PROBLEM OF THE WEEK

Solution of Problem No. 6 (Fall 2013 Series)

Problem:

Let $0 < m < n < p$, where m, n, and p are integers. Let M be a matrix with three rows and k columns, where $k \geq 2$. Suppose every column of M contains each of m, n, and p. Suppose the sum of the numbers in the top row is 20, the sum of the second row is 10, and the sum of the bottom row is 9. If the last number in the second row is p, which row has first entry n?

Solution: (by Charles Burnette, Graduate Student, Drexel University, PA)

Since each column of M has a sum of $m+n+p$, the sum of all the entries of M is $k(m+n+p)$. Yet, the sum of all the entries is also $20+10+9=39$, and so

$$k(m+n+p) = 39.$$

Now because $m \geq 1$ and $m < n < p$, the sum $m+n+p$ is at least $1+2+3=6$. The only divisors of 39 that are not smaller than 6 are 13 and 39. If $m+n+p=39$, then $k=1$, which contradicts the assumption that $k \geq 2$. So $m+n+p=13$ and $k=3$, making M a 3×3 matrix.

Focusing on the second row, we know that either m or n is missing from the row, otherwise the row sum would be $m+n+p=13$ instead of 10. We are given that entry $(2,3)$ is p. Now if the second row lacked m, then the row sum would be at least $n+n+p > m+n+p = 13$, which is impossible since the row is 10. Thus the row is lacking n. furthermore, the second row cannot have two ps, as then the row sum would again be bigger than 10. The second row is therefore $[m\ m\ p]$, and so $2m+p=10$.

Next, we know that either entry $(1,3)$ or $(3,3)$ is m. If $(1,3)$ had m, then the two remaining entries of the first row must be p in order for the sum to exceed 13. Hence $m+2p=20$, and this together with $2m+p=10$ gives $m=0$ and $p=10$, which is impossible since m is positive. Therefore m is the last number in the third row. In addition, the third row of M cannot have p, as then the row sum would be at least $p+2m=10$, which is too big. Neither of the two remaining entries can be an m since ms are already present in the first two columns. It now follows that the third row has n as its first entry.

We can actually continue and find the exact values of m, n and p. Indeed, the matrix can now be filled in:

$$M = \begin{bmatrix} p & p & n \\ m & m & p \\ n & n & m \end{bmatrix}.$$

This gives the following system of equations: $n+2p=20$, $2m+p=10$, and $m+2n=9$. Solving yeields $m=1, n=4$, and $p=8$. Hence

$$M=\begin{bmatrix} 8 & 8 & 4 \\ 1 & 1 & 8 \\ 4 & 4 & 1 \end{bmatrix}.$$

The problem was also solved by:

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