Window 4.8
Forced Response of an Underdamped System from Section 3.2

The response of an underdamped system

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$$

(with zero initial conditions) is given by (for $0 < \zeta < 1$)

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t F(\tau)e^{\zeta\omega_n\tau} \sin\omega_d(t - \tau)\, d\tau$$

where $\omega_n = \sqrt{k/m}$, $\zeta = c/(2m\omega_n)$, and $\omega_d = \omega_n\sqrt{1 - \zeta^2}$. With nonzero initial conditions this becomes

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + \frac{1}{\omega_d}e^{-\zeta\omega_n t}\int_0^t f(\tau)e^{\zeta\omega_n\tau}\sin\omega_d(t - \tau)\, d\tau$$

where $f = F/m$ and A and ϕ are constants determined by the initial conditions.

where d_i and ϕ_i must be determined by the modal initial conditions and $\omega_{di} = \omega_i\sqrt{1 - \zeta_i^2}$ as before. Note that f_i may represent a sum of forces if more than one force is applied to the system. In addition, if a force is applied to only one mass of the system, this force becomes applied to each of the modal equations (4.131) by the transformation S, as illustrated in the following example.

Example 4.6.1

Consider the simple two-degree-of-freedom system with a harmonic force applied to one mass as indicated in Figure 4.16.

For this example, let $m_1 = 9$ kg, $m_2 = 1$ kg, $k_1 = 24$ N/m, and $k_2 = 3$ N/m. Also assume that the damping is proportional with $\alpha = 0$ and $\beta = 0.1$, so that $c_1 = 2.4$ N·s/m and $c_2 = 0.3$ N·s/m. Calculate the steady-state response.

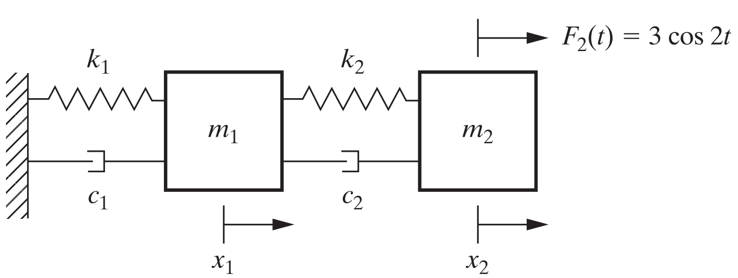

Figure 4.16 A damped two-degree-of-freedom system for Example 4.6.1.

Solution The equations of motion in matrix form become

$$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 2.7 & -0.3 \\ -0.3 & 0.3 \end{bmatrix}\dot{\mathbf{x}} + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ F_2(t) \end{bmatrix}$$

The matrices $M^{1/2}$ and $M^{-1/2}$ become

$$M^{1/2} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \qquad M^{-1/2} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$$

so that

$$\widetilde{C} = M^{-1/2}CM^{-1/2} = \begin{bmatrix} 0.3 & -0.1 \\ -0.1 & 0.3 \end{bmatrix} \quad \text{and} \quad \widetilde{K} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

The eigenvalue problem for $\widetilde{K}$ yields

$$\lambda_1 = 2 \quad \lambda_2 = 4 \quad P = 0.7071\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Hence the natural frequencies of the system are $\omega_1 = \sqrt{2}$ and $\omega_2 = 2$; the matrices $P^T\widetilde{C}P$ and $P^T\widetilde{K}P$ become

$$P^T\widetilde{C}P = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix} \quad \text{and} \quad P^T\widetilde{K}P = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

The vector $\mathbf{f}(t) = P^TM^{-1/2}B\mathbf{F}(t)$ becomes

$$\mathbf{f}(t) = \begin{bmatrix} 0.2357 & 0.7071 \\ -0.2357 & 0.7071 \end{bmatrix}\begin{bmatrix} 0 \\ F_2(t) \end{bmatrix} = 0.7071\begin{bmatrix} F_2(t) \\ F_2(t) \end{bmatrix}$$

Hence the decoupled modal equations become

$$\ddot{r}_1 + 0.2\dot{r}_1 + 2r_1 = 0.7071(3)\cos 2t = 2.1213\cos 2t$$
$$\ddot{r}_2 + 0.4\dot{r}_2 + 4r_2 = 0.7071(3)\cos 2t = 2.1213\cos 2t$$

Comparing the coefficient of $\dot{r}_i$ in each case to $2\zeta_i\omega_i$ yields

$$\zeta_1 = \frac{0.2}{2\sqrt{2}} = 0.0707$$
$$\zeta_2 = \frac{0.4}{2(2)} = 0.1000$$

Thus the damped natural frequencies become

$$\omega_{d1} = \omega_1\sqrt{1-\zeta_1^2} = 1.4106 \approx 1.41$$
$$\omega_{d2} = \omega_2\sqrt{1-\zeta_2^2} = 1.9899 \approx 1.99$$

Note that while the force F_2 is applied only to mass m_2, it becomes applied to both coordinates when transformed to modal coordinates. The modal equations for r_1 and r_2 can be solved by equation (4.131), or in this case of a simple harmonic excitation, the particular solution is given directly by equation (2.36) as

$$r_{1p}(t) = \frac{2.1213}{\sqrt{(2-4)^2 + [2(0.0707)\sqrt{2}(2)]^2}} \cos\left(2t - \tan^{-1}\frac{2(0.0707)\sqrt{2}(2)}{\sqrt{2}^2 - 2^2}\right)$$
$$= (1.040)\cos(2t + 0.1974) = 1.040\cos(2t - 2.9449)$$

Note that the argument of the arctangent function is negative $\left(\sqrt{2}^2 - 2^2 < 0\right)$ so that the fourth quadrant angle must be used (see Window 2.4), yielding 2.9449 radians. The second mode particular solution is

$$r_{2p}(t) = \frac{2.1213}{\sqrt{(4-4)^2 + (2(0.1)(2)(2))^2}} \cos\left(2t - \tan^{-1}\frac{2(0.1)(2)(2)}{2^2 - 2^2}\right)$$
$$= 2.6516\cos\left(2t - \frac{\pi}{2}\right) = 2.6516\sin 2t$$

Here r_{ip} is used to denote the particular solution of the ith modal equation. Note that $r_2(t)$ is excited at its resonance frequency but has high damping, so that the larger but finite amplitude for $r_{2p}(t)$ is not unexpected. If the transient response is ignored [it dies out per equation (2.30)], the preceding solution yields the steady-state response. The solution in the physical coordinate system is

$$\mathbf{x}_{ss}(t) = M^{-1/2}P\mathbf{r}(t) = \begin{bmatrix} 0.2357 & -0.2357 \\ 0.7071 & 0.7071 \end{bmatrix}\begin{bmatrix} 1.040\cos(2t - 2.9442) \\ 2.6516\sin 2t \end{bmatrix}$$

so that in the steady state

$$x_1(t) = 0.2451\cos(2t - 2.9442) - 0.6249\sin 2t$$
$$x_2(t) = 0.7354\cos(2t - 2.9442) + 8749\sin 2t$$

Note that even though there is a fair amount of damping in the resonant mode, the coordinates each have a large component vibrating near the resonant frequency.

□

Resonance

The concept of resonance in multiple-degree-of-freedom systems is similar to that introduced in Section 2.2 for single-degree-of-freedom systems. It is based on the idea that a harmonic driving force is exciting the system at its natural frequency, causing an unbounded oscillation in the undamped case and a response with a maximum amplitude in the damped case. However, in multiple-degree-of-freedom systems, there are n natural frequencies, and the concept of resonance is complicated by the effects of mode shapes. Basically, if a force is applied orthogonally to