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حل و بارم مسایل امتحان پایان ترم

مکانیک کوانتمی ۲ ۹۶/۴/۱۴

$$H=\frac{p_1^2}{2m}+\frac{p_2^2}{2m}+\frac{1}{4}m\omega^2(\vec{r}_1-\vec{r}_2)^2 \qquad \boxed{8} \quad -1$$

$$=\frac{P^2}{2M}+\frac{p^2}{2\mu}+\frac{1}{4}m\omega^2 r^2$$

$$\psi(r_1,r_2)=\Psi(R)\,\psi(r)=\frac{1}{(2\pi)^{3/2}}e^{\frac{i}{\hbar}\vec{P}_0\cdot\vec{R}}\,\psi(r)$$

$$\left(-\frac{\hbar^2}{2\mu}\nabla^2+\frac{1}{4}m\omega^2r^2\right)\psi(\vec{r})=E\,\psi(\vec{r})$$

$$\mu=\frac{m}{2}\ ,\quad \psi(\vec{r})=u(r)\,Y_{lm}(\theta,\varphi)$$

$$-\frac{\hbar^2}{m}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}\right)u(r)+\frac{1}{4}m\omega^2r^2u(r)=E\,u(r) \quad (۳)$$

$$\psi(R,r)=\frac{1}{(2\pi)^{3/2}}e^{\frac{i}{\hbar}\vec{P}_0\cdot\vec{R}}\,Y_{lm}(\theta,\varphi)\,u(r) \quad (۳)$$

در حالت پایه که $l=0$ است تابع موج فضایی متقارن و اسپین یگانه (پادمتقارن) است ۱

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به ازای l های زوج در حالتهای برانگیخته حالت اسپین یگانه

به ازای l های فرد " " سه گانه است

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$$E_F = \frac{\hbar^2\pi^2}{2m}\left(\frac{3n}{\pi}\right)^{2/3} = mc^2$$

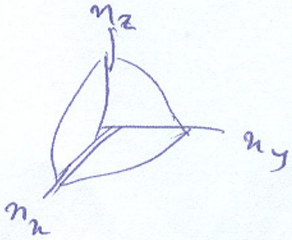
$$\Rightarrow \frac{\pi^2}{2}\left(\frac{3n}{\pi}\right)^{2/3} = \left(\frac{mc}{\hbar}\right)^2 = \frac{1}{\lambda_c^2}$$ ۳

$$\Rightarrow \left(\frac{3n}{\pi}\right)^{1/3} = \frac{\sqrt{2}}{\pi}\frac{1}{\lambda_c} \Rightarrow n = \frac{\pi}{3}\left(\frac{\sqrt{2}}{\pi}\right)^3 \frac{1}{\lambda_c^3}$$

$$n = \frac{2\sqrt{2}}{3\pi^2}\frac{1}{\lambda_c^3}$$ یعنی تقریباً در مکعبی به اضلاع λ_c یک ذره موجود است

ب-

$$E = pc = \hbar c k = \hbar c\sqrt{\frac{\pi^2}{L^2}(n_x^2+n_y^2+n_z^2)}$$



$$R^2 = n_x^2+n_y^2+n_z^2 = \frac{L^2E^2}{\hbar^2c^2\pi^2} \Rightarrow |R| = \frac{LE}{\hbar c\pi}$$

$$N = 2\times\frac{1}{8}\frac{4\pi}{3}R_F^3 = \frac{\pi}{3}\frac{L^3E_F^3}{\hbar^3c^3\pi^3} \quad *$$

$$n = \frac{N}{L^3} = \frac{E_F^3}{3\pi^2\hbar^3c^3} \Rightarrow \boxed{E_F = (3\pi^2 n)^{1/3}\hbar c}$$ ۴

ج-

$$E_{tot} = 2\times\frac{1}{8}\int E\, d^3n = \frac{1}{4}\int_0^{R_F}\left(\frac{\hbar c\pi}{L}\right)R\, d^3R$$

$$= \frac{\hbar c\pi}{4L}\times 4\pi\int_0^{R_F} R^3\, dR = \frac{\hbar c\pi^2}{4L}R_F^4$$ ۳

(از *) $\rightarrow N = \frac{\pi}{3}R_F^3 \Rightarrow E_{tot} = \frac{\hbar c\pi^2}{4L}\left(\frac{3N}{\pi}\right)^{4/3}$

$$\boxed{E_{tot} = \frac{\hbar c\pi^{2/3}}{4}(3N)^{4/3}V^{-1/3}}$$

د-

$$P_{deg} = -\frac{\partial E_{tot}}{\partial V} = \frac{1}{3}\frac{\hbar c\pi^{2/3}}{4}(3N)^{4/3}V^{4/3}$$

$$= \frac{3^{1/3}\hbar c\pi^{2/3}}{4}(n)^{4/3}$$ ۲

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$$M = \frac{Zeg_N}{2M_N} I$$

$$B = \frac{\mu_0}{4\pi}\left(\frac{3\vec{r}(\vec{r}.M) - r^2 M}{r^5} + \frac{8\pi}{3} M \delta^3(\vec{r})\right)$$

$$= \frac{\mu_0}{4\pi}\frac{Zeg_N}{2M_N}\left(\frac{3\vec{r}(\vec{r}.I) - r^2 I}{r^5} + \frac{8\pi}{3} I \delta^3(r)\right)$$

$$H_{hf} = \frac{e}{2m_e}(L+2S).B$$

$$= \frac{e}{2m_e}\frac{1}{4\pi\epsilon_0 C^2}\frac{Zeg_N}{2M_N}\left(\qquad\right).(\underbrace{L}_{=0} + 2S) \qquad l=0$$

$$= \frac{Ze^2}{4\pi\epsilon_0}\frac{g_N}{2M_N m_e c^2}\left(\frac{3(\vec{r}.\vec{S})(\vec{r}.\vec{I}) - r^2 I.S}{r^5} + \frac{8\pi}{3}\vec{I}.\vec{S}\delta^3(\vec{r})\right)$$

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$$\Delta E_{hf} = \int d^3\vec{r}\, \psi^*_{n00}(r) H_{hf} \psi_{n00}(r)$$

$$\int d\Omega (r.S)(r.I) = \int d\Omega (xS_x + yS_y + zS_z)(xI_x + yI_y + zI_z)$$

$$= \int d\Omega (x^2 S_x I_x + y^2 S_y I_y + z^2 S_z I_z) = \frac{1}{3} r^2 (S.I)\int d\Omega = \frac{4\pi}{3} r^2 (S.I)$$

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انتگرال کل اول صفر است ,

$$E_{hf} = \frac{Ze^2}{4\pi\epsilon_0}\frac{g_N}{2M_N m_e C^2}\frac{8\pi}{3}\vec{I}.\vec{S}\int d^3\vec{r}\, |\psi_{n00}(\vec{r})|^2 \delta^3(\vec{r})$$

$$= \frac{Ze^2}{4\pi\epsilon_0}\frac{g_N}{2M_N m_e C^2}\frac{8\pi}{3}|\psi_{n00}(\vec{0})|^2 I.S$$

$$= (Z\hbar c\alpha)\frac{g_N}{2M_N m_e C^2}\frac{8\pi}{3}\times\frac{4}{4\pi}\left(\frac{Z\alpha m_e c}{\hbar n}\right)^3 I.S$$

$$= \frac{4}{3}\alpha^4 m_e C^2 \left(\frac{m_e}{M_N}\right)\frac{1}{n^3}\frac{I.S}{\hbar^2}$$

$$\vec{F} = \vec{I} + \vec{S} \Rightarrow$$

$$\frac{S.I}{\hbar^2} = \frac{1}{2}\left[F(F+1) - S(S+1) - I(I+1)\right]$$

$$\Delta E_{hf} = \frac{4}{3}\alpha^4 m_e C^2 \left(\frac{m_e}{M_p}\right)$$

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$$S = \frac{1}{2}, I = \frac{1}{2} \Rightarrow \frac{S.I}{\hbar^2} = \begin{cases} \frac{1}{4} & F=1 \\ -\frac{3}{4} & F=0 \end{cases}$$

برای اتم هیدروژن , $n=1$

$$H=\frac{p^2}{2m}+\lambda x^4$$

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$$\psi(x)=N\,e^{-\frac{\alpha x^2}{2}}\quad,\quad 1=\int|\psi|^2dx=N^2\int e^{-\alpha x^2}dx=N^2\sqrt{\frac{\pi}{\alpha}}$$

$$\Rightarrow N=\left(\frac{\alpha}{\pi}\right)^{1/4}\quad\Rightarrow\quad\psi(x)=\left(\frac{\alpha}{\pi}\right)^{1/4}e^{-\frac{\alpha x^2}{2}}$$

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$$\langle H\rangle=\left\langle\frac{p^2}{2m}\right\rangle+\lambda\langle x^4\rangle$$

$$\left\langle\frac{p^2}{2m}\right\rangle=-\frac{\hbar^2}{2m}\left(\frac{\alpha}{\pi}\right)^{1/2}\int e^{-\frac{\alpha x^2}{2}}\frac{d^2}{dx^2}e^{-\frac{\alpha x^2}{2}}\,dx$$

$$=-\frac{\hbar^2}{2m}\left(\frac{\alpha}{\pi}\right)^{1/2}\int(-\alpha+\alpha^2x^2)\,e^{-\alpha x^2}\,dx$$

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$$=-\frac{\hbar^2}{2m}\left[-\alpha+\left(\frac{\alpha}{\pi}\right)^{1/2}\alpha^2\left(-\frac{d}{d\alpha}\right)\sqrt{\frac{\pi}{\alpha}}\right]=-\frac{\hbar^2}{2m}\left[-\alpha+\frac{1}{2}\alpha\right]=\boxed{\frac{\hbar^2\alpha}{4m}}$$

$$\langle x^4\rangle=\left(\frac{\alpha}{\pi}\right)^{1/2}\int x^4e^{-\alpha x^2}\,dx=\left(\frac{\alpha}{\pi}\right)^{1/2}\left(-\frac{d}{d\alpha}\right)^2\sqrt{\frac{\pi}{\alpha}}$$

$$=\alpha^{1/2}\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\alpha^{-5/2}=\frac{3}{4}\alpha^{-2}$$

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$$\Rightarrow\ \langle H\rangle=\frac{\hbar^2\alpha}{4m}+\frac{3}{4}\alpha^{-2}\lambda$$

$$\frac{d}{d\alpha}\langle H\rangle=0\ \Rightarrow\ \frac{\hbar^2}{m}-6\lambda\alpha^{-3}=0\ \Rightarrow\ \alpha=\left(\frac{\hbar^2}{6m\lambda}\right)^{-1/3}$$

$$\langle H\rangle_{min}=\frac{\hbar^2}{4m}\left(\frac{\hbar^2}{6m\lambda}\right)^{-\frac{1}{3}}+\frac{3}{4}\lambda\left(\frac{\hbar^2}{6m\lambda}\right)^{2/3}$$

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$$=\frac{1}{4}\left[6\lambda^{1/3}\left(\frac{\hbar^2}{6m}\right)^{\frac{2}{3}}+3\lambda^{1/3}\left(\frac{\hbar^2}{6m}\right)^{2/3}\right]$$

$$=\frac{9}{4}\times3^{-\frac{2}{3}}\lambda^{1/3}\left(\frac{\hbar^2}{2m}\right)^{2/3}=\frac{3^{4/3}}{4}\lambda^{\frac{1}{3}}\left(\frac{\hbar^2}{2m}\right)^{2/3}$$

$$\langle H\rangle_{min}\simeq1.082\,\lambda^{1/3}\left(\frac{\hbar^2}{2m}\right)^{2/3}$$