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Results in Mathematics

Point Symmetric 2-Structures

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Dedicated to Heinrich Wefelscheid

Abstract. We show that every symmetric 2-structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ of the class **(III)** [cf. Karzel H et al. (Result. Math., submitted)] is point symmetric, i.e. any two orthogonal chains $A, B \in \mathfrak{K}$ intersect in exactly one point and that any two points $a, b \in P$ have exactly one midpoint $m := a * b$ (with $\widetilde{m}(a) = b$ where $\widetilde{m}$ is the unique symmetry in the point m). $\widetilde{P} := \{\widetilde{p} \mid p \in P\}$ is invariant, i.e. $\forall a, b \in P : \widetilde{a} \circ \widetilde{b} \circ \widetilde{a} \in \widetilde{P}$. Therefore the pair $(P, \widetilde{P})$ is an invariant regular involution set and the loop derivation in a point $o \in P$ gives a K-loop $(P, +)$ uniquely 2-divisible.

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1. Introduction and Notations

This paper is a continuation of our investigations on symmetric 2-structures [5]. This is a generalization of *double symmetric 2-structures* (cf. [4]) which are closely related with sharply 2-transitive permutation groups [1,6]. We will use the same definitions and notations as in [5] and $\Sigma := (P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ will denote a symmetric 2-structure. We recall that for any point $p \in P$ and any $i \in \{1, 2\}$ $[p]_i$ denotes the generator of $\mathfrak{G}_i$ passing through the point p and that two points $a, b \in P$ are called *parallel* if there is an $i \in \{1, 2\}$ such that $[a]_i = [b]_i$ and *not parallel* otherwise. With $P^{(2)}$ we denote the set of all pairs of not parallel points and then (by definition of a 2-structure) to any pair $(a, b) \in P^{(2)}$ there exists exactly one chain $K \in \mathfrak{K}$—which we denote by $\overline{a, b}$—with $a, b \in K$. We collect some properties of Σ:

Theorem 1.1. $\forall A, B \in \mathfrak{K}$: *$\widetilde{A}(B) \in \mathfrak{K}$ and $\widetilde{\widetilde{A}(B)} = \widetilde{A} \circ \widetilde{B} \circ \widetilde{A}$ hence $\widetilde{\mathfrak{K}} := \{\widetilde{K} \mid K \in \mathfrak{K}\}$ is invariant.*

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In [5] we presented a classification of symmetric 2-structures based on the cardinality of the set $(p\perp K) := \{L \in \mathfrak{K} \mid p \in L \wedge L\perp K\}$ with $p \in K$ of all chains passing through the point p and which are orthogonal to the chain K. The symmetric 2-structures split into the three classes (cf. [5, Theorem 3.10]):

(I) There is a pair $(p, K) \in P \times \mathfrak{K}$ with $p \in K$ and $|(p \perp K)| > 1$.
(II) There is a pair $(p, K) \in P \times \mathfrak{K}$ with $p \in K$ and $(p \perp K) = \emptyset$.
(III) There is a pair $(p, K) \in P \times \mathfrak{K}$ with $p \in K$ and $|(p \perp K)| = 1$.

An automorphism α of 2-structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is called point reflection if α is involutory, has exactly one fixpoint p and $\alpha(B) = B$ for any chain or generator $B \in \mathfrak{B} := \mathfrak{K} \cup \mathfrak{G}_1 \cup \mathfrak{G}_2$ passing through p (cf. [5, Definition 2.1.]).
In this paper we assume that each symmetric 2-structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is of class **(III)**. This condition is equivalent to one of the following (cf. [5, Theorem. 3.13]):

1. For each pair $(x, X) \in P \times \mathfrak{K}$ with $x \in X : |(x \perp X)| = 1$.
2. For every $x \in P$ there is exactly one reflection in the point x.

We denote the only chain of the set $(p \perp K)$ by the same symbol and here we can form $(p \perp\perp K) := (p \perp (p \perp K))$. By (2) to each point $p \in P$ there corresponds exactly one reflection $\widetilde{p}$ in p. By [5, Theorem 3.4 and Corollary 3.5] we have the first statement of the following theorem:

Theorem 1.2. *If $K \in \mathfrak{K}$ and $p \in P$ then*:

1. $p \in K \Leftrightarrow \widetilde{p}(K) = K \Leftrightarrow \widetilde{K} \circ \widetilde{p}$ *is involutory* $\Leftrightarrow \exists_1 L \in \mathfrak{K} : \widetilde{K} \circ \widetilde{p} = \widetilde{L}$.
2. *If $p \in K$ and $L \in \mathfrak{K}$ with $\widetilde{K} \circ \widetilde{p} = \widetilde{L}$ then $L = (p \perp K)$.*
3. *If $p \notin K$ then $|(p \perp K)| = 1$.*
4. $\forall (a, b) \in P^{(2)} \ : \ (a \perp \overline{a, b}) \cap (b \perp \overline{a, b}) = \emptyset$.
5. $\forall a, b \in P \ : \ \widetilde{a(b)} = \widetilde{a} \circ \widetilde{b} \circ \widetilde{a}$ *hence* $\widetilde{P} := \{\widetilde{p} \mid p \in P\}$ *is invariant.*

Proof. (2): Since $\widetilde{K} \circ \widetilde{L} = \widetilde{p} \neq id, L \neq K$ and there is a $x \in L\backslash K$ hence $\widetilde{L}(x) = x, \widetilde{K}(x) \neq x$. Moreover $\widetilde{L}(\widetilde{K}(x)) = \widetilde{K}(\widetilde{L}(x)) = \widetilde{K}(x)$ hence $L = \overline{x, \widetilde{K}(x)} \perp K$ and $\widetilde{L}(p) = \widetilde{K} \circ \widetilde{p}(p) = \widetilde{K}(p) = p$, i.e. $p \in L$ and so $L = (p \perp K)$.

(3) Since $p \notin K, q := \widetilde{K}(p) \neq p$ and $(p, q) \in P^{(2)}$ hence $(p \perp K) = \{\overline{p, q}\}$.

(5) is a consequence of (1) and Theorem 1.1. □

A symmetric 2-structure of class **(III)** is called *point symmetric* if $|K \cap L| = 1$ for any $K, L \in \mathfrak{K}$ with $K\perp L$. If $a, b \in P$ are two given points then a point m is called *midpoint* of a and b if $\widetilde{m}(a) = b$. Let $a * b$ denote the set of all midpoints of a and b.

In [5] we proved that any two parallel points a and b have exactly one midpoint (cf. [5, Theorem 3.15]).

In this paper we construct for any two points x, y (not necessarily parallel) the (uniquely determined) midpoint $x * y$ (cf. Theorem 2.2). From this it follows that each symmetric 2-structure of class **(III)** is already point symmetric (cf. Theorem 2.4).

Let $\widetilde{P} := \{\widetilde{p} \mid p \in P\}$ and $\sim\colon\ P \to \widetilde{P};\ x \mapsto \widetilde{x}$. From Theorem 1.1 follows that $(P, \sim)$ is an invariant involution set and from Theorem 2.4 that $\widetilde{P}$ acts regularly on P hence $(P, \sim)$ is a *point reflection structure* and by [2, page 33 **6.1.**(3)] we have:

Theorem 1.3. *If $o \in P$ is fixed, $p' := o * p, p^+ := \widetilde{p'} \circ \widetilde{o}$ for $p \in P$ and if we set for $a, b \in P, a + b := a^+(b)$, then $(P, +)$ is a K-loop uniquely 2-divisible.*

Moreover we have (cf. Theorem 4.1.(4) for statement (1))

Theorem 1.4. *Let $a, b \in P$ and $K \in \mathfrak{K}$ then:*

1. $\widetilde{a} \circ \widetilde{ba} \circ \widetilde{b} = \widetilde{ab}$
2. *If $[a]_1 = [o]_1$ and $[b]_2 = [o]_2$ then $a + b = b + a$.*
3. *$\widetilde{K}$ is an involutory automorphism of the invariant involution set $(P, \sim)$ (in particular if $m := a * b$ then $\widetilde{K}(m) = \widetilde{K}(a) * \widetilde{K}(b)$) and an antiautomorphism of $(P, \Box)$.*
4. *If $o \in K$ then $\widetilde{K}$ is an involutory automorphism of the loop $(P, +)$ interchanging the "axises" $X := [o]_1$ and $Y := [o]_2$.*

2. Each Symmetric 2-Structure of Class (III) is Point Symmetric

Lemma 2.1. *Let $(a, b) \in P^{(2)}, c \in P$ and $\tau := \widetilde{a} \circ \widetilde{b}$ then $[c]_i \neq [\tau(c)]_i$ for $i \in \{1, 2\}$.*

Proof. (1) Let $A := (a \perp \overline{a, b}),\ B := (b \perp \overline{a, b})$ and $K = \overline{a, b}$ then by Theorem 1.2 $\widetilde{a} = \widetilde{A} \circ \widetilde{K}, \widetilde{b} = \widetilde{K} \circ \widetilde{B},\ \tau = \widetilde{A} \circ \widetilde{B},\ \tau(K) = K$ and $Fix\,\tau = Fix(\widetilde{A} \circ \widetilde{B}) = A \cap B = \emptyset$ (by Theorem 1.2.(4)). $K \cap [c]_i$ consists of exactly one point k. Assume $[c]_i = [\tau(c)]_i$ then $\tau(k) = \tau(K \cap [c]_i) = \tau(K) \cap \tau([c]_i) = K \cap [c]_i = k$, a contradiction to $Fix\,\tau = \emptyset$. $\Box$

Theorem 2.2. *Let $\Sigma := (P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be a symmetric 2-structure of class* **(III)** *and let $(a, b) \in P^{(2)}$. Then:*

1. *a and b have exactly one midpoint.*
2. *If $m = a * b$ then $ma = a * ba$ and $am = a * ab$ and moreover $\widetilde{m}(ab) = ba$.*
3. *If $m_1 = a * ab$ and $m_2 = ab * b$ then $m_2 m_1 = a * b$.*
4. *For $x, y \in P$ we have the formulas:*
   $x * y = y * x,\ (x * y)x = x * (yx),\ y(x * y) = y * (yx)$
   $x * y = (xy) * (yx) = (x * yx)(y * yx)$.
5. $[\widetilde{ab}(b)]_1 = \{\widetilde{x}(b) \mid x \in [a]_1\}, [\widetilde{ba}(b)]_2 = \{\widetilde{x}(b) \mid x \in [a]_2\}$ *and* $[b * a]_i = \{b * x \mid x \in [a]_i\}$.

Proof. By [5, Theorem 3.15.], each of the pairs of parallel points a and ab, a and ba, b and ba, and b and ab has exactly one midpoint m_1, m_2, m_1' and m_2'. If $C := \overline{a, b}$ and $D := \overline{ab, ba}$ then by the proof of [5, Theorem 3.15.], $\widetilde{C}(m_1) = m_2$,

$\widetilde{C}(m_1') = m_2'$ and $\widetilde{D}(m_1) = m_2'$, $\widetilde{D}(m_2) = m_1'$. Thus $m_2m_1, m_2'm_1' \in C$ and $m_2'm_1, m_2m_1' \in D$. We consider the map $\tau := \widetilde{m_1'} \circ \widetilde{m_1}$ and obtain: $\tau([a]_2) = \widetilde{m_1'}([ab]_2) = \widetilde{m_1'}([b]_2) = [ba]_2 = [a]_2$.

Since $(a, b) \in P^{(2)}$ we have $[m_1]_1 = [a]_1 \neq [b]_1 = [m_1']_1$. Assume $[m_1]_2 \neq [m_1']_2$ then $(m_1, m_1') \in P^{(2)}$ and we get a contradiction with Lemma 2.1 since $\tau([a]_2) = [a]_2$. Therefore $[m_1]_2 = [m_1']_2$ and in the same way, $[m_2]_1 = [m_2']_1$. This gives us $m := m_2m_1 = m_2m_1' = m_2'm_1 = m_2'm_1' \in C \cap D$, i.e. m is the midpoint of a and b and by [5, Theorem 3.14.(2)] m is unique. Moreover $\widetilde{m}(ab) = \widetilde{m}([a]_1 \cap [b]_2) = [\widetilde{m}(a)]_1 \cap [\widetilde{m}(b)]_2 = [b]_1 \cap [a]_2 = ba$. Hence (1), (2) and (3) are proved. (4) and (5) are consequences of the previous items. □

Corollary 2.3. *Let $K \in \mathfrak{K}, a, b \in K, a \neq b$ and $i \in \{1, 2\}$. Then:*

1. *If $L = (a \perp K), \{c\} = [b]_i \cap L$ and if m is the midpoint of b, c then $[m]_{3-i} = [a]_{3-i}$.*
2. *If $m = ba$ or $m = ab$ and $c = \widetilde{m}(b)$ then $K \perp \overline{a, c}$.*

Proof. (1) Let $b' := \widetilde{a}(b)$ and $i = 1$. Then $c = bb'$ and $L = \overline{bb', b'b}$. Hence $m = ba$, by Theorem 2.2. i.e. $[m]_2 = [a]_2$.

(2) We consider the case $m = ba$. Let $L = (a \perp K), \{c'\} = [b]_1 \cap L$ and m' is the midpoint of b, c'. Then, by (1), $m' = ba = m$ hence $L = \overline{a, c}$. □

Theorem 2.4. *Each symmetric 2-structure of class* **(III)** *is point symmetric.*

Proof. Let $(A, B) \in \mathfrak{K}^{2\perp}$ and $a \in A \backslash B$. By $A \perp B$, $b := \widetilde{B}(a) \in A$ hence $A = \overline{a, b}, B = \overline{ab, ba}$ and by Theorem 2.2 there is exactly one midpoint m of a and b and also of ab and ba. Therefore $\widetilde{m}(A) = A, \widetilde{m}(B) = B$ and so $m \in A \cap B$. Thus $(A, B) \in \mathfrak{K}_1^{2\perp}$. □

Corollary 2.5. *Let $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be a symmetric 2-structure of class* **(III)**. *If $(a, b) \in P^{(2)}$, $C := \overline{a, b}$, $D = \overline{ab, ba}$ and $m := a * b$ then $\{m\} = C \cap D = (ab \perp C) \cap C = (a \perp D) \cap D$, $\widetilde{C}(ab) = ba$ and $\widetilde{D}(a) = b$.*

3. Reflections in Generators

Let $\Sigma := (P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be a point symmetric 2-structure. For each $a \in P$ we have the 1-projection and the 2-projection

$\Pi_{a1} : P \to [a]_2; x \mapsto xa$ and $\Pi_{a2} : P \to [a]_1; x \mapsto ax$

and the *generator reflections*

$\widetilde{[a]_1} : P \to P;\ x \mapsto \widetilde{ax}(x)$ and $\widetilde{[a]_2} : P \to P;\ x \mapsto \widetilde{xa}(x)$

which are involutory permutations of P fixing exactly the elements of the generators $[a]_1$ and $[a]_2$, respectively. By definition, $\widetilde{[a]_1}$ is a 2-map and from Theorem 2.2(5) follows that $\widetilde{[a]_1}$ takes any 1-generator into a 1-generator, hence $\widetilde{[a]_1}$ is an automorphisms of $(P, \mathfrak{G}_1, \mathfrak{G}_2)$. $\widetilde{[a]_2}$ is an automorphism too and a 1-map. Now let $K \in \mathfrak{K}$ and $\{p\} = K \cap [a]_1$ then, by Corollary 2.3., $\widetilde{[a]_1}(K) = (p \perp K)$. Hence $\widetilde{[a]_1}$ is also an automorphism of $(P, \mathfrak{K})$ and together with [5, Theorems 3.7 and 3.8.] we have the result:

Theorem 3.1. *Let* $\Sigma := (P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ *be a point symmetric 2-structure and let* $a \in P, K \in \mathfrak{K}$ *and* $i \in \{1, 2\}$. *Then*:

1. *If* $\{p\} = K \cap [a]_i$ *then* $\widetilde{[a]_i}(K) = (p \perp K)$ *hence each reflection in a generator is involutory and contained in* $\Gamma^+(\mathfrak{K})$*, more precisely,* $\widetilde{[a]_i} \in \Gamma_{3-i}(\mathfrak{K})$ *and we have* $\widetilde{[a]_i}(K) \perp K$.
2. $\forall a \in P : \widetilde{[a]_1} \circ \widetilde{[a]_2} = \widetilde{[a]_2} \circ \widetilde{[a]_1} = \widetilde{a}$.
3. *Let* $A, B \in \mathfrak{K}$ *with* $A \perp B$ *and* $\{p\} = A \cap B$ *then* $\widetilde{[p]_i}(A) = B$.
4. *Let* $A, B \in \mathfrak{K}$ *with* $A \perp B$ *and* $\{p\} = A \cap B$ *and let* $\alpha \in \Gamma_i(\mathfrak{K})$ *with* $\alpha(A) = B$ *then* $\alpha = \widetilde{[p]_{3-i}}$.
5. *If* $A \in \mathfrak{K}, \{p\} = A \cap [a]_1, \{q\} = A \cap [a]_2, B := (p \perp A)$ *and* $C := (q \perp A)$ *then* $\widetilde{BA} \circ \widetilde{A} = \widetilde{A} \circ \widetilde{AB} = \widetilde{[a]_1}$ *and* $\widetilde{AC} \circ \widetilde{A} = \widetilde{A} \circ \widetilde{CA} = \widetilde{[a]_2}$.
6. *If* $A, B \in \mathfrak{K}$ *with* $A \perp B$ *and* $\{c\} := A \cap B$ *then* $\widetilde{AB} \in \Gamma^-(\mathfrak{K})$ *and* $\widetilde{AB} \circ \widetilde{AB} = \widetilde{A} \circ \widetilde{B} = \widetilde{c}$ *is the reflection in the point* c.
7. *If* $[a]_i = [b]_i$ *then* $(\widetilde{a} \circ \widetilde{b}) \in \Gamma_i(\mathfrak{K})$.
8. *Any involution of* $\Gamma_i(\mathfrak{K})$ *is the reflection in a generator of* $\mathfrak{G}_{3-i}$.

Proof. (2) Follows from [5, Theorem 3.8.(3)].

(3) Since $B = (p \perp A)$ we have by (1), $\widetilde{[p]_i}(A) = B$.

(4) By (1), $\widetilde{[p]_{3-i}} \in \Gamma_i(\mathfrak{K})$ and $\widetilde{[p]_{3-i}}(A) = B$. Therefore $\widetilde{[p]_{3-i}}$ and also α induce the perspectivity $[A \xrightarrow{i} B]$ or in other words, $\widetilde{[p]_{3-i}}$ and α are extensions of the perspectivity $[A \xrightarrow{i} B]$ and so by [3, Theorem 2.4.], $\alpha = \widetilde{[p]_{3-i}}$.

(5) Follows from (3) and [3, Theorem 2.9.(4),(5)].

(6) Follows from (2) and (5).

(7) By (2), $\widetilde{a} = \sigma_1 \circ \sigma_2$ and $\widetilde{b} = \sigma_2 \circ \sigma_3$ where σ_1, σ_2 and σ_3 are reflections in the generators $[a]_{3-i}, [a]_i = [b]_i$ and $[b]_{3-i}$ respectively. Hence $\widetilde{a} \circ \widetilde{b} = \sigma_1 \circ \sigma_3 \in \Gamma_i(\mathfrak{K})$. (8) Let $\alpha \in \Gamma_i(\mathfrak{K})$ be an involution, let $x \in P \backslash Fix\ \alpha, y := \alpha(x)$ and $m := x * y$. Then $\alpha(m) = m$ and since $\alpha \in \Gamma_i(\mathfrak{K}), \alpha$ fixes all elements of the generator $[m]_{3-i}$. Thus $\alpha = \widetilde{[m]_{3-i}}$. □

Let $\mathfrak{B} := \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{K}$ be the set of all *blocks*. Then $(P, \mathfrak{B})$ is an *incidence space*, i.e. any two distinct points $a, b \in P$ can be joined by exactly one block of $\mathfrak{B}$ also denoted by $\overline{a, b}$. An automorphism $\delta \in \Gamma(\mathfrak{K})$ is called a

latation if for $x \in P$ with $\delta(x) \neq x : \delta(\overline{x, \delta(x)}) = \overline{x, \delta(x)}$
dilatation if for $B \in \mathfrak{B} : \delta(B) = B$ or $\delta(B) \cap B = \emptyset$

translation if δ is the *identity* or if δ is a latation and a dilatation without fixed points.

If σ is an involutory automorphism of $(P, \mathfrak{B})$ then σ is a latation.

Corollary 3.2. *If $[a]_i = [b]_i$ then $\tau; = \widetilde{a} \circ \widetilde{b}$ is an i-map and a translation.*

Proof. By the proof of Theorem 3.1 (7), $\tau = \widetilde{a} \circ \widetilde{b} = \widetilde{[a]_{3-i}} \circ \widetilde{[b]_{3-i}} \in \Gamma_i(\mathfrak{K})$ hence τ is an i-map and so also a latation. If $K \in \mathfrak{K}$ then by Theorem 3.1 (1),

$$K \perp \widetilde{[b]_{3-i}}(K) \perp \widetilde{[a]_{3-i}} \circ \widetilde{[b]_{3-i}}(K) = \widetilde{a} \circ \widetilde{b}(K).$$

Hence by Theorem 1.2(4), $\widetilde{a} \circ \widetilde{b}(K) = K$ or $\widetilde{a} \circ \widetilde{b}(K) \cap K = \emptyset$, i.e. τ is a translation. $\square$

4. Midpoint Configurations

For two subsets $X, Y \subseteq P$ let $X * Y := \{x * y \mid x \in X,\ y \in Y\}$ and for $a \in P$ let $a * X := \{a\} * X$.

Theorem 4.1. *Let $a, b, c, p \in P$, $X, Y \in \mathfrak{G}_i$, $i \in \{1, 2\}$ and $A \in \mathfrak{K}$ with $p \notin A$. Then:*

1. *$p * X$, $X * Y \in \mathfrak{G}_i$ hence if $x \in X$, $y \in Y$ then $p * X = [p * x]_i$ and $X * Y = [x * y]_i$.*
2. *$\widetilde{p}(X) = Y \Leftrightarrow p \in X * Y$.*
3. *If $[a]_i = [b]_i$ then $[\widetilde{a}(c)]_i = [\widetilde{b}(c)]_i$ and so $\tau := \widetilde{a} \circ \widetilde{b}$ is an i-map.*
4. *$\widetilde{a} \circ \widetilde{ab} = \widetilde{ba} \circ \widetilde{b}$ is a 1-map and $\widetilde{a} \circ \widetilde{ba} = \widetilde{ab} \circ \widetilde{b}$ is a 2-map.*
5. *If $a_i := [p]_i \cap A$ and $\{p'\} := (p \perp A) \cap A$ then $p * p' \in \overline{p * a_1, p * a_2}$.*
6. *Let $[a]_i = [b]_i, a \neq b$ and $x \in [a * b]_{3-i} \backslash \{a * b\}$ then $\overline{a, x} \perp \overline{b, x}$.*

Proof. (1) Let for instance $X \in \mathfrak{G}_1, \{p'\} := [p]_2 \cap X$ and $x \in X$ then $xp = p'$ and so by Theorem 2.2.(2), $(p*x)p = p*(xp) = p*p'$ implying $p*X = [p*x]_1 = [p * p']_1 \in \mathfrak{G}_1$. If $p, q \in Y$ hence $[p]_1 = [q]_1 = Y$ and if $\{q'\} := [q]_2 \cap X$ then $[p']_1 = [q']_1 = X$ and $p*X = [p*p']_1, q*X = [q*q']_1$. Again by Theorem 2.2.(2), $[p * p']_1 = [p * q']_1 = [q * q']_1$. Therefore $Y * X = [p * p']_1 \in \mathfrak{G}_1$.

(2) "$\Rightarrow$" If $x \in X$ then $y := \widetilde{p}(x) \in \widetilde{p}(X) = Y$, hence $p = x * y \in X * Y$. "$\Leftarrow$" Let $x \in X$, $y \in Y$ with $p = x * y$, i.e. $\widetilde{p}(x) = y$. Then $\widetilde{p}(X) = \widetilde{p}([x]_i) = [\widetilde{p}(x)]_i = [y]_i = Y$.

(3) By (1) and assumption we have $b \in [a]_i = [c]_i * [\widetilde{a}(c)]_i$ and so by (2), $[\widetilde{b}(c)]_i = \widetilde{b}([c]_i) = \widetilde{a}([c]_i) = [\widetilde{a}(c)]_i$.

(4) Since $[ab]_1 = [a]_1$ and $[ba]_2 = [a]_2$, by (3), $\widetilde{a} \circ \widetilde{ab}$ and $\widetilde{ba} \circ \widetilde{b}$ are 1-maps, and $\widetilde{a} \circ \widetilde{ba}$ and $\widetilde{ab} \circ \widetilde{b}$ are 2-maps. Therefore for $x \in P$ we have $\widetilde{a} \circ \widetilde{ab}([x]_1) = \widetilde{ba} \circ \widetilde{b}([x]_1) = [x]_1$ and $\widetilde{ab}([x]_2) = \widetilde{b}([x]_2), \widetilde{a}([x]_2) = \widetilde{ba}([x]_2)$ implying $\widetilde{a} \circ \widetilde{ab}([x]_2) = \widetilde{ba} \circ \widetilde{b}([x]_2)$. Together with $x = [x]_1 \cap [x]_2$, we obtain $\widetilde{a} \circ \widetilde{ab}(x) = \widetilde{ba} \circ \widetilde{b}(x)$. Thus $\widetilde{a} \circ \widetilde{ab} = \widetilde{ba} \circ \widetilde{b}$ and in the same way $\widetilde{a} \circ \widetilde{ba} = \widetilde{ab} \circ \widetilde{b}$.

(5) Since $p = a_1a_2$ and so $\widetilde{A}(p) = a_2a_1$ we have by Corollary 2.5 $p' = a_2 * a_1$ and then by Theorem 2.2.(4), $p' = (a_2 * a_1a_2)\square(a_1 * a_1a_2) = (a_2 * p)\square(a_1 * p)$ and so $p' * p = (a_2 * p) * (a_1 * p)$ hence $p * p' \in \overline{p * a_1, p * a_2}$.

(6) Follows from Theorem 3.1.(1) since $a = [\widetilde{a * b}]_{3-i}(b)$. $\square$

Theorem 4.2. *If* $\Sigma := (P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ *is a point symmetric 2-structure and if we consider on* P *the binary operations* $a\square b := ab = [a]_1 \cap [b]_2$, $a * b$ *(= midpoint of* a *and* b*) and* $a \circ b := \widetilde{a}(b)$ *for* $a, b \in P$ *then* $(P, \square, *, \circ)$ *satisfies the following rules (let* $a, b, c \in P$*):*

1. $(P, \square), (P, *)$ *and* $(P, \circ)$ *are idempotent, i.e.* $a = aa = a * a = a \circ a$.
2. $(P, \square)$ *is associative moreover* $a(bc) = (ab)c = ac$.
3. $(P, *)$ *is commutative.*
4. $(a * b) \circ a = b$ *and* $a * (b \circ a) = b$ *hence* $(P, \circ)$ *and* $(P, *)$ *are quasigroups.*
5. $a(b * c) = (ab) * (ac)$, $(a * b)c = (ac) * (bc), a(b \circ c) = (ab) \circ (ac)$, $(a \circ b)c = (ac) \circ (bc)$ *and* $a \circ (b * c) = (a \circ b) * (a \circ c)$.

For each $B \in \mathfrak{B}, B$ *is a substructure of* $(P, *, \circ)$ *and* $(B, \circ), (B, *)$ *are idempotent quasigroups, if* $A, B \in \mathfrak{B}$ *then* $(A, *, \circ)$ *and* $(B, *, \circ)$ *are isomorphic.*

Proof. (5) By Theorem 2.2.(4) we have:

(*) $a(a * b) = a * (ab)$

and so also by using (2), $(ab) * (ac) = ((ac)b) * (ac) = (ac)(b * ac) = (ab)(b * (ac)) = a(b(b * (ac))) = a(b * (bc)) = ab(b * c) = a(b * c)$. From $b = c * (b \circ c)$ and $a(b * c) = (ab) * (ac)$ it follows that $ab = (ac) * (a(b \circ c))$. Hence $(ab) \circ (ac) = a((b \circ c)$ By Theorem 1.4.(3), $\widetilde{a}(b * c) = \widetilde{a}(b) * \widetilde{a}(c)$, i.e. $a \circ (b * c) = (a \circ b) * (a \circ c)$. Now let $A, B \in \mathfrak{B}$. If $B \in \mathfrak{K}$ and $a \in P$ then $\pi_{a,1} : B \rightarrow [a]_1$; $b \mapsto ab$ and $\pi_{a,2} : B \rightarrow [a]_2$; $b \mapsto ba$ are bijections and for $b, c \in B$ we have by (5): $\pi_{a,1}(b * c) = a(b * c) = (ab) * (ac) = \pi_{a,1}(b) * \pi_{a,1}(c), \pi_{a,2}(b * c) = (b * c)a = \pi_{a,2}(b) * \pi_{a,2}(c), \pi_{a,1}(b \circ c) = a(b \circ c) = (ab) \circ (ac) = \pi_{a,1}(b) \circ \pi_{a,1}(c)$ and $\pi_{a,2}(b \circ c) = (b \circ c)a = \pi_{a,2}(b) \circ \pi_{a,2}(c)$ hence $\pi_{a,1}$ and $\pi_{a,2}$ are isomorphisms from $(B, *, \circ)$ onto $([a]_1, *, \circ)$ and onto $([a]_2, *, \circ)$. Consequently $(A, *, \circ)$ and $(B, *, \circ)$ are isomorphic. $\square$

5. Products of i-maps and the Corresponding K-loop

For $i \in \{1, 2\}$ let $T_i := \{\widetilde{a} \circ \widetilde{b} \mid a, b \in P$ with $[a]_i = [b]_i\}$. Then by Theorem 4.1.(3), T_i is a set of i-maps.

By [5, Theorem 3.16] and [2, **6.1.**] we have

Theorem 5.1. *Let* $\Sigma := (P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ *be a point symmetric 2-structure, let* $o \in P$ *be fixed, for* $p \in P$ *let* $p' := o * p$ *and* $p^+ := \widetilde{p'} \circ \widetilde{o}$*. If we set for* $a, b \in P, a + b := a^+(b)$*, then* $(P, +)$ *is a K-loop uniquely 2-divisible.*

$(P,+)$ is the *loop derivation in the point* o (cf. [2, Definition 1]). For any two points o_1, o_2 the reflection in the midpoint $o_1 * o_2$ establishes an isomorphism between the loops derived in the points o_1 and o_2. For the following let Σ be a point symmetric 2-structure, let $o \in P$ be fixed, let $(P,+)$ be the K-loop derived in o and let $X := [o]_1,\ Y := [o]_2$.

Theorem 5.2. *Let* $\widetilde{X} \circ \widetilde{X} := \{\widetilde{a} \circ \widetilde{b} \mid a, b \in X\}, \widetilde{Y} \circ \widetilde{Y} := \{\widetilde{a} \circ \widetilde{b} \mid a, b \in Y\}, X^+ := \{x^+ \mid x \in X\}$ *and* $Y^+ := \{y^+ \mid y \in Y\}$ *then*:

1. $X^+ \subseteq T_1 = \widetilde{X} \circ \widetilde{X}$ *and* $Y^+ \subseteq T_2 = \widetilde{Y} \circ \widetilde{Y}$, *more precisely, if* $a, b \in P$ *with* $[a]_1 = [b]_1$ *then* $\widetilde{a} \circ \widetilde{b} = \widetilde{oa} \circ \widetilde{ob}$.
2. X *and* Y *are subloops of* $(P,+)$ *and* $\forall p \in P \exists_1 (x,y) \in X \times Y : p = x + y$.
3. $\forall (x,y) \in X \times Y\ x + y = y + x$ *even* $x^+ \circ y^+ = y^+ \circ x^+$.
4. *If* $o \in E \in \mathfrak{K}$ *then the loops* $(E,+), (X,+), (Y,+)$, *are isomorphic.*

Proof. (1) By the definitions of X, Y, X^+ and Y^+ and by Theorem 4.1.(3) we have $X^+ \subseteq \widetilde{X} \circ \widetilde{X} \subseteq T_1$ and $Y^+ \subseteq \widetilde{Y} \circ \widetilde{Y} \subseteq T_2$. Now let $a, b \in P$ with $[a]_1 = [b]_1$ then $oa, ob \in X, b = a\Box(ob)$ and $oa = (ob)\Box a$ hence by Theorem 4.1.(4), $\widetilde{a} \circ \widetilde{b} = \widetilde{a} \circ \widetilde{a\Box(ob)} = \widetilde{(ob)\Box a} \circ \widetilde{ob} = \widetilde{oa} \circ \widetilde{ob} \in \widetilde{X} \circ \widetilde{X}$ hence $T_1 = \widetilde{X} \circ \widetilde{X}$.

(2), (3) Let $a \in X$. Since $o \in X,\ a' = o * a \in X$ and so $\widetilde{o}(X) = \widetilde{a'}(X) = X$ implying $a + X = a^+(X) = X$. Moreover if $a, b \in X$ then $x_1 := \widetilde{o} \circ \widetilde{a'}(b) \in \widetilde{o} \circ \widetilde{a'}(X) = X$ is the solution of the equation $a + x = b$ cotained in X and since the midpoint $m := \widetilde{o}(a) * b \in X$ is contained in X also $x_2 := \widetilde{m}(o) \in X$ thus $x_2 + a = \widetilde{x_2'} \circ \widetilde{o}(a) = \widetilde{m} \circ \widetilde{o}(a) = b$ and so the solution x_2 of $x + a = b$ is contained in X. Therefore X and Y are subloops of $(P,+)$.

If we set $\{x\} := X \cap [p]_2, \{y\} := Y \cap [p]_1$ then $o = xy,\ p = yx$ and by Theorem 4.1, $[p*y]_2 = Y * [p]_2 = [o*x]_2$ imlying $p*y = (o*x)\Box y = x'y$ hence by (1), $x^+ = \widetilde{x'} \circ \widetilde{o} = \widetilde{p*y} \circ \widetilde{y}$ and we have $x + y = x^+(y) = \widetilde{p*y} \circ \widetilde{y}(y) = \widetilde{p*y}(y) = p$ and since $x'\Box y' = o$ we have by Theorem 4.1.(4), $\widetilde{x'} \circ \widetilde{o} \circ \widetilde{y'} = \widetilde{y'x'}$ implying $\widetilde{x'} \circ \widetilde{o} \circ \widetilde{y'} = \widetilde{y'} \circ \widetilde{o} \circ \widetilde{x'}$ hence $x^+ \circ y^+ = y^+ \circ x^+$. If there are $a \in X,\ b \in Y$ with $p = a + b$ then $p = x + y = \widetilde{x'} \circ \widetilde{o}(\widetilde{y'}(o)) = \widetilde{y'x'}(o) = \widetilde{b'a'}(o)$ hence $p' = o * p = y'x' = b'a'$ and so $y' = b'$ and $x' = a'$, i.e. $y = \widetilde{y'}(o) = \widetilde{b'}(o) = b$ and in the same way, $x = a$.

(4) We have $a + b = (a * o) \circ (o \circ b)$. Hence, by the proof of the last statement of Theorem 4.2, $\pi_{o,1}$ and $\pi_{o,2}$ are isomorphisms from $(E,+)$ onto $(X,+)$ and onto $(Y,+)$. $\Box$

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