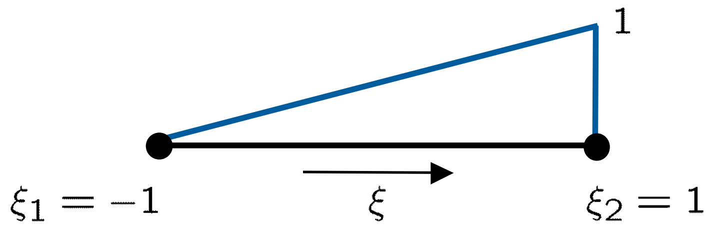


- For slender beams (L/t > 20) both theories give the same result
- For stocky beams (Lt < 10) Timoshenko beam is physically more realistic because it includes the shear deformations

Finite Element Discretization

- The weak form essentially contains β, $\beta_{,1}$, and $u_{3,1}$ and the corresponding test functions
  - C^0 interpolation appears to be sufficient, e.g. linear interpolation





$$N^1(\xi) = \frac{1}{2}(1 - \xi) \qquad N^2(\xi) = \frac{1}{2}(1 + \xi)$$

  - Interpolation of displacements and rotation angle

$$u_3 = [N^1 \; 0 \; N^2 \; 0] \begin{bmatrix} u_3^1 \\ \beta^1 \\ u_3^2 \\ \beta^2 \end{bmatrix} \qquad \beta = [0 \; N^1 \; 0 \; N^2] \begin{bmatrix} u_3^1 \\ \beta^1 \\ u_3^2 \\ \beta^2 \end{bmatrix}$$

Element Stiffness Matrix

- Shear angle

$$\gamma = u_{3,1} - \beta = \underbrace{[N^1_{,1} \;\; -N^1 \;\; N^2_{,1} \;\; -N^2]}_{B_S-\text{matrix}} \underbrace{\begin{bmatrix} u_3^1 \\ \beta^1 \\ u_3^2 \\ \beta^2 \end{bmatrix}}_{w} \Rightarrow \gamma = u_{3,1} - \beta = \sum_{K=1}^{4} B_S^K w^K$$

- Curvature

$$\kappa = -\beta_{,1} = -\underbrace{[0 \;\; N^1_{,1} \;\; 0 \;\; N^2_{,1}]}_{B_M-\text{matrix}} \begin{bmatrix} u_3^1 \\ \beta^1 \\ u_3^2 \\ \beta^2 \end{bmatrix} \Rightarrow \kappa = -\beta_{,1} = -\sum_{K=1}^{4} B_M^K w^K$$

- Test functions are interpolated in the same way like displacements and rotations
- Introducing the interpolations into the weak form leads to the element stiffness matrices
  - Shear component of the stiffness matrix

$$GAk \int_{\Omega_e} \left(u_{3,1} - \beta\right) \left(v_{3,1} - \phi\right) \, dx_1 = \sum_K \sum_L w^K v^L \underbrace{GAk \int_{\Omega_e} B_S^K B_S^L \, dx_1}_{\boldsymbol{k}_{es}}$$

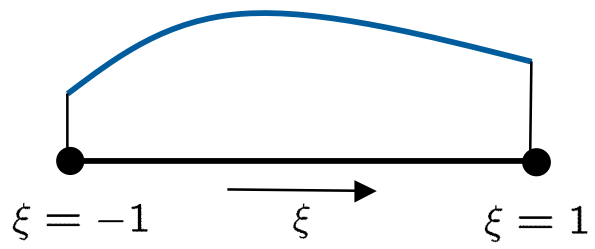
  - Bending component of the stiffness matrix

$$EI \int_{\Omega_e} \beta_{,1} \phi_{,1} \, dx_1 = \sum_K \sum_L w^K v^L \underbrace{EI \int_{\Omega_e} B_M^K B_M^L \, dx_1}_{\boldsymbol{k}_{eb}}$$

Review: Numerical Integration

- Gaussian Quadrature
  - The locations of the quadrature points and weights are determined for maximum accuracy

$$\int_{-1}^{1} f(\xi)d\xi = \sum_{i=0}^{n_{int}} f(\xi_i)w_i$$



  - n_{int}=1 $\xi_1 = 0, \; w_1 = 2$
  - n_{int}=2 $\xi_1 = \frac{-1}{\sqrt{3}}, \; \xi_2 = \frac{1}{\sqrt{3}}, \; w_1 = 1, \; w_2 = 1$
  - n_{int}=3 $\xi_1 = -\sqrt{\frac{3}{5}}, \; \xi_2 = 0, \; \xi_3 = \sqrt{\frac{3}{5}}, \; w_1 = \frac{5}{9}, \; w_2 = \frac{8}{9}, \; w_3 = \frac{5}{9}$
  - Note that polynomials with order ($2n_{int}$-1) or less are exactly integrated

- The element domain is usually different from [-1,+1) and an isoparametric mapping can be used

$$\int_{\Omega} f(x)dx = \int_{-1}^{1} f(x)x_{,\xi}d\xi$$

Stiffness Matrix of the Timoshenko Beam -1-

- Necessary number of quadrature points for linear shape functions
  - Bending stiffness: one integration point sufficient because B_M is constant
  - Shear stiffness: two integration points necessary because B_S is linear

- Element bending stiffness matrix of an element with length l_e and <u>one integration point</u>

$$\boldsymbol{k}_{eb} = \frac{EI}{l_e} \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 1 & 0 & -1 \\ & & 0 & 0 \\ \text{sym.} & & & 1 \end{bmatrix}$$

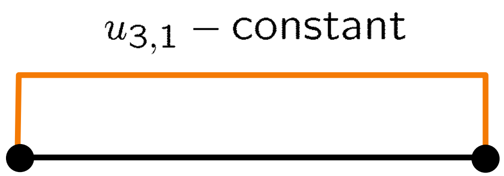
- Element shear stiffness matrix of an element with length l_e and <u>two integration points</u>

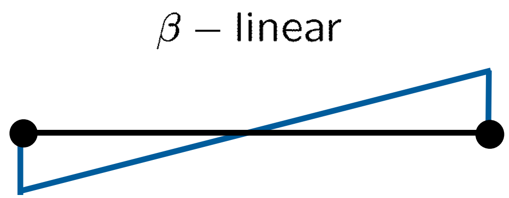
$$\boldsymbol{k}_{es} = \frac{5}{6}\frac{GA}{l_e} \begin{bmatrix} 1 & \frac{l_e}{2} & -1 & \frac{l_e}{2} \\ & \frac{l_e^2}{3} & -\frac{l_e}{2} & \frac{l_e^2}{6} \\ & & 1 & -\frac{l_e}{2} \\ \text{sym.} & & & \frac{l_e^2}{3} \end{bmatrix}$$

Limitations of the Timoshenko Beam FE

- Recap: Degrees of freedom for the Timoshenko beam

- Physics dictates that for t→0 (so-called Euler-Bernoulli limit) the shear angle has to go to zero
  - If linear shape functions are used for u_3 and β

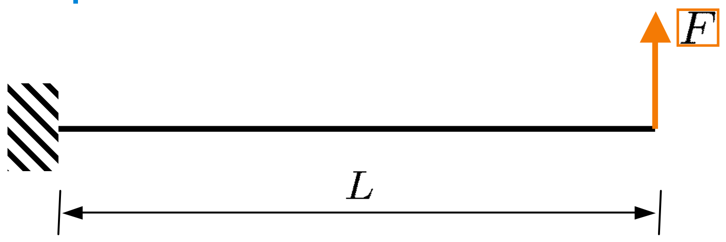


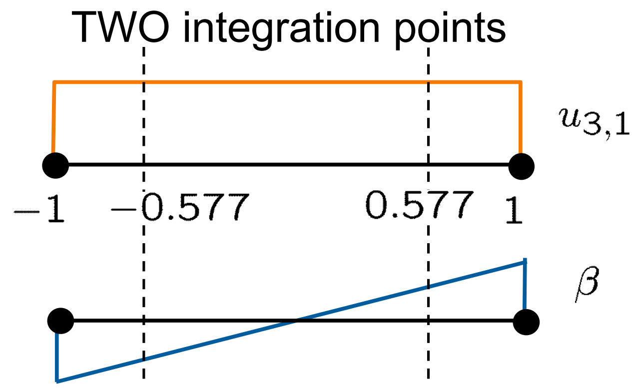


  - Adding a constant and a linear function will never give zero!
  - Hence, since the shear strains cannot be arbitrarily small everywhere, an erroneous shear strain energy will be included in the energy balance
    - In practice, the computed finite element displacements will be much smaller than the exact solution

Shear Locking: Example -1-

- Displacements of a cantilever beam





- Influence of the beam thickness on the normalized tip displacement

Thick beam

# elem.	2 point
1	0.0416
2	0.445
4	0.762
8	0.927

Thin beam

# elem.	2 point
1	0.0002
2	0.0008
4	0.0003
8	0.0013

from TJR Hughes, The finite element method.

Stiffness Matrix of the Timoshenko Beam -2-

- The beam element with only linear shape functions appears not to be ideal for very thin beams

- The problem is caused by non-matching u_3 and β interpolation
  - For very thin beams it is not possible to reproduce $\gamma = 0$

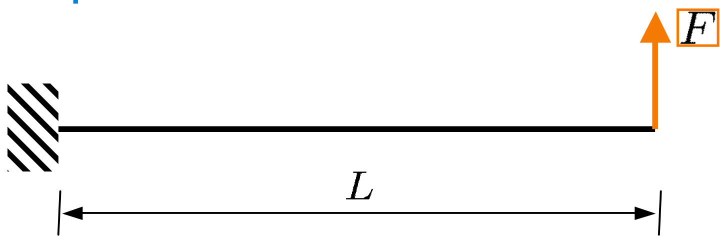
$$\gamma = u_{3,1} - \beta$$

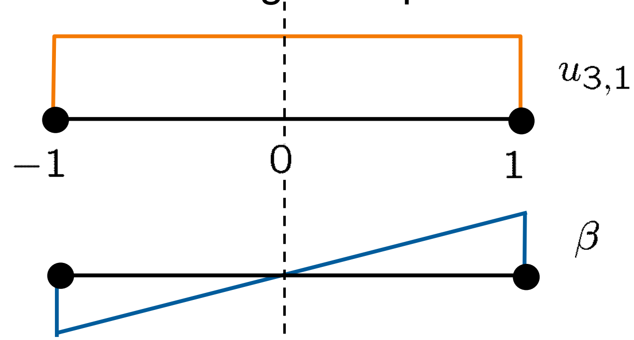
- How can we fix this problem?
  - Lets try with using only one integration point for integrating the element shear stiffness matrix
  - Element shear stiffness matrix of an element with length l_e and <u>one</u> integration points

$$\boldsymbol{k}_{es} = \frac{5}{6}\frac{GA}{l_e}\begin{bmatrix} 1 & \frac{l_e}{2} & -1 & \frac{l_e}{2} \\ & \frac{l_e^2}{4} & -\frac{l_e}{2} & \frac{l_e^2}{4} \\ & & 1 & -\frac{l_e}{2} \\ \text{sym.} & & & \frac{l_e^2}{4} \end{bmatrix}$$

Shear Locking: Example -2-

- Displacements of a cantilever beam





- Influence of the beam thickness on the normalized displacement

Thick beam

# elem.	1 point
1	0.762
2	0.940
4	0.985
8	0.996

Thin beam

# elem.	1 point
1	0.750
2	0.938
4	0.984
8	0.996

from TJR Hughes, The finite element method.

Reduced Integration Beam Elements

- If the displacements and rotations are interpolated with the same shape functions, there is tendency to lock (too stiff numerical behavior)

- Reduced integration is the most basic “engineering” approach to resolve this problem

			
Shape function order	Linear	Quadratic	Cubic
Quadrature rule	One-point	Two-point	Three-point

- Mathematically more rigorous approaches: Mixed variational principles based e.g. on the Hellinger-Reissner functional