On the simultaneous edge coloring of graphs

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Abstract

A μ–simultaneous edge coloring of graph G is a set of μ proper edge colorings of G with a same color set such that for each vertex, the sets of colors appearing on the edges incident to that vertex are the same in each coloring and no edge receives the same color in any two colorings. The μ–simultaneous edge coloring of bipartite graphs has a close relation with μ–way Latin trades. Mahdian et al. (2000) conjectured that every bridgeless bipartite graph is 2–simultaneous edge colorable. Luo et al. (2004) showed that every bipartite graphic sequence S with all its elements greater than one, has a realization that admits a 2–simultaneous edge coloring. In this paper, the μ–simultaneous edge coloring of graphs is studied. Moreover, the properties of the extermal counterexample to the above conjecture are investigated. Also, a ralation between 2–simultaneous edge coloring of a graph with a cycle double cover with certain properties is shown and using this relation, some results about 2–simultaneous edge colorable graphs are obtained.

Keywords: Simultaneous edge coloring; Cycle double cover; Oriented cycle double cover; Latin trades.

1 Introduction

In this paper all graphs we consider are finite and simple. For notations and definitions we refer to [4]. This section deals with a brief review of some concepts raleted to the main subject of the paper.

Let S be a nonempty proper subset of $V(G)$. The subset $[S,\overline{S}] = \{uv \in E(G) : u \in S, v \in \overline{S}\}$ of $E(G)$ is called an edge cut. A k-edge cut is an edge cut $[S,\overline{S}]$, where $|[S,\overline{S}]| = k$. An edge cut F, is called trivial if one of the component in $G \setminus F$ be an isolated vertex. The edge connectivity of G, $\kappa'(G)$, is the minimum k for which G has a k-edge cut and G is said to be k–edge-connected if $\kappa'(G) \geq k$. A 2–edge-connected graph is called a bridgeless graph.

A proper edge coloring of a graph G is a labeling from $E(G)$ to the color set $[l] = \{1,\ldots,l\}$ such that incident edges have different colors. The edge chromatic number of G, $\chi'(G)$, is the least l such that G admits a proper edge coloring with label set $[l]$. A k-factor of graph G is a k-regular spanning subgraph of G, and G is k-factorable if there are edge disjoint k-factors $H_1, \ldots, H_l$ such that $G = H_1 \cup \ldots \cup H_l$. Note that an r-regular graph G is 1-factorable if and only if $\chi'(G) = r$.

We use the term circuit for a connected 2-regular graph and the term cycle for a graph that all its vertices have even degrees. A cycle double cover (CDC), $\mathcal{C}$, of a graph G is a collection of its cycles such that every edge of G is contained in precisely two cycles in $\mathcal{C}$ and a k-cycle double cover (k-CDC) of G is a CDC of G such that consisting of at most k cycles of G. Note that the cycles are not necessarily distinct. A necessary condition for a graph to have a CDC is the bridgeless property. Seymour [14] in 1979 conjectured that this condition is also sufficient.

Conjecture 1. [14] (CDC conjecture) *Every bridgeless graph has a* CDC.

No counterexample to the CDC conjecture is known. It is proved that the minimal counterexample to the CDC conjecture is a bridgeless cubic graph with edge chromatic number equal to 4, which is called a snark. The CDC conjecture has many stronger forms, one of which is the following conjecture. An oriented cycle double cover (OCDC) of a graph G is a CDC of G in which every circuit can be oriented in such a way that every edge of the graph is covered by two directed circuits in two different directions.

Conjecture 2. [9] (OCDC conjecture) *Every bridgeless graph has an* OCDC.

The concept of cycle double cover has a relation with nowhere-zero flow in graphs. Some necessary relations of these two concepts are presented in what follows.
Let G be a simple graph and (D, f) be an ordered pair, where D is an orientation of $E(G)$ and f is a weight on $E(G)$ to $\mathbb{Z}$. For each $v \in V(G)$, denote

$$f^+(v) = \sum f(e) \qquad \text{and} \qquad f^-(v) = \sum f(e),$$

where the summation is taken over all directed edges of G (under the orientation D) with tails and heads, respectively, at the vertex v. An integer flow of G is an ordered pair (D, f) such that for every vertex $v \in V(G)$, $f^+(v) = f^-(v)$. The support of f, *supp*(f), is the set of the edge $e \in E(G)$ that $f(e) \neq 0$. A nowhere-zero k-flow of G is an integer flow (D, f) such that *supp*$(f) = E(G)$ and $-k < f(e) < k$, for every $e \in E(G)$ and is denoted by k-NZF.

Theorem A. [15]

(i) *If every edge of a graph G is contained in a circuit of length at most* 4, *then G admits a* 4-NZF.

(ii) *A graph G admits a* 4-NZF *if and only if G has a* 4-CDC.

(iii) *A graph G admits a* 4-NZF *if and only if G has an* OCDC *consists of four directed cycles.*

Let G be a bipartite graph with bipartition (X, Y). The bipartite degree sequence of G is the sequence $(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m)$, where $(x_1, x_2, \ldots, x_n)$ are the vertex degrees in X and $(y_1, y_2, \ldots, y_m)$ are the vertex degrees in Y. A sequence S of positive integers is called a bipartite graphic sequence if there exists a bipartite graph G whose bipartite degree sequence is S; if so then the graph G is called a realization of S.

Definition 1. [13] *A μ–simultaneous edge coloring of graph G is a set of μ proper edge colorings of G with the color set $[l]$, say $(c_1, c_2, \ldots, c_\mu)$, such that*

- *for each vertex, the sets of colors appearing on the edges incident to that vertex are the same in each coloring;*

- *no edge receives the same color in any two colorings.*

If G has a μ–simultaneous edge coloring, then G is called a μ–simultaneous edge colorable graph. The minimum l that there exists a μ–simultaneous edge coloring of G with the color set $[l]$, is called $\mu - SE$ chromatic number of G and denoted by $\chi'_{\mu-SE}(G)$.

Note that in every μ–simultaneous edge coloring of a graph G, $\mu \leq \deg_G(v)$, for every $v \in V(G)$, because every edge $e = uv \in E(G)$ admits μ different colors of colors appeared of the edge incident to v.

Observation *If G is a μ–simultaneous edge colorable graph, then $\mu \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G. Moreover,*

$$\Delta(G) \leq \chi'(G) = \chi'_{1-SE}(G) \leq \chi'_{2-SE}(G) \leq \cdots \leq \chi'_{\mu-SE}(G).$$

There are some graphs G that $\chi'(G) < \chi'_{\mu-SE}(G)$; for example in the next section we show that for graph G shown in Figure 1, $\chi'_{2-SE}(G) \leq 4$, and by a case study, it can be checked that G has no 2–simultaneous edge coloring with 3 colors. Thus, $\chi'_{2-SE}(G) = 4$ while $\chi'(G) = 3$. In this case, $\chi'_{2-SE}(G) = \Delta(G) + 1$. This is a natural question: Is this true that $\Delta(G) \leq \chi'_{2-SE}(G) \leq \Delta(G) + 1$?

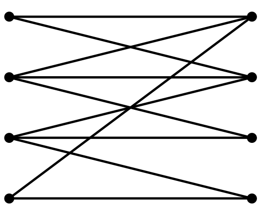

Figure 1: $\chi'(G) = 3$ and $\chi'_{2-SE}(G) = 4$.

At the 16^{th} British Combinatorial Conference (1997), Cameron introduced the concept 2–simultaneous edge coloring. He use this concept to reformulate a conjecture of Keedwell (1994) on the existence of critical partial Latin squares of given type. In fact he conjectured (called SE conjecture) that for each bipartite graphic sequence S with all its elements greater than one, there exists a 2–simultaneous edge colorable realization.

Mahdian et al. in [13] showed that the 2–simultaneous edge coloring of every bipartite graph is equivalent to an OCDC of that graph. Also, they conjectured that every bridgeless bipartite graph is 2–simultaneous edge colorable.

Theorem B. [13] *Every bipartite graph G is 2-simultaneous edge colorable if and only if G has an* OCDC.

Conjecture 3. [13] (strong SE conjecture) *Every bridgeless bipartite graph is 2–simultaneous edge colorable.*

Luo et al. in [11] showed that every bipartite graphic sequence S with all its elements greater than one, has a realization that admits a 4-NZF. Thus, by Theorems A (iii) and B, they proved that the SE conjecture is true.

In Section 2, we see the relation between μ–simultaneous edge coloring and μ–way Latin trade, also, we give some sufficient conditions for graphs to be μ–simultaneous edge colorable. In Section 3, we consider the case $\mu = 2$. First, some properties for the extermal counterexample to the strong SE conjecture are given; then, we discuss on 2–simultaneous edge coloring for general graphs and introduce some 2–simultaneous edge colorable graphs and some graphs which has no 2–simultaneous edge coloring.

2 μ–simultaneous edge coloring and μ–way Latin trade

A partial Latin square P of order n is an $n \times n$ array of elements from the set $[n] = \{1, \ldots, n\}$, where each element of $[n]$ appears at most once in each row and at most once in each column. We can represent each partial Latin square, P, as a subset of $[n] \times [n] \times [n]$,

$$P = \{(i, j; k) : \text{ element } k \text{ is located in position } (i, j)\}.$$

The set $S_P = \{(i, j) : (i, j; k) \in P,\ 1 \leq k \leq n\}$ of the partial Latin square P is called the shape of P and $|S_P|$ is called the volume of P. By $\mathcal{R}_P^i$ and $\mathcal{C}_P^j$ we mean the set of entries in row i and column j, respectively of P.

A μ–way Latin trade, $(T_1, \ldots, T_\mu)$, of volume s is a collection of μ partial Latin squares $T_1, \ldots, T_\mu$, containing exactly the same s filled cells, such that if cell (i, j) is filled, it contains a different entry in each of the μ partial Latin squares, and row i in each of the μ partial Latin squares contains, set-wise, the same symbols and column j, likewise. If $\mu = 2$, (T_1, T_2) is called a Latin bitrade. The volume spectrum $\mathcal{S}_\mu$ for all μ–way Latin trades is the set of possible volumes of μ–way Latin trades. For a survey on this topic see [3], [5], and [10].

For every μ–way Latin trade $T = (T_1, \ldots, T_\mu)$ of volume s there exists a μ–simultaneous edge colorable bipartite graph G with s edges and bipartite degree sequence $S = (|\mathcal{R}_T^1|, \ldots,$

$|\mathcal{R}_T^n|; |\mathcal{C}_T^1|, \dots, |\mathcal{C}_T^m|)$. In fact $G = (X, Y)$ is a bipartite graph, where $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$ such that for every filled cell (i, j) in T, there is an edge between x_i and y_j and the element that located in position (i, j) of T_k is the color of edge x_iy_j in the k^{th} coloring of μ–simultaneous edge coloring of G, for $1 \le k \le \mu$.

In Figure 2 a Latin bitrade, $T = (T_1, T_2)$, of volume 10 is demonstrated. (• means the cell is empty.) In fact, T is the Latin bitrade corresponding to a 2–simultaneous edge coloring of the graph G that showed in Figure 1. Therefore, $\chi'_{2-SE}(G) \le 4$.

1	2	•	•
2	4	3	•
•	1	4	3
3	•	•	1

2	1	•	•
3	2	4	•
•	4	3	1
1	•	•	3

Figure 2: $T = (T_1, T_2)$ a Latin bitrade of volume 10.

Since Luo et al. in [11] showed that each bipartite graphic sequence S with all its elements greater than 1, has a 2–simultaneous edge colorable bipartite realization, we have

$$\mathcal{S}_2 = \mathbb{N} \setminus \{1, 2, 3, 5\}.$$

Theorem C. [1, 2] *The volume spectrums for all μ–way Latin trades, $\mu = 3, 4, 5$ are*

$$\mathcal{S}_3 = \mathbb{N} \setminus ([1, 8] \cup \{10, 11, 13, 14\});$$

$$\mathcal{S}_4 = \mathbb{N} \setminus ([1, 15] \cup \{17, 18, 19, 21, 22, 26\});$$

$$\mathcal{S}_5 = \mathbb{N} \setminus ([1, 24] \cup [26, 29] \cup \{31, 32, 33, 37, 38\}).$$

Let $S = (3, 3, 3, 4; 3, 3, 3, 4)$ be a bipartite graphic sequence. By Theorem C, there is no 3–way Latin trade of volume $3 + 3 + 3 + 4 = 13$. Thus, the bipartite graph $G = (X, Y)$ with $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4\}$ and $E(G) = \{x_iy_j : 1 \le i \ne j \le 4\} \cup \{x_4y_4\}$ is not 3–simultaneous edge colorable. Note that G is a 3–edge-connected bipartite graph. Therefore, the generalization of the strong SE conjecture and SE conjecture are not true.

One can be asked the following two natural questions related to this concept.

Question 1. *Is there a positive integer s_μ such that every μ–edge-connected bipartite graph with at least s_μ edges admits a μ–simultaneous edge coloring?*

Question 2. *Is there a positive integer s_μ such that each bipartite graphic sequence $S = (x_1, \dots, x_n; y_1, \dots, y_m)$ with all its elements greater than $\mu - 1$ and $\sum_{1 \le i \le n} x_i \ge s_\mu$, there exists a μ–simultaneous edge colorable bipartite realization?*

In [6], Edmonds showed that every graphic degree sequence, with all degrees at least $\mu \geq 2$, has a μ–edge-connected realization. In [8], Hajiaghaee et al. proved that every bipartite graphic sequence, with all degrees at least 2μ ($\mu \geq 1$), has a 2μ–edge-connected realization. In the following theorem we prove a generalization of these theorems; every bipartite graphic sequence, with all elements greater than $\mu - 1$, has a μ–edge-connected bipartite realization. Therefore, if the response of Question 1 is positive, then the response of Question 2 is also positive. For this purpose we need the following theorem.

Theorem D. [13] *For every bipartite graphic sequence S with all its elements greater than one, there exists a 2–edge-connected realization.*

Theorem 1. *Every bipartite graphic sequence S with all its elements greater than $\mu - 1$, $\mu \geq 3$, has a μ–edge-connected realization.*

Proof. Let r be the maximum edge connectivity among all realizations of the bipartite graphic sequence S and $r \leq \mu - 1$. By Theorem D, $r \geq 2$. Also, let $G = (X, Y)$ be a bipartite realization of S with the edge connectivity $\kappa^{'}(G) = r$, and G has the minimum number of r-edge cuts. Assume that $F = \{e_1, e_2, \ldots, e_r\}$ is an r-edge cut of G. Therefore, $G \setminus F$ has exactly two components G_1 and G_2.
First, we show that G_1 and G_2 are bridgeless. Otherwise, without loss of generality, assume that $e = uv \in E(G_1)$ is a cut edge of G_1 and G_{11} and G_{12} are components of $G_1 \setminus \{e\}$.
If $r = 2$ and S_1 is the bipartite degree sequence of G_1, then by Theorem D, there is a bridgeless bipartite graph G'_1 with the degree sequence S_1. Thus, $G' = (G \setminus E(G_1)) \cup E(G'_1)$ is a realization of S with the same edge connectivity as G and the number of its r-edge cuts is less than the number of r-edge cuts of G, which is a contradiction.
If $r \geq 3$ and F_i is the edges between G_{1i} and G_2, $i = 1, 2$, then $F = F_1 \cup F_2$ and for some i, say $i = 2$, $|F_2| \geq 2$. Therefore, $F' = F_1 \cup \{e\}$ is an edge cut of size at most $r - 1$, which is a contradiction. Thus, G_1 and G_2 are bridgeless. Hence, in the bridgless components G_1 and G_2, every edge lies in a circuit.
Since $\delta(G) \geq \mu$, for every $v_i \in V(G_i)$, $i = 1, 2$, there exists a vertex $v'_i \in V(G_i) \cap N_G(v)$ such that $N_G(v'_i) \subseteq V(G_i)$; so, there exists a vertex $v''_i \in V(G_i) \cap N_G(v'_i)$ such that $N_G(v''_i) \subseteq V(G_i)$. Let $v_i \in V(G_i) \cap X$ and C_i be a circuit in G_i such that $e_i = v'_iv''_i \in E(C_i)$, $i = 1, 2$. Now by switching two edges e_1 and e_2 with two edges $v'_1v''_2$ and $v'_2v''_1$, we obtain a bipartite graph G' with the same degree sequence as G in which F is not an r-edge cut anymore, and no new r-edge cut is appeared. This contradicts the minimality of the number of r-edge cuts in G. Therefore, $r \geq \mu$ and this complete the proof. ∎

Mahdian et al. showed that there exists an infinite family of μ–simultaneous edge colorable graphs. In the rest of this section we consider μ–simultaneous edge colorings of complete graphs, complete bipartite graphs and some graph operations such as join and graph product.

Theorem E. [13] *Every r-regular 1-factorable graph is μ–simultaneous edge colorable for every $\mu \leq r$.*

For example every complete graph K_{2l}, $l \geq 2$, is μ–simultaneous edge colorable for every $\mu \leq 2l-1$; every complete bipartite graph $K_{n,n}$, $n \geq 2$, is μ–simultaneous edge colorable for every $\mu \leq n$; every complete multipartite graph $K_{r_1,r_2,\ldots,r_n}$, when $r_1 = \cdots = r_n = r, n \geq 2$, and rn is even, is μ–simultaneous edge colorable for every $\mu \leq (n-1)r$; and every hypercube graph Q_n, $n \geq 1$, is μ–simultaneous edge colorable for every $\mu \leq n$.

Theorem F. [1] *If* $\min\{m,n\} \geq \mu$*, then there exists an* $m \times n$ μ*-way Latin trade of volume* mn*.*

Corollary 1. *Every* $K_{n,m}$ *admits a* μ*–simultaneous edge coloring, for* $\mu \leq \min\{m,n\}$ *and* $n,m \geq 2$*. Moreover,* $\chi'_{\mu-SE}(K_{n,m}) = \max\{m,n\}$*.*

The join of two simple graphs G and H, $G \vee H$, is the graph obtained from the disjoint union of G and H by adding the edges $\{uv : u \in V(G),\ v \in V(H)\}$.

Theorem 2. *Let* G_i *be a* μ*–simultaneous edge colorable graph of order* $n_i \geq 2$*. The join graph* $G_1 \vee G_2$ *has a* μ*–simultaneous edge coloring.*

Proof. Since G_i's has a μ–simultaneous edge coloring, for $\mu \leq \min\{n_1, n_2\}$, by Corollary 1, K_{n_1,n_2} has a μ–simultaneous edge coloring. Now we define a μ–simultaneous edge coloring of $G_1 \vee G_2$ by a μ–simultaneous edge coloring of the copy G_i in $G_1 \vee G_2$ with the color set $\{1, \ldots, \chi'_{\mu-SE}(G_i)\}$, $i = 1,2$, and a μ–simultaneous edge coloring of the copy K_{n_1,n_2} in $G_1 \vee G_2$ with the color set $\{r+1, \ldots, r + \Delta(K_{n_1,n_2})\}$, where $r = \max\{\chi'_{\mu-SE}(G_1), \chi'_{\mu-SE}(G_2)\}$. ■

Proposition 1. *The complete graph* K_7 *admits a* μ*–simultaneous edge coloring, for* $\mu = 2,3$*.*

Proof. Let $V(K_7) = \{v_1, v_2, \ldots, v_7\}$ be the vertex set of K_7. The following colorings, (c_1, c_2, c_3), is a 3–simultaneous edge coloring of K_7, where c_μ, is a proper edge coloring of K_7 with color set $\{1, 2, \ldots, 7\}$, and $v_iv_j : l_1, l_2, l_3$ means $c_\mu(v_iv_j) = l_\mu$, $\mu = 1,2,3$.
$v_1v_2 : 5,7,6$; $v_1v_3 : 2,3,1$; $v_1v_4 : 3,2,7$; $v_1v_5 : 6,1,5$; $v_1v_6 : 7,6,3$; $v_1v_7 : 1,5,2$;
$v_2v_3 : 7,2,5$; $v_2v_4 : 6,1,4$; $v_2v_5 : 1,6,7$; $v_2v_6 : 4,5,2$; $v_2v_7 : 2,4,1$;
$v_3v_4 : 1,7,2$; $v_3v_5 : 4,5,3$; $v_3v_6 : 5,4,7$; $v_3v_7 : 3,1,4$;
$v_4v_5 : 7,4,1$; $v_4v_6 : 2,3,6$; $v_4v_7 : 4,6,3$;
$v_5v_6 : 3,7,4$; $v_5v_7 : 5,3,6$;
$v_6v_7 : 6,2,5$. ■

Theorem 3. *Every complete graph* K_n*, except for* $n = 2,3,5$ *and possibly for* $n = 9$ *admits a* μ*–simultaneous edge coloring, for* $\mu = 2,3$*.*

Proof. It is easy to check that K_2 and K_3 are not 2–simultaneous edge colorable. In Proposition 2, we will show that K_5, has no 2–simultaneous edge coloring. By Theorem E, K_{2l}, $l \geq 2$ admits a μ–simultaneous edge coloring, $\mu = 2,3$. Thus, by Proposition 1 and Theorem 2, $K_{11} = K_7 \vee K_4$ and $K_{13} = K_7 \vee K_6$ are μ–simultaneous edge colorable, $\mu = 2,3$. For every $n \geq 14$, we have $K_n = K_{n-4} \vee K_4$; hence by induction on n, and Theorem 2, K_n admits a μ–simultaneous edge coloring, $\mu = 2,3$. ■

The Cartesian product of two graphs G and H, denoted by $G\Box H$, is the graph with vertex set $V(G)\times V(H)$ and two vertices (u,v) and (u',v') are adjacent if and only if either $u=u'$ and $vv'\in E(H)$ or $uu'\in E(G)$ and $v=v'$.

In the following theorems we present some sufficient conditions for μ–simultaneous edge colorable of $G\Box H$ in general.

Theorem 4. *Let G and H be r-regular and s-regular graphs, respectively. If H is 1-factorable, then $G\Box H$ is μ–simultaneous edge colorable for every $\mu\le r+s$.*

Proof. Suppose that G and H are r-regular and s-regular graphs, respectively. Therefore, $G\Box H$ is an $(r+s)$-regular graph. Since H is 1-factorable, we have $\chi'(H)=\Delta(H)$ and by a theorem in [12], $\chi'(G\Box H)=\Delta(G\Box H)=r+s$. Thus by Theorem E, $G\Box H$ is μ–simultaneous edge colorable for every $\mu\le r+s$. ■

Corollary 2.

(i) *For every positive integers $n\ge 2$ and $m\ge 3$, $C_{2n}\Box C_m$ is μ–simultaneous edge colorable for every $\mu\le 4$.*

(ii) *Let G be r-regular. Then, $G\Box K_{2n}$, $n\ge 1$, is μ–simultaneous edge colorable for every $\mu\le r+2n-1$.*

Theorem 5. *Let G and H be two μ–simultaneous edge colorable graphs. The cartesian product $G\Box H$ is also μ–simultaneous edge colorable. In particular, $\chi'_{\mu-SE}(G\Box H)\le\chi'_{\mu-SE}(G)+\chi'_{\mu-SE}(H)$.*

Proof. Suppose that G and H be two μ–simultaneous edge colorable graphs. It is sufficient to consider for each copy of G in $G\Box H$ a μ–simultaneous edge coloring with color set $\{1,\ldots,\chi'_{\mu-SE}(G)\}$ and for each copy of H in $G\Box H$ a μ–simultaneous edge coloring with color set $\{\chi'_{\mu-SE}(G)+1,\chi'_{\mu-SE}(G)+2,\ldots,\chi'_{\mu-SE}(G)+\chi'_{\mu-SE}(H)\}$. Obviously, these colorings form a μ–simultaneous edge coloring of $G\Box H$. ■

The lexicographic product of two simple graphs G and H is the simple graph $G[H]$ whose vertex set is $V(G)\times V(H)$, and two vertices (u,v) and (u',v') are adjacent if and only if $uu'\in E(G)$, or $u=u'$ and $vv'\in E(H)$.

Theorem 6. *If H is μ–simultaneous edge colorable, then for every simple graph G, $G[H]$ is also μ–simultaneous edge colorable.*

Proof. Let G and H be two simple graphs, $V(G)=\{u_1,\ldots,u_m\}$, and $V(H)=\{v_1,\ldots,v_n\}$. The graph $G[H]$ consists of copies $H^1,\ldots,H^m$ of H, in which the edge between H^i and H^j are isomorph to a copy of $K_{n,n}$, whenever $u_iu_j\in E(G)$. Let J_{ij} denote the copy of $K_{n,n}$ corresponds to the edges between H^i and H^j and $c_G : E(G) \to \{1, \ldots, \chi'(G)\}$ be a proper edge coloring of G. Now define μ edge colorings of $G[H]$. For every H^i, $1 \leq i \leq m$, define a μ–simultaneous edge coloring the same as μ–simultaneous edge coloring of H by color set $\{1, \ldots, \chi'_{\mu-SE}(H)\}$. Since by Corollary 1, every $K_{n,n}$ has a μ–simultaneous edge coloring, for every J_{ij} define a μ–simultaneous edge coloring by color set $\{\chi'_{\mu-SE}(H) + (c_G(u_iu_j) - 1)n, \chi'_{\mu-SE}(H) + (c_G(u_iu_j) - 1)n + 1, \ldots, \chi'_{\mu-SE}(H) + (c_G(u_iu_j) - 1)n + (n-1)\}$. It is easy to check these colorings form a μ–simultaneous edge coloring of $G[H]$. ■

3 2–Simultaneous edge coloring

In this section we concern on the 2–Simultaneous edge coloring. First, we study the properties of the extermal counterexample to the strong SE conjecture. Then, we consider the 2–Simultaneous edge coloring for graphs in general.
If the strong SE conjecture is false, then it must have a minimal counterexample. We consider the family of counterexamples to the strong SE conjecture with maximum number of vertices among ones with minimum number of edges.

Theorem 7. *Let G be a bridgeless bipartite graph that is not 2–simultaneous edge colorable with maximum number of vertices among ones with minimum number of edges, then*

(i) *G is 2-connected;*

(ii) *$\delta(G) = 2$ and $\Delta(G) = 3$;*

(iii) *G has no nontrivial edge cut of size 2;*

(iv) *for each $v \in V(G)$, which $\deg(v) = 2$, $G - v$ is bridgeless;*

(v) *for each $v \in V(G)$, if $N(v) = \{u, w\}$, then $N(u) \cap N(w) = \{v\}$.*

Proof. Let $V(G) = X \cup Y$. By Theorem B, G is a bridgeless bipartite graph with no OCDC while every bridgeless bipartite graph G' with $|E(G')| < |E(G)|$ or $|E(G')| = |E(G)|$ and $|V(G')| > |V(G)|$ has an OCDC.

(i) Let $v \in V(G)$ be a cut vertex of G. By the minimality of G, every block B of G has an OCDC, $\mathcal{C}_{\mathrm{B}}$. Therefore,

$$\mathcal{C} = \bigcup_{B \text{ is a block of } G} \mathcal{C}_B$$

is an OCDC of G, which is a contradiction.

(ii) Let $v \in V(G)$ be a vertex of degree greater than 3. By H. Fleischner's vertex-splitting lemma [7], there exist two edges $e_1 = uv$ and $e_2 = wv \in E(G)$ such that $G \cup \{uw\} \setminus \{e_1, e_2\}$ is bridgeless. Let G' be the new graph obtained by subdividing the edge uw in vertex v'. Thus, G' is bridgeless bipartite graph such that $|V(G')| = |V(G)| + 1$ and $|E(G')| = |E(G)|$.

Therefore, G' has an OCDC, $\mathcal{C}'$. Let C_1' and C_2' be two directed circuits in $\mathcal{C}'$ that include the directed paths $uv'w$ and $wv'u$, respectively. Define $C_1 = C_1' \cup \{uv, vw\} \setminus \{uv', v'w\}$ and $C_2 = C_2' \cup \{wv, vu\} \setminus \{wv', v'u\}$. Then,

$$\mathcal{C} = \mathcal{C}' \cup \{C_1, C_2\} \setminus \{C_1', C_2'\},$$

is an OCDC of G, which is a contradiction.

If $\delta(G) \neq 2$, since G is a bridgeless graph and $\Delta(G) \leq 3$, G is 3-regular. Therefore, G is 1-factorable. Thus by Theorem E, G is 2–simultaneous colorable, which is a contradiction. Hence, $\delta(G) = 2$ and by the same reason $\Delta(G) = 3$.

(iii) Let $F = \{e_1 = ab, e_2 = cd\}$ be a disjoint vertex edge cut of G and G_1 and G_2 be two nontrivial components of $G \setminus F$ such that $a, c \in V(G_1)$. Note that the case $a = c$ or $b = d$ does not occure because if so then we get a bridge in G. We consider two following cases.

- <u>$a, d \in X$ and $b, c \in Y$.</u> Let $G_1' = G_1 \cup \{ac\}$ and $G_2' = G_2 \cup \{bd\}$. By the edge minimality of G, G_1' and G_2' have OCDCs, $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Let C_1^1 and C_1^2 be two directed circuits in $\mathcal{C}_1$ that include the directed edge ac and ca, respectively. Assume that C_2^1 and C_2^2 be two directed circuits in $\mathcal{C}_2$ that include the directed edges db and bd, respectively. Define $C_1 = C_1^1 \cup C_2^1 \cup \{ab, dc\} \setminus \{ac, db\}$ and $C_2 = C_1^2 \cup C_2^2 \cup \{ba, cd\} \setminus \{ca, bd\}$, where uv means a directed edge from u to v. Thus,

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{C_1, C_2\} \setminus \{C_1^1, C_1^2, C_2^1, C_2^2\},$$

is an OCDC of G, which is a contradiction.

- <u>$a, c \in X$ and $b, d \in Y$.</u> Let G_1' be the graph obtained from G_1 by joining a new vertex v_1 to a and c, and G_2' be the graph obtained from G_2 by joining a new vertex v_2 to b and d. By the edge minimality of G, bipartite graphs G_1' and G_2' have OCDCs, $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Let C_1^1 and C_1^2 be two directed circuits in $\mathcal{C}_1$ that include the directed paths av_1c and cv_1a, respectively. Assume that C_2^1 and C_2^2 be two directed circuits in $\mathcal{C}_2$ that include the directed paths dv_2b and bv_2d, respectively. Define $C_1 = C_1^1 \cup C_2^1 \cup \{ab, dc\} \setminus \{av_1, v_1c, dv_2, v_2b\}$ and $C_2 = C_1^2 \cup C_2^2 \cup \{ba, cd\} \setminus \{v_1a, cv_1, v_2d, bv_2\}$, where uv means a directed edge from u to v. Thus,

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{C_1, C_2\} \setminus \{C_1^1, C_1^2, C_2^1, C_2^2\},$$

is an OCDC of G, which is a contradiction.

(iv) If $\deg(v) = 2$, then every bridge in $G - v$ with one of the edges incident on v forms an nontrivial edge cut of size 2, which is a contradiction.

(v) Suppose that $N(v) = \{u, w\}$ and $v' \in (N(u) \cap N(w)) \setminus \{v\}$. By (iv) and the minimality of G, $G - v$ has an OCDC, $\mathcal{C}'$. Since $\deg_G(v') \leq 3$, without loss of generality, there exists a directed circuit $C \in \mathcal{C}'$ that include the directed edges uv' and $v'w$. Let $C_1 = C \cup \{uv, vw\} \setminus \{v'\}$ and $C_2 = vuv'wv$. Then,

$$\mathcal{C} = \mathcal{C}' \cup \{C_1, C_2\} \setminus \{C\},$$

is an OCDC of G, which is a contradiction. ■

In the rest of this section, we consider the 2–simultaneous edge coloring for graphs in general. For example the following two colorings is a 2–simultaneous edge coloring for wheel W_n, $n \geq 3$. Assume that $V(W_n) = \{u, v_1, \ldots, v_n\}$ and $E(W_n) = \{uv_i, v_iv_{i+1} (\text{mod } n) : 1 \leq i \leq n\}$. Define two edge coloring $f_j : E(W_n) \to [n]$, $j = 1, 2$, $f_1(uv_i) = i$, $f_1(v_iv_{i+1}) = i+2$, and $f_2(uv_i) = i+2$, $f_2(v_iv_{i+1}) = i + 1$, where the colors and subscripts are reduced modulo n. It is easy to check that (f_1, f_2) forms a 2–simultaneous edge coloring of W_n.

Theorem 8. *Let G be a 2–simultaneous edge colorable graph. If G' is a graph obtained from G by replacing an edge $xy \in E(G)$ with simple path $xv_1v_2 \ldots v_{2k}y$ such that $v_i \notin V(G)$, $1 \leq i \leq 2k$, then G' is also 2–simultaneous edge colorable.*

Proof. Let (f_1, f_2) be a 2–simultaneous edge coloring of G. Without loss of generality, suppose that $f_j(xy) = j$, $j = 1, 2$. Define two proper edge colorings f'_1 and f'_2 of G' as follows. $f'_j(xv_1) = f'_j(v_{2i}v_{2i+1}) = f'_j(v_{2k}y) = j$, $f'_j(v_{2i-1}v_{2i}) = j + 1 \pmod 2$, and $f'_j(e) = f_j(e)$ for $e \in E(G) \setminus \{xy\}$, $1 \leq i \leq k - 1$, and $j = 1, 2$. Therefore, (f'_1, f'_2) is a 2–simultaneous edge coloring of G'. ■

Theorem 9. *Let G be a bridgeless graph with girth at least $2k-1$, $k \geq 2$. If G is 2–simultaneous edge colorable, then $|E(G)| \geq k\chi'(G)$.*

Proof. Let (f_1, f_2) be a 2–simultaneous edge coloring of G and $f_i^j = \{e \in E(G) : f_i(e) = j\}$, $i = 1, 2$. Since $\chi'(G) \leq \chi'_{2-SE}(G)$, if $|E(G)| < k\chi'(G)$, then for some j, $1 \leq j \leq \chi'_{2-SE}(G)$, $|f_i^j| \leq k - 1$ for $i = 1, 2$. Therefore, the induced subgraph by $f_1^j \cup f_2^j$ is a union of even circuits of length at most $2k - 2$, which is a contradiction. ■

In the following, we provide a relation between 2–simultaneous edge coloring of a graph with a CDC with certain properties. Then, using this relation, we show some 2–simultaneous edge colorable graph and also some graph which has no 2–simultaneous edge coloring.

Theorem 10. *A bridgeless graph G is a 2–simultaneous edge colorable if and only if G has a* CDC, *$\mathcal{C}$, that satisfies in the following properties.*

(i) *Every circuit of $\mathcal{C}$ is an even circuit.*

(ii) *$\mathcal{C}$ has a partition to at least $\chi'(G)$ classes, such that every class is 2-regular.*

(iii) *Every circuit in $\mathcal{C}$ has a proper 2-edge coloring, such that each edge $e \in E(G)$ in different circuits admits two different colors.*

Proof. Suppose that (f_1, f_2) is a 2–simultaneous edge coloring of graph G. Let $f_i^j = \{e \in E(G) : f_i(e) = j\}$ for $1 \leq j \leq \chi'_{2-SE}(G)$ and $i = 1, 2$. The induced subgraph by $C_j = f_1^j \cup f_2^j$ is a disjoint union of even circuits, for $1 \leq j \leq \chi'_{2-SE}(G)$. Therefore, $\mathcal{C} = \{C_j : 1 \leq j \leq \chi'_{2-SE}(G)\}$ is a CDC of G with properties (i) and (ii). Now for every edge e in a circuit of C_j, let $c_j(e) = i$, where $f_i(e) = j$, $i = 1, 2$, one can see that c_j satisfies the property (iii).

Conversely, let $\mathcal{C}$ be a CDC, where $C_1, C_2, \ldots, C_t$ is a partition of $\mathcal{C}$ such that C_i, $1 \leq i \leq t$, is 2-regular and c_i is a proper edge coloring of C_i satisfies in condition (iii). Now we define two edge colorings (f_1, f_2) as follows. For every edge e, if $e \in C_j$ and $c_j(e) = i$, then set $f_i(e) = j$. By the assumption, it is clear that f_i, $i = 1, 2$, is a proper edge colorings and $f_1(e) \neq f_2(e)$ for every $e \in E(G)$. It is enough to show that the set of colors appear on the edges incident to each vertex are the same. Let v be an arbitrary vertex of G and $u \in V(G)$ be an arbitrary neighbor of v. Without loss of generality, suppose that $f_1(uv) = j$, $1 \leq j \leq t$, $uv \in C_j$ and $c_j(uv) = 1$. Since C_j is 2-regular and c_j is a proper 2-edge coloring, there exists an edge $vw \in C_j$ that $c_j(vw) = 2$. Therefore, $f_2(vw) = j$. Thus, (f_1, f_2) is a 2–simultaneous edge coloring of G. ■

If G has an even circuit decomposition, then two copies of this decomposition satisfies in three conditions of Theorem 10. Hence, G is 2–simultaneous edge colorable. In other words, by Theorem 10, an even graph, G is 2–simultaneous edge colorable if and only if G has an even circuit decomposition.

Theorem 11. *Let C be an even Hamiltonian circuit of G and $G \setminus E(C)$ be a bipartite graph. If $G \setminus E(C)$ has an* OCDC*, then G has a 2–simultaneous edge coloring.*

Proof. By Theorem B, $G \setminus E(C)$ is 2–simultaneous edge colorable. Therefore by Theorem 10, it has a CDC, $\mathcal{C}'$, of even circuits that has a partition to even 2-regular subgraphs and a proper 2-edge coloring such that each edge of $G \setminus E(C)$ admits two different colors. Now let $\mathcal{C} = \mathcal{C}' \cup \{C, C\}$. It is easily seen that, $\mathcal{C}$ satisfies in three conditions of Theorem 10. Thus, G is 2–simultaneous edge colorable. ■

By Theorem A (i) and (iii), we have the following corollary.

Corollary 3. *Let C be an even Hamiltonian circuit of G and $G \setminus E(C)$ be a bipartite graph. If every edge of $G \setminus E(C)$ is contained in a circuit of length 4 in $G \setminus E(C)$, then G is a 2–simultaneous edge colorable graph.*

Proposition 2. *The complete graph K_5 has no 2–simultaneous edge coloring.*

Proof. Let (f_1, f_2) be a 2–simultaneous edge coloring of K_5 and $f_i^j = \{e \in E(G) : f_i(e) = j\}$, $i = 1, 2$ and $1 \leq j \leq 5$. Since $\chi'(K_5) = 5$ and $|E(K_5)| = 10$, the induced subgraph by $f_1^j \cup f_2^j$ is a circuit of length 4, for $1 \leq j \leq 5$. By the isomorphic, there is exactly one CDC of K_5 with even circuits, see Figure 3. It is easy to check that the condition (iii) of Theorem 10 does not hold for this CDC, which is a contradiction. ■

Since the Petersen graph has no 4-NZF [15], we conclude the following Theorem.

Proposition 3. *The Petersen graph is not 2–simultaneous edge colorable.*

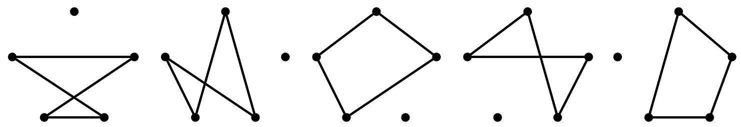

Figure 3: A CDC of K_5 with even circuits.

Proof. By the contrary, let the Petersen graph, P, is 2–simultaneous edge colorable. Let (f_1, f_2) be a 2–simultaneous edge coloring of P and $f_i^j = \{e \in E(G) : f_i(e) = j\}$, $i = 1, 2$. Thus, P has a CDC of even cicuits. Since C_6 and C_8 are only even circuits in P and $|E(P)| = 15$, without loss of generality, the induced subgraph by $f_1^j \cup f_2^j$ is a circuit of length 8 for $j = 1, 2, 3$ and the induced subgraph by $f_1^4 \cup f_2^4$ is a circuit of length 6 or the induced subgraph by $f_1^j \cup f_2^j$ is a circuit of length 6 for $j = 1, \ldots, 5$. It is easy to check that the second case is not possible. In the first case, $\mathcal{C} = \{C_j = f_1^j \cup f_2^j \ : \ 1 \leq j \leq 4\}$ is a 4-CDC of P. Therefore by Theorem A (ii), P admits a 4-NZF, while it has not [15]. ■

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