دانشگاه صنعتی اصفهان
دانشکده مهندسی حمل و نقل

پژوهش عملیاتی

مدلهای بهینه سازی شبکه

Network Optimization Models

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Some applications of Networks:

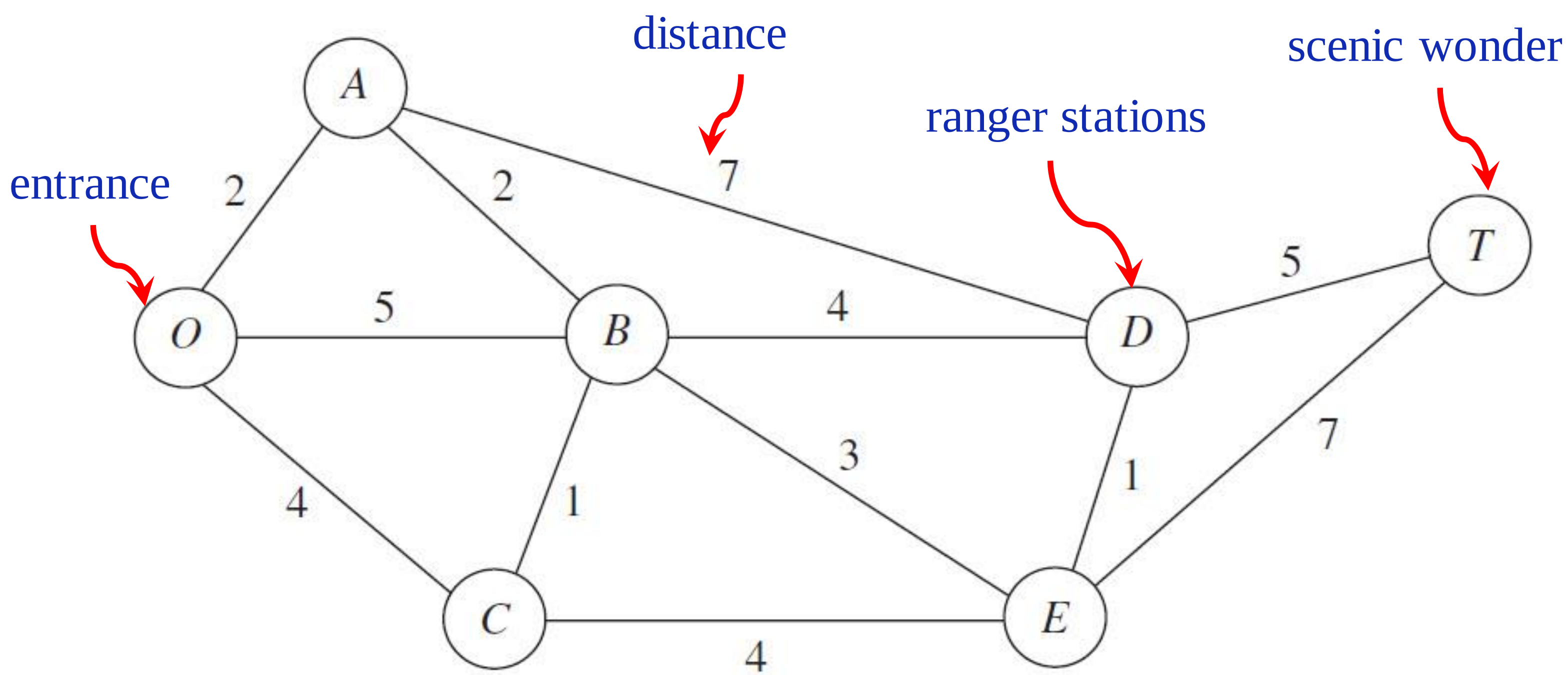
- Transportation Networks
- Electrical Networks
- Communication Networks
- Production
- Distribution
- Project Planning
- Facilities Location
- Resource Management
- Financial Planning
- ...

- shortest-path problem
- minimum spanning tree problem
- maximum flow problem
- minimum cost flow problem

Linear Programming

Example: SEERVADA PARK

A small number of trams transport sightseers from O to T and back.



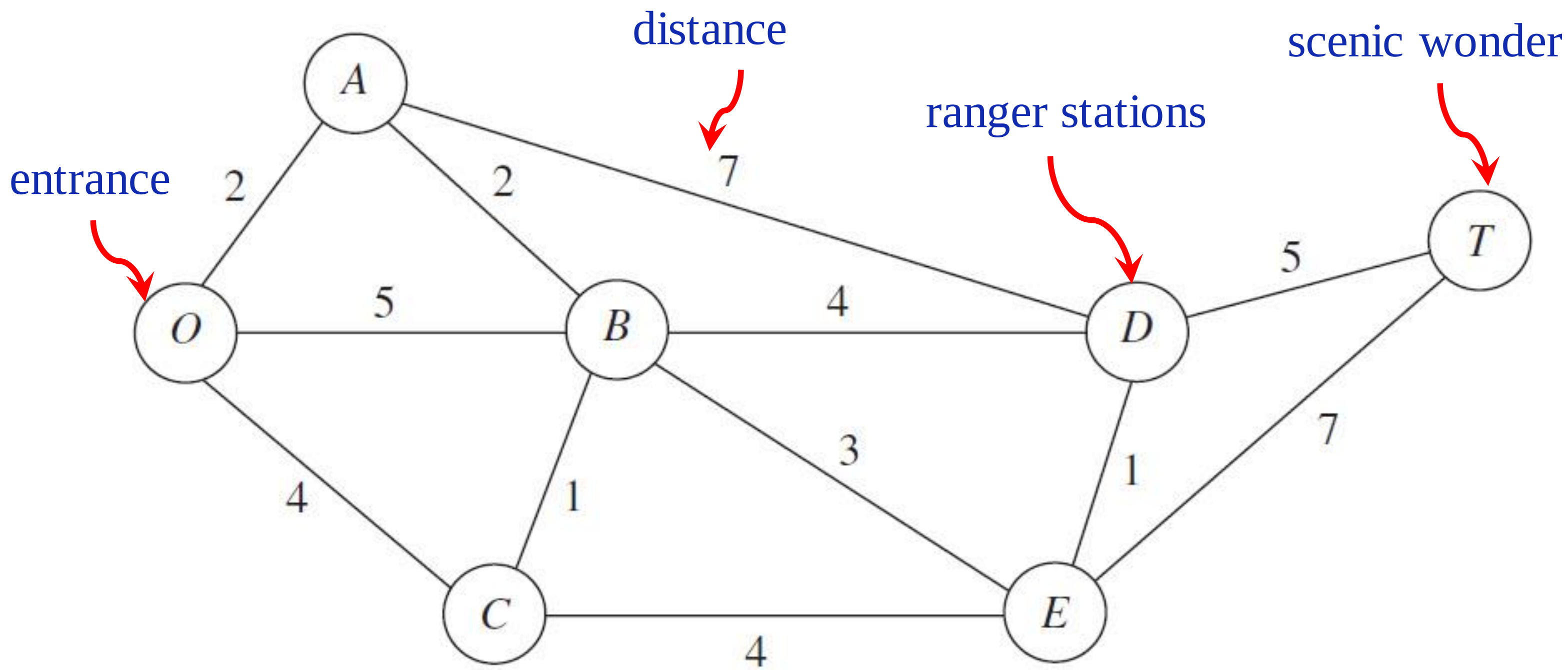
1

Which route from the park entrance to station T has the smallest total distance ?

shortest-path problem

Example: SEERVADA PARK

A small number of trams transport sightseers from O to T and back.



2

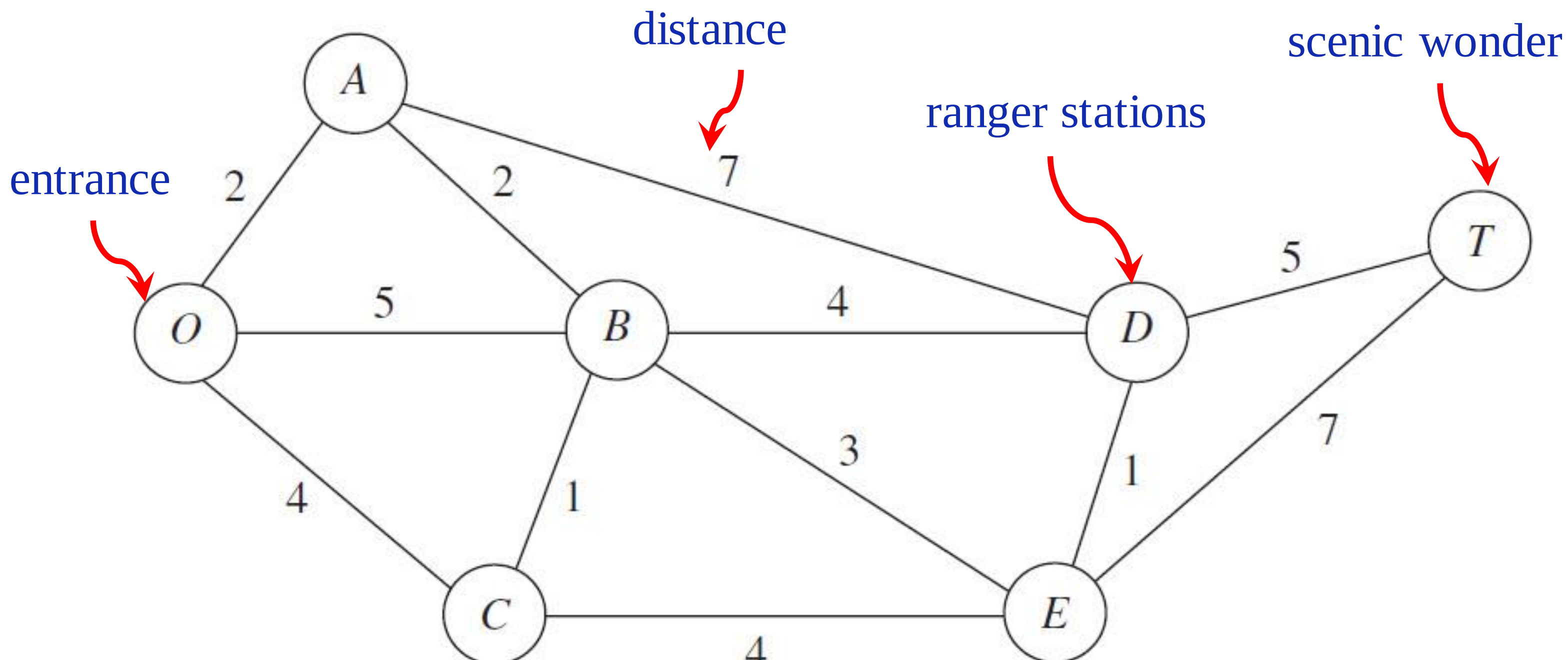
telephone lines must be installed under roads among all stations

where the lines should be laid with a minimum total number of miles of line installed ?

minimum spanning Tree problem

Example: SEERVADA PARK

A small number of trams transport sightseers from O to T and back.



3

during peak season, we want to use capacity of roads regardless of distance

how to route various trips to maximize number of trips that can be made ?

Maximum Flow Problem

Network:

a set of points and a set of lines connecting certain pairs of points.

Points: nodes or vertices

Lines: arcs or links or edges or branches

Directed arc:

flow is allowed in only 1 direction $A \rightarrow B$

Undirected arc:

flow is allowed in either direction $A \leftrightarrow B$

Directed network:

network that has only directed arcs

Undirected network:

network with mixture of directed and undirected arcs

Path (between two nodes):

A sequence of distinct arcs connecting these nodes

Directed path (from node i to node j):

A sequence of connecting arcs whose direction is toward node j, so that flow from node i to node j along this path is feasible.

Undirected path (from node i to node j):

A sequence of connecting arcs whose direction can be either toward or away from node j.

A directed path satisfies definition of undirected path, but not vice versa.

Cycle:

A path that begins and ends at the same node.

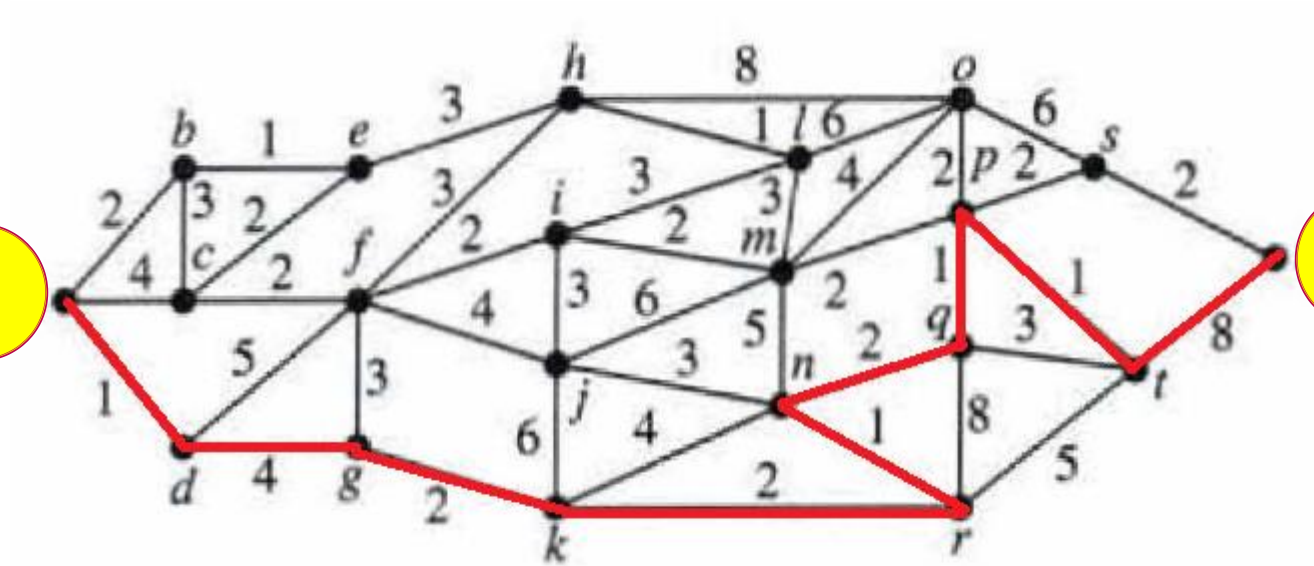
Shortest Path Problem

Aim

find a path between **two** nodes in a graph such that the sum of the costs is minimized.

Given: a directed graph (V, A) with Source node and Destination node

Source



Destination

~~Capacity~~

Variables

for each arc:
We should decide

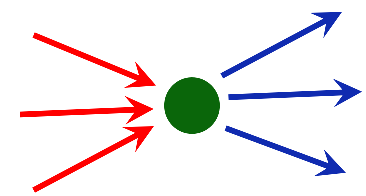
Is arc in the shortest path?

$$x_{ij} \begin{cases} 1 & \text{If arc (i,j) is part of the shortest path} \\ 0 & \text{Otherwise} \end{cases}$$

Objective Function

$$Minimize\,Cost = \sum_{(i,j)\in A} C_{ij} x_{ij}$$

Constraints



$$\sum_{k:(j,k)\in A} x_{jk} - \sum_{i:(i,j)\in A} x_{ij} = \begin{cases} 1 & j = source \\ 0 & j \neq source, destination \\ -1 & j = destination \end{cases}$$

$$x_{ij} \geq 0 \quad \forall (i,j) \in A$$

Shortest **paths** from a single source to all other nodes

Variables

x_{ij} Sum of "unit-flow"s passing from arc (i,j)

n-1 "unit-flow"s

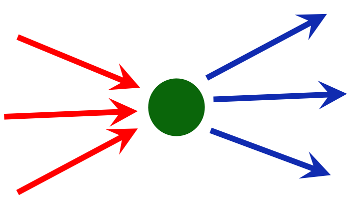
Number of nodes

Objective Function

$$Minimize\ Cost = \sum_{(i,j)\in A} C_{ij} x_{ij}$$

Constraints

$$\sum_{k:(j,k)\in A} x_{jk} - \sum_{i:(i,j)\in A} x_{ij} = \begin{cases} n-1 & j = source \\ -1 & j = All\ others \end{cases}$$



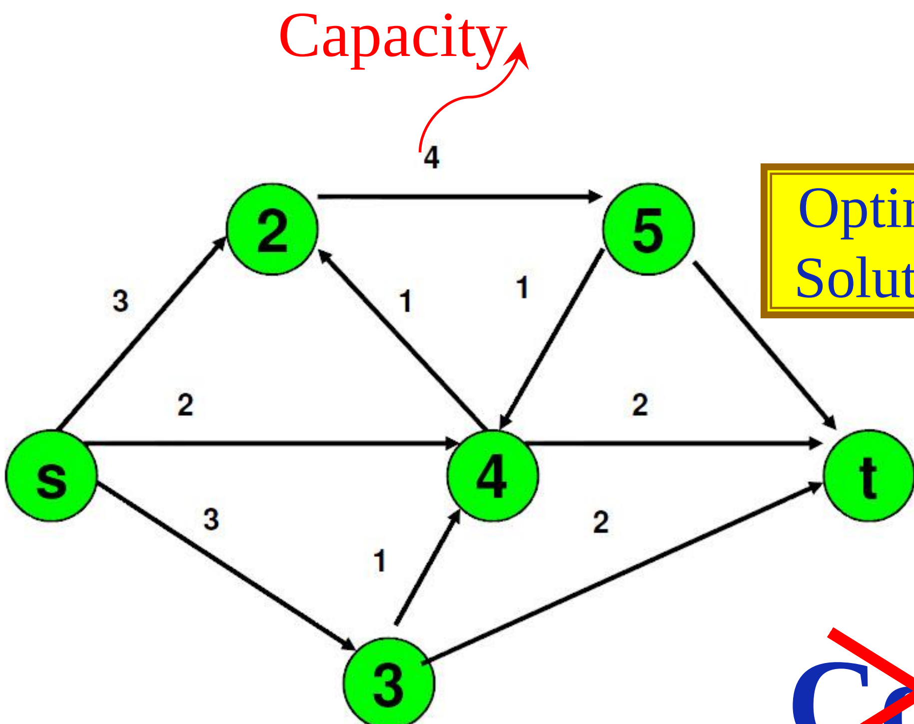
$$x_{ij} \geq 0 \quad \forall (i,j) \in A$$

Maximum Flow Problem

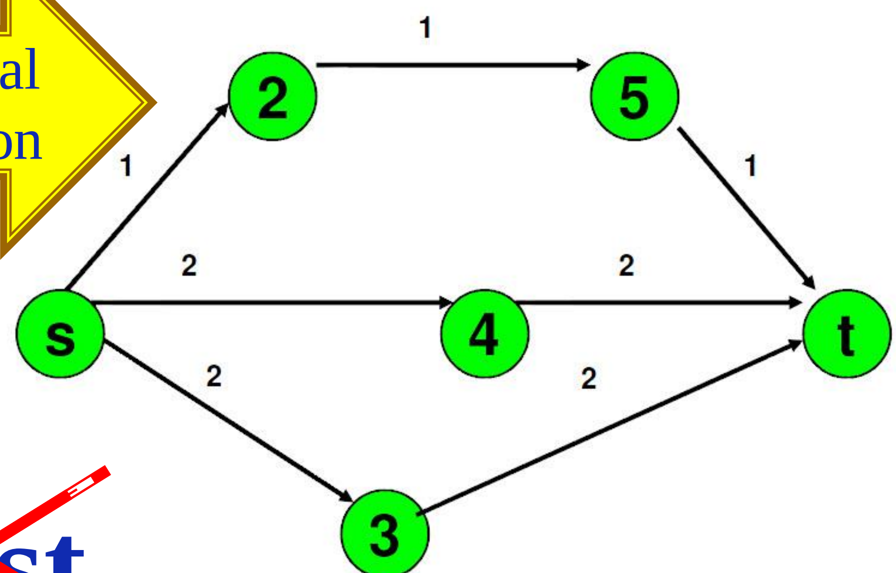
Aim

find a feasible flow through a single-source, single-sink network that is maximum

Given: a directed graph (V, A) with Source node s and Destination node t



Optimal Solution



~~Cost~~

Some applications:

- ❖ Maximize the flow through a company’s distribution network from its factories to its customers.
- ❖ Maximize the flow through a company’s supply network from its primary sellers to its factories.
- ❖ Maximize the flow of oil through a system of pipelines.
- ❖ Maximize the flow of water through a system of aqueducts.
- ❖ Maximize the flow of vehicles through a transportation network.

Variables

How much flow can be passed from source to destination?

x_{ij} **Flow passing arc (i,j)**

v **Maximum Flow passing**

Objective Function

$$Maximize\ v$$

Constraints

$$\sum_{k:(j,k)\in A} x_{jk} - \sum_{i:(i,j)\in A} x_{ij} = \begin{cases} v & j = source \\ 0 & j \neq source, destination \\ -v & j = destination \end{cases}$$

Capacity of (i,j)

$$0 \leq x_{ij} \leq u_{ij} \quad \forall (i,j) \in A$$

maximum flow problem vs. *shortest path problem*

– Similarities:

They are both pervasive in practice

They both arise as subproblems in algorithms for the ***minimum cost flow problem***.

– *Differences:*

Shortest path problems model arc costs but not arc capacities

Maximum flow problems model capacities but not costs.

Taken together, the shortest path problem and the maximum flow problem combine all the basic ingredients of network flows.

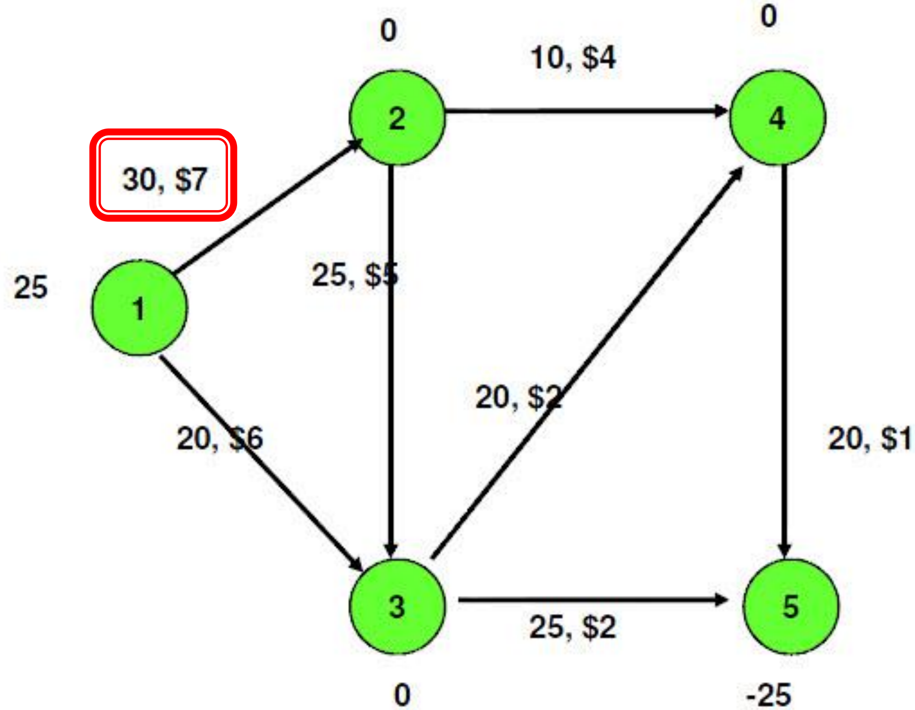
Minimum-Cost Flow Problem
(Single-Commodity Flow Problem)

Aim

find the cheapest possible way of sending a certain amount of flow through a flow network.

$G = (N, A)$: a directed network with a cost cij and a capacity uij associated with every arc $(i, j) \in A$.

b(i): node supply or demand depending on whether $b(i) > 0$ or $b(i) < 0$



The minimum cost flow problem is described below:

1. The network is a directed and connected network.
2. At least one of the nodes is a supply node.
3. At least one of the other nodes is a demand node.
4. All the remaining nodes are transshipment nodes.
5. Flow through an arc is <u>allowed only in the direction</u> indicated by the arrowhead, where the maximum amount of flow is given by the capacity of that arc. (If flow can occur in both directions, this would be represented by a pair of arcs pointing in opposite directions.)
6. The network has <u>enough arcs</u> with sufficient capacity to enable all the flow generated at the supply nodes to reach all the demand nodes.
7. The cost of the flow through each arc is <u>proportional to the amount</u> of that flow, where the cost per unit flow is known.
8. The objective is to minimize the total cost of sending the available supply through the network to satisfy the given demand.

Variables

x_{ij} **Flow passing arc (i,j)**

Objective Function

$$Minimize\,Cost = \sum_{(i,j)\in A} C_{ij} x_{ij}$$

Constraints

$$\sum_{k:(j,k)\in A} x_{jk} - \sum_{i:(i,j)\in A} x_{ij} = b(i) \quad for\,all\; j \in N$$

Supply or Demad of node i

$b_i > 0$ if node *i* is a supply node,
$b_i < 0$ if node *i* is a demand node,
$b_i = 0$ if node *i* is a transshipment node.

$$0 \le x_{ij} \le u_{ij} \quad \forall (i,j) \in A$$

Feasible solutions property:

A necessary condition to have any feasible solutions $\sum_{i=1}^{n} b_i = 0$

total flow being generated at supply nodes = total flow being absorbed at demand nodes.

If not:

either the supplies or the demands (whichever are in excess) actually represent <u>upper</u> bounds rather than exact amounts.

either a dummy destination was added to receive the excess supply

Or

a dummy source was added to send the excess demand.

Adding a dummy demand node (cij 0 arcs added from every supply node to this node)

Adding a dummy supply node (cij 0 arcs added from every demand node to this node)

Integer solutions property:

For minimum cost flow problems where every b_i and u_{ij} have integer values, all the basic variables in *every* basic feasible (BF) solution (including an optimal one) also have integer values.

The Transportation Problem. To formulate the transportation problem presented in Sec. 8.1 as a minimum cost flow problem, a supply node is provided for each source, as well as a demand node for each destination, but no transshipment nodes are included in the network. All the arcs are directed from a supply node to a demand node, where distributing x_{ij} units from source i to destination j corresponds to a flow of x_{ij} through arc i j. The cost c_{ij} per unit distributed becomes the cost c_{ij} per unit of flow. Since the transportation problem does <u>not impose upper bound</u> constraints on individual x_{ij}, all the u_{ij}.

The Assignment Problem. Since the assignment problem discussed in Sec. 8.3 is a special type of transportation problem, its formulation as a minimum cost flow problem fits into the same format. The additional factors are that

(1) the number of supply nodes equals the number of demand nodes,

(2) b_i =1 for each supply node, and

(3) b_i=1 for each demand node.

The Transshipment Problem. This special case actually includes all the general features of the minimum cost flow problem except <u>for not having (finite) arc capacities</u>. Thus, any minimum cost flow problem where each arc can carry any desired amount of flow is also called a transshipment problem.

For example, the Distribution Unlimited Co. problem shown in Fig. 9.13 would be a transshipment problem if the upper bounds on the flow through arcs A B and C E were removed.

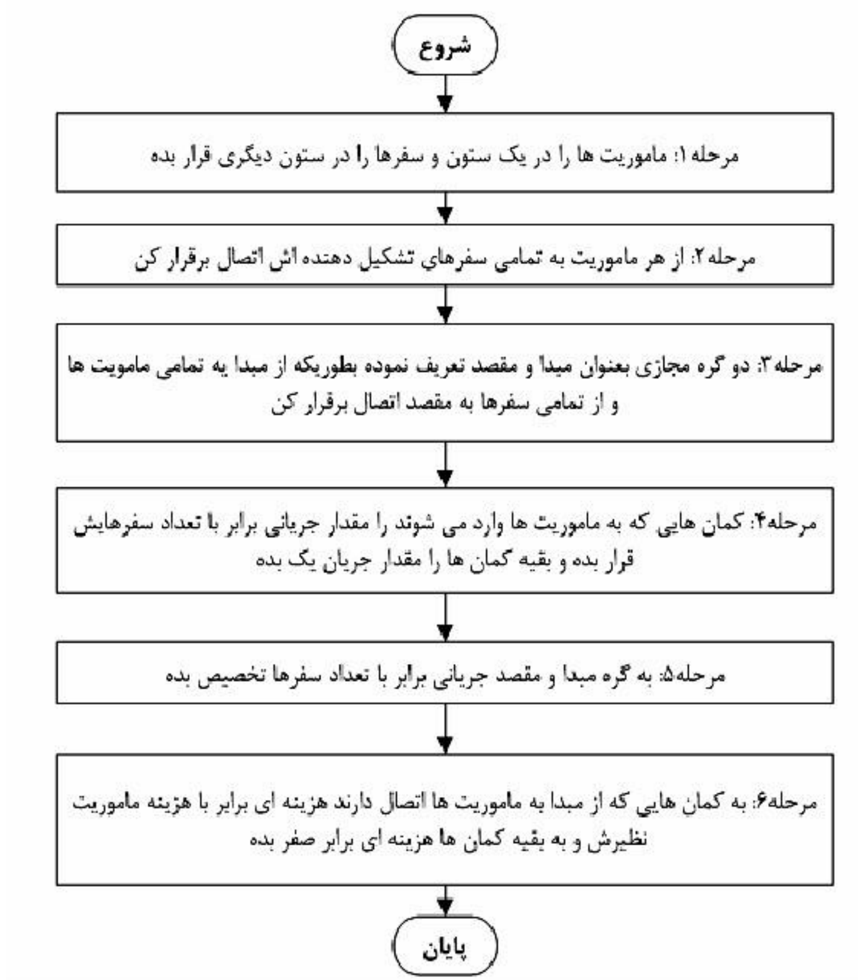
Transshipment problems frequently arise as generalizations of transportation problems where units being distributed from each source to each destination can first pass through intermediate points. These intermediate points may include other sources and destinations, as well as additional transfer points that would be represented by transshipment nodes in the network representation of the problem.

The Shortest-Path Problem. Now consider the main version of the shortest-path problem presented in Sec. 9.3 (finding the shortest path from one origin to one destination through an undirected network). To formulate this problem as a minimum cost flow problem, one supply node with a supply of 1 is provided for the origin, one demand node with a demand of 1 is provided for the destination, and the rest of the nodes are transshipment nodes. Because the network of our shortest-path problem is undirected, whereas the minimum cost flow problem is assumed to have a directed network, we replace each link with a pair of directed arcs in opposite directions (depicted by a single line with arrowheads at both ends). The only exceptions are that there is no need to bother with arcs into the supply node or out of the demand node. The distance between nodes i and j becomes the unit cost c_{ij} or c_{ji} for flow in either direction between these nodes. As with the preceding special cases, no arc capacities are imposed, so all u_{ij} .

The Maximum Flow Problem. The last special case we shall consider is the maximum
flow problem described in Sec. 9.5. In this case a network already is provided with one supply node (the source), one demand node (the sink), and various transshipment nodes, as well as the various arcs and arc capacities. Only three adjustments are needed to fit this problem into the format for the minimum cost flow problem. First, set cij 0
for all existing arcs to reflect the absence of costs in the maximum flow problem. Second, select a quantity F, which is a safe upper bound on the maximum feasible flow through the network, and then assign a supply and a demand of F to the supply node and the demand node, respectively. (Because all other nodes are transshipment nodes, they automatically
have bi 0.) Third, add an arc going directly from the supply node to the demand node and assign it an arbitrarily large unit cost of cij M as well as an unlimited arc capacity (uij). Because of this positive unit cost for this arc and the zero unit cost for all the other arcs, the minimum cost flow problem will send the maximum feasible flow through the other arcs, which achieves the objective of the maximum flow problem.

Example:

تعیین ماموریت های بهینه با استفاده از الگوریتم مینیمم هزینه- جریان برای مسأله برنامه ریزی خدمه



شکل ۳. الگوریتم تبدیل مسأله تولید ماموریت های بهینه به گراف

Example:

تعیین ماموریت های بهینه با استفاده از الگوریتم مینیمم هزینه- جریان برای مسأله برنامه ریزی خدمه

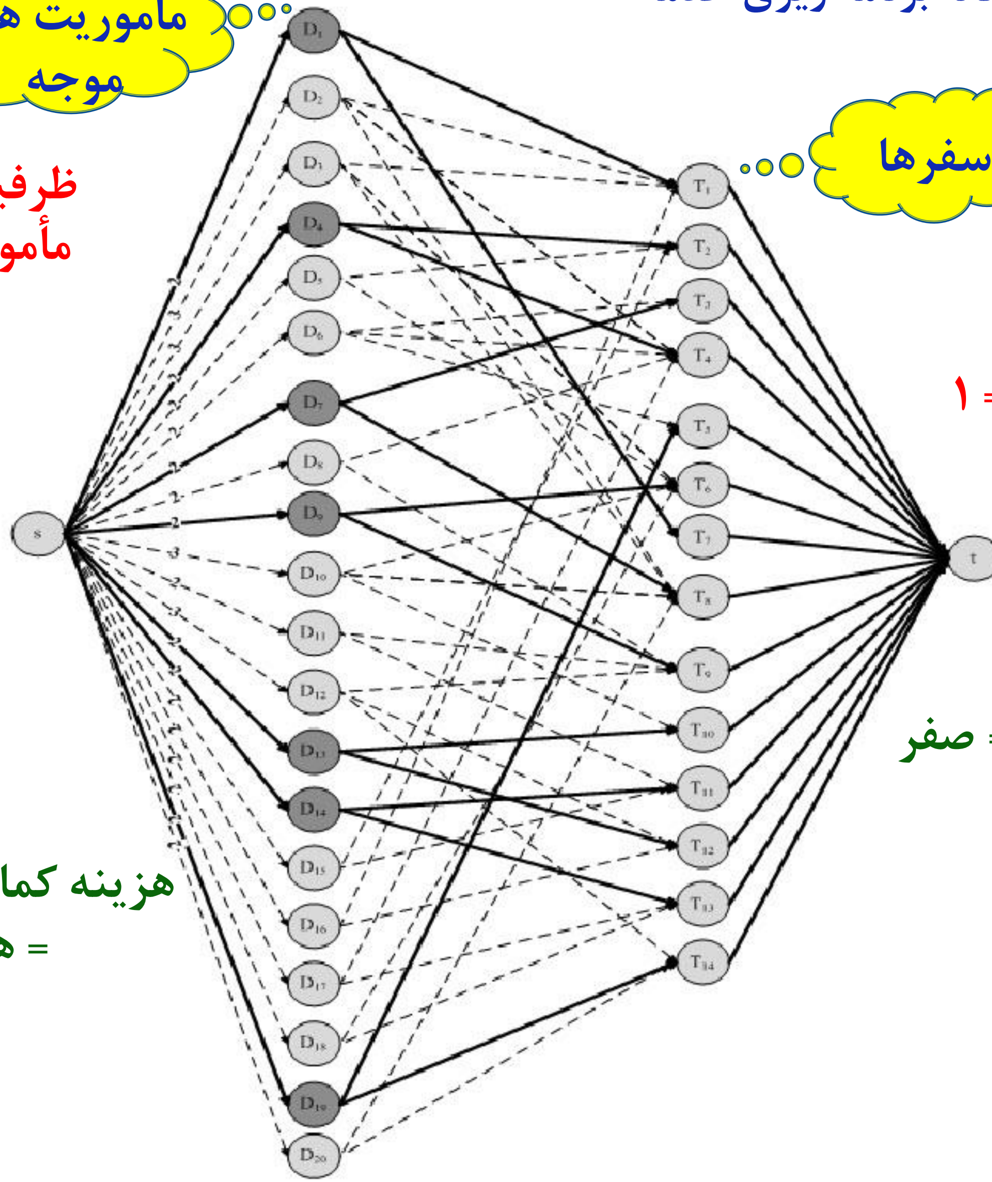
مأموریت های موجه

سفرها

ظرفیت کمانها از مبدأ به مأموریتها = تعداد سفر مأموریتها

ظرفیت سایر کمانها = ۱

تعداد کل سفرها



هزینه سایر کمانها = صفر

هزینه کمانها از مبدأ به مأموریتها = هزینه مأموریتها

Example:

9.6-3. A company will be producing the same new product at two different factories, and then the product must be shipped to two warehouses. Factory 1 can send an unlimited amount by rail to warehouse 1 only, whereas factory 2 can send an unlimited amount by rail to warehouse 2 only. However, independent truckers can be used to ship up to 50 units from each factory to a distribution center, from which up to 50 units can be shipped to each warehouse. The shipping cost per unit for each alternative is shown in the following table, along with the amounts to be produced at the factories and the amounts needed at the warehouses.

From \ To	Unit Shipping Cost: Distribution Center	Unit Shipping Cost: Warehouse 1	Unit Shipping Cost: Warehouse 2	Output
Factory 1	3	7	—	80
Factory 2	4	—	9	70
Distribution center		2	4	
Allocation		60	90	

(a) Formulate the network representation of this problem as a minimum cost flow problem.
(b) Formulate the linear programming model for this problem.

Problem 9.6-5

F. Hillier, G. J. Lieberman, “Introduction to Operations Research”, Ninth Edition, 2010.