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Vectorspacelike representation of absolute planes

Dedicated to Walter Benz on the occasion of his 75th birthday, in friendship

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Abstract. The pointset $\mathbf{E}$ of an absolute plane $(\mathbf{E}, \mathcal{G}, \alpha, \equiv)$ can be provided with a binary operation "+" such that $(\mathbf{E}, +)$ becomes a loop and for each $a \in \mathbf{E} \setminus \{o\}$ the line $[a]$ through o and a is a commutative subgroup of $(\mathbf{E}, +)$. Two elements $a, b \in \mathbf{E} \setminus \{o\}$ are called *independent* if $[a] \cap [b] = \{o\}$ and the absolute plane is called *vectorspacelike* if for any two independent elements we have $\mathbf{E} = [a] + [b] := \{x + y \mid x \in [a], y \in [b]\}$. If $(\mathbf{E}, \mathcal{G}, \alpha, \equiv)$ is singular then $(\mathbf{E}, +)$ is a commutative group and $(\mathbf{E}, \mathcal{G}, \alpha, \equiv)$ is vectorspacelike iff $(\mathbf{E}, \mathcal{G}, \alpha, \equiv)$ is Euclidean. If $(\mathbf{E}, \mathcal{G}, \alpha, \equiv)$ is a hyperbolic plane then $(\mathbf{E}, \mathcal{G}, \alpha, \equiv)$ is vectorspacelike and in the continous case if a, b are independent, each point p has a unique representation as a *quasilinear combination* $p = \alpha \cdot a + \mu \cdot b$ where $\alpha \cdot a \in [a]$and $\beta \cdot b \in [b]$ are points, α, β real numbers such that $\boldsymbol{\lambda}(o, \lambda \cdot a) = |\lambda| \cdot \boldsymbol{\lambda}(o, a)$ and $\boldsymbol{\lambda}(o, \mu \cdot b) = |\mu| \cdot \boldsymbol{\lambda}(o, b)$ and $\boldsymbol{\lambda}$ is the distance function.

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1. Introduction

After fixing two points o and e the pointset $\mathbf{E}$ of an absolute plane $(\mathbf{E}, \mathcal{G}, \alpha, \equiv)$ can be furnished with a binary operation "+" such that $(\mathbf{E}, +)$ becomes a K-loop with o as neutral element. If $\mathbf{E}^* := \mathbf{E} \setminus \{o\}$ then for each $a \in \mathbf{E}^*$ the line $[a] := \overline{o, a}$ through o and a is a commutative subgroup of the loop $(\mathbf{E}, +)$ and all these groups are isomorphic. Moreover the halfline $[a]_+ := o\wr a$ is a positive domain of the group $([a], +)$ and so by "$x < y :\Longleftrightarrow -x + y \in [a]_+$", $([a], +, <)$ becomes an ordered group. Such an ordered group $(W, +, <)$ with $W := [e]$ will be choosen as "scalar domain" and an operation "$\oplus : W \times \mathbf{E}^* \to \mathbf{E}; (w, p) \mapsto w \oplus p$" between scalars and elements of $\mathbf{E}$ introduced such that $[p] = W \oplus p$ holds.

If $(a, b) \in \mathbf{E}^* \times \mathbf{E}^*$ then the pair is called *independent* if $[a] \neq [b]$ and *direct* if $\mathbf{E} = [a] + [b] := \{x + y \mid x \in [a], y \in [b]\} = \{(u \oplus a) + (v \oplus b) \mid u, v \in W\}$. If $[a] \perp [b]$ then (a, b) is a direct pair (cf.(4.5)). We call $(\mathbf{E}, +)$ *vectorspacelike* if each independent pair is direct. We show:

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$(\mathbf{E},+)$ is vectorspacelike $\Longleftrightarrow$ to a given segment (s_1, s_2) and an acute angle α there exists a rectangular triangle (p,q,r) with $\overline{p,q} \perp \overline{q,r}$, $(q,r) \equiv (s_1,s_2)$ and $\alpha \equiv \angle(r,p,q)$ (cf.(4.7),(4.8)).

For each $n \in \mathbf{N}$ the map $n^{\cdot} : W \to W;\ x \mapsto n \cdot x = x + \cdots + x$ (n times) is a strictly isotone monomorphism of $(W,+,<)$. The set $\mathbf{N}_W := \{n \in \mathbf{N} \mid n^{\cdot}$ is surjective $\}$ contains 2 (cf.(5.1)) and the imbedding of the subring $\mathbf{Z}_W := \{\frac{m}{n} \mid m \in \mathbf{Z}, n \in \mathbf{N}_W\}$ of the field of rational numbers $\mathbf{Q}$ in $(W,+,<)$ by $\frac{m}{n} \mapsto m^{\cdot} \circ (n^{\cdot})^{-1}(e)$ is a monomorphism from $(\mathbf{Z}_W,+)$ into $(W,+)$. In this way $\mathbf{Z}_W$ will be considered as a subset of W with the operation $\cdot : \mathbf{Z}_W \times W \to W; (\frac{m}{n}, x) \mapsto \frac{m}{n} \cdot x := m^{\cdot} \circ (n^{\cdot})^{-1}(x)$. Then for $r \in \mathbf{Z}_W, r \neq 0$ the map $r^{\cdot} : W \to W; w \mapsto r \cdot w$ is contained in the set $Bet(W,+,\xi)$ of all betweenness preserving monomorphisms of $(W,+,\xi)$; $r^{\cdot}$ is isotone resp. antitone if $o < r$ resp. $r < o$. A subset $B \subseteq W$ together with an operation $\cdot : B \times W \to W$ will be called *b-ring* of $(W,+,<)$ if $(B,+,\cdot)$ is a ring containing $(\mathbf{Z}_W,+,\cdot)$ as a subring and if for each $\beta \in B^* := B \setminus \{o\}$ the map $\beta_l : W \to W; w \mapsto \beta \cdot w$ is in $Bet(W,+,\xi)$. If B is a b-ring and $\beta \in B^*$ then by a so called *rotational extension* (cf.(5.7)) β_l becomes an injection $\beta^{\cdot} : \mathbf{E} \to \mathbf{E}$; $x \mapsto \beta \cdot x$ called *B-quasidilatation* (cf. Sec. 4). For $o < \beta < e$ the quasidilatation $\beta^{\cdot}$ is a contraction hence if $x \in \mathbf{E}^*$ then $\beta \cdot x$ is a point of the open segment $]o, x[$ and if $e < \beta$ then $\beta^{\cdot}$ is an enlargement, i.e. $x \in]o, \beta \cdot x[$. For $a, b \in \mathbf{E}$ and $\lambda, \mu \in B$ the expression $\lambda \cdot a + \mu \cdot b$ is called *quasilinear B-combination*. If B is transitive, i.e. $B = W$, then $[a] + [b] = \{\lambda \cdot a + \mu \cdot b \mid \lambda, \mu \in B\}$ if $a, b \in \mathbf{E}_1$ or if $(W,+,\cdot) := (B,+,\cdot)$ is a field. In the case that $(W,+,<)$ is continuous W can be established with a multiplication "$\cdot$" such that $(W,+,\cdot,<)$ becomes an ordered field (isomorphic to the reals $\mathbf{R}$) (cf. (5.6)) and then $[a] + [b] = W \cdot a + W \cdot b = \{\lambda \cdot a + \mu \cdot b \mid \lambda, \mu \in W\}$ for all $a, b \in \mathbf{E}$.

The loop $(\mathbf{E},+)$ is a group if the absolute plane is singular. In this case $(\mathbf{E},+)$ is vectorspacelike iff $(\mathbf{E}, \mathcal{G}, \alpha, \equiv)$ is an Euclidean plane (cf. (4.6)). In the ordinary case the loop of a hyperbolic plane is vectorspacelike (cf. (6.1)).

With the theorems (5.6) and (6.1) one obtains the result of A. Greil [1]:

If $(\mathbf{E}, \mathcal{G}, \alpha, \equiv)$ is a continuous hyperbolic plane (then $\mathbf{R}$ is a b-ring) and if $a, b \in \mathbf{E}^*$ with $[a] \neq [b]$ then each point $x \in \mathbf{E}$ can be uniquely represented as a quasilinear $\mathbf{R}$-combination $x = \lambda \cdot a + \mu \cdot b$ with $\boldsymbol{\lambda}(o, \lambda \cdot a) = |\lambda| \cdot \boldsymbol{\lambda}(o, a)$ and $\boldsymbol{\lambda}(o, \mu \cdot b) = |\mu| \cdot \boldsymbol{\lambda}(o, b)$ where $\boldsymbol{\lambda}$ is the distance function (cf. Sec. 2).

2. Notations, assumptions and known results

In this paper let $(\mathbf{E}, \mathcal{G}, \alpha, \equiv)$ be an absolute plane in the sense of [6] p. 96; $\mathbf{E}$ and $\mathcal{G}$ denotes the set of *points* and *lines* respectively, α the *order-function* and $\equiv$ the *congruence*. Let $\mathcal{A}$ be the motion group of $(\mathbf{E}, \mathcal{G}, \alpha, \equiv)$. For $a \in \mathbf{E}$, $A \in \mathcal{G}$ let $\tilde{a}$ resp. $\tilde{A}$ denote the point- resp. line-reflection in a resp. in A and let $\tilde{\mathbf{E}} := \{\tilde{a} \mid a \in \mathbf{E}\}$ resp. $\tilde{\mathcal{G}} := \{\tilde{A} \mid$

$A \in \mathcal{G}\}$ be the set of all point- resp. line-reflections. If $a, b \in \mathbf{E}$ and $a \neq b$ let $\widetilde{ab}$ resp. $\widehat{ab}$ denote the (uniquely determined) point- resp. line-reflection interchanging a and b (cf. [6] (16.11), (16.12), (17.1), (17.2)) (i.e. $\widetilde{ab}$ resp. $\widehat{ab}$ is the reflection in the midpoint resp. midline of a and b (cf. [6](16.11) and p. 105)). Moreover let $\overline{a, b}$ denote the line joining a and b and let $a \between b := \{x \in \overline{a, b} \mid (a|b, x) = 1\}$[1] be the *halfline* and let $\mathcal{H} := \{a \between b \mid a, b \in \mathbf{E}, a \neq b\}$ be the set of all halflines.

By [6] (17.6),(17.9),(17.7) and $\tilde{\mathcal{G}} \subseteq \tilde{\mathcal{G}}^3$ follows:

(2.1) $\mathcal{A} = \tilde{\mathcal{G}}^2 \dot\cup \tilde{\mathcal{G}}^3$, $\tilde{\mathbf{E}} \subseteq \tilde{\mathcal{G}}^2$ and $\tilde{\mathcal{G}}^2 \trianglelefteq \mathcal{A}$ is a normal subgroup of $\mathcal{A}$ of index 2.

We call the elements of $\mathcal{A}_+ := \tilde{\mathcal{G}}^2$ *proper motions*. By [6] (17.8) and (18.3) we have:

(2.2) *Let $\varphi \in \mathcal{A}$, $a \in \mathbf{E}$ and $G \in \mathcal{G}$ then:*

(1) $\varphi \circ \tilde{G} \circ \varphi^{-1} = \widetilde{\varphi(G)}$ *hence* $\varphi \circ \tilde{\mathcal{G}} \circ \varphi^{-1} = \tilde{\mathcal{G}}$ *, i.e. $\tilde{\mathcal{G}}$ is an invariant subset consisting of involutory motions of $\mathcal{A}$ and acting transitively on* $\mathbf{E}$.

(2) $\varphi \circ \tilde{a} \circ \varphi^{-1} = \widetilde{\varphi(a)}$ *hence* $\varphi \circ \tilde{\mathbf{E}} \circ \varphi^{-1} = \tilde{\mathbf{E}}$*, i.e.* $(\mathbf{E}, \tilde{\mathbf{E}})$ *is an invariant set of involutory motions acting regularly on* $\mathbf{E}$.

(3) $\forall a, b, c, d \in \mathbf{E},\ a \neq b,\ c \neq d\ \ \exists_1\ \sigma \in \mathcal{A}_+ :\ \ \sigma(a \between b) = c \between d$*, i.e. the group of proper motions acts regularly on the set $\mathcal{H}$ of all halflines (cf. [6] (17.15)).*

From [6] (17.7.2) and (17.13.2) resp. (16.10.2) and p.105 follows:

(2.3) Let $D \in \mathcal{G}$, $a, b, c \in D$ and $p \in \mathbf{E} \setminus D$ then:

(1) $\exists\, m \in D : \tilde{a} \circ \tilde{b} \circ \tilde{c} = \tilde{m}$.

(2) $\tilde{p}(D) \cap D = \emptyset$.

The absolute planes split into two classes: the *singular planes* characterized by $\tilde{\mathbf{E}}^3 \subset \tilde{\mathbf{E}}$ and the *ordinary planes* characterized by $\tilde{\mathbf{E}}^3 \not\subset \tilde{\mathbf{E}}$

(2.4) *If* $(\mathbf{E}, \mathcal{G}, \alpha, \equiv)$ *is singular then* $\tilde{\mathbf{E}}^2$ *is a commutative normal subgroup of $\mathcal{A}$ acting regularly on* $\mathbf{E}$*. (cf. [6] (21.6))*

Now let three non collinear points $o, e_1, e_2 \in \mathbf{E}$ with $(o, e_1) \equiv (o, e_2)$ and $\overline{o, e_1} \perp \overline{o, e_2}$ be fixed as a *frame of reference*, let $\mathbf{E}_1 := \{x \in \mathbf{E} \mid (o, x) \equiv (o, e_1)\}$ and $\mathbf{E}^* := \mathbf{E} \setminus \{o\}$. For any $a \in \mathbf{E}^*$ let:

$[a] := \overline{o, a}$ the line joining o and a,
$[a]_+ := \{x \in [a] \mid (o|a, x) = 1\}$ the *halfline*,
$a^+ := \widetilde{oa} \circ \tilde{o}$ and $o^+ := id$ (let $\mathbf{E}^+ := \{a^+ \mid a \in \mathbf{E}\}$).

For $a \in \mathbf{E}_1 \setminus \{e_1\}$ let $a^\bullet := \widehat{e_1 a} \circ \widetilde{\overline{o, e_1}}$ and $e_1^\bullet := id$.

[1] $(a|b, x) := \alpha(a, b, x)$ (cf. [6] (13.9))

Then by [4] p.405:

(2.5) $(\mathbf{E}, +)$ *with* $a + b := a^+(b)$ *is a* K-loop, *i.e. a loop characterized by:*

$\forall a, b \in \mathbf{E} : a^+ \circ b^+ \circ a^+ = (a + (b + a))^+$ *and* $\tilde{o} \circ a^+ = (\tilde{o}(a))^+ \circ \tilde{o}$

Moreover:

(1) $\mathbf{E}^+$ *is a set of fixed point free proper motions of* $(\mathbf{E}, \mathcal{G}, \alpha, \equiv)$ *acting regularly on the point set* $\mathbf{E}$ *(cf.(2.2.2)).*
(2) $(\mathbf{E}, +)$ *is a group (and then even a commutative group) if and only if* $(\mathbf{E}, \mathcal{G}, \alpha, \equiv)$ *is singular; in this case* $(\mathbf{E}, +)$ *and* $(\tilde{\mathbf{E}}^2, \circ)$ *are isomorphic.*
(3) $\forall a \in \mathbf{E}^*$, $[a]$ *is a commutative subgroup of the loop* $(\mathbf{E}, +)$ *and* $[a]_+$ *a subsemigroup of* $[a]$ *with* $[a] = [a]_+ \dot\cup \{o\} \dot\cup [-a]_+$.
(4) $\mathcal{G} = \{a + [b] \mid a \in \mathbf{E}, b \in \mathbf{E}^*\}$ *and the set* $\mathcal{H}$ *of all halflines is represented by* $\mathcal{H} = \{a + [b]_+ \mid a \in \mathbf{E}, b \in \mathbf{E}^*\}$.
(5) *If* $(\mathbf{E}, \mathcal{G}, \alpha, \equiv)$ *is ordinary then* $\forall a \in \mathbf{E}^*, \forall \sigma \in Aut(\mathbf{E}, +) : [a] = \{x \in \mathbf{E} \mid a^+ \circ x^+ = x^+ \circ a^+\}$, $\sigma([a]) = [\sigma(a)]$ *and* $Aut(\mathbf{E}, +) \leq Aut(\mathbf{E}, \mathcal{G})$.

Proof. "(5)" $\forall x \in \mathbf{E} : \sigma \circ a^+ \circ \sigma^{-1}(x) = \sigma(a + \sigma^{-1}(x)) = \sigma(a) + x = (\sigma(a))^+(x)$ hence $\sigma \circ a^+ \circ \sigma^{-1} = (\sigma(a))^+$ and so $(\sigma(a))^+ \circ (\sigma(x))^+ = \sigma \circ a^+ \circ x^+ \circ \sigma^{-1} = (\sigma(x))^+ \circ (\sigma(a))^+ \iff a^+ \circ x^+ = x^+ \circ a^+$.

Consequently $\sigma([a]) = [\sigma(a)]$. Since $\sigma(a + [b]) = \sigma(a) + \sigma([b]) = \sigma(a) + [\sigma(b)]$ we have $\sigma \in Aut(\mathbf{E}, \mathcal{G})$. □

From [6] (16.12) and (19.1) we obtain the first part of the following theorem:

(2.6) $(\mathbf{E}_1, \bullet)$ *with* $a \bullet b := a^\bullet(b)$ *is a commutative group with the neutral element* e_1, *isomorphic to the rotation group in* o *and for* $a \in \mathbf{E}_1$ *and* $b \in \mathbf{E}^*$ *we have:*

(1) $a^\bullet \circ b^+ = (a^\bullet(b))^+ \circ a^\bullet$, *i.e.* $a^\bullet \in Aut(\mathbf{E}, +)$ *hence* $\mathbf{E}_1^\bullet := \{a^\bullet \mid a \in \mathbf{E}_1\} \leq Aut(\mathbf{E}, +)$.
(2) $a^\bullet([b]) = [a^\bullet(b)]$, $a^\bullet([b]_+) = [a^\bullet(b)]_+$, *i.e. the automorphism* $a^\bullet$ *maps the commutative subgroup* $[b]$ *of the loop* $(\mathbf{E}, +)$ *onto the subgroup* $[a^\bullet(b)]$, *in particular* $a^\bullet([e_1]) = [a]$.
(3) $|[b]_+ \cap \mathbf{E}_1| = 1$.
(4) $\forall b, c \in \mathbf{E}^* \exists_1 m \in \mathbf{E}_1 : [c]_+ = m^\bullet([b]_+) = [m^\bullet(b)]_+$.
(5) *For* $a, b \in \mathbf{E}$ *let* $\delta_{a,b} := ((a + b)^+)^{-1} \circ a^+ \circ b^+$ *and let* $d_{a,b} := \delta_{a,b}(e_1)$ *then* $\delta_{a,b} = d^\bullet_{a,b}$ *and* $a^+ \circ b^+ = (a + b)^+ \circ d^\bullet_{a,b}$.
(6) $\mathbf{E}^+ \lhd_Q \mathbf{E}_1^\bullet = \mathcal{A}_+$ *is the quasidirect product of the loop* $(\mathbf{E}, +)$ *and the commutative group* $(\mathbf{E}_1, \bullet)$*: If* $\sigma \in \mathcal{A}_+$, $a := \sigma(o)$ *and* $b := (a^+)^{-1} \circ \sigma(e_1)$ *then* $b \in \mathbf{E}_1$ *and* $\sigma = a^+ \circ b^\bullet$ *and if* $a, b \in \mathbf{E}$, $c, d \in \mathbf{E}_1$ *then* $(a^+ \circ c^\bullet) \circ (b^+ \circ d^\bullet) = (a + c^\bullet(b))^+ \circ ((d_{a,c^\bullet(b)}) \bullet c \bullet d)^\bullet$.

Proof. "(1)" Since $(o, e_1) \equiv (o, a)$ the midline of e_1 and a contains the point o (cf.[6] (16.12), (16.13)) hence $\widehat{e_1 a}(o) = o$ and so $a^\bullet(o) = o$. Therefore by (2.2.2), $a^\bullet \circ \tilde{o} \circ (a^\bullet)^{-1} = \widetilde{a^\bullet(o)} = \tilde{o}$ and $a^\bullet \circ \widetilde{ob} \circ (a^\bullet)^{-1} = (\widetilde{a^\bullet(o) a^\bullet(b)}) = \widetilde{oa^\bullet(b)}$ implying $a^\bullet \circ b^+ \circ (a^\bullet)^{-1} = (\widetilde{oa^\bullet(b)}) \circ \tilde{o} = (a^\bullet(b))^+$. □

3. Measurement and polar coordinates

Let $W := [e_1]$, $W_+ := [e_1]_+$ and $\mathbf{E}_+ := \{x \in \mathbf{E} \mid (W|e_2, x) = 1\}$. According to [6] (13.3) there is a total order relation "<" on W such that $o < e_1$ and for all $\{x, y, z\} \in \binom{W}{3}$ holds: $(x|y, z) = -1 \iff y < x < z$ or $z < x < y$.

From the excelent paper of D. Gröger (cf. [2] §2) we obtain:

(3.1) *Between the commutative group* $(W, +)$ *(cf.(2.5.3)) and the ordered set* $(W, <)$ *there are the following relations :*

1. $\forall a \in W$, $\tilde{a}_{|W}$ *is an antiton permutation of* $(W, <)$.
2. $\forall a \in W$, $a^+_{|W}$ *is an isoton permutation , i.e.* $(W, +, <)$ *is an ordered commutative group.*
3. W_+ *is a positive domain hence for* $a, b \in W$: $a < b \iff -a + b \in W_+$. □

By (2.6.4) to any $x \in \mathbf{E}^*$ there exists exactly one $m \in \mathbf{E}_1$ with $m^\bullet([x]_+) = [e_1]_+ = W_+$. Therefore the map

$$|\ | : \mathbf{E} \rightarrow W_+ \cup \{o\} \quad ; \quad x \mapsto \begin{cases} m^\bullet(x) & \text{if } x \neq o \\ o & \text{if } x = o, \end{cases}$$

called *absolute value*, is welldefined and we have:

(3.2) $\forall x, y \in \mathbf{E} : \quad |x| = |y| \iff (o, x) \equiv (o, y)$. □

Using the loop operation of $(\mathbf{E}, +)$ we define:

$$\boldsymbol{\lambda} : \mathbf{E} \times \mathbf{E} \rightarrow W_+ \cup \{o\}; (a, b) \mapsto \boldsymbol{\lambda}(a, b) := |-a + b|$$

and call $\boldsymbol{\lambda}(a, b)$ the *distance* of the points a and b. Since the maps a^+ are also motions we can summarize the results of ([2] (2.5), (2.6), (2.7)) and state:

(3.3) *Let* $a, b, c, d \in \mathbf{E}$ *and* $\varphi \in \mathcal{A}$ *then :*

1. $(a, b) \equiv (c, d) \iff \boldsymbol{\lambda}(a, b) = \boldsymbol{\lambda}(c, d)$
2. $\boldsymbol{\lambda}(\varphi(a), \varphi(b)) = \boldsymbol{\lambda}(a, b) = \boldsymbol{\lambda}(b, a)$
3. $\boldsymbol{\lambda}(a, b) = o \iff a = b$
4. *If* (a, b, c) *is a rectangular triangle with* $\overline{a, c} \perp \overline{b, c}$ *then* $\boldsymbol{\lambda}(a, c) < \boldsymbol{\lambda}(a, b)$.
5. (triangular inequality) $\boldsymbol{\lambda}(a, b) \leq \boldsymbol{\lambda}(a, c) + \boldsymbol{\lambda}(b, c)$ *and* $\boldsymbol{\lambda}(a, b) = \boldsymbol{\lambda}(a, c) + \boldsymbol{\lambda}(b, c) \iff c \in [a, b]$. □

From (3.3.4) follows:

(3.4) *For $A \in \mathcal{G}$ and $x \in \mathbf{E}$ let $x_A := (x \perp A) \cap A$ be the foot of x on A then for $a \in A$, $\boldsymbol{\lambda}(x, x_A) \leq \boldsymbol{\lambda}(x, a)$ and $\boldsymbol{\lambda}(x, A) := \boldsymbol{\lambda}(x, x_A)$ is called the* distance *from the point x to the line A.* □

If $p, q \in \mathbf{E}$, $A \in \mathcal{G}$ and $u \in W_+$ are given with $p \neq q$ and $q \notin A$, let

$D(A, q) := \{x \in \mathbf{E} \mid \boldsymbol{\lambda}\,(x, A) = \boldsymbol{\lambda}\,(q, A) \wedge (A|q, x) = 1\}$ resp.

$D(A; u) := \{x \in \mathbf{E} \mid \boldsymbol{\lambda}(x, A) = u\}$

be the *equidistant* of A through q resp. the set of all points having the distance u from A. If $\boldsymbol{\lambda}(q, A) = u$ then $D(A; u) = D(A, q) \,\dot{\cup}\, D(A, \tilde{A}(q))$.

In the absolute plane $(\mathbf{E}, \mathcal{G}, \alpha, \equiv)$ we introduce an *orientation* $Or : \Delta \to \{1, -1\}$; $(a, b, c) \mapsto Or(a, b, c)$, i.e. a function defined on the set Δ of all triangles by:

Let $\sigma \in \mathcal{A}_+$ be the proper motion uniquely determined by $\sigma(\overset{\to}{a,b}) = W_+$ (cf. (2.2.3)) then $Or(a, b, c) := (W|e_2, \sigma(c))$.

We say (a, b, c) is *positively oriented* if $Or(a, b, c) = 1$ otherwise *negatively*.

The orientation Or induces a *cyclic order* ω on $\mathbf{E}_1$ turning the commutative group $(\mathbf{E}_1, \bullet)$ in a *cyclic ordered group* $(\mathbf{E}_1, \bullet, \omega)$ by:

For $\{a, b, c\} \in \binom{\mathbf{E}_1}{3}$ we have $(a, b, c) \in \Delta$ and therefore we set $\omega(a, b, c) := Or(a, b, c)$.

Now we can introduce a measure for angles : if $\alpha = \angle(b, a, c) = (\overset{\to}{ab}, \overset{\to}{ac})$ is an angle let again $\sigma \in \mathcal{A}_+$ with $\sigma(\overset{\to}{ab}) = W_+$ then $\boldsymbol{\mu}\,(\alpha) := [\sigma(c)]_+ \cap \mathbf{E}_1$ is called the *measure* of α.

Analogously to (3.3) we have:

(3.5) *Let $\gamma := \angle(a, c, b)$ be an angle, let $d \in \mathbf{E} \setminus \{o\}$ with $(\overline{c, d}|a, b) = -1$ then $\boldsymbol{\mu}(\gamma) = \boldsymbol{\mu}(\angle(a, c, d)) \bullet \boldsymbol{\mu}(\angle(d, c, b))$.* □

Moreover for any $x \in \mathbf{E}^*$ let $\xi := [x]_+ \cap \mathbf{E}_1$. Then the pair $(|x|, \xi) \in W_+ \times \mathbf{E}_1$ is called the *polar coordinates* of x, and the function $pc : \mathbf{E}^* \to W_+ \times \mathbf{E}_1; x \mapsto (|x|, [x]_+ \cap \mathbf{E}_1)$ is a bijection; for if $\xi \in \mathbf{E}_1$ and $w \in W_+$ are given then $x := \xi^\bullet(w)$ is exactly the point with the polar coordinates (w, ξ).

4. Direct sums and direct pairs

Since for each $a \in \mathbf{E}_1$ the motion $a^\bullet = \widehat{e_1 a} \circ \tilde{W}$ is an automorphism of the loop $(\mathbf{E}, +)$ we set $\bullet : \mathbf{E}_1 \times \mathbf{E} \to \mathbf{E}; (a, x) \mapsto a \bullet x := a^\bullet(x)$ and call the elements of $\mathbf{E}_1$ *multipliers*. To each $p \in \mathbf{E}^*$ we associate the multiplier $p_1 := [p]_+ \cap \mathbf{E}_1$ then:

(4.1) $\forall a, b \in \mathbf{E}_1, \forall x, y \in \mathbf{E}, \forall p \in \mathbf{E}^*$:

(1) $e_1 \bullet x = x$, $|a \bullet x| = |x|$ and $(a \bullet p)_1 = a \bullet p_1$
(2) $a \bullet (x + y) = a \bullet x + a \bullet y$
(3) $a \bullet (b \bullet x) = (a \bullet b) \bullet x$
(4) $\mathbf{E}_1 \bullet x := \{a \bullet x \mid a \in \mathbf{E}_1\}$ is a circle with center o passing through x
(5) $p = p_1 \bullet |p|$, i.e. $(|p|, p_1)$ are the polar coordinates of p
(6) $-e_1 \in \mathbf{E}_1$, $(-e_1)^\bullet = \tilde{o}$ and $(-e_1) \bullet x = -x$.
(7) $a \bullet [p] = [a \bullet p]$. □

We call the commutative group $(W, +)$ *scalar domain* and their elements *scalars* and introduce between W and $\mathbf{E}^*$ by:

$$\oplus : W \times \mathbf{E}^* \to \mathbf{E}; (w, p) \mapsto w \oplus p := p_1 \bullet (w + |p|) = p_1 \bullet w + p$$

an operation which has the properties:

(4.2) *For all* $u, v \in W$, *for all* $p \in \mathbf{E}^*$:

(1) $o \oplus p = p$
(2) $((u + v) \oplus p) + p = (u \oplus p) + (v \oplus p)$
(3) *If* $u \geq o$ *then* $|u \oplus p| = u + |p|$
(4) $W \oplus p = [p]$, $W_+ \oplus p = [p]_+$. □

If $a, b \in \mathbf{E}^*$ and $u, v \in W$ then the expression $(u \oplus a) + (v \oplus b)$ shall be called *scalar combination* of a and b. Then:

(4.3) *For all* $a, b \in \mathbf{E}^*$, *for all* $u, v \in W$, *for all* $c \in \mathbf{E}_1$: $c \bullet (u \oplus a) = u \oplus (c \bullet a)$, $c \bullet ((u \oplus a) + (v \oplus b)) = (u \oplus (c \bullet a)) + (v \oplus (c \bullet b))$.

(4.4) *Let* $a, c \in \mathbf{E}^*$ *with* $[a] \neq [c]$ *and let* $b \in [a] \setminus \{a\}$ *then:*

(1) $[a] \cap (a + [c]) = \{a\}$
(2) $(b + [c]) \cap (a + [c]) = \emptyset$
(3) $\forall p \in \mathbf{E}$ *there is at most one pair* $(x, y) \in [a] \times [c]$ *such that* $p = x + y$ *, i.e. there is at most one pair* (u, v) *of scalars such that* $p = (u \oplus a) + (v \oplus c)$ *is a scalar combination of* a *and* c.

Proof. "(1)" : By assumption $[a] \cap [c] = \{o\}$, since $[a]$ is a subgroup of the loop $(\mathbf{E}, +)$ and a^+ a permutation we have: $\{a\} = (a + [a]) \cap (a + [c]) = [a] \cap (a + [c])$.

"(2)": Let $a' := Fix\, \widetilde{oa}$, $b' := Fix\, \widetilde{ob}$ hence $\widetilde{a'} = \widetilde{oa}$, $\widetilde{b'} = \widetilde{ob}$ and a' resp. b' is the midpoint of $\{o, a\}$ resp. $\{o, b\}$. By $b \in [a]$ follows $o, a', b' \in [a]$ hence by (2.3.1) there is a $d' \in [a]$ with $\widetilde{d'} = \widetilde{b'} \circ \widetilde{a'} \circ \tilde{o} = \widetilde{ob} \circ \widetilde{oa} \circ \tilde{o}$. Since $\tilde{o}([c]) = [c]$ we obtain by (2.3.2) :

$(b + [c]) \cap (a + [c]) = \widetilde{ob}([c]) \cap \widetilde{ob} \circ \widetilde{oa}([c])) = \widetilde{ob}([c] \cap \widetilde{d'}([c]) \neq \emptyset \Longleftrightarrow$

$[c] \cap \widetilde{d'}([c]) \neq \emptyset \Longleftrightarrow d' \in [a] \cap [c] = \{o\} \Longleftrightarrow \widetilde{oa} = \widetilde{ob} \Longleftrightarrow a = b.$

Since $a \neq b$, (2) is valid.

"(3)": Assume there are $(x, y), (x', y') \in [a] \times [c]$ with $p = x + y = x' + y'$ and $x \neq x'$ then for instance $x \neq o$ and so $x' \in [a] = [x]$. Thus $p \in (x + [c]) \cap (x' + [c])$ and by (2), $(x + [c]) \cap (x' + [c]) = \emptyset$, a contradiction. Hence $x = x'$ and so $y = y'$. □

A pair $(a, b) \in \mathbf{E} \times \mathbf{E}$ is called a *direct pair* if $[a] + [b] := \{x + y \mid x \in [a], y \in [b]\} = \mathbf{E}$ or equivalently if $\mathbf{E} = (W \oplus a) + (W \oplus b)$. Then by (4.4.3) for every direct pair (a, b) the loop $(\mathbf{E}, +)$ is representable as *direct sum* of the commutative subgroups $[a]$ and $[b]$, i.e. each element $p \in \mathbf{E}$ is uniquely representable as a scalar combination of a and b.

(4.5) *Let* $a, b \in \mathbf{E}^*$ *with* $[a] \perp [b]$ *then* (a, b) *is a direct pair.*

Proof. Let $p \in \mathbf{E}$, $x := (p \perp [a]) \cap [a]$ then $[a] \perp [b]$, $x + [a] = \widetilde{ox}([a]) = [a]$ and $x + [b] = \widetilde{ox}([b])$ imply $[a] \perp (x + [b])$ hence $p \in (p \perp [a]) = x + [b]$ and so there is a $y \in [b]$ with $p = x + y$. □

We call the K-loop $(\mathbf{E}, +)$ of an absolute plane *vectorspacelike* if for all $a, b \in \mathbf{E}^*$ with $[a] \neq [b]$, (a, b) is a direct pair.

(4.6) *The K-loop of a singular plane is vectorspacelike if and only if the plane is Euclidean.*

Proof. By (2.5.2) $(\mathbf{E}, +)$ is a commutative group. Let $a, b \in \mathbf{E}^*$ with $[a] \neq [b]$ and let $p = x + y$ with $x \in [a]$ and $y \in [b]$ then $p = (x + [b]) \cap (y + [a]) = (p + [b]) \cap (p + [a])$.

Therefore (a, b) is a direct pair if for all $p \in \mathbf{E}$ holds:

$(p + [a]) \cap [b] \neq \emptyset$ and $(p + [b]) \cap [a] \neq \emptyset$. Clearly, if the parallelaxiom is valid then this condition is satisfied:

for let $u, x, y, z \in \mathbf{E}$ with $y, z \neq o$ then $(u + [y]) \parallel (x + [z]) \quad \Longleftrightarrow \quad [y] = [z]$.

If the parallelaxiom is not satisfied then by (2.2.2) there exist lines C, $[a]$, $[b]$ with $[a] \neq [b]$ and $C \cap ([a] \cup [b]) = \emptyset$. If $C = d + [c]$ then at least one of the statements $[c] \neq [a]$ or $[c] \neq [b]$ is true for instance $[c] \neq [a]$ .Since $(d + [c]) \cap [a] = C \cap [a] = \emptyset$ it follows that (a, c) is not a direct pair. □

Next we consider the case $(a, b) \in \mathbf{E}^* \times \mathbf{E}^*$ with $[a] \neq [b]$ and $[a] \not\perp [b]$. Then there are $a_1 \in [a] \cap \mathbf{E}_1$, $b_1 \in [b] \cap \mathbf{E}_1$ such that $\gamma := \angle(b_1, o, a_1)$ is an acute angle hence $\boldsymbol{\mu}(\gamma) = a_1^{-1} \bullet b_1 \in \mathbf{E}_1$ with $\omega(e_1, \boldsymbol{\mu}(\gamma), e_2) = 1$. We show:

(4.7) *For* $a, b \in \mathbf{E}_1$ *with* $\omega(e_1, a^{-1} \bullet b, e_2) = 1$ *the following statements are equivalent:*

(1) (a, b) is a direct pair

(2) $\forall w \in W_+ : [b] \cap D([a]; w) \neq \emptyset$
(3) $\forall w \in W_+$ exists a rectangular triangle $\Delta = (p, q, r)$ with $\overline{p, q} \perp \overline{r, q}$, $\boldsymbol{\mu}(\angle(r, p, q)) = a^{-1} \bullet b$ and $\boldsymbol{\lambda}(r, q) = w$.

Proof. By (2.6) and (4.3), $(a^{-1})^{\bullet}$ is a proper motion and an automorphism of $(\mathbf{E}, +, W; \oplus)$. Therefore we may assume $a = e_1$ and $\omega(e_1, b, e_2) = 1$.

"(1) $\Rightarrow$ (2), (3)". Let $w \in W_+$ be given. Since (a, b) is a direct pair there are uniquely determined scalars $u, v \in W$ such that $e_2 \bullet w = (u \oplus e_1) + (v \oplus b) = (u + e_1) + (v \oplus b)$.

Since $w > o$ and $\omega(e_1, b, e_2) = 1$ we have $v > o$ and $u + e_1 < o$. We consider the triangle $\Delta := (o, -(u + e_1), (v \oplus b))$ which has the properties:

1. Since $u + e_1 \in W$ and $o \neq u + e_1$ we have $\overline{o, -(u + e_1)} = W = [e_1]$ and $(-(u + e_1))^+$ is a proper motion fixing the line $[e_1]$. Since $[e_1] \perp [e_2]$ also the lines $[e_1]$ and $(-(u + e_1))^+([e_2]) = -(u + e_1) + [e_2]$ are orthogonal. The line $-(u + e_1) + [e_2]$ contains the points $-(u + e_1)$ and $-(u \oplus e_1) + e_2 \bullet w = -(u \oplus e_1) + ((u \oplus e_1) + (v \oplus b)) = v \oplus b$. Therefore Δ is rectangular with $\overline{o, -(u + e_1)} \perp \overline{-(u + e_1), v \oplus b}$ and so $-(u + e_1)$ is the orthogonal projection of $(v \oplus b)$ onto $[e_1]$. Hence: $\boldsymbol{\lambda}(v \oplus b, [e_1]) = \boldsymbol{\lambda}(-(u + e_1), v \oplus b) = |-(u + e_1) - (v \oplus b)| = |-e_2 \bullet w| = |w| = w$ implying $v \oplus b \in D([e_1]; w)$, i.e. $[b] \cap D([e_1]; w) \neq \emptyset$ and (2) is proved. Finally since $v > o$ and $-(u + e_1) > o$ we have $[v \oplus b]_+ = [b]_+$ and $[-(u + e_1)]_+ = [e_1]_+$ hence $\angle(v \oplus b, o, -(u + e_1)) = \angle(b, o, e_1)$ and so $\boldsymbol{\mu}(\angle(b, o, e_1)) = b$, i.e. also (3) is proved. "(2) $\Rightarrow$ (1)". Let $p \in \mathbf{E}$ be given. If $p \in [e_1]$ then $p = p + o$ with $o \in [b]$. Therefore let $p \notin [e_1]$. Then by assumption (2) there is exactly one $v \in W$ such that $\{v \oplus b\} = [b] \cap D([e_1], p)$. Let $p_W := (p \perp [e_1]) \cap [e_1]$ and $(v \oplus b)_W := (v \oplus b \perp [e_1]) \cap [e_1]$, then $-p_W + p = -(v \oplus b)_W + (v \oplus b) \in [e_2]$ and since $(W, +)$ is a commutative group there is exactly one $u \in W$ such that $p_W = (u \oplus e_1) + (v \oplus b)_w = (u \oplus e_1)^+(v \oplus b)_W$. Consequently: $p = p_W^+ \circ (-(v \oplus b)_W)^+(v \oplus b) = (u \oplus e_1)^+ \circ ((v \oplus b)_W)^+ \circ (-(v \oplus b)_W)^+(v \oplus b) = (u \oplus e_1)^+(v \oplus b) = (u \oplus e_1) + (v \oplus b)$. $\square$

From (4.7) follows:

(4.8) The K-loop of an absolute plane is vectorspacelike if and only if : $\exists A \in \mathcal{G}$ and $a \in A : \forall G \in \mathcal{G} \setminus \{A\}$ with $a \in G, \forall x \in \mathbf{E} \setminus A : G \cap D(A, x) \neq \emptyset$.

5. b-Rings, rotational extensions and quasidilatations

Quasidilatations for the K-loop of an absolute geometry were introduced in [4]. In order to define them we consider firstly the ordered commutative group$(W, +, <)$ (cf.(3.1)).

Let ξ denote the betweenness relation on W corresponding to $<$, let $Iso(W, +, <)$ resp. $Bet(W, +, \xi)$ be the set of all endomorphisms of the group $(W, +)$ which are strictly isotone resp. which preserve the betweenness relation ξ on W, let: $\nu : W \to W;\ x \mapsto -x := \tilde{o}(x)$ and let $Mon(W,+)$ be the set of all monomorphisms of $(W,+)$. Then $Bet(W,+,\xi) = Iso(W,+,<) \dot{\cup}\ \nu \circ Iso(W,+,<) \subseteq Mon(W,+)$ where $\nu \circ Iso(W,+,<)$ is the set of all antitone monomorphisms.

$(Bet(W,+,\xi),\circ)$, $(Iso(W,+,<),\circ)$ and $(Iso(W,+,<),+)$ are semigroups. The automorphism groups $Aut(W,+,\xi)$ resp. $Aut(W,+,<)$ are subgroups of $(Bet(W,+,\xi),\circ)$ resp. $(Iso(W,+,<),\circ)$ and $Aut(W,+,\xi) = Aut(W,+.<) \dot{\cup} \nu \circ Aut(W,+,<)$.

We show:

(5.1) $(W,+)$ *is uniquely divisible by 2 : for* $a \in W$ *let* $\frac{1}{2}a$ *be the midpoint of* o *and* a *then* $\frac{1}{2}a \in W$ *and* $\frac{1}{2}a + \frac{1}{2}a = a$.

Proof. Let $a' := \frac{1}{2}a$ then $a' + a' = \widetilde{oa'} \circ \tilde{o}(a') = \tilde{a'} \circ \widetilde{oa'}(a') = \tilde{a'}(o) = \widetilde{oa}(o) = a$ and if $a = b + b = \widetilde{ob} \circ \tilde{o}(b) = \tilde{b} \circ \widetilde{ob}(b) = \tilde{b}(o)$ then b is the midpoint of o and a hence $b = a'$. ☐

Since $(W,+,<)$ is an ordered commutative group , $(W,+)$ is a $\mathbf{Z}$- module such that $\forall n \in \mathbf{Z}^* := \mathbf{Z} \setminus \{0\}$, the map $n^{\cdot} : W \to W; x \mapsto n \cdot x$ is a monomorphism where $n^{\cdot}$ is isotone if $n \in \mathbf{N}$ and antitone if $-n \in \mathbf{N}$. By (5.1) , $2^{\cdot}$ is even an automorphism with $(2^{\cdot})^{-1}(x) = \frac{1}{2}x$.

Therefore:

(5.2) *Let* $\mathbf{P}_W := \{p \in \mathbf{P} \mid p^{\cdot} \in SymW\}$ *be the set of all prime numbers* p *such that* $p^{\cdot}$ *is even an automorphism of* $(W,+)$, *let* $\mathbf{N}_W$ *be the set of all natural numbers which are products of prime numbers of* $\mathbf{P}_W$ *and let* $\mathbf{Z}_W := \{\frac{m}{n} \mid m \in \mathbf{Z}, n \in \mathbf{N}_W\}$ *be the subring of the field* $\mathbf{Q}$ *consisting of all fractions where the denominator is an element of* $\mathbf{N}_W$. *Then:*

(1) $2 \in \mathbf{P}_W$ *(by (5.1)) and* $\mathbf{Z}_2 := \{m \cdot 2^{-n} \mid m \in \mathbf{Z}, n \in \mathbf{N} \cup \{0\}\} \subseteq \mathbf{Z}_W$.
(2) $\forall\, r = \frac{m}{n} \in \mathbf{Z}_W^*$ *the map* $r^{\cdot} = m^{\cdot} \circ ((n^{\cdot})^{-1}$ *is a monomorphism of* $(W,+)$ *and* $r^{\cdot}$ *is strictly isotone resp. antitone if* $r > 0$ *resp.* $r < 0$.
(3) *If* $r := \frac{m}{n}$ *is a unit of* $\mathbf{Z}_W$ *hence if* $m \in \mathbf{N}_W$ *then* $r^{\cdot}$ *is an automorphism of* $(W,+)$.
(4) $(-e_1)^{\bullet}_{|W} = (-1)^{\cdot}$ *is an antitone automorphism of* $(W,+)$.
(5) $\mathbf{Z}_W$ *is a subring of* $End(W,+)$ *with* $\mathbf{Z}_W^* := \mathbf{Z}_W \setminus \{0\} \subseteq Bet(W,+,\xi)$.

(5.3) *If* $\mathbf{P}_W = \mathbf{P}$ *, i.e. for each* $n \in \mathbf{N}$, $n^{\cdot}$ *is a permutation of* W *then* $\mathbf{Z}_W = \mathbf{Q}$ *and* $(W,+)$ *is a* $\mathbf{Q}$*-module, i.e.* $(W,\mathbf{Q})$ *is a vectorspace.*

A subring B of the endomorphismring $End(W,+)$ is called *b-ring of* $(W,+)$ if $\mathbf{Z}_W \subseteq B$ and $B^* := B \setminus \{0\} \subseteq Bet(W,+,\xi))$.

By (5.2.5) $\mathbf{Z}_W$ is a b-ring of $(W,+,<)$.

Now let B be a b-ring of $(W, +, <)$. Then $B^* := B \setminus \{o\} \subseteq Bet(W, +, \xi) \subseteq Mon(W, +)$ implies that B^* is a subsemigroup of $(Mon(W, +), \circ)$ and so the map $\iota : B \to W; \beta \mapsto \beta(e_1)$ is injective. If $\beta_i \in B,\ i \in \{1, 2\}$ and $b_i := \beta_i(e_1)$ then $\beta_1 + \beta_2 \in B$ and so $b_1 + b_2 = \beta_1(e_1) + \beta_2(e_1) = (\beta_1 + \beta_2)(e_1)$. Therefore ι is a monomorphism from $(B, +)$ into $(W, +)$ hence $\iota(B)$ a subgroup of $(W, +)$ isomorphic with $(B, +)$. We identify always B and $\iota(B)$ and if for $\beta \in B$ and $b := \iota(\beta) = \beta(e_1)$ we set $b^{\cdot} := \beta$ and define :

$\cdot : B \times W \to W; (b, w) \mapsto b \cdot w := b^{\cdot}(w)$.

If $B = W$ then the b-ring B is called *transitive*.

(5.4) *Let $(B, +, \circ)$ be a b-ring of $(W, +, <)$. Then for $a, b \in B^*$ and $x, y \in W$ we have : $e_1 \in B$, $e_1^{\cdot} = id$, $e_1 \cdot x = x$, $a \cdot e_1 = e_1 \cdot a = a, a \cdot b = a^{\cdot}(b) = a^{\cdot} \circ b^{\cdot}(e_1) \in B$ hence $(a \cdot b)^{\cdot} = a^{\cdot} \circ b^{\cdot}$ and $a \cdot (b \cdot x) = (a \cdot b) \cdot x, (a + b) \cdot x = a \cdot x + b \cdot x, a \cdot (x + y) = a \cdot x + a \cdot y, a \cdot x = a \cdot y \Longleftrightarrow x = y$ and $a \cdot x = b \cdot x \Longleftrightarrow a = b$ or $x = o$.* □

This shows: $((W, +), B, \cdot)$ is a *nearfield* in the sense of H.Zassenhaus (cf.[9],[3] p.2) (i.e. $(W, +)$ is a group , $B \subseteq W$ with $B^* \neq \emptyset$ and if $a, b \in B$ then $a^{\cdot}(b) \in B$ and $(a^{\cdot}(b))^{\cdot} = a^{\cdot} \circ b^{\cdot}$, i.e. $(B, \cdot)$ is a semigroup , if $x \in W^*$ with $a^{\cdot}(x) = b^{\cdot}(x)$ then $a = b$ and $B^{*\cdot} := \{b^{\cdot} \mid b \in B^*\}$ is a subgroup of the automorphism group $Aut(W, +)$.)[2] Moreover $B_+ := B \cap W_+$ is a subsemigroup of $(B^*, \cdot)$ and $B_+ \cdot W_+ = W_+$.

(5.5) If $(B, +, \circ)$ is a transitive b-ring of $(W, +, <)$ hence $B := \iota(B) = B(e_1) = W$ then $(W, +, \cdot)$ is a complete nearfield even a field and $(W, +, \cdot, <)$ is an ordered field.

Proof. $B^* \subseteq Bet(W, +, \xi) \subseteq Mon(W, +)$, $B^*(e) = W^*$ and (5.4) imply that $(W^*, \cdot)$ is a group hence by (5.4) $(W, +, \cdot)$ is a field and so if $a \in W^*$ then $a^{\cdot}$ is an automorphism of $(W, +)$.

Consequently $B^* \subseteq Aut(W, +, \xi) = Aut(W, +, <) \,\dot{\cup}\, \nu \circ Aut(W, +, <)$.

Let $a < b$ and $o < c$. Then $c^{\cdot} \in Aut(W, +, \xi)$, $o < e_1$ and $c^{\cdot}(e_1) = c$ imply $c^{\cdot} \in Aut(W, +, <)$ and therefore $c \cdot a = c^{\cdot}(a) < c^{\cdot}(b) = c \cdot b$. Moreover $a < b$ hence $o < -a + b$ implies $(-a + b)^{\cdot} \in Aut(W, +, <)$ and so $o < (-a + b)^{\cdot}(c) = (-a + b) \cdot c$. Since $(W, +, \cdot)$ is a field we obtain $o < -a \cdot c + b \cdot c$, i.e. $a \cdot c < b \cdot c$. □

REMARK. If $(W, +, <)$ is an archimedian ordered group then (by the theorem of O. Hölder) $(W, +)$ is isomorphic to a subgroup of $(\mathbf{R}, +)$ (resp. to $(\mathbf{R}, +)$). Therefore:

(5.6) If $(W, +, <)$ is continuous then $(\mathbf{R}, +, \cdot)$ is a transitive b-ring of $(W, +, <)$ and $(W, +)$ can be provided with a multiplication "$\cdot$" such that $(W, +, \cdot)$ is a field isomorphic to $(\mathbf{R}, +, \cdot)$.

[2] Zassenhaus calls a nearfield *complete* if $B = W$. Today the notion "nearfield" is used for complete nearfields in the sense of Zassenhaus.

We call a map $\varphi : \mathbf{E} \to \mathbf{E}$ *rotational* (homogenous) if : $\forall a \in \mathbf{E}_1 : \varphi \circ a^\bullet = a^\bullet \circ \varphi$.

A rotational map φ fixes o and is completely determined by its restriction $\varphi_{|W_+}$: for if $x = x_1 \bullet |x| \in \mathbf{E}^*$ is given by its polar coordinates then $\varphi(x) = \varphi(x_1 \bullet |x|) = \varphi(x_1^\bullet \circ |x|) = x_1^\bullet \circ \varphi(|x|) = x_1^\bullet \circ \varphi_{|W_+}(|x|)$ and since $W = \{o\} \dot\cup W_+ \dot\cup (-e_1) \bullet W_+$ and $[x] = x_1 \bullet W = \{o\} \dot\cup x_1 \bullet W_+ \dot\cup x_1 \bullet (-e_1) \bullet W_+$ we have : $\varphi([x]) = x_1 \bullet \varphi(W) = \{o\} \dot\cup x_1 \bullet \varphi(W_+) \dot\cup x_1 \bullet (-e_1) \bullet \varphi(W_+)$.

Conversely:

(5.7) *Any map* $\psi : W_+ \to \mathbf{E}$ *can be uniquely extended to a rotational map* $\bar\psi : \mathbf{E} \to \mathbf{E}$ *by* $\bar\psi(x) = \bar\psi(x_1 \bullet |x|) := x_1^\bullet(\psi(|x|))$ *for all* $x \in \mathbf{E}^*$. $\bar\psi$ *is then called* rotational extension *of* ψ.

Proof. We have to show that $\bar\psi$ is rotational. Let $a \in \mathbf{E}_1$ and $x = x_1 \bullet |x| \in \mathbf{E}^*$ then $a \bullet x_1 \in \mathbf{E}_1$ (cf. (2.6)) and $(a \bullet x_1)^\bullet = a^\bullet \circ x_1^\bullet$ hence $\bar\psi \circ a^\bullet(x) = \bar\psi \circ a^\bullet(x_1^\bullet(|x|)) = \bar\psi((a \bullet x_1)^\bullet(|x|) = (a \bullet x_1)^\bullet(\psi(|x|)) = a^\bullet \circ x_1^\bullet(\psi(|x|)) = a^\bullet \circ \bar\psi(x)$. □

If A is an arbitrary set then any two maps $\varphi, \psi \in Map(A, \mathbf{E})$ from A into the loop $(\mathbf{E}, +)$ can be added with the help of the loop operation "+" by: $(\varphi + \psi)(x) := \varphi(x) + \psi(x)$ for $x \in A$.

Then $\varphi + \psi \in Map(A, \mathbf{E})$ and so $(Map(A, \mathbf{E}), +)$ is also a loop. The properties of the loop $(\mathbf{E}, +)$ pass on $(Map(A, \mathbf{E}), +)$, i.e. in our case $(Map(A, \mathbf{E}), +)$ is a K-loop too. For $A = \mathbf{E}$ we set $Map(\mathbf{E}) := Map(\mathbf{E}, \mathbf{E})$. In this case with $\varphi, \psi, \chi \in Map(\mathbf{E})$ also $\varphi \circ \psi \in Map(\mathbf{E})$, (i.e. $(Map(\mathbf{E}), \circ)$ is a semigroup) and $(\varphi + \psi) \circ \chi = \varphi \circ \chi + \psi \circ \chi$. This shows that $(Map(\mathbf{E}), +, \circ)$ is a (right) K-loop-nearring (cf.[8]).

Let $\mathcal{R}(\mathbf{E}, o) := \{\varphi \in Map(\mathbf{E}) \mid \forall a \in \mathbf{E}_1 : \varphi \circ a^\bullet = a^\bullet \circ \varphi\}$ be the set of all *rotational maps* of the loop $(\mathbf{E}, +)$, let $\mathcal{R}(\mathbf{E}, [\ ]) := \{\varphi \in \mathcal{R}(\mathbf{E}, o) \mid \forall x \in \mathbf{E}$ with $\varphi(x) \neq o : \varphi([x]) \subseteq [\varphi(x)]\}$, $\mathcal{R}(\mathbf{E}, [[\ ]]) := \{\varphi \in \mathcal{R}(\mathbf{E}, o) \mid \forall x \in \mathbf{E}^* : \varphi(x) \subseteq [x]\}$ and $\mathcal{R}(W, o) := \{\varphi \in Map(W) \mid \nu \circ \varphi = \varphi \circ \nu\}$. Then we can show:

(5.8)

1. $\mathcal{R}(\mathbf{E}, o)$ is a subloop-nearring of the K-loop-nearring $(Map(\mathbf{E}), +, \circ)$.
2. $\mathcal{R}(W, o)$ is a subnearring of the nearring $(Map(W), +, \circ)$.
3. $(\mathcal{R}(\mathbf{E}, [[\ ]]), \circ) \leq (\mathcal{R}(\mathbf{E}, [\ ]), \circ) \leq (\mathcal{R}(\mathbf{E}), \circ)$ and $\mathbf{E}_1^\bullet \leq (\mathcal{R}(\mathbf{E}, [\ ]), \circ)$.
4. $\mathcal{R}(\mathbf{E}, [[\ ]])$ is a subnearring of $(\mathcal{R}(\mathbf{E}, [\ ]), +, \circ)$ and $(\mathcal{R}(\mathbf{E}, [[\ (]]), +, \circ)$ is isomorphic to $(\mathcal{R}(W, o), +, \circ)$: The map $\iota : \mathcal{R}(W, o) \to \mathcal{R}(\mathbf{E}, [[\ ]])$; $\varphi \mapsto \overline{(\varphi|W_+}$ (where $\overline{\varphi|W_+}$ denotes the rotational extension of the restriction $\varphi|W_+$) is an isomorphism from $(\mathcal{R}(W, o), +, \circ)$ onto $(\mathcal{R}(\mathbf{E}, [\ ]), +, \circ)$.
5. The endomorphismring $End(W, +)$ is a subring of the nearring $\mathcal{R}(W, o), +, \circ)$ and so $En(\mathbf{E}, o) := \iota(End(W, +))$ is a subring of the nearring $(\mathcal{R}(\mathbf{E}, [[\ ]]), +, \circ)$. The

elements φ of $En(\mathbf{E}, o)$ are rotational maps characterized by: If $x, y \in \mathbf{E}$ with $[x] = [y]$ then $\varphi(x + y) = \varphi(x) + \varphi(y)$.

Proof. Let $\varphi, \psi \in \mathcal{R}(\mathbf{E}, +)$, $a \in \mathbf{E}_1$, $x \in \mathbf{E}$ and observe $a^\bullet \in Aut(\mathbf{E}, +)$ (cf.(2.6.1)) then $(\varphi + \psi) \circ a^\bullet(x) = \varphi \circ a^\bullet(x) + \psi \circ a^\bullet(x) = a^\bullet(\varphi(x)) + a^\bullet(\psi(x)) = a^\bullet(\varphi(x) + \psi(x)) = a^\bullet \circ (\varphi + \psi)(x)$ and $(\varphi \circ \psi) \circ a^\bullet = \varphi \circ a^\bullet \circ \psi = a^\bullet \circ (\varphi \circ \psi)$. Hence $\varphi + \psi, \varphi \circ \psi \in \mathcal{R}(\mathbf{E}, o)$. This shows (1).

Since with $(W, +)$ also $(Map(W), +)$ is a commutative group, $(Map(W), +, \circ)$ is a nearring and with the previous arguments, (2) is proved.

By (4.1.6) $-\varphi = (-e_1)^\bullet \circ \varphi \in \mathcal{R}(\mathbf{E}, o)$. Now assume moreover $\varphi, \psi \in \mathcal{R}(\mathbf{E}, [\ ])$ and $\varphi \circ \psi(x) \neq o$. Then (since $\varphi(o) = o$) $\psi(x) \neq o$ and so $\varphi(\psi([x])) \subseteq \varphi([\psi(x)]) \subseteq [\varphi(\psi(x))] = [\varphi \circ \psi(x)]$. If even $\varphi, \psi \in \mathcal{R}(\mathbf{E}, [[\ ]])$ and $x \in \mathbf{E}^*$ then $\varphi(\psi([x])) \subseteq \varphi([x]) \subseteq [x]$ and $(\varphi + \psi)([x]) = \{(\varphi + \psi)(y) = \varphi(y) + \psi(y) \mid y \in [x]\} \subseteq [x] + [x] \subseteq [x]$, i.e.$\varphi \circ \psi, \varphi + \psi \in \mathcal{R}(\mathbf{E}, [[\ ]])$. Moreover by (4.1.7), $a^\bullet([x]) = [a^\bullet(x)]$ hence (3) is completely proved.

If $\psi \in \mathcal{R}(\mathbf{E}, [[\ ]])$ then $\psi(W) = \psi([e_1]) \subseteq [e_1] = W$, $\psi(o) = o$ and if $w \in W$ then $\psi(-w) = \psi(\nu(w)) = \psi((-e)^\bullet(w) = (-e)^\bullet \circ \psi(w) = \nu(\psi(w))$ hence $\varphi := \psi|W \in \mathcal{R}(W, \circ)$ and so φ is completely determined by $\varphi|W_+$ and by (5.7) we have firstly $\psi = \overline{\varphi|W_+}$ and secondly that ι is injective and surjective. Clearly if $\varphi, \psi \in \mathcal{R}(W, o)$ and $x \in \mathbf{E}^*$ then by (5.7) and $x_1^\bullet \in Aut(\mathbf{E}, +)$, $\overline{\varphi|W_+}(x) + \overline{\psi|W_+}(x) = x_1^\bullet(\varphi(|x|) + x_1^\bullet(\psi(|x|) = x_1^\bullet(\varphi(|x|) + \psi(|x|)) = x_1^\bullet \circ (\varphi + \psi)(|x|) = \overline{(\varphi + \psi)|W_+}(x)$, i.e. $\overline{\varphi|W_+} + \overline{\psi|W_+} = \overline{(\varphi + \psi)|W_+}$. Furthermore $\overline{(\varphi \circ \psi)|W_+}(x) = x_1^\bullet(\varphi \circ \psi)(|x|) = x_1^\bullet \circ (\psi(|x|))_1^\bullet(\varphi(|\psi(|x|)|))$ and observing (5.7), $\overline{\varphi|W_+} \circ \overline{\psi|W_+}(x) = \overline{\varphi|W_+}(x_1^\bullet \circ \psi(|x|)) = x_1^\bullet \circ \overline{\varphi|W_+}(\psi(|x|)) = x_1^\bullet \circ (\psi(|x|))_1^\bullet(\varphi(|\psi(|x|)|))$. Thus ι is an isomorphism.

Since $(W, +)$ is a commutative group the map $\nu : W \to W; w \mapsto -w$ is an automorphism of $(W, +)$ hence $\nu \in End(W, +)$ and if $\varphi \in End(W, +)$ and $x \in W$ then $\varphi \circ \nu(x) = \varphi(-x) = -\varphi(x) = \nu \circ \varphi(x)$ hence $End(W, +) \leq (\mathcal{R}(W, o), +, \circ)$. The other statements of (5) are a consequence of (4).

Now let B be a b-ring of $(W, +, <)$ and let $\lambda \in B$ be the rotational extension of the leftmultiplication $\lambda_l : W \to W; w \mapsto \lambda \cdot w$ (cf. (5.4)) hence $\lambda^\cdot : \mathbf{E} \to \mathbf{E}; x = x_1 \bullet |x| \mapsto x_1 \bullet (\boldsymbol{\lambda} \cdot |x|)$ is called a *B-quasidilatation*. By [4]p.407 follows:

(5.9) *Let B be a b-ring of $(W, +, <)$, let U be the set of units of $(B, +, \cdot)$ and let $\mathcal{F} := \{[x] \mid x \in \mathbf{E}^*\}$. Then $(\mathbf{E}, +, \mathcal{F}, B, \cdot)$ is a structure where $(\mathbf{E}, +, \mathcal{F})$ is a loop with an incidence fibration and $\cdot : B \times \mathbf{E} \to \mathbf{E}$; $(\lambda, x) \mapsto \lambda \cdot x := \lambda^\cdot(x)$ is a map such that for all $\lambda, \mu \in B$, for all $X \in \mathcal{F}$ and for all $a, b \in \mathbf{E}$ the following hold:*

(1) $\lambda \cdot a = o \Leftrightarrow \lambda = 0$ *or* $a = o$.
(2) *If* $\lambda \in U$, *then* $\lambda \cdot \mathbf{E} = \mathbf{E}$ *and* $\lambda \cdot X = X$.

(3) $(\lambda \cdot \mu) \cdot a = \lambda \cdot (\mu \cdot a)$, $(\lambda + \mu) \cdot a = \lambda \cdot a + \mu \cdot a$.
(4) *If* $a, b \in X$ *then* $\lambda \cdot (a + b) = \lambda \cdot a + \lambda \cdot b$.

(5.10) *For* $\lambda \in \mathbf{Z}_W \setminus \{0, 1\}$ *the* $\mathbf{Z}_W$*-quasidilatation* $\lambda^{\cdot}$ *is a collination of* $(\mathbf{E}, \mathcal{G})$ *if and only if* $(\mathbf{E}, \mathcal{G}, \alpha, \equiv)$ *is singular.*

Let B be a b-ring of our absolute plane , i.e. of $(W, +, <)$. If $a, b \in \mathbf{E}$ and $\lambda, \mu \in B$ then the expression $\lambda \cdot a + \mu \cdot b$ shall be called *quasilinear B-combination* or shortly *q-linear B-combination*.

(5.11) *If* $(W, +, <)$ *possesses a transitive b-ring (i.e. by* (5.5)*, W can be turned in an ordered field* $(W, +, \cdot, <)$*) then for all* $a, b \in \mathbf{E}^*$ *with* $[a] \neq [b]$ *each element* $x \in [a] + [b]$ *can be written uniquely as a quasilinear W-combination of a and b, i.e.* $\exists_1 (\alpha, \beta) \in W \times W$: $x = \alpha \cdot a + \beta \cdot b$. □

6. Hyperbolic planes

Among the absolute planes the hyperbolic planes $(\mathbf{E}, \mathcal{G}, \alpha, \equiv)$ are characterized by the following axiom (cf. [6]p.149):

(H) $\forall G \in \mathcal{G},\ \forall p \in \mathbf{E} \setminus G\ \ \exists H \in \mathcal{G}$ with $p \in H \wedge H \parallel_h G$

where $H \parallel_h G$ is defined by: Let $P := (p \perp G)$ then: $G \cap H = \emptyset$, $\tilde{P}(H) \neq H, \forall x \in \mathbf{E} \setminus (H \cup \tilde{P}(H))$ with $(H|x, \tilde{P}(x)) = 1 : \overline{p, x} \cap G \neq \emptyset$.

In [6] it is shown that there is a one-to-one correspondence between the hyperbolic planes and the commutative Euclidean fields $(K, +, \cdot)$. A commutative field is *Euclidean* if $K^{(2)} := \{x^2 \mid x \in K^* := K \setminus \{o\}\}$ is a positive domain. For $a \in K^*$ let $sgn\ a = 1$ if $a \in K^{(2)}$ and $sgn\ a = -1$ if $a \notin K^{(2)}$. Starting from a commutative Euclidean field $(K, +, \cdot)$ one can obtain the corresponding hyperbolic plane in the following way:

Let $(\mathcal{M}, +, \cdot)$ be the ring of all 2×2-matrices $A = (a_{ij})$ (with $a_{ij} \in K$) over the Euclidean field , let $E = (\delta_{ij})$ be the identity matrix and let $(K, +, \cdot)$ be imbedded in $(\mathcal{M}, +, \cdot)$ via the map : $K \to \mathcal{M}; u \mapsto u \cdot E$. For $A, B \in \mathcal{M}$ let :

$$\hat{A} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}, \quad A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix},$$

$A^{\square}(B) := A \cdot B \cdot A^T$ and $f(A, B) := A \cdot \hat{B} + B \cdot \hat{A}$. Then $detA = A \cdot \hat{A} = \frac{1}{2} f(A, A)$ and $TrA = A + \hat{A}$.

We denote by $\mathcal{S} := \{X \in \mathcal{M} \mid X^T = X\}$ the set of all symmetric matrices of $\mathcal{M}$ and consider the subset $\mathbf{E} := \mathcal{S}^{1,+} := \{S \in \mathcal{S} \mid S \cdot \hat{S} = 1 \wedge S + \hat{S} > 0\}$ as *point-set* of the hyperbolic plane.

For $G \in \mathcal{S}^{-1} := \{S \in \mathcal{S} \mid S \cdot \hat{S} = -1\}$ let $\underline{G} := \{X \in \mathcal{S}^{1+} \mid f(X, G) = 0\}$ and let $\mathcal{G} := \{\underline{G} \mid G \in \mathcal{S}^{-1}\}$ be the set of *lines*

NOTE: for $G, H \in \mathcal{S}^{-1}$: $\underline{G} = \underline{H} \Longleftrightarrow H \in \{G, -G\}$.

The *congruence* $\equiv$ is given by:

If $A, B, C, D \in \mathbf{E}$ then: $(A, B) \equiv (C, D) :\Longleftrightarrow f(A, B) = f(C, D)$

And the *order* α is defined by:

If $A, B \in \mathbf{E}$, $G \in \mathcal{S}^{-1}$ and $A, B \notin \underline{G}$ then: $(\underline{G}|A, B) := sgn(f(A, G) \cdot f(B, G))$.

In [6] and [5] it is shown that $(\mathbf{E}, \mathcal{G}, \alpha, \equiv)$ is a hyperbolic plane.

If $A \in \mathbf{E} = \mathcal{S}^{1+}$ and $G \in \mathcal{S}^{-1}$ then the reflection in the point A and in the line $\underline{G}$ is given by:

$\tilde{A} : \mathbf{E} \to \mathbf{E};\ X \mapsto A^{\square}(\hat{X}) = A \cdot \hat{X} \cdot A^T$ and

$\tilde{\underline{G}} : \mathbf{E} \to \mathbf{E};\ X \mapsto G^{\square}(\hat{X}) = G \cdot \hat{X} \cdot G^T$ and the foot by

$A_G := (A \perp \underline{G}) \cap \underline{G} = (2 + f(A, G)^2)^{-\frac{1}{2}}(A + G \cdot \hat{A} \cdot G)$.

Let $(o, e_1, e_2) := (E, \frac{1}{2}\left(\begin{smallmatrix}4 & 0\\ 0 & 1\end{smallmatrix}\right), \frac{1}{4}\left(\begin{smallmatrix}5 & 3\\ 3 & 5\end{smallmatrix}\right))$ be our frame of reference and let "$\Diamond$" denote the K-loop operation corresponding to the point E. Then $[e_i] = \underline{G_i}$ where $G_1 = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$, $G_2 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ hence $W := [e_1] := \{\left(\begin{smallmatrix}x & 0\\ 0 & x^{-1}\end{smallmatrix}\right) \mid x \in K^{(2)}\}$ and $W_+ := \{\left(\begin{smallmatrix}x & 0\\ 0 & x^{-1}\end{smallmatrix}\right) \mid x \in K^{(2)} : 1 < x\}$. For $A := \left(\begin{smallmatrix}a & 0\\ 0 & a^{-1}\end{smallmatrix}\right)$, $B := \left(\begin{smallmatrix}b & 0\\ 0 & b^{-1}\end{smallmatrix}\right) \in W$ we have:

$$\widetilde{EA} = \begin{pmatrix}\sqrt{a} & 0\\ 0 & \sqrt{a^{-1}}\end{pmatrix}^{\square} \circ \wedge \text{ hence } A^{\Diamond} = \widetilde{EA} \circ \tilde{E} = \begin{pmatrix}\sqrt{a} & 0\\ 0 & \sqrt{a^{-1}}\end{pmatrix}^{\square} \text{ and}$$
$$A \Diamond B = \begin{pmatrix}\sqrt{a} & 0\\ 0 & \sqrt{a^{-1}}\end{pmatrix}\begin{pmatrix}b & 0\\ 0 & b^{-1}\end{pmatrix}\begin{pmatrix}\sqrt{a} & 0\\ 0 & \sqrt{a}^{-1}\end{pmatrix} = \begin{pmatrix}ab & 0\\ 0 & (ab)^{-1}\end{pmatrix}.$$

Therefore the map $\varphi : (K^{(2)}, \cdot) \to (W, \Diamond);\quad x \mapsto \left(\begin{smallmatrix}x & 0\\ 0 & x^{-1}\end{smallmatrix}\right)$ is an isomorphism of the multiplicative group of all squares of the Euclidean field $(K, +, \cdot)$ onto the scalar domain $(W, \Diamond)$ and so $(W, \Diamond)$ can be identified with the group $(K^{(2)}, \cdot)$.

Then the absolute value $|\ | : \mathbf{E} \to W_+ \cup \{o\} = \{\lambda \in K^{(2)} \mid 1 \le \lambda\}$ is given by: $|X| = \frac{1}{2}(TrX + \sqrt{(TrX)^2 - 4})$.

Now we can prove:

(6.1) *The K-loop of a hyperbolic plane is vectorspacelike.*

Proof. By (4.8) we may consider the line $\underline{G_1}$ and the point E. The lines passing through E are given by the set of matrices

$$\mathcal{S}^{-1}(E) := \left\{ \begin{pmatrix} u & v \\ v & -u \end{pmatrix} \mid u, v \in K : u^2 + v^2 = 1 \right\},$$

and for $X = \begin{pmatrix} x & y \\ y & x^{-1}(1+y^2) \end{pmatrix} \in \mathbf{E} \setminus \underline{G_1}$ we have $y \neq 0$ and

$$D(\underline{G_1}, X) = \left\{ \begin{pmatrix} \lambda^2 x & y \\ y & \lambda^{-2}x^{-1}(1+y^2) \end{pmatrix} \mid \lambda \in K^* \right\}$$
$$= \left\{ \begin{pmatrix} \lambda x & y \\ y & \lambda^{-1}x^{-1}(1+y^2) \end{pmatrix} \mid \lambda \in K^{(2)} \right\}. \qquad (6.1)$$

Now let $U = \begin{pmatrix} u & v \\ v & -u \end{pmatrix} \in \mathcal{S}^{-1}(E)$ with $\underline{U} \neq \underline{G_1}$, i.e. $v \neq 0$. Then

$\underline{U} \cap D(\underline{G_1}, X) = \{ \begin{pmatrix} \lambda x & y \\ y & \lambda^{-1}x^{-1}(1+y^2) \end{pmatrix} \mid \lambda \in K^{(2)} : \; (*)\ \lambda^2 ux + 2yv - ux^{-1}(1+y^2) = 0 \}$.

The equation (*) has a solution if the discriminant $d = u^2(1+y^2) + y^2v^2 \in K^{(2)}$. But since $(K, +, \cdot)$ is an Euclidean field and since $y, v \neq 0$ we have $d \in K^{(2)}$. Thus the criterion (4.8) is fulfilled and any hyperbolic plane is vectorspacelike. □

From (5.6),(5.11) and (6.1) we obtain the result of A. Greil [1]:

(6.2) *Let* $(\mathbf{E}, \mathcal{G}, \alpha, \equiv)$ *be the classical hyperbolic plane (i.e. also the continuity axiom is assumed), let* $o \in \mathbf{E}$ *be fixed, let* $(\mathbf{E}, +)$ *be the corresponding K-loop and let* $a, b \in \mathbf{E} \setminus \{o\}$ *with* $\overline{o,a} \neq \overline{o,b}$ *then each point* $p \in \mathbf{E}$ *can be written uniquely as a quasilinear* $\mathbf{R}$*-combination of a and b, i.e.:* $\forall p \in \mathbf{E}\ \ \exists_1 (\alpha, \beta) \in \mathbf{R} \times \mathbf{R} : p = \alpha \cdot a + \beta \cdot b$.

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