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The defect in an invariant reflection structure

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Dedicated to Prof. Walter Benz on his 80th birthday.

Abstract. The defect function [introduced in Karzel and Marchi (Results Math 47:305–326, 2005)] of an invariant reflection structure (P, I) is strictly connected to the precession maps of the corresponding K-loop $(P, +)$, therefore it permits a classification of such structures with respect to the algebraic properties of their K-loop. In the ordinary case (i.e. when the K-loop is not a group) we define, by means of products of three involutions, four different families of blocks denoted, respectively, by $\mathcal{L}_G, \mathcal{L}, \mathcal{B}_G, \mathcal{B}$ (cf. Sect. 4) so that we can provide the reflection structure with some appropriate incidence structure. On the other hand we consider in $(P, +)$ two types of centralizers and recognize a strong connection between them and the aforesaid blocks: actually we prove that all the blocks of (P, I) can be represented as left cosets of suitable centralizers of the loop $(P, +)$ (Theorem 6.1). Finally we give necessary and sufficient conditions in order that the incidence structures $(P, \mathcal{L}_G)$ and $(P, \mathcal{L})$ become linear spaces (cf. Theorem 8.6).

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Introduction

In this paper we start from the abstract situation of an *invariant reflection structure* (P, I) in the sense of [2]. Then for any two elements $a, b \in P$ there is exactly one permutation $\widetilde{a,b} \in I \subseteq SymP$ interchanging a and b (if $a = b$ we set $\widetilde{a} := \widetilde{a,a}$) and so we can also define the defect functions

$$\rho : P^3 \to SymP; (a, b, c) \mapsto \rho_{a;b,c} := \widetilde{a,c} \circ \widetilde{b,c} \circ \widetilde{a,b}$$

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and

$$\delta : P^3 \to SymP; (a,b,c) \mapsto \delta_{a;b,c} := \widetilde{a} \circ \rho_{a;b,c}$$

which allow us to split the invariant reflection structures, in the same way as the absolute geometries (cf. [8]), into two classes: the singular and the ordinary ones, characterized, respectively, by the property that δ is constant or not.

In the singular case one can prove that the set $I \circ I := \{\alpha \circ \beta \mid \alpha, \beta \in I\}$ is a commutative regular subgroup of $SymP$ (cf. (3.6.1)).

Firstly we collect some basic properties introducing the defect functions, in connection with the theory of K-loops (cf. Sects. 1–3). From Sect. 4 on, we will focus on the ordinary invariant reflection structures only. We define different kinds of blocks in the reflection structure (cf. Sect. 4), in view of constructing suitable incidence geometries admitting the original set of involutions as a regular set of automorphisms (cf. Proposition 4.5). Moreover, we prove (cf. Theorem 6.1) that all the blocks of (P, I) can be represented as left cosets of suitable centralizers of the associated K-loop $(P, +)$. In the main Theorem 8.6 conditions on reflection structures and their corresponding K-loops are summarized, which yield linear spaces.

1. Notations and properties of reflection structures

Let $P \neq \emptyset$ be a set and $I \subseteq J := \{\sigma \in SymP \mid \sigma^2 = id\}$, then we call the pair (P, I) an *involution set* and denote by $G := \langle I \rangle$ and $D := \langle I \circ I \rangle$ the subgroups of $SymP$ generated by I and $I \circ I$, respectively.

For $a, b \in P, \alpha, \beta \in G$ we define $\widetilde{a,b} := \{\iota \in I \mid \iota(a) = b\}$, $\widetilde{a} := \widetilde{a,a}$ and $\alpha \equiv \beta : \Leftrightarrow \exists \gamma \in G : \gamma \circ \alpha = \beta \circ \gamma$. An involution set (P, I) is called a *reflection structure* when it is *regular*, i.e. if $\forall a, b \in P : |\widetilde{a,b}| = 1$, and *invariant* if $\forall \alpha \in I : \alpha \circ I \circ \alpha = I$.

Proposition 1.1. *For a reflection structure (P, I) and for any $a, b, c, \dots, p \in P$ and $\alpha, \beta \in I$, the following properties hold true:*

(1) $\widetilde{a,b} = \widetilde{b,a}$ *and the function*

$$\sim : \binom{P}{2} \to I; \{a, b\} \mapsto \widetilde{a,b}$$

is surjective.

(2) *If* $c' := \widetilde{a,b}\,(c)$ *then* $\widetilde{a,b} = \widetilde{c,c'}$*; in particular, if* $e \in Fix\ \widetilde{a,b}$ *then* $\widetilde{e} = \widetilde{a,b}$ *and if* $\widetilde{a,b} \neq \widetilde{c,d}$ *then* $Fix\ (\widetilde{a,b} \circ \widetilde{c,d}) = \emptyset$.

(3) $a \in Fix\ \alpha$ *if and only if* $\alpha = \widetilde{a}$. *Hence* $\widetilde{P} := \{\widetilde{x} \mid x \in P\} = \{\xi \in I \mid Fix\ \xi \neq \emptyset\}$ *and furthermore* $\overline{P} := \{\overline{x} := Fix\ \widetilde{x} \mid x \in P\}$ *forms a partition of P. Moreover, if (P, I) is invariant then* $\widetilde{\alpha(p)} = \alpha \circ \widetilde{p} \circ \alpha$, *furthermore* $\alpha(Fix\ \widetilde{p}) = Fix\ \widetilde{\alpha(p)}, \alpha(Fix\ \beta) = Fix(\alpha \circ \beta \circ \alpha)$. *Finally, if* $a, b \in Fix\ \widetilde{c}$ *then* $\widetilde{a,b}\ (Fix\ \widetilde{c}) = Fix\ \widetilde{c}$.

(4) *If* $id \in I$ *then: (i) If* $\alpha \neq id$*:* $Fix\ \alpha = \emptyset$, *(ii)* $\widetilde{p} = id$ *for all* $p \in P$, *(iii) if* $a \neq b : Fix\ \widetilde{a,b} = \emptyset$.

(5) *$I \circ I$ is fixed point free and any pair $(P, \alpha \circ I)$ is a regular and fixed point free permutation set.*

Proof. (3). The first part is clear. Let (P, I) be invariant, then $\alpha \circ \widetilde{p} \circ \alpha \in I$ and $\alpha \circ \widetilde{p} \circ \alpha(\alpha(p)) = \alpha(p)$, therefore $\alpha \circ \widetilde{p} \circ \alpha = \widetilde{\alpha(p)}$. Moreover, $x \in \alpha\ (Fix\ \widetilde{p}) \Longleftrightarrow \alpha(x) \in Fix\ \widetilde{p} \Longleftrightarrow \widetilde{p}\ (\alpha(x)) = \alpha(x) \Longleftrightarrow \alpha \circ \widetilde{p} \circ \alpha(x) = x \Longleftrightarrow x \in Fix\ (\alpha \circ \widetilde{p} \circ \alpha) = Fix\ \widetilde{\alpha(p)}$.

Finally, $\widetilde{a, b} \in Fix\ \widetilde{c}$ implies $\widetilde{a} = \widetilde{b} = \widetilde{c}$ and so $\widetilde{a, b}\ (Fix\ \widetilde{c}) = \widetilde{a, b}\ (Fix\ \widetilde{a}) = Fix\ \widetilde{\widetilde{a, b}\ (a)} = Fix\ \widetilde{b} = Fix\ \widetilde{c}$. □

Remark 1.2. As a consequence of (1.1.3), in an invariant reflection structure (P, I), if $|Fix\ \widetilde{a}| = 1$ for some $a \in P$ then, for all $p \in P, |Fix\ \widetilde{p}| = 1$, i.e. $\overline{p} = \{p\}$.

We observe that in a general involution set (P, I), each element $\alpha \in I$ defines a partition of P in 1- and 2-sets $\{\{x, \alpha(x)\} \mid x \in P\}$. Therefore a permutation $\sigma \in SymP$ is an automorphism of the involution set (P, I) if and only if it preserves the collection of all these partitions, which means that, for any $\alpha \in I$, the set $\{\{\sigma(x), \sigma(\alpha(x))\} \mid x \in P\} = \{\{\sigma(x), (\sigma \circ \alpha \circ \sigma^{-1})(\sigma(x))\} \mid x \in P\} = \{\{y, (\sigma \circ \alpha \circ \sigma^{-1})(y)\} \mid y \in P\}$ has to be again such a partition, i.e. there has to be a $\beta \in I$ with $\sigma \circ \alpha \circ \sigma^{-1} = \beta$.

Consequently, the normalizer $N := N(I) := \{\sigma \in SymP \mid \sigma \circ I \circ \sigma^{-1} = I\}$ of I in $SymP$ coincides with the *automorphism group* $Aut(P, I)$ of the involution set.

Proposition 1.3. *If (P, I) is an invariant reflection structure then $I \subseteq G \subseteq N$ and $D \trianglelefteq G$.*

Proposition 1.4. *For an invariant reflection structure (P, I) the following properties hold:*

(1) $\forall \sigma \in Aut(P, I), \forall a, b \in P : \widetilde{\sigma(a), \sigma(b)} = \sigma \circ \widetilde{a, b} \circ \sigma^{-1}$ *and* $\widetilde{\sigma(a)} = \sigma \circ \widetilde{a} \circ \sigma^{-1}$.

(2) $\forall a, b, c, d \in P : \widetilde{a, b} \circ \widetilde{c, d} \circ \widetilde{a, b} = \left(\widetilde{\widetilde{a, b}(c), \widetilde{a, b}(d)}\right)$.

(3) *If $\sim : P \to I; p \mapsto \widetilde{p}$ is injective , $\gamma \in J \cap \langle I \rangle$, $x \in P$ with $Fix(\widetilde{x, \gamma(x)}) \neq \emptyset$ and if $m \in Fix(\widetilde{x, \gamma(x)})$ then $\gamma(m) = m$.*

Proof. The first two statements follow from the invariance of I and (1.1.2) since $(\sigma \circ \widetilde{a, b} \circ \sigma^{-1})(\sigma(b)) = \sigma(a)$.
(3). By assumption $\widetilde{m} = \widetilde{x, \gamma(x)}$ hence $\widetilde{m}(x) = \gamma(x)$ and $\widetilde{m}(\gamma(x)) = x$. Since $\gamma \in J$, $\widetilde{\gamma(m)}(\gamma(x)) = \gamma \circ \widetilde{m} \circ \gamma^{-1}(\gamma(x)) = \gamma \circ \widetilde{m}(x) = \gamma^2(x) = x$ and we obtain $\widetilde{m} = \widetilde{\gamma(m)}$ by the regularity of I, hence $\gamma(m) = m$ by our assumption of injectivity. □

A permutation $\rho \in D = \langle I \circ I \rangle$ will be called a *rotation* if $Fix\ \rho \neq \emptyset$. Let $D_r := \{\rho \in D \mid Fix\ \rho \neq \emptyset\}$ be the set of all rotations. By (1.3) $D_r \subseteq N$.

2. The defect function

If (P, I) is a reflection structure, following [3] (where the definition is slightly different from the original one given in [5]), we call the map

$$\delta : P^3 \longrightarrow SymP; (a, b, c) \mapsto \delta_{a;b,c} := \widetilde{a} \circ \widetilde{a,c} \circ \widetilde{b,c} \circ \widetilde{a,b}$$

the *defect* of (P, I). Since $\delta_{a;b,c} \in D$ and $a \in Fix(\delta_{a;b,c})$, $\delta_{a;b,c}$ is a rotation.

The defect δ of an invariant reflection structure (P, I) has the following properties (cf. also [5], (3.2), (3.8.2), (3.8.3), (3.8.5)):

Proposition 2.1. *Let $a, b, c, d \in P$ and $\sigma \in N = Aut(P, I)$, then:*

(1) $\sigma \circ \delta_{a;b,c} \circ \sigma^{-1} = \delta_{\sigma(a);\sigma(b),\sigma(c)}$; *if* $b' := \widetilde{o,a}(b)$, $c' := \widetilde{o,a}(c)$ *then* $\delta_{a;b,c} = \widetilde{o,a} \circ \delta_{o;b',c'} \circ \widetilde{o,a}$.
(2) $\delta_{a;b,c} = \widetilde{a,c} \circ \widetilde{c} \circ \widetilde{b,c} \circ \widetilde{a,b} = \widetilde{a,c} \circ \widetilde{b,c} \circ \widetilde{b} \circ \widetilde{a,b} = \widetilde{a,c} \circ \widetilde{b,c} \circ \widetilde{a,b} \circ \widetilde{a}$, $\widetilde{a} \circ \delta_{a;b,c} = \delta_{a;b,c} \circ \widetilde{a}$.
(3) $\delta_{a;c,d} \circ \delta_{a;b,c} \circ \delta_{a;d,b} = \widetilde{a,d} \circ \delta_{d;b,c} \circ \widetilde{a,d}$.
(4) *If* $|\{a, b, c\}| \le 2$ *then* $\delta_{a;b,c} = id$.
(5) $\delta_{a;c,b} = (\delta_{a;b,c})^{-1} = \widetilde{a,c} \circ \delta_{c;b,a} \circ \widetilde{a,c}$.

Proof. (1) and (2) follow, respectively, from (1.4.1) and (1.4.2), and the definition of δ.
(3). By (2) we obtain

$$\begin{aligned}\delta_{a;c,d} \circ \delta_{a;b,c} \circ \delta_{a;d,b} &= \widetilde{a,d} \circ \widetilde{c,d} \circ \widetilde{a,c} \circ \widetilde{a,c} \circ \widetilde{b,c} \circ \widetilde{a,b} \circ \widetilde{a,b} \circ \widetilde{b,d} \\ &\quad \circ \widetilde{d} \circ \widetilde{a,d} \\ &= \widetilde{a,d} \circ \widetilde{c,d} \circ \widetilde{b,c} \circ \widetilde{b,d} \circ \widetilde{d} \circ \widetilde{a,d} \\ &= \widetilde{a,d} \circ \delta_{d;b,c} \circ \widetilde{a,d}.\end{aligned}$$

(4) is immediate if $a = b$ or $a = c$; if $b = c$, by (1.4.2) we have:

$$\delta_{a;b,b} = \widetilde{a} \circ \widetilde{a,b} \circ \widetilde{b} \circ \widetilde{a,b} = \widetilde{a} \circ \widetilde{a} = id.$$

(5), (6). We set in (3) $d = b$, or, respectively, $c = a$, and obtain by (4): $\delta_{a;c,b} \circ \delta_{a;b,c} = \widetilde{a,b} \circ \widetilde{a,b} = id$, or, respectively, $\delta_{a;d,b} = \widetilde{a,d} \circ \delta_{d;b,a} \circ \widetilde{a,d}$. □

Proposition 2.2. *δ satisfies the following "addition property":*

$$\forall a, b, c, d \in P \text{ with } \delta_{b;c,d} = id : \delta_{a;c,d} \circ \delta_{a;b,c} = \delta_{a;b,d}.$$

Proof. By (2.1.6), with $\delta_{b;c,d} = id$ also $\delta_{d;b,c} = id$ and so (2.2) is a consequence of (2.1.3) and (2.1.5). □

With respect to the defect one can split the invariant reflection structures into two classes:

(P, I) is called *singular* if $\delta(P^3) = \{id\}$ and *ordinary* if $\delta(P^3) \neq \{id\}$.

3. Loop derivation

We recall (cf. e.g. [2]) that, given an invariant reflection structure (P, I), after fixing an element $o \in P$ we can define a *loop-operation*

$$+ : P \times P \to P \ ; \ (a, b) \mapsto a + b := \widetilde{o, a} \circ \widetilde{o}(b).$$

So $(P, +)$ is a loop with o as neutral element, and denoting by $-a$ the right opposite of an element $a \in P$, one has $-a = \widetilde{o} \circ \widetilde{o, a}(o) = \widetilde{o}(a)$, hence $\widetilde{o} = \widetilde{-a, a}$.

Note that, fixing another element $o' \in P$ as base point, we obtain, by the regularity and the invariance of the reflection structure, an isomorphic loop $(P, +')$.

Moreover, considering the map $a^+ : P \to P \ ; \ x \mapsto a^+(x) := a + x$, for any $a \in P$, we see that $a^+ = \widetilde{o, a} \circ \widetilde{o}$ and get immediately:

$$P^+ := \{a^+ \mid a \in P\} = I \circ \widetilde{o} \subseteq D.$$

From the definition of $+$, it follows by (1.3, 1.4):

Proposition 3.1. *If $(P, +)$ is the loop derived (in a point $o \in P$) from the invariant reflection structure (P, I) then $P^+ \subseteq Aut(P, I) = N(I)$ and*

$$Aut(P, +) = \{\sigma \in Aut(P, I) \mid \sigma(o) = o\} \subseteq Aut(P, I)$$

hence, for any $\sigma \in Aut(P, +)$ and $a \in P$ we have : $(\sigma(a))^+ = \sigma \circ a^+ \circ \sigma^{-1}$.

(For a detailed proof, cf., e.g. [7, Sec.7]).

In a loop $(P, +)$ the *precession map* $\delta_{a,b} := ((a + b)^+)^{-1} \circ a^+ \circ b^+$, for $a, b \in P$, such that for any $x \in P$: $a + (b + x) = (a + b) + \delta_{a,b}(x)$ (cf. e.g. [9]), gives the deviation of the loop-operation from associativity.

We recall some algebraic properties of the loop associated to an invariant reflection structure.

A loop $(P, +)$ is called a *K-loop*, or *Bruck loop* if it satisfies one of the following equivalent conditions:

($\mathbf{K}_1$) $\forall a, b \in P : a^+ \circ b^+ \circ a^+ = (a + (b + a))^+$ (*Bol condition*) and if $\nu : P \to P; x \mapsto -x, \nu \in Aut(P, +)$.

($\mathbf{K}_2$) $\forall a, b \in P : \delta_{a,b} = \delta_{a,b+a} = \delta_{a+b,b} \in Aut(P, +)$ and $\nu \in Aut(P, +)$.

Moreover, for a K-loop it holds:

$$\forall a, b \in P : (a^+)^{-1} = (-a)^+, \ a + (-a + b) = b \quad \text{and} \quad a - b = b \ \Leftrightarrow \ a = b + b.$$

In fact, since by ($\mathbf{K}_1$), $2b - b = (2b)^+(-b) = b^+ \circ o^+ \circ b^+(-b) = b$, we get : $b = a - b = 2b - b \iff a = 2b$.

The relationship between invariant reflection structures and K-loops is clarified by the following proposition (cf. [2]):

Proposition 3.2. *The loop $(P,+)$ derived from an invariant reflection structure (P,I) is a K-loop.*

Conversely, if $(P,+)$ is a K-loop, denoting for any $p\in P$, by $p^\circ := p^+\circ\nu$ and by $P^\circ := \{p^\circ \mid p\in P\}$, then (P,P°) is an invariant reflection structure.

Between the δ-functions (rotations images of the defect) of an invariant reflection structure (P,I) and the δ-functions (precession maps) of the K-loop $(P,+)$ derived from (P,I) in a point o there is the following connection:

Proposition 3.3. *Let $a,b\in P$ then it holds:*

(1) $\delta_{a,b}=\delta_{o;-b,a+b}$ *and if* $b=c-a$ *then* $\delta_{o;a,b}=\delta_{c,-a}$.
(2) $\delta_{a;b,c}=a^+\circ\delta_{-a+c,\ a-b}\circ(a^+)^{-1}$.
(3) $a+b=\delta_{a;b,o}\ \circ\ \delta_{b;o,a}\ (b+a)$, *equivalently* $\delta_{a;o,b}(a+b)=\delta_{b;o,a}(b+a)$.

Proof. The proof of (1) can be obtained directly from (4.4.1) of [5].

(2). Since $\widetilde{o,a}(x)=a-x$ for any $x\in P$, by (2.1.1) we have $\delta_{a;b,c}=\widetilde{o,a}\circ\delta_{o;a-b,a-c}\circ\widetilde{o,a}=a^+\circ\delta_{o;-a+b,-a+c}\circ(a^+)^{-1}$ for $\tilde{o}(x)=-x$, and this is equal, by part (1), to $a^+\circ\delta_{d,a-b}\circ(a^+)^{-1}$ with $-a+c=d+(a-b)$. Finally, using $(\mathbf{K}_2)$ we are done.
(3) is an easy consequence of (2.1.2). □

Let $(P,+)$ be a loop. Every $n\in\mathbf{Z}$ defines a map

$$n^{\cdot}: P\to P\ ;\ x\mapsto n\cdot x := nx := (x^+)^n(o).$$

The loop $(P,+)$ is called *n-divisible*, or *divisible by n* when $n^{\cdot}$ is surjective, and *uniquely n-divisible* when $n^{\cdot}$ is bijective.

A K-loop $(P,+)$ is *left power alternative* (cf. [9]), that is, $(na)^+=(a^+)^n$ for all $a\in P, n\in\mathbf{Z}$ hence $na+ma=(a^+)^n\circ(a^+)^m(o)=(a^+)^{(n+m)}(o)=(n+m)a$ for all $a\in P, n,m\in\mathbf{Z}$, that is, P is *left associative.*

So we can state the following proposition:

Proposition 3.4. *Let $(P,+)$ be a K-loop and $a\in P$, then $\mathbf{Z}a := \{na\mid n\in\mathbf{Z}\}$ is a cyclic subgroup of the loop $(P,+)$.*

The order of the cyclic group $\mathbf{Z}a$ is called the *order* of the loop element a.

Let us consider again an invariant reflection structure (P,I) and the associated K-loop (P,+). It turns out that $id\in I$ if and only if the loop $(P,+)$ has *exponent* 2, that is, each $a\in P\backslash\{o\}$ has order 2.

For $a\in P$ let $\{a\}' := (2^{\cdot})^{-1}(a)=\{x\in P\mid x+x=a\}$ be the preimage of a with respect to the map $2^{\cdot}$; then:

Proposition 3.5. *Let $id\notin I$ and $a\in P^* := P\backslash\{o\}$. Then:*

(1) $\{a\}'=Fix\ \widetilde{o,a}$.
(2) $\widetilde{a}=\widetilde{o,2a}$.
(3) $\{2a\}'=Fix\ \widetilde{a}$.

Proof.

(1) By (1.1.2) and (1.1.4), $x \in Fix\widetilde{o,a}$ implies $\widetilde{x} = \widetilde{o,a} \neq id$ which is equivalent to $a = \widetilde{x}(o)$; then, by (1.4.2): $a = \widetilde{x}(o) = \widetilde{x,o} \circ \widetilde{o} \circ \widetilde{x,o}(o) = x^+ \circ x^+(o) = x^+(x) = x + x = 2x$.

(2) By (1.4.2) $2a = a^+ \circ a^+(o) = \widetilde{o,a} \circ \widetilde{o} \circ \widetilde{o,a}(o) = \widetilde{a}(o)$ and then $\widetilde{a} = \widetilde{o,2a}$.

(3) $x \in \{2a\}'$ means $x + x = 2a$ which is equivalent, by (1) and (2), to $2x = \widetilde{a}(o)$, i.e. $\widetilde{x}(o) = \widetilde{a}(o)$ and by the regularity: $\widetilde{x} = \widetilde{a}$ and so $Fix\ \widetilde{x} = Fix\ \widetilde{a}$.

□

An invariant reflection structure (P, I) such that $\forall \alpha \in I : |Fix\alpha| = 1$ is called *invariant point reflection structure*. This implies, by (1.1.3) and (1.1.4), $I = \widetilde{P}$ and $id \notin I$, while the converse is not true. Furthermore (by the regularity of the set I on P), the condition $|Fix\alpha| = 1$ for all $\alpha \in P$ is equivalent to the existence, for any $x, y \in P$, of a unique $a' \in P$ such that $\widetilde{a'} = \widetilde{x,y}$ and this, in turn, is equivalent to saying that, for any $a \in P$ there is a unique $a' \in P$ such that $\widetilde{a'} = \widetilde{o,a}$, i.e. by (3.5), $2a' = a$. This implies $2a \neq o$ for any $a \in P^*$, i.e. $\{o\}' = \{id\}$.

Thus (P, I) is an invariant point reflection structure if and only if the corresponding K-loop is uniquely 2-divisible.

According to (3.2) there is a one to one correspondence between invariant reflection structures and K-loops where the singular ones correspond with the commutative groups. More precisely we have:

Proposition 3.6. *Let (P, I) be an invariant reflection structure and $(P, +)$ the corresponding derived K-loop.*

I. *The following statements are equivalent:*
   (1) $\delta(P^3) = id$.
   (2) $I \circ \widetilde{P} \circ I \subseteq I$.
   (3) $I \circ I \circ I \subseteq I$.
   (4) $I \circ I$ *is a commutative regular subgroup of* $Sym(P)$.
   (5) $\forall a, b \in P : a^+ \circ b^+ = (a + b)^+$.
   (6) $(P, +)$ *is a commutative group, isomorphic to* $I \circ I$.

II. *The following statements are equivalent and a consequence of the statements of I:*
   (1) $\forall a, b, c \in P : \delta_{a;b,c} = \delta_{c;b,a}$.
   (2) $I \circ \widetilde{P} \circ I \subseteq J$.
   (3) $\forall a, b \in P : a^+ \circ b^+ = b^+ \circ a^+$.

Proof. I. See, e.g. [1].
II. (1) ⇒(2): Let $\alpha, \gamma \in I$, $\widetilde{b} \in \widetilde{P}$, $a := \alpha(b)$ and $c := \gamma(b)$. Then $\alpha = \widetilde{a,b}$, $\gamma = \widetilde{b,c}$ and $\delta_{a;b,c} = \widetilde{a,c} \circ \gamma \circ \widetilde{b} \circ \alpha = \delta_{c;b,a} = \widetilde{a,c} \circ \alpha \circ \widetilde{b} \circ \gamma$ hence $\alpha \circ \widetilde{b} \circ \gamma \in J$.
(2) ⇒ (3): $a^+ \circ b^+ = (\widetilde{o,a} \circ \widetilde{o} \circ \widetilde{o,b}) \circ \widetilde{o} = (\widetilde{o,b} \circ \widetilde{o} \circ \widetilde{o,a}) \circ \widetilde{o} = b^+ \circ a^+$.

(3) ⇒ (1): From

$$\delta_{a;o,b} = \widetilde{a,b} \circ \widetilde{o,b} \circ \widetilde{o} \circ \widetilde{o,a} = \widetilde{a,b} \circ b^+ \circ a^+ \circ \widetilde{o}$$
$$= \widetilde{b,a} \circ a^+ \circ b^+ \circ \widetilde{o} = \delta_{b;o,a}$$

we obtain like above $\delta_{a;c,b} = \delta_{b;c,a}$ for all $a, b, c \in P$.

Furthermore the statements of II are a consequence of the statements of I since, e.g. I.2 implies II.2. □

4. Blocks and three reflection axioms in (P, I)

In this and in the following sections, let (P, I) be an *ordinary* invariant reflection structure with $id \notin I$.

We consider the following *three reflection axioms*, for $i \in \{0, 1, 2, 3\}$:

$$\forall \alpha_1, \alpha_2, \alpha_3 \in I \text{ with } |\{h \in \{1,2,3\} \mid Fix\ \alpha_h \neq \emptyset\}| \geq i : \qquad (\mathbf{R_i})$$
$$\alpha_1 \circ \alpha_2 \circ \alpha_3 \in J \Longrightarrow \alpha_1 \circ \alpha_2 \circ \alpha_3 \in I.$$

Furthermore:

$$\forall a, b, c \in P : \widetilde{a} \circ \widetilde{b} \circ \widetilde{c} \in J \Longleftrightarrow \widetilde{a} \circ \widetilde{b} \circ \widetilde{c} \in \widetilde{P}\ ; \qquad (\overline{\mathbf{R_3}})$$

and

$$\forall a, b, c \in P : \widetilde{b,a} \circ \widetilde{a} \circ \widetilde{a,c} \in J \Longleftrightarrow \widetilde{a,b} \circ \widetilde{b} \circ \widetilde{b,c} \in J. \qquad (\mathbf{R}')$$

We observe:

Proposition 4.1. $(\mathbf{R}_0) \Longrightarrow (\mathbf{R}_1) \Longrightarrow (\mathbf{R}_2) \Longrightarrow (\mathbf{R}_3)$; $(\mathbf{R}_1) \Longrightarrow (\mathbf{R}')$ *and, if* $\forall \alpha_1, \alpha_2, \alpha_3 \in I\ \ Fix(\alpha_1 \circ \alpha_2 \circ \alpha_3) \neq \emptyset$ *then* $(\mathbf{R}_3) \Longleftrightarrow (\overline{\mathbf{R_3}})$.

Proof. If $\widetilde{b,a} \circ \widetilde{a} \circ \widetilde{a,c} \in J$ then by $a \in Fix\ \widetilde{a}$ and $\widetilde{b,a} \circ \widetilde{a} \circ \widetilde{a,c}(c) = b$, $(\mathbf{R}_1)$ implies $\widetilde{b,a} \circ \widetilde{a} \circ \widetilde{a,c} = \widetilde{b,c}$ and since $b = \widetilde{a,b}\ (a)$ implies $\widetilde{b} = \widetilde{a,b} \circ \widetilde{a} \circ \widetilde{a,b}$ we have $\widetilde{a,b} \circ \widetilde{b} \circ \widetilde{b,c} = \widetilde{a} \circ \widetilde{a,b} \circ \widetilde{b,c} = \widetilde{a,c} \in I \subseteq J$.

The last statement is a direct consequence of (1.1.3). □

We define the following types of *blocks*, for all $a, b \in P$ with $a \neq b$:

- $\dddot{a,b} := \{x \in P \mid \widetilde{a,b} \circ \widetilde{a} \circ \widetilde{a,x} \in J\}$, $\mathcal{L}_G := \left\{ \dddot{x,y} \mid \{x,y\} \in \binom{P}{2} \right\}$ and $\mathcal{L}_g := \left\{ \dddot{x,y} \in \mathcal{L}_G \mid Fix\ \widetilde{x,y} \neq \emptyset \right\} \subseteq \mathcal{L}_G$,
- $\overline{a,b} := \{x \in P \mid \widetilde{a,b} \circ \widetilde{a} \circ \widetilde{a,x} \in I\}$ and $\mathcal{L} := \left\{ \overline{x,y} \mid \{x,y\} \in \binom{P}{2} \right\}$,
- $\overline{\dddot{a,b}} := \{x \in P \mid \widetilde{a} \circ \widetilde{b} \circ \widetilde{x} \in J\}$ and $\mathcal{B}_G := \left\{ \overline{\dddot{x,y}} \mid \{x,y\} \in \binom{P}{2} \right\}$,
- $\overline{\overline{a,b}} := \{x \in P \mid \widetilde{a} \circ \widetilde{b} \circ \widetilde{x} \in \widetilde{P}\}$ and $\mathcal{B} := \left\{ \overline{\overline{x,y}} \mid \{x,y\} \in \binom{P}{2} \right\}$.

Then

Proposition 4.2. *For all* $a, b \in P$ *with* $a \neq b$:

(1) $\{a, b, \widetilde{a}(b), \widetilde{b}(a)\} \subseteq \overline{a,b} = \overline{b,a} \subseteq \overset{\cdots}{a,b} \cap \overset{\cdots}{b,a}$, $\overline{a,b} = \{x \in P \mid \delta_{x;a,b} = id\}$ *and* $\overset{\cdots}{x,y} = \overset{\cdots}{y,x}$ *for all* $x, y \in P$ *with* $x \neq y \iff (P, I)$ *satisfies* $(\mathbf{R}')$.

(2) $\overset{\cdots}{a,b} = \overset{\cdots}{b,a}$; $\overline{\overline{a,b}} = \overline{\overline{b,a}}$ *and* $\overline{a,b} \subseteq \overset{\cdots}{a,b} \subseteq \overline{\overline{a,b}} \subseteq \overset{\cdots}{a,b}$.

(3) $\forall c \in \overset{\cdots}{a,b}$: (*i*) $\widetilde{c}(\overset{\cdots}{a,b}) = \overset{\cdots}{a,b}$ *and* $\widetilde{c}(\overline{\overline{a,b}}) = \overline{\overline{a,b}}$, (*ii*) $\widetilde{c}(a) \in \overset{\cdots}{a,\widetilde{b}(a)}$,
(*iii*) $\{\widetilde{c}(a) \mid c \in \overset{\cdots}{a,b}\} \subseteq \overset{\cdots}{a,b} \cap \overset{\cdots}{a,\widetilde{b}(a)}$.

Proof.

(1) By $I \subseteq J$, $\overline{a,b} \subseteq \overset{\cdots}{a,b}$ and since $\widetilde{a,b} \circ \widetilde{a} \circ \widetilde{a,x}(x) = b$ we have: $x \in \overline{a,b} \iff \widetilde{a,b} \circ \widetilde{a} \circ \widetilde{a,x} = \widetilde{b,x} \iff id = \widetilde{b,x} \circ \widetilde{a,b} \circ \widetilde{a} \circ \widetilde{a,x} = \delta_{x;a,b}$ (by (2.1.2)). Hence $\overline{a,b} := \{x \in P \mid \delta_{x;a,b} = id\}$. Now by (2.1.5) follows $\overline{a,b} = \overline{b,a}$ and by (2.1.4), $a, b \in \overline{a,b}$. Since $\widetilde{\widetilde{a}(b)} = \widetilde{a} \circ \widetilde{b} \circ \widetilde{a}$ (by (1.4.2)), $\widetilde{\widetilde{a}(b); b} = \widetilde{a}$ (by (1.1.2)) and $\widetilde{\widetilde{a}(b), a} = \widetilde{\widetilde{a}(b), \widetilde{a}(a)} = \widetilde{a} \circ \widetilde{b,a} \circ \widetilde{a}$ (by (1.4.1)) , we have: $\delta_{\widetilde{a}(b);a,b} = (\widetilde{a} \circ \widetilde{b} \circ \widetilde{a}) \circ \widetilde{a} \circ \widetilde{a,b} \circ \widetilde{a} \circ \widetilde{a,b} \circ \widetilde{a} = \widetilde{a} \circ \widetilde{b} \circ \widetilde{b} \circ \widetilde{a} = id$. Hence $\widetilde{a}(b), \widetilde{b}(a) \in \overline{a,b}$.

(2) $x \in \overset{\cdots}{a,b} \iff \widetilde{a} \circ \widetilde{b} \circ \widetilde{x} \in J \iff \widetilde{a} \circ \widetilde{b} \circ \widetilde{x} = \widetilde{x} \circ \widetilde{b} \circ \widetilde{a} \iff \widetilde{x} \circ \widetilde{a} \circ \widetilde{b} = \widetilde{b} \circ \widetilde{a} \circ \widetilde{x} \iff \widetilde{b} \circ \widetilde{a} \circ \widetilde{x} \in J \iff x \in \overset{\cdots}{b,a}$. $x \in \overline{\overline{a,b}} \iff \widetilde{a} \circ \widetilde{b} \circ \widetilde{x} = \widetilde{x} \circ \widetilde{b} \circ \widetilde{a} \in \widetilde{P} \iff \widetilde{b} \circ \widetilde{a} \circ \widetilde{x} \in \widetilde{b} \circ \widetilde{a} \circ \widetilde{P} \circ \widetilde{a} \circ \widetilde{b} = \widetilde{P}$ (by (1.1.3)) $\iff x \in \overline{\overline{b,a}}$.

Furthermore, since $\overline{a,b} \subseteq \overset{\cdots}{a,b}$ by (1) and $\overline{\overline{a,b}} \subseteq \overset{\cdots}{a,b}$ by definition, we have only to prove $\overset{\cdots}{a,b} \subseteq \overline{\overline{a,b}}$. Let $x \in \overset{\cdots}{a,b}$ hence $\widetilde{a,b} \circ \widetilde{a} \circ \widetilde{a,x} \in J$. Since $b = \widetilde{a,b}\,(a)$ implies $\widetilde{b} = \widetilde{a,b} \circ \widetilde{a} \circ \widetilde{a,b}$ we have $\widetilde{a} \circ \widetilde{b} \circ \widetilde{x} = \widetilde{a} \circ \widetilde{a,b} \circ (\widetilde{a} \circ \widetilde{a,b} \circ \widetilde{a,x}) \circ \widetilde{a} \circ \widetilde{a,x} = (\widetilde{a} \circ \widetilde{a,b} \circ \widetilde{a,x}) \circ \widetilde{a,b} \circ \widetilde{a} \circ \widetilde{a} \circ \widetilde{a,x} = \widetilde{a,x} \circ \widetilde{a,b} \circ \widetilde{a} \circ \widetilde{a,b} \circ \widetilde{a,x} \in \widetilde{P}$ by (1.1.3)

(3) (i) Let $x \in \overset{\cdots}{a,b}$ and $y \in \overline{\overline{a,b}}$ then $\widetilde{a} \circ \widetilde{b} \circ \widetilde{c}$, $\widetilde{a} \circ \widetilde{b} \circ \widetilde{x} \in J, \widetilde{a} \circ \widetilde{b} \circ \widetilde{y} \in \widetilde{P}$ and by (1.4.1) $\widetilde{\widetilde{c}(x)} = \widetilde{c} \circ \widetilde{x} \circ \widetilde{c}$. Hence $\widetilde{a} \circ \widetilde{b} \circ \widetilde{\widetilde{c}(x)} = \widetilde{a} \circ \widetilde{b} \circ \widetilde{c} \circ \widetilde{x} \circ \widetilde{c} = \widetilde{c} \circ \widetilde{b} \circ \widetilde{a} \circ \widetilde{x} \circ \widetilde{c} = \widetilde{c} \circ \widetilde{x} \circ \widetilde{a} \circ \widetilde{b} \circ \widetilde{c} = \widetilde{c} \circ \widetilde{x} \circ \widetilde{c} \circ \widetilde{b} \circ \widetilde{a} = \widetilde{\widetilde{c}(x)} \circ \widetilde{b} \circ \widetilde{a} \in J$, i.e. $\widetilde{c}(x) \in \overset{\cdots}{a,b}$. Since $\widetilde{a} \circ \widetilde{b} \circ \widetilde{y} = \widetilde{y} \circ \widetilde{b} \circ \widetilde{a} \in \widetilde{P}$ $\Leftrightarrow \widetilde{y} \circ (\widetilde{y} \circ \widetilde{b} \circ \widetilde{a}) \circ \widetilde{y} = \widetilde{b} \circ \widetilde{a} \circ \widetilde{y} \in \widetilde{P}$ we obtain : $\widetilde{a} \circ \widetilde{b} \circ \widetilde{\widetilde{c}(y)} = \widetilde{a} \circ \widetilde{b} \circ \widetilde{c} \circ \widetilde{y} \circ \widetilde{c} = \widetilde{c} \circ (\widetilde{b} \circ \widetilde{a} \circ \widetilde{y}) \circ \widetilde{c} \in \widetilde{P}$, i.e. $\widetilde{c}(y) \in \overline{\overline{a,b}}$.

(ii) We have $\widetilde{a} \circ \widetilde{b} \circ \widetilde{c} \in J, \widetilde{a, \widetilde{b}(a)} = \widetilde{b}$ and $\widetilde{a, \widetilde{c}(a)} = \widetilde{c}$ hence $\widetilde{a, \widetilde{b}(a)} \circ \widetilde{a} \circ \widetilde{a, \widetilde{c}(a)} = \widetilde{b} \circ \widetilde{a} \circ \widetilde{c} \in J$, i.e. $\widetilde{c}(a) \in \overset{\cdots}{a, \widetilde{b}(a)}$.

(iii) is a consequence of (i) and (ii).

□

Between the blocks of $\mathcal{L}_G, \mathcal{L}_g, \mathcal{L}$ and $\mathcal{B}_G, \mathcal{B}$ we can find the following relations that descend directly from the definitions and Proposition 4.2:

Proposition 4.3. *Consider the sets of blocks* $\mathcal{L}_G$, $\mathcal{L}_g$, $\mathcal{L}$ *and* $\mathcal{B}_G$, $\mathcal{B}$, *then:*

(1) $\forall L \in \mathcal{L}\ \exists L' \in \mathcal{L}_G : L \subseteq L'$ *and* $\forall B \in \mathcal{B}\ \exists B' \in \mathcal{B}_G : B \subseteq B'$.

(2) *If* $(\bar{\mathbf{R}}_3)$ *is satisfied then* $\mathcal{B} = \mathcal{B}_G$.
(3) *If* $(\mathbf{R}_2)$ *is satisfied then* $\mathcal{L}_g \subseteq \mathcal{L}$ *and* $\mathcal{B} = \mathcal{B}_G$.
(4) *If* $(\mathbf{R}_1)$ *is satisfied then* $\mathcal{L}_g \subseteq \mathcal{L} = \mathcal{L}_G$, *i.e. for all* $\{a, b\} \in \binom{P}{2} : \overset{\cdots}{a, b} = \overline{a, b}$.

For the blocks of $\mathcal{B}_G$ and $\mathcal{B}$ we can prove the following:

Proposition 4.4. *Let* $a, b \in P$ *with* $a \neq b$, *then:*

(1) $x \in \overline{\overset{\cdots}{a, b}} \Rightarrow Fix\ \widetilde{x} \subseteq \overline{\overset{\cdots}{a, b}}$ *and* $\widetilde{x}(b) \in \overline{\overset{\cdots}{a, b}}$.
(2) $\overline{\overset{\cdots}{a, b}} = \overline{\overset{\cdots}{a, \widetilde{a}(b)}} \subseteq \overline{\overset{\cdots}{b, \widetilde{a}(b)}}$.
(3) $x \in \overline{\overline{a, b}} \Rightarrow Fix\ \widetilde{x} \subseteq \overline{\overline{a, b}}$ *and* $\widetilde{x}(b) \in \overline{\overline{a, b}}$.
(4) $\overline{\overline{a, b}} = \overline{\overline{a, \widetilde{a}(b)}} \subseteq \overline{\overline{b, \widetilde{a}(b)}}$.

Proof. (1), (2) (in the same way (3), (4)). Since $\widetilde{\widetilde{a}(b)} = \widetilde{a} \circ \widetilde{b} \circ \widetilde{a}$ and $x \in \overline{\overset{\cdots}{a, b}}$ $\iff$ $\widetilde{a} \circ \widetilde{b} \circ \widetilde{x} \in J$, we have firstly:
$\widetilde{a} \circ \widetilde{b} \circ \widetilde{\widetilde{x}(b)} = \widetilde{a} \circ \widetilde{b} \circ \widetilde{x} \circ \widetilde{b} \circ \widetilde{x} = \widetilde{x} \circ \widetilde{b} \circ \widetilde{a} \circ \widetilde{b} \circ \widetilde{x} \in J$ hence $\widetilde{x}(b) \in \overline{\overset{\cdots}{a, b}}$, secondly:
$\widetilde{a} \circ \widetilde{\widetilde{a}(b)} \circ \widetilde{x} = \widetilde{a} \circ \widetilde{a} \circ \widetilde{b} \circ \widetilde{a} \circ \widetilde{x} = \widetilde{b} \circ \widetilde{a} \circ \widetilde{x}$. Hence by (4.2.2), $\overline{\overset{\cdots}{a, b}} = \overline{\overset{\cdots}{a, \widetilde{a}(b)}}$ and thirdly:
$\widetilde{b} \circ \widetilde{\widetilde{a}(b)} \circ \widetilde{x} = \widetilde{b} \circ \widetilde{a} \circ \widetilde{b} \circ \widetilde{a} \circ \widetilde{x} = \widetilde{b} \circ \widetilde{a} \circ \widetilde{x} \circ \widetilde{a} \circ \widetilde{b} \in J$. Hence $\overline{\overset{\cdots}{a, b}} \subseteq \overline{\overset{\cdots}{b, \widetilde{a}(b)}}$. □

Finally, considering the action of the automorphism group of (P, I) on the sets of blocks, from (1.4) we obtain:

Proposition 4.5. *For any* $\sigma \in N = Aut(P, I)$ *and for any* $\{a, b\} \in \binom{P}{2}$*:*

(1) $\sigma(\overset{\cdots}{a, b}) = \overset{\cdots}{\sigma(a), \sigma(b)}$ *: hence* $N \subseteq Aut(P, \mathcal{L}_G)$.
(2) $\sigma(\overline{a, b}) = \overline{\sigma(a), \sigma(b)}$*: hence* $N \subseteq Aut(P, \mathcal{L})$.
(3) $\sigma(\overline{\overset{\cdots}{a, b}}) = \overline{\overset{\cdots}{\sigma(a), \sigma(b)}}$ *: hence* $N \subseteq Aut(P, \mathcal{B}_g)$.
(4) $\sigma(\overline{\overline{a, b}}) = \overline{\overline{\sigma(a), \sigma(b)}}$*: hence* $N \subseteq Aut(P, \mathcal{B})$.
(5) *Moreover, if* $o \in P$ *and* $c := \widetilde{o, a}(b)$ *then* $\overset{\cdots}{a, b} = \widetilde{o, a}(\overset{\cdots}{o, c})$, $\overline{a, b} = \widetilde{o, a}(\overline{o, c})$, $\overline{\overset{\cdots}{a, b}} = \widetilde{o, a}(\overline{\overset{\cdots}{o, c}})$ *and* $\overline{\overline{a, b}} = \widetilde{o, a}(\overline{\overline{o, c}})$.

If $a \in P$ is fixed, a *bundle in* a is defined as a subset of blocks of the form $\mathcal{L}(a) := \{\overline{a, x} \mid x \in P \backslash \{a\}\}$, $\mathcal{L}_G(a) := \{\overset{\cdots}{a, x} \mid x \in P \backslash \{a\}\}$, $\mathcal{B}_G(a) := \{\overline{\overset{\cdots}{a, x}} \mid x \in P \backslash \{a\}\}$ or $\mathcal{B}(a) := \{\overline{\overline{a, x}} \mid x \in P \backslash \{a\}\}$.

Then $\mathcal{L} = \bigcup\{\mathcal{L}(a) \mid a \in P\}$ is covered by the bundle set $\{\mathcal{L}(a) \mid a \in P\}$. In the same way, $\mathcal{L}_G = \bigcup\{\mathcal{L}_G(a) \mid a \in P\}, \mathcal{B} = \bigcup\{\mathcal{B}(a) \mid a \in P\}$ and $\mathcal{B}_G = \bigcup\{\mathcal{B}_G(a) \mid a \in P\}$. By (4.5.5) we see that the set I of involutions acts transitively on each set of bundles of the same type.

5. Centralizers

Now let $o \in P$ be fixed, let $P^* := P\backslash\{o\}$ and let $(P,+)$ be the K-loop derived from (P,I) in o. On P^* we consider two binary relations and two types of centralizers:

$$\beta_G := \{(a,b) \in (P^*)^2 \mid a^+ \circ\ b^+ = b^+ \circ\ a^+\} = \{(a,b) \in (P^*)^2 \mid \widetilde{o,a}\ \circ\ \widetilde{o}\ \circ\ \widetilde{o,b} \in J\},$$
$$\beta := \{(a,b) \in (P^*)^2 \mid a^+ \circ\ b^+ \in P^+\} = \{(a,b) \in (P^*)^2 \mid \widetilde{o,a}\ \circ\ \widetilde{o}\ \circ\ \widetilde{o,b} \in I\};$$

and for $a \in P^*$:

$$[a]_+ := \{x \in P \mid a^+ \circ x^+ = x^+ \circ a^+\} \text{ and let } \mathcal{F}_+ := \{[p]_+ \mid\ p \in P^*\},$$
$$[a] := \{x \in P \mid a^+ \circ x^+ \in P^+\} \text{ and let } \mathcal{F} := \{[p] \mid\ p \in P^*\}.$$

Proposition 5.1. *Let $a \in P^*$ such that $2a \neq o$ and $\sigma \in Aut(P,+)$; then:*

(1) $[a]_+ = \overset{\cdots\cdots}{o,a}$, $[a] = \overline{o,a}$, *hence* $\mathcal{F}_+ = \mathcal{L}_G(o)$ *and* $\mathcal{F} = \mathcal{L}(o)$ *are bundles in* o, $[a] \subseteq [a]_+$, $\sigma([a]_+) = [\sigma(a)]_+$ *and* $\sigma([a]) = [\sigma(a)]$.
(2) $[a]_+ = [-a]_+ = -[a]_+$, $[a] = [-a] = -[a]$.
(3) $\forall n \in \mathbf{N} : n \cdot [a]_+ \subseteq [a]_+$ *and* $n \cdot [a] \subseteq [a]$.
(4) $\forall x \in [a]_+ : x + ([a]_+ + x) = [a]_+$, $\forall x \in [a] : x + ([a] + x) = [a]$.
(5) $a + [a] = [a] = [a] + a$.
(6) $2[a]_+ \subseteq [a]$, $[a]_+ \subseteq [2a]$, $2a + [a]_+ = [a]_+$ *and* $2a + [a] = [a]$.

Proof.

(1) Since $a^+ = \widetilde{o,a} \circ \widetilde{o}$,
$x \in [a]_+ \iff \widetilde{o,a} \circ \widetilde{o} \circ \widetilde{o,x} \circ \widetilde{o} = \widetilde{o,x} \circ \widetilde{o} \circ \widetilde{o,a} \circ \widetilde{o} \iff \widetilde{o,a} \circ \widetilde{o} \circ \widetilde{o,x} \in J$ $\iff x \in \overset{\cdots\cdots}{o,a}$ and
$x \in [a] \iff \exists y \in P$ with $a^+ \circ x^+ = y^+$, i.e. $\widetilde{o,a} \circ \widetilde{o} \circ \widetilde{o,x} = \widetilde{o,y} \in I$ $\iff x \in \overline{o,a}$.
Now, let $x \in [a]$, hence $a^+ \circ x^+ = d^+ = \widetilde{o,d} \circ \widetilde{o} \Leftrightarrow a^+ \circ x^+ \circ \widetilde{o} = \widetilde{o,a} \circ \widetilde{o} \circ \widetilde{o,x} = \widetilde{o,d} = \widetilde{o,x} \circ \widetilde{o} \circ \widetilde{o,a} = x^+ \circ a^+ \circ \widetilde{o} \Rightarrow a^+ \circ x^+ = x^+ \circ a^+ \Leftrightarrow x \in [a]_+$.
Finally, by (3.1) we obtain : $x \in [a] \Leftrightarrow a^+ \circ x^+ = (a+x)^+ \Leftrightarrow \sigma \circ a^+ \circ \sigma^{-1} \circ \sigma \circ x^+ \circ \sigma^{-1} = (\sigma(a))^+ \circ (\sigma(x))^+ = \sigma \circ (a+x)^+ \circ \sigma^{-1} = (\sigma(a+x))^+ = (\sigma(a)+\sigma(x))^+ \Leftrightarrow \sigma(x) \in [\sigma(a)]$, i.e. $\sigma([a]) = [\sigma(a)]$; furthermore $x \in [a]_+ \Leftrightarrow a^+ \circ x^+ = x^+ \circ a^+ \Leftrightarrow \sigma \circ a^+ \circ \sigma^{-1} \circ \sigma \circ x^+ \circ \sigma^{-1} = (\sigma(a))^+ \circ (\sigma(x))^+ = (\sigma(x))^+ \circ (\sigma(a))^+$, i.e. $\sigma([a]_+) = [\sigma(a)]_+$.

(2) In a K-loop $x \in [a]_+ \Leftrightarrow a^+ \circ x^+ = x^+ \circ a^+ \Leftrightarrow x^+ \circ (a^+)^{-1} = (a^+)^{-1} \circ x^+ \Leftrightarrow x^+ \circ (-a)^+ = (-a)^+ \circ x^+ \Leftrightarrow x \in [-a]_+$.
Let $d := a+x$ then : $x \in [a] \Leftrightarrow a^+ \circ x^+ = (a+x)^+ = d^+ \Leftrightarrow x^+ \circ (-a)^+ = x^+ \circ (a^+)^{-1} = (a^+)^{-1} \circ d^+ \circ (a^+)^{-1} = (-a)^+ \circ d^+ \circ (-a)^+ \in P^+$ (by $(\mathbf{K}_1)$) $\Leftrightarrow x \in [-a]$.

(3) Let $x \in [a]_+$. By definition $(nx)^+ = (x^+)^n$ and so $(nx)^+ \circ a^+ = (x^+)^n \circ a^+ = a^+ \circ (x^+)^n = a^+ \circ (nx)^+$, i.e. $nx \in [a]_+$. If $x \in [a]$ and $n = 2m$ then $(nx)^+ \circ a^+ = ((x^+)^m)^2 \circ a^+ = (x^+)^m \circ a^+ \circ (x^+)^m \in P^+$ (by $(\mathbf{K}_1)$), i.e. $nx \in [a]$; if $n = 2m+1$ then $(nx)^+ \circ\ a^+ = (x^+)^m \circ\ (x^+ \circ a^+) \circ (x^+)^m = (mx)^+ \circ (x+a)^+ \circ (mx)^+ \in P^+$ thus also here $nx \in [a]$.

(4) Let $x, y \in [a]_+$ then by $(\mathbf{K}_1)$ $(x+(y+x))^+ \circ a^+ = x^+ \circ y^+ \circ x^+ \circ a^+ = a^+ \circ x^+ \circ y^+ \circ x^+ = a^+ \circ (x+(y+x))^+$. Hence $x + ([a]_+ + x) \subseteq [a]_+$. By (2) we have $-x \in [a]_+$ and by the previous considerations, $z := -x+(y-x) \in [a]_+$. Since $z^+ = (x^+)^{-1} \circ y^+ \circ (x^+)^{-1}$ and $x+(z+x) = (x+(z+x))^+(o) = x^+ \circ z^+ \circ x^+(o)$ we have $x+(z+x) = y^+(o) = y$ showing $[a]_+ \subseteq x + ([a]_+ + x)$.
Now let $x, y \in [a]$, then $a^+ \circ (x+(y+x))^+ = a^+ \circ x^+ \circ y^+ \circ x^+ = x^+ \circ a^+ \circ y^+ \circ x^+ = x^+ \circ (a+y)^+ \circ x^+ = (x+((a+y)+x))^+ \in P^+$ hence $x + ([a]+x) \subseteq [a]$. This time $z := -x+(y-x) \in [a]$ implying $y = x+(z+x) \in x+([a]+x)$ hence $[a] \subseteq x+([a]+x)$.

(5) Let $x \in [a]$, i.e. $a^+ \circ x^+ = (a+x)^+$. Then firstly : $(a+x)^+ \circ a^+ = a^+ \circ x^+ \circ a^+ = (a+(x+a))^+$ and this means $a+x \in [a]$ hence $a+[a] \subseteq [a]$ and secondly $a^+ \circ (x+a)^+ = a^+ \circ x^+ \circ a^+ = (a+(x+a))^+$, i.e. $x+a \in [a]$ hence $[a]+a \subseteq [a]$. Now let $b \in [a]$ and $x \in P$ be the solution of $a+x = a^+(x) = b$, then by (2), $x = (a^+)^{-1}(b) = -a+b \in -a+[a] = -a+[-a] \subseteq [-a] = [a]$, thus $a+[a] = [a]$. Finally, $[a]+a \subseteq [a]$ and $a+[a] = [a]$ and by (4) $[a] = a+([a]+a) \subseteq a+[a] = [a]$ thus $[a]+a = [a]$.

(6) Let $x \in [a]_+$ then, by $(\mathbf{K}_1)$ firstly $(2x)^+ \circ a^+ = x^+ \circ x^+ \circ a^+ = x^+ \circ a^+ \circ x^+ = (x+(a+x))^+ \in P^+$, i.e. $2x \in [a]$, secondly $(2a)^+ \circ x^+ = a^+ \circ a^+ \circ x^+ = a^+ \circ x^+ \circ a^+ = (a+(x+a))^+ \in P^+$, i.e. $x \in [2a]$, and thirdly $2a+x = (2a)^+ \circ x^+(o) = (a+(x+a))^+(o) = a+(x+a)$ hence $(2a+x)^+ = a^+ \circ x^+ \circ a^+$. Therefore $a^+ \circ (2a+x)^+ = a^+ \circ (a^+ \circ x^+ \circ a^+) = a^+ \circ x^+ \circ a^+ \circ a^+ = (2a+x)^+ \circ a^+$, i.e. $2a+[a]_+ \subseteq [a]_+$ and so $[a]_+ = 2a+(-2a+[a]_+) = 2a+(-2a+[-a]_+) \subseteq 2a+[-a]_+) = 2a+[a]_+ \subseteq [a]_+$. From (5) we obtain $2a+[a] = (2a)^+([a]) = a^+ \circ a^+([a]) = a^+([a]) = [a]$.

□

Proposition 5.2. *For* $a, b \in P^*$: $(a,b) \in \beta \iff a^+ \circ b^+ = (a+b)^+ \iff \delta_{a,b} = id \Longrightarrow (a,b) \in \beta_G$.

Proof. $a^+ \circ b^+(o) = a^+(b) = a+b = (a+b)^+(o)$ thus $a^+ \circ b^+ \in P^+$ implies $a^+ \circ b^+ = (a+b)^+$ which is equivalent to $\delta_{a,b} = id$ and furthermore implies, as in the proof of (5.1.1), $a^+ \circ b^+ = b^+ \circ a^+$, i.e. $(a,b) \in \beta_G$. □

For $A \subseteq P$ we define $A' := (2\,\cdot\,)^{-1}(A) = \{x \in P \mid 2x \in A\}$ and $A'' := (2\,\cdot\,)^{-1}(2A) = \{x \in P \mid 2x \in 2A\}$.

Besides the sets of centralizers $\mathcal{F}_+$ and $\mathcal{F}$, we consider $\mathcal{F}'_+ := \{([2a]_+)' \mid a \in P^*\}$ and $\mathcal{F}' := \{[2a]' \mid a \in P^*\}$.

Then (5.1) has the following supplements:

Proposition 5.3. *Let* (P, I) *be such that* $2x \neq o$ *for any* $x \in P^*$*, let* $a \in P^*$ *and* $A \subseteq P$*; then:*

(1) $A \subseteq A' \iff 2A \subseteq A$.
(2) $2[a] \subseteq 2[a]_+ \subseteq [a] \subseteq [a]_+ \subseteq [2a] \subseteq [2a]_+ \subseteq [2a]' \subseteq ([2a]_+)'$.
(3) $[2a]' = \overline{\overline{o,a}} \subseteq \overset{\cdots\cdots}{\overline{o,a}} = ([2a]_+)'$ *hence* $\mathcal{F}' = \mathcal{B}(o)$ *and* $\mathcal{F}'_+ = \mathcal{B}_G(o)$.

(4) $b \in [a]_+ \Rightarrow a \in [2b]$.
(5) $2\,[2a]' \subseteq 2\,([2a]_+)' \subseteq [2a]' \subseteq ([2a]_+)'$ *hence* $2\,\overline{\overline{o,a}} \subseteq 2\,\overset{\cdots\cdots}{o,a} \subseteq \overline{\overline{o,a}}$, *i.e.* $\overline{\overline{o,a}} \subseteq \overset{\cdots\cdots}{o,a} \subseteq (\overline{\overline{o,a}})'$.

Proof.

(2) Except for $[2a]_+ \subseteq [2a]'$ all inclusions follow by (5.1.6) and (5.1.1). Let $x \in [2a]_+$ hence $x^+ \circ (2a)^+ = (2a)^+ \circ x^+$ and so by $(\mathbf{K}_1)$ $(2x)^+ \circ (2a)^+ = x^+ \circ x^+ \circ (2a)^+ = x^+ \circ (2a)^+ \circ x^+ \in P^+$, i.e. $(2x) \in [2a]$, i.e. $x \in [2a]'$.

(3) By (3.5.2) $\widetilde{a} = \widetilde{o,2a}$, therefore: $x \in \overset{\cdots\cdots}{o,a} = \overset{\cdots\cdots}{a,o} \Leftrightarrow \widetilde{a} \circ \widetilde{o} \circ \widetilde{x} \in J \Leftrightarrow (2a)^+ \circ (2x)^+ = \widetilde{a} \circ \widetilde{o} \circ \widetilde{x} \circ \widetilde{o} = \widetilde{x} \circ \widetilde{o} \circ \widetilde{a} \circ \widetilde{o} = (2x)^+ \circ (2a)^+ \Leftrightarrow 2x \in [2a]_+$ hence $x \in \overset{\cdots\cdots}{o,a} \Leftrightarrow x \in ([2a]_+)'$. Furthermore $x \in \overline{\overline{o,a}}(= \overline{\overline{a,o}}) \Leftrightarrow \widetilde{a} \circ \widetilde{o} \circ \widetilde{x} \in \widetilde{P} \Leftrightarrow (2a)^+ \circ (2x)^+ \in \widetilde{P} \circ \widetilde{o} = P^+ \Leftrightarrow 2x \in [2a] \Leftrightarrow x \in [2a]'$ and the inclusion is stated in (2).

(4) $b \in [a]_+$ implies $b^+ \circ a^+ = a^+ \circ b^+$ and so $a^+ \circ (2b)^+ = a^+ \circ b^+ \circ b^+ = b^+ \circ a^+ \circ b^+ \in P^+$, i.e. $a \in [2b]$.

(5) Let $x \in ([2a]_+)' = \overset{\cdots\cdots}{o,a}$ (cf. (3)) hence $\widetilde{o} \circ \widetilde{a} \circ \widetilde{x} \in J$. Since by (3.5.2), $\widetilde{x} = \widetilde{o,2x}$ and so $(\widetilde{x} \circ \widetilde{o} \circ \widetilde{x})(2x) = 2x$, we have $\widetilde{2x} = \widetilde{x} \circ \widetilde{o} \circ \widetilde{x}$ hence $\widetilde{o} \circ \widetilde{a} \circ \widetilde{2x} = \widetilde{o} \circ \widetilde{a} \circ \widetilde{x} \circ \widetilde{o} \circ \widetilde{x} = \widetilde{x} \circ \widetilde{a} \circ \widetilde{x} \in \widetilde{P}$, i.e. $2x \in \overline{\overline{o,a}} = [2a]'$.

□

Proposition 5.4. *Let $A \subseteq P$ then:*

(1) $A' = \bigcup\{Fix\ \widetilde{o,a} \mid a \in A\}$ *and* $A \subseteq A'' = \bigcup\{Fix\ \widetilde{a} \mid a \in A\}$.
(2) *For $a \in P$, if $a \neq o \neq 2a$ then* $([2a]_+)' = \overset{\cdots\cdots}{o,a} = (\overset{\cdots\cdots}{o,a})'' = (([2a]_+)')''$ *and* $[2a]' = \overline{\overline{o,a}} = (\overline{\overline{o,a}})'' = ([2a]')''$.

Proof.

(1) is a direct consequence of (3.5.1) and (3.5.3) since $A'' = (2A)'$.

(2) By (3.5.2) $\widetilde{x} = \widetilde{y} \Leftrightarrow 2y = 2x$ and by (5.3.3) $x \in ([2a]_+)' = \overset{\cdots\cdots}{o,a} \Leftrightarrow \widetilde{o} \circ \widetilde{a} \circ \widetilde{x} \in J$ and $x \in [2a]' = \overline{\overline{o,a}} \Leftrightarrow \widetilde{o} \circ \widetilde{a} \circ \widetilde{x} \in \widetilde{P}$. Therefore, by (1), (1.1.2) and (5.3.2), $(([2a]_+)')'' = \bigcup\ \{Fix\ \widetilde{x} \mid x \in ([2a]_+)'\} = \bigcup\ \{Fix\ \widetilde{x} \mid \widetilde{o} \circ \widetilde{a} \circ \widetilde{x} \in J\} = \overset{\cdots\cdots}{o,a}$ and $([2a]')'' = \bigcup\ \{Fix\ \widetilde{x} \mid x \in [2a]'\} = \bigcup\ \{Fix\ \widetilde{x} \mid \widetilde{o} \circ \widetilde{a} \circ \widetilde{x} \in \widetilde{P}\} = \overline{\overline{o,a}}$.

□

6. Representation of blocks by centralizers

As we have seen in (5.1.1) and (5.3.3), there is a strict relation between the different kinds of blocks through a fixed point o of the invariant reflection structure (P, I) and the centralizers of the corresponding K-loop derivation in o. With the following Theorem we show that all the blocks of (P, I) can be represented as left cosets of suitable centralizers of the loop $(P, +)$:

Theorem 6.1. *Let $a, b \in P, a \neq b$ and $c := a - b$ then:*

(1) $\overset{\cdots\cdots}{a,b} = a + [c]_+ = a + [-a+b]_+$, *hence* $\mathcal{L}_G = \{x + [y]_+ \mid x \in P,\ y \in P^*\}$ *and* $\mathcal{L}_G(a) = a + \mathcal{F}_+$.

(2) $\overline{a,b} = a + [c] = a + [-a+b] = b + [-b+a] = \overline{b,a}$, *hence* $\mathcal{L} = \{x + [y] \mid x \in P,\ y \in P^*\}$ *and* $\mathcal{L}(a) = a + \mathcal{F}$.

(3) $\overset{\overline{\cdots\cdots}}{a,b} = a + \overline{\overset{\cdots\cdots}{o,c}} = a + ([2c]_+)' = b + ([2(-b+a)]_+)' = \overset{\overline{\cdots\cdots}}{b,a}$, *hence* $\mathcal{B}_G = \{x + ([2y]_+)' \mid x \in P,\ y \in P^*\}$ *and* $\mathcal{B}_G(a) = a + \mathcal{F}'_+$.

(4) $\overline{\overline{a,b}} = a + \overline{\overline{o,c}} = a + [2c]' = b + [2(-b+a)]' = \overline{\overline{b,a}}$, *hence* $\mathcal{B} = \{x + [2y]' \mid x \in P,\ y \in P^*\}$ *and* $\mathcal{B} = a + \mathcal{F}'$.

Proof. By the definition of $+, \widetilde{o,a} = a^+ \circ \widetilde{o}$ (cf. Sect. 3) hence $c = \widetilde{o,a}(b) = a - b$, $-c = -a + b$ and so by (4.5.5), (5.1.2) and (5.1.1), $\overset{\cdots\cdots}{a,b} = a^+ \circ \widetilde{o}\ (\overset{\cdots\cdots}{o,c}) = a^+ \circ \widetilde{o}\ ([c]_+) = a + [c]_+ = a + [-c]_+$, $\overline{a,b} = a + [c] = a + [-c]$, while by (4.5.5), (4.2.3) and (5.3.3), $\overset{\overline{\cdots\cdots}}{a,b} = a^+ \circ \widetilde{o}\ (\overline{\overset{\cdots\cdots}{o,c}}) = a^+(\overline{\overset{\cdots\cdots}{o,c}}) = a + ([2c]_+)'$. The same holds for $\overline{\overline{a,b}}$. □

If a loop $(P, +)$ is provided with a structure Σ we call $(P, +, \Sigma)$ a Σ-*loop* if $P^+ \subseteq Aut(P, \Sigma)$. By (4.5) we have:

Proposition 6.2. *If Σ is one of the structures $\mathcal{L}_G, \mathcal{L}_g, \mathcal{L}$ or $\mathcal{B}_G, \mathcal{B}$ defined in Sect. 3 then $(P, +, \Sigma)$ is a Σ-loop.*

7. Ternary equivalence relations

An involution set (P, I) is called a *point reflection set* if we claim :

$$\forall p \in P \ :\ |\widetilde{p}| = 1 \text{ and } Fix\ \widetilde{p} = \{p\}. \qquad (\mathbf{p})$$

Then $\widetilde{P} := \{\widetilde{p} \mid p \in P\} \subseteq I$.

Now let (P, I) be a *point reflection set* . Let us define on the set P the *ternary relations*

- $\kappa_G := \{(a, b, c) \in P^3 \mid \widetilde{a} \circ \widetilde{b} \circ \widetilde{c} \in J\}$ and
- $\kappa := \{(a, b, c) \in P^3 \mid \widetilde{a} \circ \widetilde{b} \circ \widetilde{c} \in \widetilde{P}\}$.

Remark: $\kappa \subseteq \kappa_G$.

We recall (cf. [6]) that a ternary relation $\rho \subseteq P^3$ is called *ternary equivalence relation* if $\forall a, b, c, d \in P$ the following conditions are satisfied:

(t1) (*reflexivity*) $(a, a, b), (a, b, a), (a, b, b) \in \rho$,
(t2) (*symmetry*) if $(a, b, c) \in \rho$ then $(b, a, c), (c, b, a) \in \rho$ and
(t3) (*transitivity*) if $a \neq b$ and $(a, b, c), (a, b, d) \in \rho$ then $(b, c, d) \in \rho$.

If ρ is a ternary equivalence relation on a set P then any two distinct elements $a, b \in P$ determine an equivalence class $\lfloor a, b \rfloor := \{x \in P \mid (a, b, x) \in \rho\}$. So, defining the set $\mathfrak{R}$ of all ternary equivalence classes determined by ρ, we can state:

Proposition 7.1. $(P, \mathfrak{R})$ *is a linear space.*

Proof. The proof can be obtained directly by comparing the definition of a ternary equivalence relation and the definition of a linear (or *incidence*) space (cf, e.g. [8, Sec. 1]). □

A point reflection set (P, I) may obey the *general three reflection axiom* (cf. [8], p. 115 (18.8)):

$$\forall a, b, x, y, z \in P, \text{ with } a \neq b \text{ and } \widetilde{a} \circ \widetilde{b} \circ \widetilde{x},\ \widetilde{a} \circ \widetilde{b} \circ \widetilde{y},\\ \widetilde{a} \circ \widetilde{b} \circ \widetilde{z} \in J :\ \widetilde{x} \circ \widetilde{y} \circ \widetilde{z} \in \widetilde{P}, \qquad (\mathbf{DS})$$

or one of its alterations:

($\mathbf{DS}'$) $\forall a, b, x, y \in P$ with $a \neq b$ and $\widetilde{a}\circ\widetilde{b}\circ\widetilde{x},\ \widetilde{a}\circ\widetilde{b}\circ\widetilde{y} \in J\ :\ \widetilde{b}\circ\widetilde{x}\circ\widetilde{y} \in J.$
($\mathbf{DS}''$) $\forall a, b, x, y \in P$ with $a \neq b$ and $\widetilde{a}\circ\widetilde{b}\circ\widetilde{x},\ \widetilde{a}\circ\widetilde{b}\circ\widetilde{y} \in \widetilde{P}\ :\ \widetilde{b}\circ\widetilde{x}\circ\widetilde{y} \in \widetilde{P}.$

We see immediately that **(DS)** implies **(DS′)** and **(DS″)**.

Proposition 7.2. *Let* (P, I) *be a point reflection set; then*

- κ_G *is a ternary equivalence relation if and only if* (P, I) *satisfies* ($\mathbf{DS}'$).
- κ *is a ternary equivalence relation if and only if* (P, I) *is invariant and satisfies* ($\mathbf{DS}''$).

The point reflection set (P, I) is called *exchange point reflection set* if (P, I) satisfies ($\mathbf{DS}$).

Proposition 7.3. *If* (P, I) *is an exchange point reflection set then:*

(1) $\forall a, b, c \in P$ *with* $a \neq b$ *and* $\widetilde{a} \circ \widetilde{b} \circ \widetilde{c} \in J\ :\ \widetilde{a} \circ \widetilde{b} \circ \widetilde{c} \in \widetilde{P}$ *(hence* $\kappa = \kappa_G$*).*
(2) $(P, \widetilde{P})$ *is an invariant point reflection set.*
(3) *The relation* $\kappa_G = \kappa$ *is a ternary equivalence relation.*

Now we consider an invariant reflection structure (P, I) and define the following ternary relations on the point set P:

$\sigma_G := \{(a, b, c) \in P^3 \mid \widetilde{a, b} \circ \widetilde{b} \circ \widetilde{b, c} \in J\}$ and

$\sigma := \{(a, b, c) \in P^3 \mid \widetilde{a, b} \circ \widetilde{b} \circ \widetilde{b, c} \in I\}$.

Clearly $\sigma \subseteq \sigma_G, \sigma_G$ and σ are reflexive, σ is symmetric and σ_G is symmetric if, and only if, (P, I) satisfies ($\mathbf{R}'$) (cf. (4.2.1)).

Moreover, if σ_G is transitive then it is immediate to see that it is also symmetric, hence a ternary equivalence relation, and $\mathcal{L}_G$ is the set of the corresponding ternary equivalence classes since $\sigma_G = \{(a, b, x) \in P^3 \mid x \in \overset{\cdots\cdots}{b, a}\}$. Similarly, if σ is transitive then $\mathcal{L}$ is the set of the corresponding ternary equivalence classes since $\sigma = \{(a, b, x) \in P^3 \mid x \in \overline{b, a}\}$. Therefore by (7.1):

Proposition 7.4. *$(P, \mathcal{L}_G)$, respectively, $(P, \mathcal{L})$, is a linear space if and only if σ_G, respectively, σ, is a ternary equivalence relation on P.*

From (1.1.3) we obtain (recall $\overline{p} = Fix\widetilde{p}$ and $\overline{P} = \{\overline{p} \mid p \in P\}$):

Proposition 7.5. *Let (P, I) be an invariant reflection structure then:*

(1) *$\forall \alpha \in I$, $\forall p \in P$: $\overline{\alpha(p)} = \alpha(\overline{p})$ and so $\overline{\alpha} : \overline{P} \to \overline{P}$; $\overline{p} \mapsto \overline{\alpha(p)}$ is a permutation of $\overline{P}$ (hence $\overline{\alpha} \in Sym\ \overline{P}$) in particular if $|\overline{P}| \geq 2$ and $a \in P$ then $\overline{\widetilde{a}}$ is an involution of $\overline{P}$ with $Fix\ \overline{\widetilde{a}} = \{\overline{a}\}$.*

(2) *For $\overline{\widetilde{P}} := \{\overline{\widetilde{p}} \mid p \in P\}$ the pair $(\overline{P}, \overline{\widetilde{P}})$ is an invariant point reflection set.*

If (P, I) is an invariant reflection structure then $(\overline{P}, \overline{\widetilde{P}})$ is called the *corresponding point reflection set* and we can define, as observed before, the following ternary relations:

$\overline{\kappa_G} := \{(\overline{a}, \overline{b}, \overline{c}) \in \overline{P}^3 \mid \widetilde{a} \circ \widetilde{b} \circ \widetilde{c} \in J\}$ and

$\overline{\kappa} := \{(\overline{a}, \overline{b}, \overline{c}) \in \overline{P}^3 \mid \widetilde{a} \circ \widetilde{b} \circ \widetilde{c} \in I\}$.

These are well defined since by (1.1.3) $\widetilde{u} = \widetilde{v} \iff v \in \overline{u}$. By (4.4.1) and (4.4.2), if $B \in \mathcal{B}_G$, or $B \in \mathcal{B}$, and $x \in B$ then $\overline{x} \subseteq B$. Therefore we set $\overline{B} := \{\overline{x} \mid x \in B\}$ and $\overline{\mathcal{B}} := \{\overline{B} \mid B \in \mathcal{B}\}$ and $\overline{\mathcal{B}_G} := \{\overline{B} \mid B \in \mathcal{B}_G\}$; thus we can formulate the theorem:

Proposition 7.6. *Let (P, I) be an invariant reflection structure and assume that σ_G, respectively, σ, is a ternary equivalence relation on P. Then, if $(\overline{P}, \overline{\widetilde{P}})$ is the corresponding point reflection set, it holds:*

(1) *$\overline{\kappa_G}$, respectively, $\overline{\kappa}$, is a ternary equivalence relation on $\overline{P}$.*

(2) *$\overline{\mathcal{B}_G}$, respectively, $\overline{\mathcal{B}}$, is the set of equivalence classes with respect to the ternary equivalence relation $\overline{\kappa_G}$, respectively, $\overline{\kappa}$.*

(3) *$(\overline{P}, \overline{\mathcal{B}_G})$, respectively $(\overline{P}, \overline{\mathcal{B}})$, is a linear space and, for any $\alpha \in I$, the map $\overline{\alpha} : \overline{P} \to \overline{P}$; $\overline{p} \mapsto \overline{\alpha(p)}$ is an automorphism of $(\overline{P}, \overline{\mathcal{B}_G})$, respectively, of $(\overline{P}, \overline{\mathcal{B}})$.*

(4) *$(\overline{P}, \overline{\widetilde{P}})$ is an invariant point reflection set satisfying $(\mathbf{DS}')$, respectively $(\mathbf{DS}'')$.*

Proof. By definition and (7.5.2) the relations $\overline{\kappa_G}$ and $\overline{\kappa}$ are reflexive and symmetric. It remains the transitivity. Let $a, b, c, d \in P$ such that $\overline{a} \neq \overline{b}$ and $(\overline{a}, \overline{b}, \overline{c})$, $(\overline{a}, \overline{b}, \overline{d}) \in \overline{\kappa_G}$, or, respectively, $\in \overline{\kappa}$, hence $\widetilde{a} \circ \widetilde{b} \circ \widetilde{c}$, $\widetilde{a} \circ \widetilde{b} \circ \widetilde{d} \in J$, or, respectively, $\in I$. By (1.1.2), $\widetilde{a} = \widetilde{\widetilde{a}(b), b}$ then we obtain $\widetilde{a} \circ \widetilde{b} \circ \widetilde{c} = \widetilde{\widetilde{a}(b), b} \circ \widetilde{b} \circ \widetilde{b, \widetilde{c}(b)} \in J$, or resp. $\in I$, and so $(\widetilde{a}(b), b, \widetilde{c}(b))$, $(\widetilde{a}(b), b, \widetilde{d}(b)) \in \sigma_G$, or resp. $\in \sigma$. Now, $\overline{a} \neq \overline{b}$ implies $\widetilde{a}(b) \neq b$ and so the transitivity of σ_G, or resp. σ, forces $(b, \widetilde{c}(b), \widetilde{d}(b)) \in \sigma_G$, or resp. $\in \sigma$, and the symmetry, $(\widetilde{c}(b), b, \widetilde{d}(b)) \in \sigma_G$, or resp. $\in \sigma$. Consequently, $\widetilde{c} \circ \widetilde{b} \circ \widetilde{d} = \widetilde{\widetilde{c}(b), b} \circ \widetilde{b} \circ \widetilde{b, \widetilde{d}(b)} \in J$, or resp. $\in I$, i.e. $(\overline{c}, \overline{b}, \overline{d}) \in \overline{\kappa_G}$, or resp. $\in \overline{\kappa}$.

The remaining parts follow from the definitions of $\mathcal{B}_G$ and $\mathcal{B}$, and from (7.1) and (7.2). $\square$

Finally we remark that an invariant reflection structure (P, I) and the corresponding point reflection set $(\overline{P}, \overline{\widetilde{P}})$ coincide if, and only if (P, I) is an invariant point reflection structure (i.e. $\forall \alpha \in I : |Fix\alpha| = 1$) and, by (1.1.2) and (4.2.2), we have the following

Proposition 7.7. *Let (P, I) be an exchange invariant point reflection structure (i.e. the general three reflection axiom $(\mathbf{DS})$ is fulfilled) then $\forall a, b \in P, a \neq b$: $\overline{a,b} = \overset{\cdots\cdots}{a,b} \subseteq \overline{\overline{a,b}} = \overline{\overset{\cdots\cdots}{a,b}}$. Furthermore $(\mathbf{R_3})$ and $(\overline{\mathbf{R_3}})$ are fulfilled.*

8. Exchange conditions for centralizers

In this section let (P, I) be an invariant reflection structure, let $o \in P$ be fixed, let $P^* := P \backslash \{o\}$ and let $(P, +)$ be the K-loop derived from (P, I) in o.
We will consider the following *exchange conditions* (cf. [10] for $(\mathbf{E})$ and $(\mathbf{E}_K)$):

$(\mathbf{E}_G)$ $\forall\ a, b \in P^*$ if $b \in [a]_+$ then $[b]_+ = [a]_+$
$(\mathbf{E})$ $\forall\ a, b \in P^*$ if $b \in [a]$ then $[b] = [a]$
$(\mathbf{E}_K)$ $\forall\ a, b, c \in P^*$ if $\delta_{a,b} = \delta_{a,c} = id$ then $\delta_{b,c} = id$

The first two exchange conditions, expressed in terms of the two kinds of centralizers introduced in Sect. 5, have also their counterpart in terms of the corresponding binary relations β_G and β (introduced in Sect. 5 too), furthermore, by (5.2), $(\mathbf{E}_K)$ is an equivalent version of **(E)** in terms of the δ-functions of the loop $(P, +)$ (the equivalence of these two conditions was also shown in [10]). So we can state:

Proposition 8.1. *The binary relation β_G, respectively, β, is an equivalence relation if and only if $(\mathbf{E}_G)$, respectively,* **(E)**, *is satisfied; moreover the exchange conditions $(\mathbf{E})$ and $(\mathbf{E}_K)$ are equivalent.*

Proposition 8.2. *For the K-loop $(P, +)$ derived in a point $o \in P$ from the invariant reflection structure (P, I), the following hold:*

(1) *If $(P, +)$ satisfies* **(E)** *then for all $a \in P^*$, $([a], +)$ is a commutative subgroup of $(P, +)$ (cf. [10]) and $\mathcal{F}$ is the set of all equivalence classes of the relation β. Moreover $[a]_+ = [a]$ for all $a \in P^*$ such that $2a \neq o$.*
(2) *If $(P, +)$ satisfies $(\mathbf{E}_G)$ then for all $a \in P^*$, $2[a]_+ + [a]_+ = [a]_+$ and $\mathcal{F}_+$ is the set of all equivalence classes of the relation β_G.*

Proof.

(1) Let $b \in [a], b \neq o$ then by $(\mathbf{E})$ and (5.15.1.5), $b + [a] = b + [b] = [b] = [a]$ hence $[a] + [a] \subseteq [a]$ and by (5.1.2) $-[a] = [-a] = [a]$. Now if $x, y, z \in [a]$ and $x \neq o$ then $y \in [x]$, i.e. since $[a] \subseteq [a]_+$ by (5.1.1), $x^+ \circ y^+ = (x + y)^+ = y^+ \circ x^+ = (y + x)^+$ hence $x + y = y + x$ and $x + (y + z) = x^+ \circ y^+(z) = (x + y)^+(z) = (x + y) + z$. Thus $([a], +)$ is a

commutative subgroup of $(P,+)$. If $2a \neq o$ then by **(E)**, $[a]=[2a]$ and hence, by (5.3.2), $[a] \subseteq [a]_+ \subseteq [2a]=[a]$, i.e. $[a]_+ = [a]$.

(2) Let $x \in [a]_+ \backslash \{o\}$ then $[x]_+ = [a]_+$ and by (5.1.6), $2x + [a]_+ = 2x + [x]_+ = [x]_+ = [a]_+$.

□

A loop $(P,+)$ is called an *exchange loop* if the exchange condition $(\mathbf{E})$ is satisfied. From (8.2.1) we obtain, for an exchange K-loop with $2a \neq o$ for all $a \in P^*$, that $[a]_+ = [a]$, for all $a \in P^*$, hence also the condition $(\mathbf{E}_G)$ is satisfied.

If (P,I) is an invariant point reflection structure (hence $I = \widetilde{P}$), then we recall that the corresponding K-loop $(P,+)$ (derived in any fixed point o) is uniquely 2-divisible and, in particular, $2a \neq o$ for all $a \neq o$.

So, we can state:

Proposition 8.3. *If* (P,I) *is an invariant point reflection structure satisfying the exchange condition* $(\mathbf{E}_G)$ *then* $\forall c \in P^* : ([2c]_+)' = [c]_+$ *and this implies (cf. (6.1))* $\forall a,b \in P$, *with* $a \neq b$, $\overset{\cdots\cdots}{a,b} = \overline{\overset{\cdots\cdots}{a,b}}$.

Proof. By (5.3.2), $[c]_+ \subseteq ([2c]_+)'$ and $2c \in [c]_+$; then by $(\mathbf{E}_G)$, $[2c]_+ = [c]_+$ and by definition of $([2c]_+)', x \in ([2c]_+)' \Leftrightarrow 2x \in [2c]_+ = [c]_+$. Hence if $x \in ([2c]_+)'$ and $x \neq o$ then $x \in [2x]_+ = [2c]_+ = [c]_+$, i.e. $([2c]_+)' = [c]_+$. □

As a consequence, we see that in the case of an invariant point reflection structure the general three reflection axioms $(\mathbf{DS})$, $(\mathbf{DS'})$, $(\mathbf{DS''})$ introduced in Sect. 7 may be compared with the three reflection axioms $(\mathbf{R_i})$ and $(\overline{\mathbf{R_3}})$ introduced in Sect. 4. In fact we have the following results.

Proposition 8.4. *Let* (P,I) *be an invariant point reflection structure. Then, between the three reflection axioms* $(\mathbf{R_3})$ *and* $(\overline{\mathbf{R_3}})$, *the general three reflection axioms* $(\mathbf{DS}), (\mathbf{DS'}), (\mathbf{DS''})$ *and the exchange conditions* $(\mathbf{E}_G), (\mathbf{E})$ *the following connections hold:*

(1) $(\mathbf{DS}) \Rightarrow (\mathbf{R_3}) \Leftrightarrow (\overline{\mathbf{R_3}})$.
(2) $(\mathbf{DS'}) \Leftrightarrow (\mathbf{E}_G)$.
(3) $(\mathbf{DS''}) \Leftrightarrow (\mathbf{E})$.

In particular, $(\mathbf{DS''}) \Rightarrow (\mathbf{DS'})$ *since* $(\mathbf{E}) \Rightarrow (\mathbf{E}_G)$.

Proof.

(1) Since $I = \widetilde{P}$ the statements $(\mathbf{R_3})$ and $(\overline{\mathbf{R_3}})$ are the same. Let $\widetilde{a} \circ \widetilde{b} \circ \widetilde{c} \in J$. If $a = b$ then $\widetilde{a} \circ \widetilde{b} \circ \widetilde{c} = \widetilde{c} \in \widetilde{P}$. If $a \neq b$ then $\widetilde{a} \circ \widetilde{b} \circ \widetilde{a}$, $\widetilde{a} \circ \widetilde{b} \circ \widetilde{b} = \widetilde{a}$, together with $\widetilde{a} \circ \widetilde{b} \circ \widetilde{c} \in J$, imply by $(\mathbf{DS}), \widetilde{a} \circ \widetilde{b} \circ \widetilde{c} \in \widetilde{P} = I$. Hence $(\mathbf{DS}) \Longrightarrow (\overline{\mathbf{R_3}})$

(2) (in the same way 3.) First assume $(\mathbf{DS'})$ and let $a,b,x \in P^*$ with $b,x \in [a]_+ = \overset{\cdots\cdots}{o,a} = \{x \in P \mid \widetilde{o,a} \circ \widetilde{o} \circ \widetilde{o,x} \in J\}$ (cf. Sect. 4 and (5.1.1)) and $a \neq b$. Hence by (3.5.1) and (3.5.2) $\widetilde{o,a} \circ \widetilde{o} \circ \widetilde{o,b} = \widetilde{a'} \circ \widetilde{o} \circ \widetilde{b'}$, $\widetilde{o,a} \circ \widetilde{o} \circ \widetilde{o,x} = \widetilde{a'} \circ \widetilde{o} \circ \widetilde{x'} \in J$ and $a \neq b$, and so by $(\mathbf{DS'})$,

$\widetilde{o} \circ \widetilde{b'} \circ \widetilde{x'} \in J \Longrightarrow \widetilde{o,b} \circ \widetilde{o} \circ \widetilde{o,x} = \widetilde{b'} \circ \widetilde{o} \circ \widetilde{x'} \in J \Longrightarrow x \in [b]_+$.
Therefore $(\mathbf{DS'}) \Rightarrow (\mathbf{E}_G)$.
Now we assume $(\mathbf{E}_G)$. For $a, b, x, y \in P$ with $a \neq b$ and $c := a - b$ we have by (4.2.2) and (8.3):
(*) $\widetilde{a} \circ \widetilde{b} \circ \widetilde{x} \in J \Longleftrightarrow \widetilde{b} \circ \widetilde{a} \circ \widetilde{x} \in J \Longleftrightarrow x \in \overline{\overset{\cdots\cdots}{a,b}} = \overline{\overset{\cdots\cdots}{b,a}} = a + ([2c]_+)' = a + [c]_+$ (cf. (6.1.3)) $\Longleftrightarrow -a + x \in [c]_+ \Longleftrightarrow -b + x \in [c]_+$. Therefore $\widetilde{a} \circ \widetilde{b} \circ \widetilde{x} \in J$ and $\widetilde{a} \circ \widetilde{b} \circ \widetilde{y} \in J$ implies by (*) $-b + x, -b + y \in [c]_+$. If $x = y$ then $\widetilde{b} \circ \widetilde{x} \circ \widetilde{y} = \widetilde{b} \in I \subseteq J$. Hence, suppose $x \neq y$, so we may assume $-b + x \neq o$. Then by $(\mathbf{E}_G)$, $-b + y \in [c]_+ = [-b + x]_+ \overset{(*)}{\Longleftrightarrow} \widetilde{b} \circ \widetilde{x} \circ \widetilde{y} \in J$ showing $(\mathbf{E}_G) \Rightarrow (\mathbf{DS'})$.

□

By (8.3) and (8.4), we can improve the first part of Proposition (7.7) by the following

Proposition 8.5. *Let (P, I) be an exchange invariant point reflection structure (i.e. the general three reflection axiom* $(\mathbf{DS})$ *is fulfilled) then $\forall a, b \in P, a \neq b$: $\overline{a,b} = \overset{\cdots\cdots}{a,b} = \overline{\overline{a,b}} = \overline{\overset{\cdots\cdots}{a,b}}$.*

Finally we can prove the main Theorem:

Theorem 8.6. *Let σ_G and σ be the ternary relations defined for (P, I) on P according to Sect. 7, let β_G and β be the binary relations defined for $(P, +)$ on P^* according to Sect. 5 and let $\mathcal{L}_G$, $\mathcal{L}$ be the sets of blocks defined in P according to Sect. 4. Then:*

(1) *The following conditions are equivalent:*
- *σ_G is a ternary equivalence relation,*
- *β_G is a binary equivalence relation,*
- *the exchange condition* $(\mathbf{E}_G)$ *is fulfilled,*
- *$(P, \mathcal{L}_G)$ is a linear space.*

(2) *The following conditions are equivalent:*
- *σ is a ternary equivalence relation,*
- *β is a binary equivalence relation,*
- *the exchange condition* $(\mathbf{E})$ *is fulfilled,*
- *$(P, \mathcal{L})$ is a linear space.*

Proof.

(1) We already remarked (see Sect. 7) that the ternary relation σ_G is reflexive by definition and that its symmetry is a consequence of the transitivity. Moreover, since $\beta_G = \{(a, b) \in (P^*)^2 \mid \widetilde{o,a} \circ \widetilde{o} \circ \widetilde{o,b} \in J\}$ is trivially reflexive and symmetric, in order to prove the equivalence of the first two statements we have only to prove that the transitive property holds for β_G if and only if it holds for σ_G. In fact, by definition $\sigma_G = \{(a, b, c) \in P^3 \mid \widetilde{a,b} \circ \widetilde{b} \circ \widetilde{b,c} \in J\}$, then $(a, x) \in \beta_G$ if and only if $(a, o, x) \in \sigma_G$, so $(a, b), (b, c) \in \beta_G$ implies $(a, c) \in \beta_G$ if and only if (a, o, b) and $(b, o, c) \in \sigma_G$ implies $(a, o, c) \in \sigma_G$. Now, if we consider any

triple $(x, p, y) \in P^3$ and define $x' := \widetilde{o, p}(x)$ and $y' := \widetilde{o, p}(y)$, from (1.4.1) we deduce that $(x, p, y) \in \sigma_G$ if and only if $(x', o, y') \in \sigma_G$ and this completes the proof.

By (8.1) β_G is an equivalence relation if and only if the exchange condition ($\mathbf{E}_G$) is satisfied, and this proves the equivalence of the second and third sentence.

Finally, by (7.4) also the equivalence between the first and the last statement is valid.

(2) In this case the proof of the equivalence of the first two sentences is like as in the previous case and even shorter since the relation σ is automatically symmetric. For the equivalence of the second and third statement we refer to (8.1) and, for the equivalence of the first and the last one, to (7.4).

□

We conclude by remarking that if $(P, \mathfrak{G}, \equiv, \alpha)$ is an ordinary absolute geometry and $\widetilde{P} := \{\tilde{p} \mid p \in P\}$ is the set of all point reflections then (cf. [4]):

- The pair $(P, \widetilde{P})$ is an *exchange invariant point reflection structure* (cf. Sects. 3 and 7), so that $\kappa = \{(a, b, c) \in P^3 \mid \tilde{a} \circ \tilde{b} \circ \tilde{c} \in \widetilde{P}\}$ is a ternary equivalence relation and $\mathfrak{G}$ is the set of all equivalence classes of κ.
- If $(P, +)$ is the K-loop obtained by fixing a point $o \in P$ and setting $a + b = \widetilde{a'} \circ \tilde{o}(b)$, where a' denotes the midpoint of o and a, then $(P, +)$ is an exchange uniquely 2-divisible K-loop and the set of lines $\mathfrak{G}$ can be written in the form $\mathfrak{G} = \{a + [b] \mid a, b \in P,\ b \neq o\}$. In particular, if $a, b \in P$ with $a \neq b$ then $\overline{a, b} = \overset{\cdots\cdots}{a, b} = \underline{\overset{\cdots\cdots}{a, b}} = \overline{\overline{a, b}} = a + [b - a]$ is the unique line of $\mathfrak{G}$ joining the points a and b.

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