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Webs Related to K-Loops and Reflection Structures

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Abstract. We give a characterization of webs $(\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G})$ which are related to A_l-loops, weak K-loops, K-loops and reflection structures. We also obtain a geometric proof of KREUZER'S result that the concept of K-loop is equivalent to that of Bruck loop.

1 Introduction

By the works of G. BOL and W. BLASCHKE [1], K. REIDEMEISTER [18] and G. THOMSEN [19] we know that there is a correspondence between loops and webs (cf. Theorem 4.1). In the last years the so called K-loops gained particular interest (cf. [3, 4, 5, 7, 8, 11, 13, 14, 20, 21]). The notion of a K-loop $(E, +)$ is defined among the loops as follows.

For $a, b \in E$, let $a^+ \colon E \to E; x \mapsto a + x$, $\delta_{a,b} := \big((a+b)^+\big)^{-1} \circ a^+ \circ b^+$, let $-a \in E$ be defined by $a + (-a) = 0$ and let $\nu\colon E \to E; x \mapsto -x$ be the negative map. The loop $(E, +)$ is called an *A_l-loop* if for all $a, b \in E$ the permutation $\delta_{a,b}$ is an automorphism of the loop $(E, +)$, i.e. $\delta_{a,b} \in \mathrm{Aut}(E, +)$, a *weak K-loop* if moreover $\delta_{a,-a} = \mathrm{id}$ and a *K-loop* if furthermore $\nu \in \mathrm{Aut}(E, +)$ (automorphic inverse property) and $\delta_{a,b} = \delta_{a,b+a}$ for all $a, b \in E$.

Recently it has been proved in [13] by A. KREUZER that the concept of a K-loop is equivalent to that of a Bruck loop. A *Bruck loop* $(E, +)$ is a *Bol loop*, i.e. a loop satisfying the *Bol identity*

$$a^+ \circ b^+ \circ a^+ = \big(a + (b + a)\big)^+, \quad \forall a, b \in E,$$

which, moreover, satisfies the automorphic inverse property (cf. [13]).

K-loops are closely related to invariant reflection structures. A triple $(\mathcal{P}, {}^\circ; 0)$ consisting of a non-empty set $\mathcal{P}$, a fixed element $0 \in \mathcal{P}$ and a map ${}^\circ\colon \mathcal{P} \to J := \{\sigma \in \mathrm{Sym}\,\mathcal{P} \mid \sigma^2 = \mathrm{id}\}; x \mapsto x^\circ$ such that:

B1 $\forall a \in \mathcal{P}\colon\ a^\circ(0) = a$

is called a *reflection structure* and an *invariant reflection structure* if moreover

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B2 $\forall a, b \in \mathcal{P}:\ a^\circ \circ b^\circ \circ a^\circ = \big(a^\circ \circ b^\circ(a)\big)^\circ$

is satisfied. By [4] we have

(1.1) *Let* $(\mathcal{P}, +)$ *be a right loop, (i.e. for all* $a, b \in \mathcal{P}$ *the equation* $a + x = b$ *has a unique solution* $x \in \mathcal{P}$ *and there is a* $0 \in \mathcal{P}$ *with* $a + 0 = 0 + a = a$*.) and for* $a \in \mathcal{P}$ *let* $a^\circ := a^+ \circ \nu$*. Then:*

(i) *If* $(\mathcal{P}, +)$ *has the property*

$$\forall a, b \in \mathcal{P}:\ a - (a - b) = b, \tag{$*$}$$

then $(\mathcal{P}, ^\circ; 0)$ *is a reflection structure;*

(ii) *If* $(\mathcal{P}, +)$ *is a K-loop, then* $(\mathcal{P}, ^\circ; 0)$ *is an invariant reflection structure.*

By [7] the converse is also true

(1.2) *Let* $(\mathcal{P}, ^\circ; 0)$ *be a reflection structure and for* $a, b \in \mathcal{P}$ *let* $a^+ := a^\circ \circ 0^\circ$ *and* $a + b = a^+(b)$*. Then:*

(i) $(\mathcal{P}, +)$ *is a right loop with* $(*)$*;*

(ii) *If* $(\mathcal{P}, ^\circ; 0)$ *is invariant, then* $(\mathcal{P}, +)$ *is a K-loop.*

Remark. In [7] the following statements of Theorem 6.1 were proved completely (cf. [7], (6.1)(3) and (4)) :

(i) $0^\circ \circ \mathcal{P}^\circ \circ 0^\circ = \mathcal{P}^\circ \iff \nu \in \mathrm{Aut}(\mathcal{P}, +)$;

(ii) $(\mathcal{P}, ^\circ; 0)$ is invariant $\Rightarrow (\mathcal{P}, +)$ is a weak K-loop with $\nu \in \mathrm{Aut}(\mathcal{P}, +)$.

In order to show (ii) in **(1.2)** we have still to prove the property:

$$\forall a, b \in \mathcal{P}: \delta_{a,b} = \delta_{a,b+a}. \tag{1}$$

This can be done in the following way by modifying the proof of [8], (3.3):

Proof. Let $a, b \in \mathcal{P}$, $c := a+b = a^\circ \circ 0^\circ(b)$, $d := b+a$ and $e := a+(b+a) = a+d$. Then

$$c^\circ \circ a^\circ \circ 0^\circ \circ b^\circ(0) = c^\circ(c) = 0, \tag{2}$$

$$d = d^\circ(0) = b^\circ \circ 0^\circ \circ a^\circ(0) \tag{3}$$

and

$$e = e^\circ(0) = a^\circ \circ 0^\circ \circ d^\circ(0) \overset{(3)}{=} a^\circ \circ 0^\circ \circ b^\circ \circ 0^\circ \circ a^\circ(0).$$

By **B1** and **B2** this equation implies

$$e^\circ = a^\circ \circ 0^\circ \circ b^\circ \circ 0^\circ \circ a^\circ. \tag{4}$$

Again, since $b^\circ \circ 0^\circ \circ a^\circ \circ c^\circ \circ a^\circ \circ 0^\circ \circ b^\circ(0) \overset{(2)}{=} b^\circ \circ 0^\circ \circ a^\circ(0) \overset{(3)}{=} d$ we obtain by **B1** and **B2**:

$$d^\circ = b^\circ \circ 0^\circ \circ a^\circ \circ c^\circ \circ a^\circ \circ 0^\circ \circ b^\circ. \tag{5}$$

Now

$$\delta_{a,b+a} = (e^+)^{-1} \circ a^+ \circ d^+ = 0^\circ \circ e^\circ \circ a^\circ \circ 0^\circ \circ d^\circ \circ 0^\circ$$

$$\overset{(4),(5)}{=} 0^\circ \circ (a^\circ \circ 0^\circ \circ b^\circ \circ 0^\circ \circ a^\circ) \circ a^\circ \circ 0^\circ \circ (b^\circ \circ 0^\circ \circ a^\circ \circ c^\circ \circ a^\circ \circ 0^\circ \circ b^\circ) \circ 0^\circ$$

$$= 0^\circ \circ c^\circ \circ a^\circ \circ 0^\circ \circ b^\circ \circ 0^\circ = (c^+)^{-1} \circ a^+ \circ b^+ = \delta_{a,b}.$$

□

The purpose of this paper is to characterize the structure of webs corresponding to A_l-loops, weak K-loops, K-loops and reflection structures. Our main results are stated in (3.2), (3.3), (4.2), (5.1), (6.4): By the proofs of (1.3), (3.2), (3.3) and (4.2) we have a purely geometric proof of Kreuzer's result ([13]) that Bruck loops and K-loops are the same. The most important step in the proof is that the Bol identity and the automorphic inverse property imply that the loop is an A_l-loop. A geometric proof of this result is also contained in [2].

2 Basic concepts concerning nets and chain-nets related to K-loops

Let $\mathcal{P}$ be a non-empty set and let $\mathcal{G}_1$ and $\mathcal{G}_2$ be subsets of the power set of $\mathcal{P}$; the elements of $\mathcal{P}$, respectively of $\mathcal{G}_1$ and $\mathcal{G}_2$ will be called *points*, respectively *generators*. The triple $(\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2)$ is called a *net*, if for each $X \in \mathcal{G}_1 \cup \mathcal{G}_2$, $|X| \geq 2$ and if the following two conditions are valid:

N1 For each point $x \in \mathcal{P}$, for each $i \in \{1, 2\}$ there is exactly one generator $G \in \mathcal{G}_i$ with $x \in G$; such generator will be denoted by $[x]_i$.

N2 Any two generators X_1 and X_2 of distinct classes $\mathcal{G}_1$ and $\mathcal{G}_2$ intersect in exactly one point.

Let $J := \{\alpha \in \operatorname{Sym} \mathcal{P} \mid \alpha^2 = \mathrm{id}\}$ and $J^* := J \setminus \{\mathrm{id}\}$ (= set of all involutions). We denote by $\Gamma := \operatorname{Aut}(\mathcal{P}, \mathcal{G}_1 \cup \mathcal{G}_2)$ the group of all permutations χ of $\mathcal{P}$ with the property:

$$\forall X \in \mathcal{G}_1 \cup \mathcal{G}_2 \colon \; \chi(X) \in \mathcal{G}_1 \cup \mathcal{G}_2.$$

Clearly, for each $\chi \in \operatorname{Aut}(\mathcal{P}, \mathcal{G}_1 \cup \mathcal{G}_2)$ and for each $x \in \mathcal{P}$ we have either

(1) $\chi([x]_1) = [\chi(x)]_1$ and $\chi([x]_2) = [\chi(x)]_2$ or
(2) $\chi([x]_1) = [\chi(x)]_2$ and $\chi([x]_2) = [\chi(x)]_1$.

Let $\Gamma^+ := \operatorname{Aut}(\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2)$, respectively Γ^- be the set of all automorphisms of type (1), respectively (2). If $\Gamma^- \neq \emptyset$ then Γ^+ is a normal subgroup of Γ of index 2.

For the point set $\mathcal{P}$ of our net $(\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2)$ we introduce the following binary operation:

$$\Box\colon \; \mathcal{P} \times \mathcal{P} \to \mathcal{P}; \; (x, y) \mapsto x \Box y := [x]_1 \cap [y]_2$$

A subset $S \subset \mathcal{P}$ is called a *subnet* if $\forall x, y \in S : x \Box y \in S$.

(2.1) *If $\mathcal{N}$ denotes the set of all subnets, then $\mathcal{N}$ is $\cap$-closed and for the associated closure operation $X^\Box := \bigcap \{N \in \mathcal{N} \mid X \subseteq N\}$ for $X \subset \mathcal{P}$ we have:*

$$X^\Box = X \Box X := \{x \Box y \mid x, y \in X\}.$$

Proof. Let $x, y, x', y' \in X$, then $(x\Box y)\Box(x'\Box y') = x\Box y'$. $\Box$

(2.2) $\Gamma^+ = \mathrm{Aut}(\mathcal{P}, \Box)$ *and* Γ^- *is the set of all antiautomorphisms of* $(\mathcal{P}, \Box)$.

Proof. Let $x, y \in \mathcal{P}$, $\alpha \in \Gamma^+$ and $\beta \in \Gamma^-$, then $\alpha(x\Box y) = \alpha([x]_1 \cap [y]_2) = [\alpha(x)]_1 \cap [\alpha(y)]_2 = \alpha(x)\Box\alpha(y)$ and $\beta(x\Box y) = \beta([x]_1 \cap [y]_2) = [\beta(x)]_2 \cap [\beta(y)]_1 = \beta(y)\Box\beta(x)$. Now let $\alpha \in \mathrm{Aut}(\mathcal{P}, \Box)$ and let β be an antiautomorphism of $(\mathcal{P}, \Box)$. Then $[x]_1 = x\Box\mathcal{P}$, $[x]_2 = \mathcal{P}\Box x$ and so $\alpha([x]_1) = \alpha(x)\Box\alpha(\mathcal{P}) = \alpha(x)\Box\mathcal{P} = [\alpha(x)]_1$, $\alpha([x]_2) = \mathcal{P}\Box\alpha(x) = [\alpha(x)]_2$, $\beta([x]_1) = \mathcal{P}\Box\beta(x) = [\beta(x)]_2$, $\beta([x]_2) = \beta(x)\Box\mathcal{P} = [\beta(x)]_1$. $\Box$

A subset $C \subset \mathcal{P}$ is called a *chain* of the net $(\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2)$ if the following condition holds:

N3 $\forall X \in \mathcal{G}_1 \cup \mathcal{G}_2 \colon |X \cap C| = 1$;

Let $\mathfrak{C}$ be the set of all chains of $(\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2)$. If $\mathfrak{C} \neq \emptyset$ and $C \in \mathfrak{C}$, then $\forall X \in \mathcal{G}_1 \cup \mathcal{G}_2$:

$$|C| = |X| = |\mathcal{G}_1| = |\mathcal{G}_2| \quad \text{and} \quad |\mathcal{P}| = |\mathcal{G}_1|^2.$$

(2.3) *For each* $C \in \mathfrak{C}$ *let*

$$\widetilde{C}\colon \mathcal{P} \to \mathcal{P};\ x \mapsto \big[[x]_1 \cap C\big]_2 \cap \big[[x]_2 \cap C\big]_1$$

and let $\widetilde{\mathfrak{C}} := \{\widetilde{C} \mid C \in \mathfrak{C}\}$, *then we have:*

(1) $\widetilde{\mathfrak{C}} \subset \Gamma^-$ *and* $\widetilde{\mathfrak{C}}^2 \subset \Gamma^+$;

(2) $\widetilde{C} \circ \widetilde{C} = \mathrm{id}$ *and* $\mathrm{Fix}\,\widetilde{C} = C$, *i.e.,* $\sim\colon \mathfrak{C} \to \Gamma^-;\ X \mapsto \widetilde{X}$ *is an injection.*

(2.4) *Let* $\alpha \in \Gamma^-$. *If* $\alpha \in J^*$, *then* $\mathrm{Fix}\,\alpha \in \mathfrak{C}$; *if* $\mathrm{Fix}\,\alpha \in \mathfrak{C}$, *then* $\widetilde{\mathrm{Fix}\,\alpha} = \alpha$.

Proof. Let $X \in \mathcal{G}_1 \cup \mathcal{G}_2$ for instance $X \in \mathcal{G}_1$. Then $\alpha(X) \in \mathcal{G}_2$, since $\alpha \in \Gamma^-$ and therefore $c := X \cap \alpha(X)$ is a point. If $\alpha \in J^*$ then $\alpha(c) = c$ and c is the only fixed point of α contained in X. Hence $\mathrm{Fix}\,\alpha \in \mathfrak{C}$. Now let $C := \mathrm{Fix}\,\alpha \in \mathfrak{C}, ; x \in \mathcal{P}$ and $x_i := [x]_i \cap C$ $(i \in \{1, 2\})$. Then $x = x_1\Box x_2$, $\alpha(x_i) = x_i$ and since $\alpha \in \Gamma^-$, $\alpha(x) = \alpha(x_1\Box x_2) = \alpha(x_2)\Box\alpha(x_1) = x_2\Box x_1 = \widetilde{C}(x)$ by (2.2). $\Box$

(2.5) $\forall A, B, C \in \mathfrak{C}$ *we have:*

(1) $\widetilde{A}(B) \in \mathfrak{C}$;

(2) $\widetilde{\widetilde{A}(B)} = \widetilde{A} \circ \widetilde{B} \circ \widetilde{A}$;

(3) $\mathrm{Fix}(\widetilde{A} \circ \widetilde{B}) = (A \cap B)^{\Box}$;

(4) $\widetilde{A} \circ \widetilde{B} \circ \widetilde{C} \in J^* \Leftrightarrow \widetilde{A} \circ \widetilde{B} \circ \widetilde{C} \in \widetilde{\mathfrak{C}}$;

(5) $\widetilde{A}|_{\mathcal{G}_1} = \widetilde{B}|_{\mathcal{G}_1} \iff A = B$.

Proof. (1): If $X \in \mathcal{G}_1 \cup \mathcal{G}_2$, then $|\widetilde{A}(B) \cap X| = \big|\widetilde{A}\big(B \cap \widetilde{A}(X)\big)\big| = |B \cap \widetilde{A}(X)| = 1$ since $\widetilde{A} \circ \widetilde{A} = id$, thus $\widetilde{A}(B) \in \mathfrak{C}$.

(2): From $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{A} \circ \widetilde{B} \circ \widetilde{A} \in \Gamma^-$, $\mathrm{Fix}(\widetilde{A} \circ \widetilde{B} \circ \widetilde{A}) = \widetilde{A}\big(\mathrm{Fix}(\widetilde{B})\big) = \widetilde{A}(B)$, we obtain by (2.4) $\widetilde{A} \circ \widetilde{B} \circ \widetilde{A} = \widetilde{\widetilde{A}(B)}$.

(3): Let $\alpha := \widetilde{A} \circ \widetilde{B}$ and let $x, y \in A \cap B$, then $\alpha \in \Gamma^+$ by (2.3, (1)). $x, y \in \mathrm{Fix}\,\alpha$ by (2.3, (2)) and so by (2.2) $\alpha(x\Box y) = \alpha(x)\Box\alpha(y) = x\Box y$, i.e. by (2.1), $(A \cap B)^{\Box} = (A \cap B)\Box(A \cap B) \subseteq \mathrm{Fix}\,\alpha$. Now let $x \in \mathrm{Fix}\,\alpha$ then $\widetilde{A}(x) = \big[[x]_1 \cap A\big]_2 \cap \big[[x]_2 \cap$

$A]_1 = \widetilde{B}(x) = \big[[x]_1 \cap B\big]_2 \cap \big[[x]_2 \cap B\big]_1$. Therefore $\big[[x]_1 \cap A\big]_2 = \big[[x]_1 \cap B\big]_2$ and $\big[[x]_2 \cap A\big]_1 = \big[[x]_2 \cap B\big]_1$. This implies $a := [x]_1 \cap [[x]_1 \cap A]_2 = [x]_1 \cap A = [x]_1 \cap B \in A \cap B$ and $b := [x]_2 \cap A = [x]_2 \cap B \in A \cap B$, and so $x = a\Box b$, i.e. $\operatorname{Fix}\alpha \subseteq (A \cap B)^{\Box}$.
(4): "$\Rightarrow$" Let $\alpha := \widetilde{A} \circ \widetilde{B} \circ \widetilde{C} \in J^*$. By (2.3, (1)) $\alpha \in \Gamma^-$ and so by (2.4) $\alpha = \widetilde{\operatorname{Fix}\alpha} \in \widetilde{\mathcal{C}}$. □

Two chains $A, B \in \mathcal{C}$ are called *orthogonal* and denoted by $A \perp B$, if $A \neq B$ and $\widetilde{A}(B) = B$. This relation is symmetric since $\widetilde{A}(B) = B$ implies by (2.5, (2)) $\widetilde{B} = \widetilde{\widetilde{A}(B)} = \widetilde{A} \circ \widetilde{B} \circ \widetilde{A}$, hence $\widetilde{A} = \widetilde{B} \circ \widetilde{A} \circ \widetilde{B} = \widetilde{\widetilde{B}(A)}$ and so $A = \widetilde{B}(A)$ by (2.3, (2)).

Let $A^{\perp} := \{X \in \mathcal{C} \mid X \perp A\}$. Now we are going to consider subsets $\mathcal{S}$ of the set $\mathcal{C}$ of chains satisfying certain conditions. $\mathcal{S} \subset \mathcal{C}$ is called *transitive*, respectively *regular* if

T $\forall A, B \in \mathcal{S}: \exists C \in \mathcal{S}: \widetilde{C}(A) = B$, respectively

$\overline{\mathbf{T}}$ $\forall A, B \in \mathcal{S}: \exists_1 C \in \mathcal{S}: \widetilde{C}(A) = B$,

symmetric if

S $\forall A, B \in \mathcal{S}: \widetilde{A}(B) \in \mathcal{S}$.

The quadruple $(\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2, \mathcal{S})$ is called a *web* if $\mathcal{S}$ satisfies the following condition

N1′ $\forall x \in \mathcal{P} \exists_1 [x]_3 \in \mathcal{S}$ with $x \in [x]_3$.

Theorem 2.6. *Let $\mathcal{L} \subset \mathcal{C}$ be a symmetric and regular set of chains and let $O \in \mathcal{L}$ be fixed. For each $A \in \mathcal{L}$ let $A' \in \mathcal{L}$ such that $\widetilde{A'}(O) = A$ (cf. $\overline{\mathbf{T}}$) and for all $A, B \in \mathcal{L}$ let $A \oplus B := \widetilde{A'} \circ \widetilde{O}(B)$. Then $(\mathcal{L}, \oplus)$ is a K-loop.*

Proof. By (2.3, (2)) and (2.5, (1)) for each $A \in \mathcal{L}$, $\widetilde{A}$ induces an involutory permutation on the set $\mathcal{C}$, and since $\mathcal{L}$ is symmetric, we have $\widetilde{A}(\mathcal{L}) = \mathcal{L}$. Therefore we can consider $\widetilde{\mathcal{L}} := \{\widetilde{L}|_{\mathcal{L}} : L \in \mathcal{L}\}$ as a subset of $J^*_{\mathcal{L}} := \{\sigma \in \operatorname{Sym}\mathcal{L} \mid \sigma^2 = \mathrm{id} \neq \sigma\}$. Since $\mathcal{L}$ is regular, the map $^\circ : \mathcal{L} \to J^*_{\mathcal{L}} : X \mapsto X^\circ := \widetilde{X'}|_{\mathcal{L}}$ is an injection, i.e. $(\mathcal{L}, ^\circ; 0)$ is a reflection structure. Since $\mathcal{L}$ is symmetric, $(\mathcal{L}, ^\circ; 0)$ is invariant by (2.5, (2)). Therefore, our Theorem 2.6 is a consequence of (1.2). □

Finally, we consider a correspondence between chain–nets and permutation groups (cf. [10], 15.1). We assume $\mathcal{C} \neq \emptyset$ and fix an element $E \in \mathcal{C}$. For each $C \in \mathcal{C}$ let

$$\widehat{C} : E \to E; \quad x \mapsto \big[[x]_1 \cap C\big]_2 \cap E$$

then $\widehat{C}$ is a permutation of E, and if $\gamma \in \operatorname{Sym} E$, the set $C(\gamma) := \{x\Box\gamma(x) \mid x \in E\}$ is a chain.

(2.7) *Let $J(E) := \{\sigma \in \operatorname{Sym} E \mid \sigma^2 = \mathrm{id}\}$, $J^*(E) = J(E) \setminus \{id\}$, $\gamma \in \operatorname{Sym} E$, $a \in E$, $b := \gamma(a)$ and $C := C(\gamma)$, then:*

(1) *$a\Box b \in C$ and $\gamma(b) = a \iff b\Box a \in C$;*
(2) *$\gamma \in J(E) \iff C \perp E$ or $C = E$;*
(3) *If $\gamma \in J(E)$ then $\gamma = \widetilde{C}|_E$;*

(4) $\forall \alpha, \beta \in \operatorname{Sym} E\colon C(\alpha) \perp C(\beta) \iff \alpha^{-1} \circ \beta \in J^*(E)$.

(2.8) (Extension Theorem) *For $\sigma \in \operatorname{Sym} E$ let*

$$\sigma_1 : \begin{cases} \mathcal{P} := E\Box E & \longrightarrow \mathcal{P} \\ x\Box y & \longmapsto \sigma(x)\Box y \end{cases}, \quad \sigma_2 : \begin{cases} \mathcal{P} & \longrightarrow \mathcal{P} \\ x\Box y & \longmapsto x\Box\sigma(y) \end{cases}$$

and $\overline{\sigma} = \sigma_1 \circ \sigma_2$ and let $S := C(\sigma)$. Then

(1) $\forall \gamma \in \operatorname{Sym} E$*:* $\sigma_1(C(\gamma)) = C(\gamma \circ \sigma^{-1})$, $\sigma_2\big(C(\gamma)\big) = C(\sigma \circ \gamma)$, $\overline{\sigma}\big(C(\gamma)\big) = C(\sigma \circ \gamma \circ \sigma^{-1})$*, i.e.,* $\sigma_1, \sigma_2, \overline{\sigma} \in \operatorname{Aut}(\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2, \mathfrak{C}, \perp)$*;*

(2) $\forall \sigma, \tau \in \operatorname{Sym} E$*:* $(\sigma \circ \tau)_1 = \sigma_1 \circ \tau_1$, $(\sigma \circ \tau)_2 = \sigma_2 \circ \tau_2$, $\overline{\sigma \circ \tau} = \overline{\sigma} \circ \overline{\tau}$*;*

(3) *If* $\sigma \in J(E)$*, then* $\overline{\sigma} = \widetilde{S} \circ \widetilde{E} = \widetilde{E} \circ \widetilde{S}$*;*

(4) *If* $S \perp E$ *or* $S = E$*, then* $\overline{(\widetilde{S}|_E)} = \widetilde{S} \circ \widetilde{E} = \widetilde{E} \circ \widetilde{S}$*;*

(5) *Let* $A, B, C, D \in E^\perp$*, then* $\widetilde{A} \circ \widetilde{B} \circ \widetilde{C} = \widetilde{D} \iff \widetilde{A}|_E \circ \widetilde{B}|_E \circ \widetilde{C}|_E = \widetilde{D}|_E$.

Proof. (1): $\sigma_1\big(C(\gamma)\big) = \sigma_1(\{x\Box\gamma(x) \mid x \in E\}) = \{\sigma(x)\Box\gamma \circ \sigma^{-1} \circ \sigma(x) \mid x \in E\} = C(\gamma \circ \sigma^{-1})$ and $\sigma_1\big(C(\alpha)\big) = C(\alpha \circ \sigma^{-1}) \perp \sigma_1\big(C(\beta)\big) = C(\beta \circ \sigma^{-1}) \overset{(2.7,(4))}{\iff} \sigma \circ \alpha^{-1} \circ \beta \circ \sigma^{-1} \in J^*(E) \iff \alpha^{-1} \circ \beta \in J^*(E) \overset{(2.7,(4))}{\iff} C(\alpha) \perp C(\beta)$.

(3): By (2.7, (3)), $\sigma = \widetilde{S}|_E$ hence $\overline{\sigma}(x\Box y) = \sigma(x)\Box\sigma(y) = \widetilde{S}(x)\Box\widetilde{S}(y) \overset{(2.3,(1))}{=} \widetilde{S}(y\Box x) = \widetilde{S} \circ \widetilde{E}(x\Box y)$, i.e. $\overline{\sigma} = \widetilde{S} \circ \widetilde{E}$.

(4): is a consequence of (3) and (2.7, (3)) and (5) follows from (4). □

3 Chain nets associated with reflection structures

In this section, let $(E, {}^\circ; 0)$ be a reflection structure and $(\mathcal{P} := E \times E, \mathcal{G}_1, \mathcal{G}_2, \mathfrak{C})$ (with $\mathcal{G}_1 := \{\{x\} \times E \mid x \in E\}$ und $\mathcal{G}_2 := \{E \times \{x\} \mid x \in E\}$) the chain net corresponding to the symmetric group $\operatorname{Sym} E$ with the identifications $x = (x, x) = x\Box x$ for $x \in E$, hence $0 = (0, 0)$ and $E = \{(x, x) \mid x \in E\}$.

Since $E^\circ := \{a^\circ \mid a \in E\} \subset J(E)$ and $a^\circ(0) = a$ by **(B1)**, we have $a^\circ \in J^*(E)$ if $a \neq 0$. For $0 \in E$ we have the two cases, $0^\circ \in J^*(E)$ and $0^\circ = id$.
From (2.7, (3)), (2.8, (3)) we obtain:

(3.1) *For $a \in E$ let $a^c := C(a^\circ) = \{x\Box a^\circ(x) \mid x \in E\} \in \mathfrak{C}$ be the graph of the map a° and $\widetilde{a} := \widetilde{a^c}$ the reflection in the chain a^c. Then*

(1) $\widetilde{a} = \overline{a^\circ} \circ \widetilde{E} = \widetilde{E} \circ \overline{a^\circ}$, $a^\circ = \widetilde{a}|_E$, $a^c \in E^\perp \cup \{E\}$ *and* $\widetilde{a} \circ \widetilde{b} \circ \widetilde{c} = \widetilde{E} \circ \overline{a^\circ \circ b^\circ \circ c^\circ} = \overline{a^\circ \circ b^\circ \circ c^\circ} \circ \widetilde{E}$*, in particular* $\widetilde{a}(b^c) = \widetilde{a} \circ \widetilde{b} \circ \widetilde{a} = \widetilde{E} \circ \overline{a^\circ \circ b^\circ \circ a^\circ}$ *and* $\widetilde{a}(b^c) = C(a^\circ \circ b^\circ \circ a^\circ) \subset E^\perp \cup \{E\}$*, hence* $\widetilde{a}(b^c) \in E^c := \{a^c \mid a \in E\} \iff a^\circ \circ b^\circ \circ a^\circ \in E^\circ$*;*

(2) *If* $0^\circ \neq \mathrm{id}$*, then* $E^c \subset E^\perp$ *and* $\overline{E^\circ}(0^c) := \{\overline{a^\circ}(0^c) \mid a \in E\} = E^\sim(0^c) := \{\widetilde{a}(0^c) \mid a \in E\} = \{C(a^\circ \circ 0^\circ \circ a^\circ) \mid a \in E\} \subset E^\perp$*;*

(3) *If* $0^\circ = \mathrm{id}$*, then* $E = 0^c \in E^c \subset E^\perp \cup \{E\}$, $\widetilde{0} = \widetilde{E}$ *and* $\overline{E^\circ}(0^c) = E^\sim(0^c) = \{E\}$*;*

(4) $\forall x \in [0]_1 \cup [0]_2 \quad \exists_1 a^c \in E^c : x \in a^c$*;*

(5) $\bigcup E^c = \mathcal{P} \iff E^\circ$ *acts transitively on* E*;*

(6) $(\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2, E^c)$ *is a web* $\iff E^\circ$ *acts regularly on* E*;*

(7) $\overline{E^\circ}(0^c) \subset E^c \iff \forall a \in E : a^\circ \circ 0^\circ \circ a^\circ \in E^\circ \Rightarrow E \subset \bigcup E^c$*;*

(8) $a^\circ \circ E^\circ \circ a^\circ = E^\circ \iff \widetilde{a} \in \operatorname{Aut}(\mathcal{P}, \mathcal{G}_1 \cup \mathcal{G}_2, E^c)$*;*

(9) $(E, ^\circ; 0)$ *is invariant* $\Longleftrightarrow$ E^c *is symmetric;*

(10) *If* $(E, ^\circ; 0)$ *is invariant, then* E° *acts regularly on* E *and* $(\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2, E^c)$ *is a symmetric web such that there is an* $F \in \mathcal{C}$ *with* $E^c \subset F^\perp \cup \{F\}$ *and we have for each* $a \in E$ *the commutative diagramm*

$$\begin{array}{ccc} 0 & \xrightarrow{c} & 0^c \\ {\scriptstyle (a^\circ \circ 0^\circ(a))^\circ}\big\downarrow & & \big\downarrow{\scriptstyle \overline{a^\circ}} \\ a^\circ \circ 0^\circ(a) & \longrightarrow & \left(a^\circ \circ 0^\circ(a)\right)^c \end{array}$$

(11) **(Immersion theorem)** *If* $(E, ^\circ; 0)$ *is invariant and if* $\operatorname{Fix} a^\circ \neq \emptyset$ *for each* $a \in E$, *then* $\operatorname{Fix} a^\circ$ *consists of a single element* $a' \in E$ *and* $(E^c, ^\square, 0^c)$ *with*

$$(a^c)^\square : \begin{cases} E^c & \rightarrow \quad E^c \\ x^c & \mapsto \quad \widetilde{a'}(x^c) = C(a'^\circ \circ x^\circ \circ a'^\circ) \end{cases}$$

is a reflection structure isomorphic to $(E, ^\circ; 0)$.

If we call $(E, ^\circ; 0)^c := (\mathcal{P} := E \times E, \mathcal{G}_1, \mathcal{G}_2, E^c)$ the *chain-derivation* of the reflection structure $(E, ^\circ; 0)$ then we can state the following characterization theorems:

(3.2) *Let* $(E, ^\circ; 0)$ *be an invariant reflection structure,* $\mathcal{W} := (\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2; \mathcal{G}) := (E, ^\circ; 0)^c$ *and* $\mathcal{C}$ *the set of chains of* $(\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2)$. *Then* $\mathcal{W}$ *is a web,* $\mathcal{G}$ *is symmetric and there is an* $E \in \mathcal{C}$ *such that* $\mathcal{G} \subset E^\perp \cup \{E\}$.

If $(\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2; \mathcal{G})$ is a chain net such that:

(O1) $\exists E \in \mathcal{C}\colon \mathcal{G} \subset E^\perp \cup \{E\}$

is satisfied, then we call $(\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2; \mathcal{G})^E := (E; \widetilde{\mathcal{G}}|_E)$ the *reflection derivation in* E and if moreover

(O2) $\exists 0 \in E\colon \forall x \in E \quad \exists_1 x^g \in \mathcal{G}\colon 0 \square x \in x^g$

is valid, then the map

$$\circ : \begin{cases} E & \rightarrow \quad \operatorname{Sym} E \\ x & \mapsto \quad x^\circ := \widetilde{x^g} \end{cases}$$

is an injection and $(\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2; \mathcal{G})^{E,0} := (E, ^\circ; 0)$ is a reflection structure.

(3.3) *Let* $\mathcal{W} = (\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2; \mathcal{G})$ *be a web such that* **(O1)** *is satisfied and* $\mathcal{G}$ *is symmetric. Then for each* $0 \in E$, $(E, ^\circ; 0) := (\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2; \mathcal{G})^{E,0}$ *is an invariant reflection structure and moreover* $\mathcal{W}$ *is isomorphic to* $(E, ^\circ; 0)^c$.

4 Applications to K-loops

In this section let $(\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2)$ be a net, $\mathcal{C}$ the set of all chains of $(\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2)$ and $\mathcal{G}$ a subset of $\mathcal{C}$ such that there is a generator $Y \in \mathcal{G}_1$ satisfying the condition:

N1′ For each $y \in Y$ there is exactly one $G \in \mathcal{G}$ with $y \in G$; we set $[y]_3 := G$.

Then fixing a point $0 \in Y$ the chain $E := [0]_3$ can be turned in a right loop $(E, +)$ with the neutral element 0: For all $a, b \in E$ let

$$a^+ \colon E \to E; \quad x \mapsto \big[[Y \cap [a]_2]_3 \cap [x]_1\big]_2 \cap E$$

and $a + b := a^+(b)$. We set $\mathcal{W} := (\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2; \mathcal{G})$ and $\mathcal{W}^{0+} := (E, +)$ and call this the *loop derivation* in the point 0. This derivation exists for each point $0 \in \mathcal{P}$ where $[0]_1$ satisfies **N1**$'$. If on the other hand $(E, +)$ is a right loop with neutral element 0, $a^+ \colon E \to E$; $x \mapsto a + x$ for $a \in E$ and $E^+ := \{a^+ \mid a \in E\}$ then the chain derivation $(E, +)^c := (E, E^+)^c$ gives us a chain net $(\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2; \mathcal{G})$ ($\mathcal{G} := \{C(a^+) \mid a \in E\}$) where $[0\square 0]_1$ satisfies **N1**$'$. Clearly $\big((E, +)^c\big)^{0+} = (E, +)$ if 0 denotes the point $0\square 0$, and $(\mathcal{W}^{0+})^c = \mathcal{W}$ if for $0 \in \mathcal{P}$, $[0]_1$ satisfies **N1**$'$.

We have (cf. [9], (2.5), [10], p. 81):

(4.1) *If $\mathcal{W} = (\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2; \mathcal{G})$ is a web, then for each $0 \in \mathcal{P}$, $\mathcal{W}^{0+}$ is a loop with the neutral element 0; if $(E, +)$ is a loop, then $(E, +)^c$ is a web.*

By (1.1) and (1.2) there is a one to one correspondence between reflection structures $(E, ^\circ; 0)$ and right loops $(E, +)$ satisfying the condition $(*)$ of (1.1): If $(E, ^\circ; 0)$ is given, then we set $(E, ^\circ; 0)^+ := (E, +)$ where $a + b := a^\circ \circ 0^\circ(b)$ and if we start from $(E, +)$, we set $(E, +)^\circ := (E, ^\circ; 0)$ where $a^\circ := a^+ \circ \nu$ and $\nu \colon E \longrightarrow E$; $x \mapsto -x$. Here we have $\big((E, +)^\circ\big)^+ = (E, +)$ and $\big((E, ^\circ; 0)^+\big)^\circ = (E, ^\circ; 0)$.

(4.2) *Let $(E, +)$ be a right loop, $\mathcal{W} = (\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2; \mathcal{G}) := (E, +)^c$ and $0^c := C(\nu) = \{x\square(-x) \mid x \in E\}$. Then:*

(1) *The following statements are equivalent:*
   (i) *$(E, +)$ satisfies $(*)$ of (1.1);*
   (ii) *$\mathcal{G} \subset (0^c)^\perp \cup \{0^c\}$.*

(2) *Equivalent are:*
   (i) *$(E, +)$ is a right loop satisfying the Bol condition: $\forall a, b \in E \colon a^+ \circ b^+ \circ a^+ = (a^+(b))^+$;*
   (ii) *$(E, +)$ is a Bol loop;*
   (iii) *$\mathcal{G}$ is symmetric;*
   (iv) *$\mathcal{W}$ is a Bol web.*

(3) *Equivalent are:*
   (i) *$(E, +)$ is a K-loop;*
   (ii) *$\mathcal{G}$ is symmetric and $\mathcal{G} \subset (0^c)^\perp \cup \{0^c\}$;*
   (iii) *$\mathcal{W}$ is a Bol web with the additional property: $\exists A \in \mathcal{C} \colon \mathcal{G} \subset A^\perp \cup \{A\}$.*

Proof. This theorem is a consequence of (1.1), (1.2), (3.1), (3.2) and (3.3). We have only in (2) to show that $\mathcal{W}$ is a web since then the symmetry is equivalent to the property that each Bol configuration closes. Let $x \in \mathcal{P}$ be given, $x_1 := [x]_2 \cap [0]_1$, $x_2 := [x]_1 \cap [x_1]_3$, $x_3 := [0]_1 \cap [x_2]_2$. Then since $\mathcal{G}$ is symmetric, $X := \widetilde{[x_1]_3}([x_3]_3) \in \mathcal{G}$ and since $\widetilde{[x_1]_3}(x_3) = x$, we have $x \in X$. Suppose there is a further $U \in \mathcal{G}$ with $x \in U$, then $x_3 = \widetilde{[x_1]_3}(x) \in \widetilde{[x_1]_3}(U) \in \mathcal{G}$ with $x_3 \in [0]_1$, hence $\widetilde{[x_1]_3}(U) = [x_3]_3$ by **N1**$'$ and so $U = X$. □

Remark. Let $(E,^\circ;0)$ be a reflection structure, $(E,+) := (E,^\circ;0)^+$, $(\mathcal{P},\mathcal{G}_1,\mathcal{G}_2;\mathcal{G}) := (E,^\circ;0)^c$ and $(\mathcal{P},\mathcal{G}_1,\mathcal{G}_2;\mathcal{H}) := (E,+)^c$, then $(0^\circ)_1(\mathcal{G}) = \mathcal{H}$ (cf. (2.8)).

5 Web configurations related to properties of K-loops

Here let $\mathcal{W} = (\mathcal{P},\mathcal{G}_1,\mathcal{G}_2;\mathcal{G})$ be a web and for $x \in \mathcal{P}$ let $[x]_3 \in \mathcal{G}$ with $x \in [x]_3$. We remark that the closing of web configurations characterizing certain classes of webs can be expressed elegantly by using reflections in elements of $\mathcal{G}$. The condition **RE** If $a \in \mathcal{P}$, $b_i \in [a]_i$, $c_{ij} := [b_i]_j \cap [b_j]_i$ for $i, j \in \{1,2,3\}$ with $i \neq j$, then $[c_{12}]_3 \cap [c_{23}]_1 \cap [c_{31}]_2 \neq \emptyset$

which characterizes the *Reidemeister webs* can be written in the form:

RE′ If $A, B, C, D \in \mathcal{G}$ with $\mathrm{Fix}(\widetilde{A}\circ\widetilde{B}\circ\widetilde{C}\circ\widetilde{D}|_{\mathcal{G}_1}) \neq \emptyset$, then $\widetilde{A}\circ\widetilde{B}\circ\widetilde{C}\circ\widetilde{D}|_{\mathcal{G}_1} = id_{\mathcal{G}_1}$.

Proof. Let $X \in \mathcal{G}_1$, $a := A \cap X$, $b_1 := D \cap X$, $b_2 := [a]_2 \cap B$ and $c_{12} := [b_1]_2 \cap [b_2]_1$. Then $\widetilde{D}(X) = \widetilde{D}([a]_1) = [b_1]_2$, $\widetilde{B}\circ\widetilde{A}(X) = \widetilde{B}([a]_2) = [b_2]_1$ and we have: $\widetilde{A}\circ\widetilde{B}\circ\widetilde{C}\circ\widetilde{D}(X) = X \iff \widetilde{C}\circ\widetilde{D}(X) = \widetilde{C}([b_1]_2) = \widetilde{B}\circ\widetilde{A}(X) = [b_2]_1 \iff c_{12} \in C \iff C = [c_{12}]_3$.
We assume $[c_{12}]_3 = C$. Let $Y \in \mathcal{G}_1, b_3 := Y \cap A = Y \cap [a]_3, c_{13} := [b_1]_3 \cap [b_3]_1 = D \cap Y$ and $c_{23} := [b_2]_3 \cap [b_3]_2 = B \cap [b_3]_2$. Then $\widetilde{A}\circ\widetilde{B}\circ\widetilde{C}\circ\widetilde{D}(Y) = Y \iff \widetilde{C}\circ\widetilde{D}(Y) = \widetilde{C}([c_{13}]_2) = \widetilde{B}\circ\widetilde{A}(Y) = \widetilde{B}([b_3]_2) = [c_{23}]_1 \iff [c_{12}]_3 \cap [c_{23}]_1 \cap [c_{13}]_2 \neq \emptyset$. This shows the equivalence of **RE** and **RE′**. □

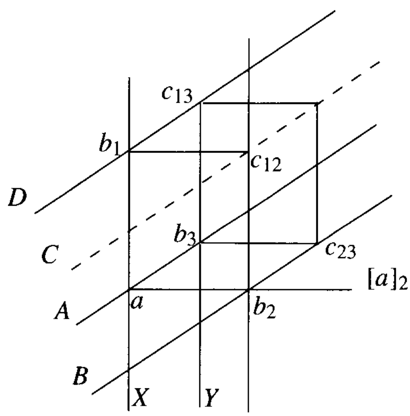

FIGURE 1.

If we add in **RE**, respectively in **RE′** the assumption $c_{12} \in [a]_3$, respectively $B = D$ then we obtain the conditions **BO**, respectively **BO′** characterizing the *Bol webs* which are equivalent to (cf. [15], (1.2), [2]):

BO″ $\mathcal{G}$ is symmetric.

The stronger assumption $c_{12} = b_3$ in **RE** leads to the condition **HEX** describing the *hexagonal webs*, a condition which is equivalent to

HEX′ If $X \in \mathrm{Fix}(\widetilde{A}\circ\widetilde{B}\circ\widetilde{C}\circ\widetilde{B}|_{\mathcal{G}_1})$, then $\widetilde{B}\circ\widetilde{C}(X) \in \mathrm{Fix}(\widetilde{A}\circ\widetilde{B}\circ\widetilde{C}\circ\widetilde{B}|_{\mathcal{G}_1})$.

If $p \in \mathcal{P}$ is fixed and **HEX** is valid for $c_{12} = b_3 = p$, then we denote this condition by **HEX(p)** and call $\mathcal{W}$ *hexagonal with respect to* p. **HEX(p)** can be expressed by:

HEX(p)′ If $[p]_1 \in \operatorname{Fix}(\widetilde{A} \circ \widetilde{[p]_3} \circ \widetilde{C} \circ \widetilde{[p]_3}|_{\mathcal{G}_1})$, then $\widetilde{[p]_3} \circ \widetilde{C}([p]_1) \in \operatorname{Fix}(\widetilde{A} \circ \widetilde{[p]_3} \circ \widetilde{C} \circ \widetilde{[p]_3}|_{\mathcal{G}_1})$.

(5.1) *Let* $0 \in \mathcal{P}$ *be fixed, let* $E := [0]_3$ *and let* $(E, +) := \mathcal{W}^{0+}$ *be the derived loop and let* $N := C(v) = \{x\square(-x) \mid x \in E\} \in \mathfrak{C}$. *For each* $a \in E$ *let* $a^+ : E \to E;$ $x \mapsto a + x$, $-a$, *respectively* $\sim a$ *be defined by* $a + (-a) = 0$, *respectively* $\sim a + a = 0$. *For* $x \in E$ *let* x *be identified with* $[x]_1$ *and let* $\overline{x} := [0\square x]_3$ *and* $\widetilde{x} := \widetilde{\overline{x}}$. *Moreover let* $a, b \in E$, $c := a + b$ *and* $d := b + a$. *Then:*

(1) $\forall x \in E$: $-x = \sim x \iff$ **HEX(0)** $\iff E \in N^{\perp} \cup \{N\} \iff \widetilde{v} = \widetilde{E} \circ \widetilde{N}$;
(2) $\delta_{a,a} = \mathrm{id} \iff (a + a)^+ = a^+ \circ a^+ \iff \widetilde{a}(E) \in \mathcal{G}$;
(3) $\forall x \in E$: $\delta_{x,-x} = x^+ \circ (-x)^+ = id \iff \widetilde{E} \in \operatorname{Aut}(\mathcal{P}, \mathcal{G})$;
(4) **BO″** $\Rightarrow \widetilde{E} \in \operatorname{Aut}(\mathcal{P}, \mathcal{G}) \Rightarrow$ **HEX(0)**;
(5) $\delta_{a,b} = \widetilde{c} \circ \widetilde{a} \circ \widetilde{E} \circ \widetilde{b}|_{\mathcal{G}_1}$;
(6) $\delta_{a,b+a} = (\delta_{b,a})^{-1} \iff a^+ \circ b^+ \circ a^+ = (a + (b + a))^+ \iff \widetilde{a} \circ \widetilde{E}(\widetilde{b}) \in \mathcal{G}$;
(7) $\delta_{a,b} = (\delta_{b,a})^{-1} \iff \widetilde{b} \circ \widetilde{E} \circ \widetilde{a} \circ \widetilde{c} \circ \widetilde{a} \circ \widetilde{E} \circ \widetilde{b} \circ \widetilde{d}|_{\mathcal{G}_1} = \mathrm{id}_{\mathcal{G}_1}$;
(8) $\delta_{a,b} = \delta_{a,b+a} \iff \widetilde{c} \circ \widetilde{a} \circ \widetilde{E} \circ \widetilde{b} \circ \widetilde{d} \circ \widetilde{E} \circ \widetilde{a} \circ \widetilde{d}|_{\mathcal{G}_1} = \mathrm{id}_{\mathcal{G}_1}$;
(9) $\delta_{a,b} \in \operatorname{Aut}(E, +) \iff \forall X \in \mathcal{G}, \exists X' \in \mathcal{G}$: $\widetilde{X'} \circ \widetilde{E} \circ \widetilde{c} \circ \widetilde{a} \circ \widetilde{E} \circ \widetilde{b} \circ \widetilde{E} \circ \widetilde{X} \circ \widetilde{b} \circ \widetilde{E} \circ \widetilde{a} \circ \widetilde{c}|_{\mathcal{G}_1} = \mathrm{id}_{\mathcal{G}_1}$.

6 Point-reflections related to a web and the negative map of the corresponding loop

In this section let $\mathcal{W} = (\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$ be a web, then $(\mathcal{P}, \mathcal{G}_1, \mathcal{G}_3)$, respectively $(\mathcal{P}, \mathcal{G}_2, \mathcal{G}_3)$ is a net and $\mathcal{G}_2$, respectively $\mathcal{G}_1$ can be considered as a chain-set. Therefore for $A \in \mathcal{G}_2$, respectively $A \in \mathcal{G}_1$ we define (according to (2.3)) the map $\widetilde{A} : \mathcal{P} \to \mathcal{P};\ x \mapsto [[x]_i \cap A]_j \cap [[x]_j \cap A]_i$ with $\{i, j\} = \{1, 3\}$, respectively $\{i, j\} \in \{2, 3\}$.

We call a pair $(0, \sigma) \in \mathcal{P} \times S_3$ (S_3 denotes the symmetric group of three elements) a *frame of reference* and the bijection

$$\mathcal{P} \to [0]_{\sigma(3)} \times [0]_{\sigma(3)}; \quad x \mapsto \big([x]_{\sigma(1)} \cap [0]_{\sigma(3)}, [x]_{\sigma(2)} \cap [0]_{\sigma(3)}\big)$$

the corresponding *coordinatization function*. In order to simplify our considerations we discuss only the case $\sigma = \mathrm{id}$ and $E := [0]_3$. Then $\mathcal{P} = E\square E = \{x\square y \mid x, y \in E\}$ and x, y are the coordinates of the point $x\square y$.

For each $q \in \mathcal{P}$ and for each $\sigma \in S_3$ we define now a permutation of the line $[q]_{\sigma(3)}$ fixing the point q by

$$q_\sigma : [q]_{\sigma(3)} \to [q]_{\sigma(3)}; \quad x \mapsto \big[[[x]_{\sigma(2)} \cap [q]_{\sigma(1)}]_{\sigma(3)} \cap [q]_{\sigma(2)}\big]_{\sigma(1)} \cap [q]_{\sigma(3)}$$

which we will call a *turn* of $[q]_{\sigma(3)}$ about q.

Remark. $\mathcal{W}$ is hexagonal with respect to q if and only if $q_\sigma \circ q_\sigma = \mathrm{id}$.

Now we extend the turn q_σ to a permutation of $\mathcal{P}$ via (2.8). In order to answer the question whether $\overline{q_\sigma} \in \operatorname{Aut}(\mathcal{P}, \mathcal{G}_{\sigma(3)})$ we need the following *bend-configuration* with respect to a point $q \in \mathcal{P}$ and a permutation $\sigma \in S_3$:

BE$(q;\sigma)$ Let $\sigma \in S_3$, $\tau_i \in S_3 \setminus A_3$ with $\tau_i(1) = i$ (hence $\tau_1 = (23)$, $\tau_2 = (12)$, $\tau_3 = (13)$),

and let

$$\tau_{\sigma(i)q}(p) := \big[[[p]_{\sigma(i)} \cap [q]_{\sigma\circ\tau_i(2)}]_{\sigma\circ\tau_i(3)} \cap [q]_{\sigma(i)}\big]_{\sigma\circ\tau_i(2)} \cap [q]_{\sigma\circ\tau_i(3)}.$$

We say that the bend-configuration closes if for all $p \in \mathcal{P}$:

$$\bigcap_{i\in\{1,2,3\}} \big[\tau_{\sigma(i)q}(p)\big]_i \neq \emptyset.$$

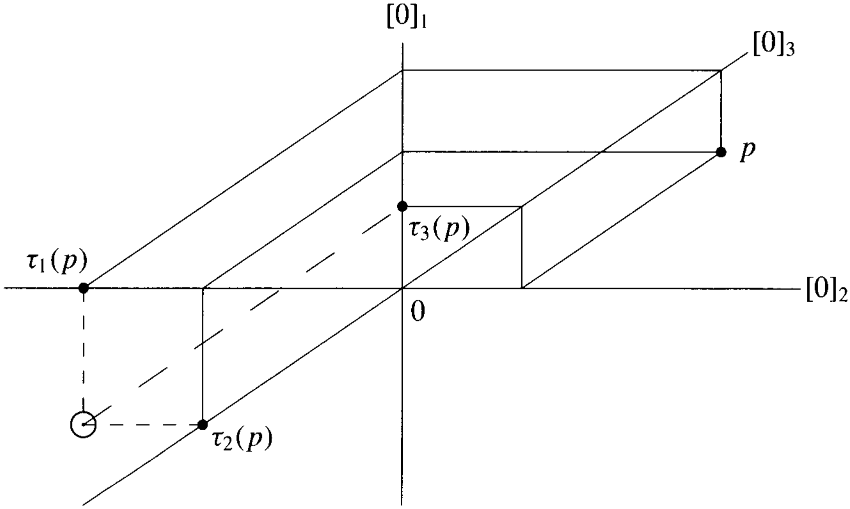

FIGURE 2. **BE**(0; id)

(6.1) (Characterization Theorem) *For $q \in \mathcal{P}$ and $\sigma \in S_3$ the following statements are equivalent:*

(1) $\overline{q_\sigma} \in \operatorname{Aut}(\mathcal{P}, \mathcal{G}_{\sigma(3)})$;
(2) *The bend-configuration* **BE**$(q;\sigma)$ *with respect to q and σ closes.*

Proof. We may assume $\sigma = \mathrm{id}$, $q = 0$, $E = [0]_3$. We set $\tau_i := \tau_{i0}$ ($i \in \{1,2,3\}$). Let $A \in \mathcal{G}_3$, $a \in E$ such that $a\Box 0 \in A$ and $A' := [\overline{0_{\mathrm{id}}}(a\Box 0)]_3 = [0_{\mathrm{id}}(a)\Box 0]_3$. For all $p := x_1\Box x_2$ with $x_1, x_2 \in E$, we have by definition $\overline{0_{\mathrm{id}}}(p) = [\tau_1(p)]_1 \cap [\tau_2(p)]_2$ and $p \in A \iff \tau_3(p) \in A'$. Then $\overline{0_{\mathrm{id}}}(A) \in \mathcal{G}_3 \iff \overline{0_{\mathrm{id}}}(A) = A' \iff \forall p = x_1\Box x_2 \in A\colon\ [\tau_1(p)]_1 \cap [\tau_2(p)]_2 \in A' := [\tau_3(p)]_3$. □

(6.2) *Let $q \in \mathcal{P}$, $\sigma \in S_3$, $\omega \in A_3$, $\tau \in S_3 \setminus A_3$, then we have:*

(1) *If $\tau\circ\sigma(1) = \sigma(2)$, then $q_\sigma \circ q_{\tau\circ\sigma} = \mathrm{id}_{[q]_{\sigma(3)}}$ and $\overline{q_\sigma} \circ \overline{q_{\tau\circ\sigma}} = \mathrm{id}$;*

(2) *If* $\overline{q_\sigma} \in \operatorname{Aut}(\mathcal{P}, \mathcal{G}_{\sigma(3)})$, *then* $\overline{q_{\omega\circ\sigma}} = \overline{q_\sigma}$ *and* $\overline{q_{\tau\circ\sigma}} = \overline{q_\sigma}^{-1}$ *(i.e. if the bend-configuration with respect to q closes for one permutation* $\sigma_0 \in S_3$ *then it closes for all permutations* $\sigma \in S_3$*);*

(3) *If* $\overline{q_\sigma} \in \operatorname{Aut}(\mathcal{P}, \mathcal{G}_{\sigma(3)})$ *and* $q_\sigma \circ q_\sigma = \mathrm{id}$, *then* $\overline{q_\sigma} \in \operatorname{Aut}(\mathcal{P}, \mathcal{G}_{\sigma(1)}, \mathcal{G}_{\sigma(2)}, \mathcal{G}_{\sigma(3)}) \cap J$. *In this case we call* $\overline{q_\sigma}$ *the point-reflection in q related to the web.*

Proof.
(1): Without loss of generality we may assume $q = 0$ and $\sigma = \mathrm{id}$. One verifies by the definition of turn that for each $x \in [0]_3$: : $0_{\mathrm{id}} \circ 0_{(12)}(x) = x$ and this implies $\overline{0_{id}} \circ \overline{0_{(12)}} = \mathrm{id}$.

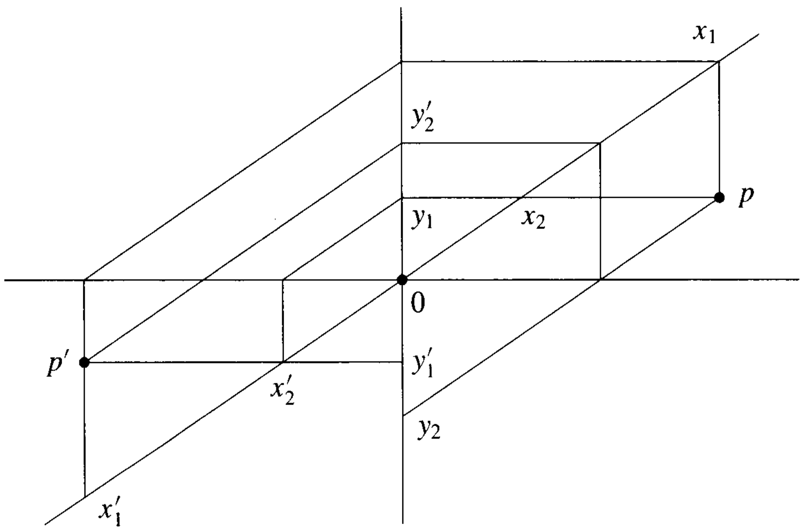

FIGURE 3.

(2): We may assume $\omega = (123)$. Then Fig.2 shows that $\overline{0_{id}} = \overline{0_{(123)}}$ is equivalent to the fact that the bend-configuration with respect to 0 and $\sigma = \mathrm{id}$ closes. Therefore by (6.1) and (1) the statements of (2) are valid. □

Now let $0 \in \mathcal{P}$ be fixed, $E := [0]_3$, $(E, +) := \mathcal{W}^{0+}$ and $N := C(\nu) = \{x\Box(-x) \mid x \in E\}$ (cf. (4.2)). Then $N \in \mathfrak{C}$ by $\nu \in \operatorname{Sym} E$ and we can state:

(6.3) *The following statements* (1), (2), (3) *are equivalent:*

(1) $\nu \in \operatorname{Aut}(E, +)$*;*

(2) *The bend-configuration* **BE**(0; id) *with respect to* 0 *and* $\sigma = \mathrm{id}$ *closes;*

(3) $\overline{\nu} = \overline{0_{\mathrm{id}}} \in \operatorname{Aut}(\mathcal{P}, \mathcal{G}_3)$.

Under the assumption $\widetilde{E} \in \operatorname{Aut}(\mathcal{P}, \mathcal{G}_3)$ *also* (1), (2), (3) *and* (4) *are equivalent:*

(4) $\mathcal{G}_3 \subset N^\perp \cup \{N\}$.

Proof.
"(1) $\Longleftrightarrow$ (2)": The map $E \times E \to \mathcal{P}$, $(a, b) \mapsto p := b\Box(a + b)$ is a bijection and we have $\tau_1(p) = (-b)\Box 0 = \nu(b)\Box 0$, $\tau_2(p) = -(a + b) = \nu(a + b)$, $\tau_3(p) =$

$0\Box(-a) = 0\Box\nu(a)$. $\nu(a)+\nu(b) = [\overline{\nu(a)}\cap[\nu(b)]_1]_2\cap E = \big[[\tau_3(p)]_3\cap[\tau_1(p)]_1\big]_2\cap$ E hence $\nu(a)+\nu(b) = \nu(a+b) \iff [\tau_2(p)]_2 \ni [\tau_3(p)]_3 \cap [\tau_1(p)]_1$.

"(2) $\iff$ (3)": cf. (6.1).

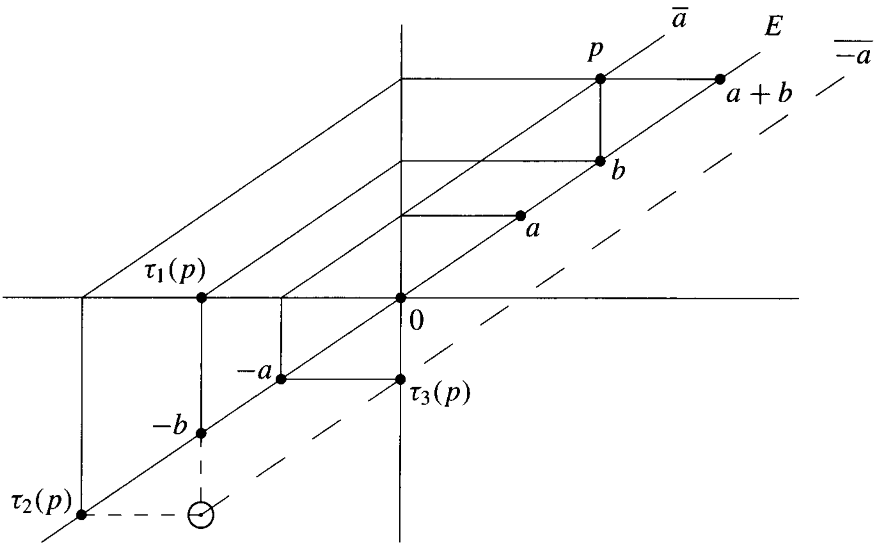

FIGURE 4.

Now let $\widetilde{E} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_3)$. Then $\overline{\nu} = \widetilde{E} \circ \widetilde{N}$ by (5.1, (4)) and (5.1, (1)), and so: $\overline{\nu} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_3) \iff \widetilde{N} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_3)$. Clearly (4) $\Rightarrow \widetilde{N} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_3)$. Now let $\widetilde{N} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_3)$ and $X = \overline{x} \in \mathcal{G}_3$. Then $0\Box x$, $(-x)\Box 0 \in X$, hence $\widetilde{N}(0\Box x) = \widetilde{E} \circ \overline{\nu}(0\Box x) = \widetilde{E}(0\Box(-x)) = (-x)\Box 0 \in \widetilde{N}(X) \in \mathcal{G}_3$ and so $X = [(-x)\Box 0]_3 = \widetilde{N}(X)$, i.e. $\mathcal{G}_3 \subset N^{\perp} \cup \{N\}$. □

Remark. Theorem (4.6) of [2] proves that the loop corresponding to a 3-web satisfying the Bol condition, such that the bend-configuration closes, satisfies the automorphic inverse property. Theorem (6.3) proves the equivalence between the automorphic inverse property in a loop (not necessary a Bol loop) corresponding to a 3-web and the closure of the bend-configuration.

From (4.2, (3)) and (6.3) we obtain the result:

Theorem 6.4. *$(E, +)$ is a K-loop if and only if $\mathcal{G}_3$ is symmetric and the bend-configuration* **BE**(0; id) *with respect to* 0 *and* $\sigma = \mathrm{id}$ *closes.*

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