PROBLEM OF THE WEEK
Solution of Problem No. 1 (Fall 2013 Series)

Problem:

Let $a_n > 0$, $n \geq 0$. Call k "good" if $k \geq 1$ and $a_k > \frac{1}{2} a_{k-1}$. Also call 0 good.

Show $\sum_{k=0}^{\infty} a_k$ converges if $\sum_{\text{good } k} a_k$ converges.

Solution: (by David Stoner)

For each nonnegative integer k, define $f(k)$ to be the least positive integer greater than k which is good. If no such integers exist, take $f(k) = \infty$. Define $S_k = \sum_{i=k}^{f(k)-1} a_i$ (this is $\sum_{i=k}^{\infty} a_i$ if $f(k) = \infty$.)

Lemma: When k is good, $S_k \leq 2a_k$.

Proof: If $f(k) = k+1$, then $S_k = a_k$ and the result is clearly true. Otherwise, for each good k, by definition all integers n with $k < n < f(k)$ satisfy $a_n \leq \frac{a_{n-1}}{2}$. By repeated applications of this, we have:

$$a_{k+i} \leq \frac{a_k}{2^i}$$

for integers $0 < i < f(k) - k$. This means that $S_k = a_k + \sum_{i=1}^{f(k)-k-1} a_{k+i} \leq a_k + \sum_{i=1}^{\infty} \frac{a_k}{2^i} = 2a_k$ as desired. So the lemma is proved. Now note that since a_0 is good, we have $\sum_{k=0}^{\infty} a_k = \sum_{\text{good } k} S_k$. But $0 < \sum_{\text{good } k} S_k \leq \sum_{\text{good } k} 2a_k$ by the lemma, and the latter converges by assumption. Therefore, $\sum_{k=0}^{\infty} a_k$ converges as desired.

The problem was also solved by:

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