Small oriented cycle double cover of graphs

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Abstract

A small oriented cycle double cover (SOCDC) of a bridgeless graph G on n vertices is a collection of at most $n-1$ directed cycles of the symmetric orientation, G_s, of G such that each arc of G_s lies in exactly one of the cycles. It is conjectured that every 2-connected graph except two complete graphs K_4 and K_6 has an SOCDC. In this paper, we study graphs with SOCDC and obtain some properties of the minimal counterexample to this conjecture.

Keywords: Cycle double cover, Small cycle double cover, Oriented cycle double cover, Small oriented cycle double cover.

1 Introduction

We denote by G a finite undirected graph with vertex set V and edge set E with no loops or multiple edges. The symmetric orientation of G, denoted by G_s, is an oriented graph obtained from G by replacing each edge of G by a pair of opposite directed arcs. An even graph (odd graph) is a graph such that each vertex is incident to an even (odd) number of edges. A directed even graph is a graph such that for each vertex its out-degree equals to its in-degree. A cycle (a directed cycle) is a minimal non-empty even graph (directed even graph). We denote every directed cycle C and directed path P on n vertices with vertex set $\{v_1, \dots, v_n\}$ and directed edge set $E(C) = \{v_iv_{i+1}, v_nv_1 : 1 \le i \le n-1\}$ and $E(P) = \{v_iv_{i+1} : 1 \le i \le n-1\}$ by $C = [v_1, \dots, v_n]$, and $P = (v_1, \dots, v_n)$, respectively.

A cycle double cover (CDC) $\mathcal{C}$ of a graph G is a collection of cycles in G such that every edge of G belongs to exactly two cycles of $\mathcal{C}$. Note that the cycles are not necessarily distinct. It can be easily seen that a necessary condition for a graph to have a CDC is that the graph has no cut edge which is called a bridgeless graph. Seymour [17] in 1979 conjectured that every bridgeless graph has a CDC. No counterexample to the CDC conjecture is known. It is proved that the minimal counterexample to the CDC conjecture is a bridgeless cubic graph with edge chromatic number equal to 4, which is called a snark.

A small cycle double cover (SCDC) of a graph on n vertices is a CDC with at most $n-1$ cycles. There exist simple graphs of order n for which any CDC requires at least $n-1$ cycles (e.g., $K_n, n \geq 3$). Furthermore, no simple bridgeless graph of order n is known to require more than $n-1$ cycles in a CDC. Note that clearly it is false if not restricted to simple graphs. Bondy [3] conjectured that every simple bridgeless graph has an SCDC. For more results on the CDC conjecture see [7, 19].

The CDC conjecture has many stronger forms. In this paper, we consider the oriented version of these conjectures.

An oriented cycle double cover (OCDC) is a CDC in which every cycle can be oriented in such a way that every edge of the graph is covered by two directed cycles in two different directions.

Conjecture 1.1 [8] (Oriented CDC conjecture) *Every bridgeless graph has an* OCDC*.*

No counterexample to this conjecture is known. It is clear that the validity of the OCDC conjecture implies the validity of the CDC conjecture. While there is a CDC of the Petersen graph that can not be oriented in such a way that forms an OCDC.

Definition 1.2 *A* small oriented cycle double cover (SOCDC) *of a graph on n vertices is an* OCDC *with at most $n-1$ directed cycles.*

A perfect path double cover (PPDC) of a graph G is a collection $\mathcal{P}$ of paths in G such that each edge of G belongs to exactly two members of $\mathcal{P}$ and each vertex of G occurs exactly twice as an end of a path in $\mathcal{P}$ [2]. In [11] it is proved that every simple graph has a PPDC.

An oriented perfect path double cover (OPPDC) of a graph G is a collection of directed paths in the symmetric orientation G_s such that each arc of G_s lies in exactly one of the paths and each vertex of G appears just once as a beginning and just once as end of a directed path. Maxová and Nešetřil in [14] showed that two complete graphs K_3 and K_5 have no OPPDC and in [13], they conjectured every connected graph except K_3 and K_5 has an OPPDC.

The join of two simple graphs G and H, $G \vee H$, is the graph obtained from the disjoint union of G and H by adding the edges $\{uv : u \in V(G),\ v \in V(H)\}$.

The existence of a PPDC for graphs in general is equivalent to the existence of an SCDC for the bridgeless graph obtained by joining a new vertex to all other vertices [2]. The following theorem denotes a relation between OPPDC and SOCDC.

Theorem 1.3 [14] *Let G be a connected graph. The graph G has an* OPPDC *if and only if $G \vee K_1$ has an* SOCDC.

In the following theorem a list of some families of graphs that admit an OPPDC is provided. Therefore by Theorem 1.3, the join of graphs satisfying at least one of the conditions in below and K_1 admit an SOCDC.

Theorem 1.4 [1, 14] *Let $G \neq K_3$ be a graph. In each of the following cases, G has an* OPPDC.

(i) *G is a union of two arbitrary trees.*

(ii) *G is an odd graph.*

(iii) *G has no adjacent vertices of degree greater than two.*

(iv) *G is a 2-connected graph of order n and $|E(G)| \leq 2n - 1$.*

(v) *$G = L(T)$, for some tree T.*

(vi) *$G = L(H)$, where the degree of no adjacent vertices in H have the same parity.*

(vii) *G is a graph with $\Delta(G) \leq 4$ and $\delta(G) \leq 3$.*

(viii) *G is a separable 4-regular graph. (A separable graph is a graph contains cut vertex.)*

In what follows we have three sections. Section 2 deals with certain families of graphs with a small oriented cycle double cover. It is conjectured that every 2-connected graph except two complete graphs K_4 and K_6 has an SOCDC. In Section 3, we study the properties of the minimal counterexample to this conjecture. Finally in Section 4, some more relations between OPPDC and SOCDC are given.

2 The small oriented cycle double cover

The natural question is that which simple bridgeless graphs of order n have an OCDC with at most $n - 1$ cycles (SOCDC)?

Since K_3 and K_5 have no OPPDC, by Theorem 1.3, K_4 and K_6 have no SOCDC. It is known that every $K_{2n-1}, n \geq 4$, has an OPPDC [1], thus by Theorem 1.3, every $K_{2n},\ n \geq 4$, has an SOCDC. Moreover, every K_{2n+1} has an SOCDC, since K_{2n+1} has a Hamiltonian cycle decomposition [18].

The following observation shows that if every block of a graph G has an SOCDC, then G has also an SOCDC.

Observation 2.1 *If $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v\}$ which G_i is a graph with an* SOCDC*, $i = 1, 2$; then G also has an* SOCDC*.*

Moreover, Observation 2.1 directly concludes the following corollaries. A block graph is a graph for which each block is a clique.

Corollary 2.2 *Every block graph with no block of order* $2, 4$ *and* 6 *has an* SOCDC*.*

Since the line graph of every tree is a block graph, the following result obtained which is an oriented version of existence of SCDC of line graph of trees [12].

Corollary 2.3 *If T is a tree without vertices of degree* $2, 4$ *or* 6*, then $L(T)$ has an* SOCDC*.*

In the following proposition, we construct some graphs with no SOCDC. In fact, we show that the difference $|\mathcal{C}| - (n - 1)$ could be large enough for every OCDC, $\mathcal{C}$ of some bridgeless graph of order n.

Let $V(K_4) = \{v_1, v_2, v_3, v_4\}$. The collection $\mathcal{C} = \{[v_1, v_2, v_4], [v_2, v_1, v_3], [v_3, v_4, v_2], [v_4, v_3, v_1]\}$ is an OCDC of K_4. Since K_4 has six edges, if $\mathcal{C}$ is an arbitrary OCDC of K_4, then $|\mathcal{C}| \leq (2 \times 6)/3 = 4$. Thus, every OCDC of K_4 is of size 4.

Let $V(K_6) = \{v_1, \ldots, v_6\}$. The collection $\mathcal{C} = \{[v_1, v_2, v_3, v_4, v_5, v_6], [v_2, v_6, v_3, v_5, v_4], [v_1, v_5, v_2, v_4, v_3], [v_1, v_4, v_6, v_2, v_5], [v_1, v_6, v_5, v_3, v_2], [v_1, v_3, v_6, v_4]\}$ is an OCDC of K_6 of size 6.

Proposition 2.4 *For every integer $r \geq 1$, there exists a bridgeless graph G of order n such that every* OCDC *of G has $(n - 1) + r$ directed cycles.*

Proof. Let P be a path of length r with $V(P) = \{v_1, \ldots, v_{r+1}\}$ and $E(P) = \{v_i v_{i+1} : 1 \leq i \leq r\}$. Assume that G is a graph obtained from P by replacing each edge $v_i v_{i+1}$ of P with a clique K_4, say K_4^i, where $V(K_4^i) = \{v_i, v_i', v_{i+1}, v_{i+1}'\}$, $1 \leq i \leq r$. Every OCDC of G is decomposable to r OCDC of K_4. Moreover, every OCDC of K_4 has four cycles. Therefore, every OCDC of G has $4r$ cycles. Note that $|V(G)| = 3r + 1$, thus every OCDC of G has $(|V(G)| - 1) + r$ cycles. ■

This fact motivates us to present the following conjecture.

Conjecture 2.5 (SOCDC conjecture) *Every simple* 2*-connected graph except* K_4 *and* K_6 *admits an* SOCDC*.*

The above conjecture has a close relation to the following conjecture.

Conjecture 2.6 [4] (Hajós' conjecture) *If* G *is a simple, even graph of order* n*, then* G *can be decomposed into* $\lfloor (n-1)/2 \rfloor$ *cycles.*

If the Hajós' conjecture holds, then every even graph has an SOCDC obtained by taking two copies of the cycles used in its decomposition, in two opposite directions.

An edge of a graph G is said to be contracted if it is deleted and its two ends are identified. A minor of G is a graph obtained from G by deletions of vertices, and deletions and contractions of edges. The graph obtained from K_6 by deleting an edge is denoted K_6^-. A K_6^--minor free graph is a graph that does not contain K_6^- as a minor.

As the Hajós' conjecture is true for even graphs with maximum degree four [5], planar graphs [16], projective graphs (a projective graph is a graph G which is embeddable on the projective plane.), and K_6^--minor free graphs [4], these graphs have an SOCDC.

Proposition 2.7 *Let* G *be an even graph. In each of the following cases,* G *has an* SOCDC*.*

(i) $\Delta(G) = 4$.

(ii) G *is planar.*

(iii) G *is a projective graph.*

(iv) G *is* K_6^-*-minor free.*

Klimmek [9] proved that every even line graph of order n has a cycle decomposition into $\lfloor (n-1)/2 \rfloor$ cycles, thus the Hajós' conjecture holds for such graphs. Since a line graph, $L(G)$, is even if and only if every component of G is either even or odd, the line graph of every even graph and of every odd graph has an SOCDC.

Proposition 2.8 *If* G *is an even or an odd graph, then* $L(G)$ *has an* SOCDC*.*

The following proposition considers another class of graphs with OCDC which also has SOCDC.

Proposition 2.9 *If* G *has an* OCDC*,* $\mathcal{C}$*, and the girth of* G*,* $g(G)$*, is greater than average degree,* $\bar{d}(G)$*, then* $\mathcal{C}$ *is also an* SOCDC *of* G*.*

Proof. Let $\mathcal{C}$ be an OCDC of G. Note that each edge of G is covered twice by elements of $\mathcal{C}$, therefore,

$$g(G)|\mathcal{C}| \leq \sum_{C\in\mathcal{C}} |E(C)| = 2|E(G)| = \sum_{v\in V(G)} d(v) = |V(G)|\bar{d}(G).$$

Since $g(G) > \bar{d}(G)$, we have $|\mathcal{C}| \leq |V(G)| - 1$. Hence, $\mathcal{C}$ is an SOCDC of G. ■

It can be proved that an OCDC for planar graphs can be obtained from their planar embedding and some planar graph has also SOCDC.

Proposition 2.10 *Every bridgeless planar graph G with $|E(G)| < 2|V(G)|-2$, has an* SOCDC.

Proof. Let G be a bridgeless planar graph. Since we can orient the edges of each face of G in such a way that the collection of the boundary of its faces, $\mathcal{F}$, is an OCDC. By Euler's formula, $|\mathcal{F}| = 2+|E(G)|-|V(G)|$. Since $|E(G)| < 2|V(G)|-2$, we conclude $|\mathcal{F}| < |V(G)|$. Hence, G has an SOCDC. ■

Since in every simple triangle-free planar graph G with at least three vertices, $|E(G)| \leq 2|V(G)| - 4$, we obtain the following corollary.

Corollary 2.11 *Let G be a bridgeless planar graph. If G is triangle-free, then G admits an* SOCDC.

The following proposition presents an SOCDC for the well-known non-planar triangle-free graphs.

Proposition 2.12 *Every $K_{n,m}$, $n, m \geq 2$, has an* SOCDC.

Proof. Assume that $V(K_{n,m}) = \{v_1, \dots, v_n; w_1, \dots, w_m\}$, $n \leq m$. Let

$$C_i = [v_1, w_i, v_2, w_{i+1}, v_3, w_{i+2}, \dots, v_{n-1}, w_{i+n-2}, v_n, w_{i+n-1}],$$

be a directed cycle, where subscripts are reduced modulo m. It is easy to check that $\mathcal{C} = \{C_i : 1 \leq i \leq m\}$ is an SOCDC of $K_{n,m}$, $n, m \geq 2$. ■

Let G be a simple graph and (D, f) be an ordered pair where D is an orientation of $E(G)$ and f is a weight on $E(G)$ to $\mathbb{Z}$. For each $v \in V(G)$, denote

$$f^+(v) = \sum f(e) \qquad \text{and} \qquad f^-(v) = \sum f(e),$$

where the summation is taken over all directed edges of G (under the orientation D) with tails and heads, respectively, at the vertex v. An integer flow of G is an ordered pair (D, f) such that for every vertex $v \in V(G)$, $f^+(v) = f^-(v)$. A nowhere-zero k-flow of G is an integer flow (D, f) such that $0 < |f(e)| < k$, for every edge $e \in E(G)$ and is denoted by k-NZF [19].

Theorem 2.13 [19] *Every cubic graph G admits a* 4-NZF *if and only if* $\chi'(G) = 3$.

Theorem 2.14 [19] *A graph G admits a* 4-NZF *if and only if G has an* OCDC *consists of four directed even subgraphs.*

The following theorem concludes from Theorems 2.13 and 2.14.

Theorem 2.15 *Every cubic graph with edge chromatic number* 3 *admits an* OCDC*.*

Theorem 2.16 [10] *If $\mathcal{C}$ is a* CDC *of a cubic graph G of order n, then* $|\mathcal{C}| \leq n/2+2$.

Since from every OCDC of a graph a CDC for the graph is obtained, we have the following corollary.

Corollary 2.17 *Every* OCDC*, $\mathcal{C}$, of a cubic graph of order $n \geq 6$, is an* SOCDC*.*

The following corollary concludes directly from Theorem 2.15 and Corollary 2.17.

Corollary 2.18 *Every cubic graph with edge chromatic number* 3*, $G \neq K_4$, has an* SOCDC*.*

3 The minimal counterexample to the SOCDC conjecture

If the CDC conjecture is false, then it must have a minimal counterexample. In this section, we study the properties of the minimal counterexample to the SOCDC conjecture.

Observation 3.1 *If G is a graph with an* SOCDC *and G' is the graph obtained from G by subdividing one edge of G, then G' also admits an* SOCDC*.*

Corollary 3.2 *Let G be the minimal counterexample to the* SOCDC *conjecture, then the minimum degree of G is at least* 3*.*

Theorem 3.3 *The minimal counterexample to the* SOCDC *conjecture is* 3*-connected.*

Proof. Let G, the minimal counterexample to the SOCDC conjecture be a 2-connected graph of order n with vertex cut $\{v_1, v_2\}$ and $G = G_1 \cup G_2$, where $V(G_1) \cap V(G_2) = \{v_1, v_2\}$ and $|V(G_i)| = n_i$, $i = 1, 2$. Assume that $G_i \cup \{v_1v_2\}$ has an SOCDC, $\mathcal{C}_{\text{i}}$, i = 1, 2. Let $C_i^j, j = 1, 2$, be the two directed cycles in $\mathcal{C}_i$, $i = 1, 2$, which include the directed edge v_jv_{j+1}, where subscripts are reduced modulo 2. In each of the following cases, we show that G admits an SOCDC, which is a contradiction.

(I) If $v_1v_2 \in E(G)$, then we define

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{C_1^1 \Delta C_2^2\} \setminus \{C_1^1, C_2^2\}.$$

The collection $\mathcal{C}$ is an OCDC of G, where

$$\begin{aligned}|\mathcal{C}| = |\mathcal{C}_1| + |\mathcal{C}_2| - 1 &\leq (n_1 - 1) + (n_2 - 1) - 1 \\ &\leq (n_1 + n_2) - 3 \\ &\leq (n + 2) - 3 = n - 1.\end{aligned}$$

If $G_1 \cup \{v_1v_2\} = K_4$ with $V(K_4) = \{v_1, v_2, v_3, v_4\}$, and $G_2 \cup \{v_1v_2\}$ has an SOCDC, say $\mathcal{C}_2$, then let $C_1 = [v_1, v_2, v_4]$, $C_2 = [v_1, v_4, v_3, v_2]$, $C_3 = C_2^1 \cup (v_1, v_3, v_4, v_2) \setminus \{v_1v_2\}$, and $C_4 = C_2^2 \cup (v_2, v_3, v_1) \setminus \{v_2v_1\}$. Therefore,

$$\mathcal{C} = \mathcal{C}_2 \cup \{C_1, C_2, C_3, C_4\} \setminus \{C_1^1, C_2^2\}$$

is an SOCDC of G.

If $G_1 \cup \{v_1v_2\} = K_6$ with $V(K_6) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, and $G_2 \cup \{v_1v_2\}$ has an SOCDC, say $\mathcal{C}_2$, then let $C_1 = [v_1, v_2, v_4, v_6, v_3, v_5]$, $C_2 = [v_1, v_3, v_6, v_2]$, $C_3 = [v_1, v_4, v_2, v_5, v_6]$, $C_4 = [v_1, v_5, v_2, v_3, v_4]$, $C_5 = C_2^1 \cup (v_1, v_6, v_5, v_4, v_3, v_2) \setminus \{v_1v_2\}$, and $C_6 = C_2^2 \cup (v_2, v_6, v_4, v_5, v_3, v_1) \setminus \{v_2v_1\}$. Therefore,

$$\mathcal{C} = \mathcal{C}_2 \cup \{C_1, C_2, C_3, C_4, C_5, C_6\} \setminus \{C_1^1, C_2^2\}$$

is an SOCDC of G.

If $G_1 \cup \{v_1v_2\} = G_2 \cup \{v_1v_2\} = K_4$ or $G_1 \cup \{v_1v_2\} = K_4$ and $G_2 \cup \{v_1v_2\} = K_6$ or $G_1 \cup \{v_1v_2\} = G_2 \cup \{v_1v_2\} = K_6$, then by Theorem 1 in [1], $G \setminus v_1$ admits an OPPDC, thus by Theorem 1.3, G has an SOCDC.

(II) If $v_1v_2 \notin E(G)$, then we define

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{C_1^1 \Delta C_2^2, C_1^2 \Delta C_2^1\} \setminus \{C_1^1, C_1^2, C_2^1, C_2^2\}.$$

The collection $\mathcal{C}$ is an OCDC of G, where $|\mathcal{C}| \leq n - 2$.
Furthermore, if $G_1 \cup \{v_1v_2\} = K_4$ or K_6, $v_1v_2 \notin E(G)$, and $G_2 \cup \{v_1v_2\}$ has an SOCDC, by the similar argument in above using the given SOCDC for K_4 and K_6 of size 4 and 6, an SOCDC for G is obtained.

If $G_1 \cup \{v_1v_2\} = G_2 \cup \{v_1v_2\} = K_4$ with $V(G_1) = \{v_1, v_2, v_3, v_4\}$ and $V(G_2) = \{v_1, v_2, v_5, v_6\}$, then

$\mathcal{C} = \{[v_1, v_4, v_3, v_2, v_5, v_6], [v_1, v_5, v_2, v_3], [v_1, v_3, v_4, v_2, v_6, v_5], [v_1, v_6, v_2, v_4]\}$

is an SOCDC of G.

If $G_1 \cup \{v_1v_2\} = K_4$ with $V(G_1) = \{v_1, v_2, v_3, v_4\}$ and $G_2 \cup \{v_1v_2\} = K_6$ $V(G_2) = \{v_1, v_2, v_5, v_6, v_7, v_8\}$, then

$\mathcal{C} = \{[v_1, v_6, v_5, v_7, v_8, v_2, v_3], [v_1, v_3, v_4, v_2, v_8], [v_1, v_7, v_6, v_8, v_5, v_2, v_4], [v_1, v_5, v_8, v_7, v_2, v_6], [v_1, v_8, v_6, v_2, v_7, v_5], [v_1, v_4, v_3, v_2, v_5, v_6, v_7]\}$

is an SOCDC of G.

If $G_1 \cup \{v_1v_2\} = G_2 \cup \{v_1v_2\} = K_6$ with $V(G_1) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $V(G_2) = \{v_1, v_2, v_7, v_8, v_9, v_{10}\}$, then

$\mathcal{C} = \{[v_1, v_6, v_4, v_5, v_3, v_2, v_7, v_9, v_8, v_{10}], [v_1, v_3, v_5, v_4, v_6, v_2, v_{10}, v_8, v_9, v_7], [v_1, v_4, v_3, v_6, v_5, v_2, v_9, v_{10}, v_7, v_8], [v_1, v_5, v_6, v_3, v_4, v_2, v_8, v_7, v_{10}, v_9], [v_1, v_8, v_2, v_4], [v_1, v_{10}, v_2, v_6], [v_1, v_9, v_2, v_5], [v_1, v_7, v_2, v_3]\}$

is an SOCDC of G.

$\blacksquare$

Corollary 3.4 *The minimal counterexample to the* SOCDC *conjecture is* 3*-edge-connected.*

An edge cut F, is called trivial if one of the component in $G \setminus F$ be an isolated vertex.

Theorem 3.5 *The minimal counterexample to the* SOCDC *conjecture has no non-trivial edge cut of size* 3.

Proof. Let G be the minimal counterexample to the SOCDC conjecture. We know that G is 2-connected and 3-edge-connected. Assume that G has a non-trivial edge cut of size 3. We consider the following cases.

(I) $G = G_1 \cup G_2 \cup \{u_1v_1, u_2v_2, u_3v_3\}$, where $G_1 \cap G_2 = \emptyset$, the vertices u_i are distinct vertices of G_1, and the vertices v_i are distinct vertices of G_2, $i = 1, 2, 3$.

Denote by H_i the graph obtained by contracting the subgraph G_{i+1} to a single vertex w_i, $i = 1, 2$, where subscripts are reduced modulo 2. Since $\deg(w_i) = 3$, $H_i \neq K_6$, $i = 1, 2$. By the minimality of G, H_i has an SOCDC or $H_i = K_4$. Therefore, H_i has an OCDC, $\mathcal{C}_i$, $i = 1, 2$. Let C_i^j, $j = 1, 2, 3$, be the three directed cycles in $\mathcal{C}_i$ which include w_i, $i = 1, 2$, where without loss of generality, we assume that C_1^j includes directed path (u_{j-1}, w_1, u_{j+1}), and C_2^j includes directed path (v_{j+1}, w_2, v_{j-1}), where subscripts are reduced modulo 3, $j = 1, 2, 3$. Let $P_i^j = C_i^j \setminus w_i$, $i = 1, 2$, $j = 1, 2, 3$. Define $C^j = P_1^j \cup P_2^j \cup \{u_{j-1}v_{j-1}, v_{j+1}u_{j+1}\}$, $\mathcal{C}' = \{C^j : j = 1, 2, 3\}$, and $\mathcal{C}'' = \{C_i^j : i = 1, 2,\ j = 1, 2, 3\}$. Thus, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}' \setminus \mathcal{C}''$ is an OCDC of G, where $|\mathcal{C}| = |\mathcal{C}_1| + |\mathcal{C}_2| - 3$. Note that every OCDC of K_4 has 4 cycles, therefore, in both cases $|\mathcal{C}| \leq |V(G)| - 1$, which is a contradiction.

(II) $G = G_1 \cup G_2 \cup \{u_1v_1, u_1v_2, u_2v_3\}$, where $G_1 \cap G_2 = \emptyset$, the vertices u_1 and u_2 are distinct vertices of G_1, and the vertices v_i are distinct vertices of G_2, $i = 1, 2, 3$.

Denote by H_i the graph obtained by contracting the subgraph G_{i+1} to a single vertex w_i, $i = 1, 2$, and removing the multiple edge in H_1, where subscripts are reduced modulo 2. Since $\deg(w_i) = 2$ or 3, $H_1 \neq K_4$ and $H_i \neq K_6$, $i = 1, 2$. By the minimality of G, H_i has an SOCDC or $H_2 = K_4$. Therefore, H_i has an OCDC, $\mathcal{C}_i$, $i = 1, 2$. Let C_1^1 and C_1^2 be two directed cycles in $\mathcal{C}_1$ which include w_1, where without loss of generality, we assume that C_1^j includes directed path (u_j, w_1, u_{j+1}), where subscripts are reduced modulo 2, $j = 1, 2$, and C_2^k, $k = 1, 2, 3$, be the three directed cycles in $\mathcal{C}_2$ which include w_2, where without loss of generality, we assume that C_2^k includes directed path (v_k, w_2, v_{k-1}), where subscripts are reduced modulo 3, $k = 1, 2, 3$. Let $P_1^j = C_1^j \setminus w_1$, $j = 1, 2$, and $P_2^k = C_2^k \setminus w_2$, $k = 1, 2, 3$. Define $C^1 = P_1^1 \cup P_2^3 \cup \{u_1v_2, v_3u_2\}$, $C^2 = P_1^2 \cup P_2^1 \cup \{u_2v_3, v_1u_1\}$, and $C^3 = P_2^2 \cup \{u_1v_1, v_2u_1\}$. Let $\mathcal{C}' = \{C^1, C^2, C^3\}$, and $\mathcal{C}'' = \{C_1^1, C_1^2, C_2^1, C_2^2, C_2^3\}$. Thus, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}' \setminus \mathcal{C}''$ is an OCDC of G, where $|\mathcal{C}| = |\mathcal{C}_1| + |\mathcal{C}_2| - 2$. Note that every OCDC of K_4 has 4 cycles, therefore, in both cases $|\mathcal{C}| \leq |V(G)| - 1$, which is a contradiction.

(III) $G = G_1 \cup G_2 \cup \{u_1v_1, u_1v_2, u_2v_2\}$, where $G_1 \cap G_2 = \emptyset$, the vertices u_1 and u_2 are distinct vertices of G_1, and the vertices v_1 and v_2 are distinct vertices of G_2.

Denote by H_i the graph obtained by contracting the subgraph G_{i+1} to a single vertex w_i, $i = 1, 2$, and removing the multiple edges, where subscripts are reduced modulo 2. Since $\deg(w_i) = 2$, $H_i \neq K_4$ or K_6, $i = 1, 2$. By the minimality of G, H_i has an SOCDC, $\mathcal{C}_i$, $i = 1, 2$.

Let C_i^j, $j = 1, 2$, be the two directed cycles in $\mathcal{C}_i$ which include w_i, $i = 1, 2$, where without loss of generality, we assume that C_1^j includes directed path (u_j, w_1, u_{j+1}), and C_2^j includes directed path (v_j, w_2, v_{j+1}), where subscripts are reduced modulo 2, $j = 1, 2$. Let $P_i^j = C_i^j \setminus w_i$, $i = 1, 2$, $j = 1, 2$. Define $C^1 = P_1^1 \cup P_2^2 \cup \{u_1v_1, v_2u_2\}$, $C^2 = P_1^2 \cup \{u_2v_2, v_2u_1\}$, and $C^3 = P_2^1 \cup \{u_1v_2, v_1u_1\}$. Let $\mathcal{C}' = \{C^1, C^2, C^3\}$, and $\mathcal{C}'' = \{C_1^1, C_1^2, C_2^1, C_2^2\}$. Thus, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}' \setminus \mathcal{C}''$ is an OCDC of G, where $|\mathcal{C}| = |\mathcal{C}_1| + |\mathcal{C}_2| - 1$. Therefore, $|\mathcal{C}| \leq |V(G)| - 1$, which is a contradiction. ■

The properties of the minimal counterexample to the OPPDC conjecture is studied in [1, 13], and it is shown that this counterexample is a 2-connected and 3-edge-connected graph with minimum degree at least 4. Regarding to the relation between existence of an OPPDC for a graph and an SOCDC (Theorem 1.3), the following relation between the order and the number of edges of these two minimal counterexamples can be obtained.

Assume that G_p and G_c are the minimal counterexamples to the OPPDC conjecture and the SOCDC conjecture, respectively. Let G'_p be a graph obtained from G_p by joining a new vertex to all vertex of G_p. By Theorem 1.3, G'_p has no SOCDC. Note that G_p is connected, so G'_p is 2-connected. Thus, G'_p is a counterexample to the SOCDC conjecture. Therefore by the minimality of G_c,

$$|V(G_c)| + |E(G_c)| \leq |V(G'_p)| + |E(G'_p)| = 2|V(G_p)| + |E(G_p)| + 1.$$

Since $\delta(G_c) \geq 3$,

$$|V(G_c)| \leq \frac{2}{5}(2|V(G_p)| + |E(G_p)| + 1).$$

4 SOCDC **and the Cartesian product**

In [15] infinite classes of graphs with an SCDC are obtained using the Cartesian product of graphs. In this section, we proved the similar results in the oriented version.

The Cartesian product of two graphs G and H, denoted by $G\Box H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices (u, v) and (x, y) are adjacent if and only if either $u = x$ and $vy \in E(H)$ or $ux \in E(G)$ and $v = y$.

Theorem 4.1 *If G has an* OPPDC*, then $G\Box P_2$ has an* SOCDC *with $|V(G)|$ directed cycles. Furthermore, if G has an* SOCDC*, then $G\Box P_n$, $n \geq 3$, has an* SOCDC *with at most $|V(G\Box P_n)| - 1$ directed cycles.*

Proof. The proof is similar to the proof of Theorem 1 in [15]. ■

Now we need a theorem about the existence of OPPDC for the Cartesian product of graphs.

Theorem 4.2 [1] *If G and H have an* OPPDC*, then $G\Box H$ also has an* OPPDC*.*

The following corollary follows directly from Theorems 4.1 and 4.2.

Corollary 4.3 *If G has an* OPPDC*, then for all $l \geq 2$, $G^l\Box P_2$ has an* SOCDC*, where $G^l = \overbrace{G\Box \cdots \Box G}^{l\ times}$.*

Corollary 4.4 *Every hypercube graph Q_n, $n \geq 2$, has an* SOCDC*.*

With the similar argument as above, the following theorems, which are the oriented version of some results in [15] for SCDC, can be proved.

Theorem 4.5 *If G has an* OPPDC *and an* SOCDC*, then for any tree, T, $G\Box T$ has an* SOCDC*.*

Theorem 4.6 *If G has an* OPPDC*, then for all $k \geq 2$, $G\Box C_{2k}$ has an* SOCDC*. Furthermore, if G has an* SOCDC*, then $G\Box C_{2k-1}$ has an* SOCDC*.*

The following corollary concludes directly from Theorems 4.2 and 4.6.

Corollary 4.7 *If G has an* OPPDC*, then for all $k, l \geq 2$, $G^l\Box C_{2k}$ has an* SOCDC*, where $G^l = \overbrace{G\Box\cdots\Box G}^{l\ times}$.*

Theorem 4.8 *If G has an* SOCDC*, then for $n \geq 2|V(G)|+1$, $G\Box C_n$ has an* SOCDC*.*

Proof. Let $\mathcal{F}$ be the union of the corresponding SOCDC's of copies of G and the corresponding SOCDC's of copies of C_n. Since every OCDC of C_n has two directed cycles, and $n \geq 2|V(G)|+1$, we have $|\mathcal{F}| \leq n(|V(G)|-1)+2|V(G)| \leq n(|V(G)|-1)+(n-1) \leq n|V(G)|-1 = |V(G\Box C_n)|-1$. Therefore, $\mathcal{F}$ is an SOCDC of $G\Box C_n$. ■

Acknowledgments

The authors thank Professor C.Q. Zhang for reading the manuscript and for his helpful suggestions.

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