PROBLEM OF THE WEEK
Solution of Problem No. 8 (Fall 2013 Series)

Problem:

Let $\vec{v}$ and $\vec{v}_1, \vec{v}_2, \ldots$ be vectors in three space. Suppose $|\vec{v}| = 1$ and $|\vec{v}_n| \geq 1$ for $n \geq 1$, where the absolute value signs represent the length of the vector involved. Prove

$$|\vec{v}_n| + |\vec{v}| - |\vec{v}_n + \vec{v}| \longrightarrow 0 \quad \textbf{as} \quad n \to \infty$$

if and only if there exists a sequence of positive scalars r_n such that

$$|r_n \vec{v}_n - \vec{v}| \longrightarrow 0 \quad \textbf{as} \quad n \to \infty.$$

Solution: (by David Stoner, Student, South Aiken High School, S. Carolina)

Let θ_n and d_n denote the angle between v and v_n, and the value $|v_n|$ respectively. Note that the second condition is true if and only if the sequence of scalars which minimize the given quantity work. Note that for the second condition to hold true, θ_n must clearly become eventually acute. After this point, the minimum value of the second quantity is $|v| \sin \theta_n$ for each n. Therefore, the second condition is true iff $\theta_n \to 0$. It remains to show that $|v| + |v_n| - |v + v_n| \to 0$ iff $\theta_n \to 0$. Note that by the law of cosines, $|v| + |v_n| - |v + v_n| = 1 + d_n - \sqrt{1 + d_n^2 + 2d_n \cos \theta_n}$. Now:

$$1 + d_n - \sqrt{1 + d_n^2 + 2d_n \cos \theta_n} = \frac{2d_n(1 - \cos \theta_n)}{1 + d_n + \sqrt{1 + d_n^2 + 2d_n \cos \theta_n}}.$$

Let this quantity be A_n. Then, using $d_n \geq 1$:

$$A_n \geq \frac{2d_n(1 - \cos \theta_n)}{d_n + 1 + d_n + 1} = \left(\frac{d_n}{d_n + 1}\right)(1 - \cos \theta_n) \geq \frac{1 - \cos \theta_n}{2}$$

$$A_n \leq \frac{2d_n(1 - \cos \theta_n)}{d_n + d_n} = 1 - \cos \theta_n.$$

Hence $A_n \to 0$ iff $1 - \cos \theta_n \to 0$ iff $\theta_n \to 0$ as desired.

The problem was also solved by:

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