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ON K-LOOPS OF FINITE ORDER

To the memory of Hans Zassenhaus

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Abstract. In this note we undertake an axiomatic investigation of K-loops (or gyrogroups, as A.A. Ungar used to name them) and provide new construction methods for finite K-loops. It is shown how, more or less, the axioms are independent from each other. Especially **(K6)** is independent as A.A. Ungar already had conjectured.
We begin with right loops $(L,\oplus)$ and add step by step further properties. So the connection between K-loops, Bol-loops, Bruck-loops and the homogeneous loops of Kikkawa became clear. The smallest examples of proper K-loops possess 8 elements; there are exactly 3 non-isomorphic of these.
At last it is shown that one gets quite naturally a Frobenius-group as a quasidirect product of a K-loop $(L,\oplus)$ and a group D of automorphisms of $(L,\oplus)$ if D is fixed point free except from 0.

1. INTRODUCTION

In order to describe sharply 2-transitive groups, H. Karzel introduced in [9] the notion of a neardomain $(F,\oplus,\cdot)$(cf. [33]). The crucial difficulty of a neardomain is the additive structure $(F,\oplus)$, which need not be associative. A neardomain $(F,\oplus,\cdot)$ with an associative addition is already a nearfield. In many notes neardomains are investigated (cf. [10, 13, 14, 15, 16, 17, 19, 33, 34]), but until today no example of a proper neardomain is known. To obtain partial results, W. Kerby and H. Wefelscheid considered separately the additive structure $(F,\oplus)$ and called such loops K-loops (see definition in section 2), but since they could not find a proper example of a K-loop no further theoretical investigations were done. The interest on K-loops has been revived grown since A. A. Ungar 1988 has found a famous physical example.

A.A. Ungar investigated the relativistic addition $\oplus$ of the velocities $\mathbb{R}^3_c := \{ v \in \mathbb{R}^3 : |v| < c\}$

(c speed of light). He showed that $(\mathbb{R}_c^3, \oplus)$ is a non-associative and non-commutative loop with characteristic automorphisms, the socalled Thomas rotations. He proved that for any two velocities $a, b \in \mathbb{R}_c^3$ there is a Thomas rotation $\delta_{a,b}$ fulfilling $a \oplus (b \oplus x) = (a \oplus b) \oplus \delta_{a,b}(x)$ (cf. [29, 30, 31]). H. Wefelscheid recognized then that $(\mathbb{R}_c^3, \oplus)$ is a K-loop.

But there is also a close connection between K-loops and Bruck loops, which is discovered first by G. Kist [19]. Bruck loops are Bol loops satisfying the automorphic inverse property **(K5)** (cf. section 2). Bol loops are introduced by G. Bol in 1937 in order to coordinatize webs (cf. [1]), and are investigated in later years in many papers [2 to 6, 12, 25 to 28]. The examples in Bol's paper are due to Zassenhaus. Bol also seemed having had difficulties in finding proper examples. In [19, §1.3] G. Kist remarks, that already from Lemma 6 of G. Glauberman [6] one can deduce that every finite Bruck loop of odd order satisfies **(K3)**. As a generalisation it is proved in [21, Theorem 1] that every Bruck loop with no element of order 2 is a K-loop. Hence many examples of Bruck loops turns out to be examples of K-loops (cf. [12, 25]). Other examples are given by A. Kreuzer in [21, 22].

In this note we consider right loops with the axiom **(K3)**, not necessarily finite, and add step by step the other axioms of a K-loop if necessary for stronger results. In section 2 a comprehensive list of properties of right loops is presented. Let D denote the group of automorphisms of a right loop $(L, \oplus)$ with **(K3)** and **(K4)**, generated by the maps $\delta_{a,b}$. Then one can introduce on $G := L \times D$ a group operation (cf. **(3.5)**). In section 3 we consider the inverse of that process, starting with a group G and an exact decomposition $G = K \cdot A$. In section 4 a new construction method for a loop operation on the set $G \times H$ for commutative groups G, H is given. In section 5 and 6 we consider finite examples of K-loops constructed with that method. If any automorphism $\delta \in D \setminus \{id\}$ of a K-loop $(L, \oplus)$ has only the fixed point 0, then L determines a Frobenius group. In the finite case that fact implies that for proper finite K-loops there always does exist an automorphism $\tau \in D \setminus \{id\}$ having fixed point distinct form 0 (cf. section 7).

2. DEFINITIONS AND PROPERTIES OF LOOPS

Let L be a set with a binary operation $\oplus$. We call $(L, \oplus)$ a **right loop** if **(K1r)**, **(K2)** are valid , and a **loop**, if **(K1r)**, **(K1ℓ)** and **(K2)** are valid for all $a, b \in L$.

(K1 r) The equation $a\oplus x = b$ has a unique solution $x\in L$.

(K1 ℓ) The equation $y\oplus a = b$ has a unique solution $y\in L$.

(K2) There is a two-sided neutral element $0\in L$ with $a\oplus 0 = a = 0\oplus a$.

In the following let $(L,\oplus)$ always denote a right loop. By **(K1 r)**, for $a\in L$ the map

$$\lambda_a : L\to L,\ x\to\lambda_a(x) := a\oplus x$$

is bijective. Therefore for $a,b\in L$ also $\delta_{a,b} := \lambda_{a\oplus b}^{-1}\circ\lambda_a\circ\lambda_b$ is a bijection.

(2.1) For all $a,b\in L$, the map $\delta_{a,b}: L\to L,\ x\to\delta_{a,b}(x)$ is a bijection with the properties $a\oplus(b\oplus x) = (a\oplus b)\oplus\delta_{a,b}(x)$, $\delta_{a,b}(0) = 0$, and $\delta_{a,0} = \delta_{0,a} = id$.

Proof. Clearly $(a\oplus b)\oplus\delta_{a,b}(x) = \lambda_{a\oplus b}\circ\lambda_{a\oplus b}^{-1}\circ\lambda_a\circ\lambda_b(x) = a\oplus(b\oplus x)$. Since $a\oplus b = a\oplus(b\oplus 0) = (a\oplus b)\oplus\delta_{a,b}(0)$, by **(K1 r)** $\delta_{a,b}(0) = 0$. Since $a\oplus b = a\oplus(0\oplus b) = (a\oplus 0)\oplus\delta_{a,0}(b) = a\oplus\delta_{a,0}(b)$ for every $b\in L$ it follows that, $\delta_{a,0} = id$, and by $a\oplus b = 0\oplus(a\oplus b) = (0\oplus a)\oplus\delta_{0,a}(b) = a\oplus\delta_{0,a}(b)$, also $\delta_{0,a} = id$.

(2.2) For $a,b,x,y\in L$:

(a) $(a\oplus b)\oplus x = a\oplus(b\oplus\delta_{a,b}^{-1}(x))$.

(b) $\delta_{a,b}(x\oplus y) = \delta_{a,b}(x)\oplus\delta_{a\oplus b,\delta_{a,b}(x)}^{-1}\circ\delta_{a,b\oplus x}\circ\delta_{b,x}(y)$.

(c) For every automorphism α of $(L,\oplus)$, $\alpha\circ\delta_{a,b}\circ\alpha^{-1} = \delta_{\alpha(a),\alpha(b)}$.

Proof. **(a)**. $a\oplus(b\oplus\delta_{a,b}^{-1}(x)) = (a\oplus b)\oplus\delta_{a,b}\circ\delta_{a,b}^{-1}(x) = (a\oplus b)\oplus x$.

(b). $a\oplus\big(b\oplus(x\oplus y)\big) = (a\oplus b)\oplus\delta_{a,b}(x\oplus y)$, and on the other hand $a\oplus\big(b\oplus(x\oplus y)\big) =$ $a\oplus\big((b\oplus x)\oplus\delta_{b,x}(y)\big) = \big(a\oplus(b\oplus x)\big)\oplus\delta_{a,b\oplus x}\circ\delta_{b,x}(y) = \big((a\oplus b)\oplus\delta_{a,b}(x)\big)\oplus\delta_{a,b\oplus x}\circ\delta_{b,x}(y) \overset{(a)}{=}$ $(a\oplus b)\oplus\big(\delta_{a,b}(x)\oplus\delta_{a\oplus b,\delta_{a,b}(x)}^{-1}\circ\delta_{a,b\oplus x}\circ\delta_{b,x}(y)\big)$. Hence by **(K1 r)** $\delta_{a,b}(x\oplus y) =$ $\delta_{a,b}(x)\oplus\delta_{a\oplus b,\delta_{a,b}(x)}^{-1}\circ\delta_{a,b\oplus x}\circ\delta_{b,x}(y)$

(c). For $a\in L$, we have $\alpha\circ\lambda_a\circ\alpha^{-1}(x) = \alpha(a\oplus\alpha^{-1}(x)) = \alpha(a)\oplus x = \lambda_{\alpha(a)}(x)$. Hence $\alpha\circ\delta_{a,b}\circ\alpha^{-1} =$ $\alpha\circ\lambda_{a\oplus b}^{-1}\circ\alpha^{-1}\circ\alpha\circ\lambda_a\circ\alpha^{-1}\circ\alpha\circ\lambda_b\circ\alpha^{-1} = \lambda_{\alpha(a)\oplus\alpha(b)}^{-1}\circ\lambda_{\alpha(a)}\circ\lambda_{\alpha(b)} = \delta_{\alpha(a),\alpha(b)}$.

(2.3) $\delta_{a,b}$ is an automorphism of $(L,\oplus)$, if and only if $\delta_{a\oplus b,\delta_{a,b}(x)}\circ\delta_{a,b} = \delta_{a,b\oplus x}\circ\delta_{b,x}$.

Proof. By **(K1 r)** and (2.2.b) $\delta_{a,b}(x)\oplus\delta_{a,b}(y) \overset{!}{=} \delta_{a,b}(x\oplus y) = \delta_{a,b}(x)\oplus\delta_{a\oplus b,\delta_{a,b}(x)}^{-1}\circ\delta_{a,b\oplus x}\circ\delta_{b,x}(y)$ iff $\delta_{a,b}(y) = \delta_{a\oplus b,\delta_{a,b}(x)}^{-1}\circ\delta_{a,b\oplus x}\circ\delta_{b,x}(y)$ for every $y\in L$, i.e., $\delta_{a,b} = \delta_{a\oplus b,\delta_{a,b}(x)}^{-1}\circ\delta_{a,b\oplus x}\circ\delta_{b,x}$.

We consider the following properties for $a, b \in L$:

(I) If $a \oplus b = 0$, then $b \oplus a = 0$.

(K3) $\delta_{a,b}$ is an automorphism of $(L, \oplus)$.

(K4) If $a \oplus b = 0$ then $\delta_{a,b} = id$.

(K4') $\delta_{a,a} = id$.

(K5) $(\ominus a) \oplus (\ominus b) = \ominus (a \oplus b)$ (**Automorphic inverse property**)

(K6) $\delta_{a,b} = \delta_{a, b \oplus a}$.

(KB) $a \oplus \big(b \oplus (a \oplus c)\big) = \big(a \oplus (b \oplus a)\big) \oplus c$ (**Bol-identity**).

If **(I)** is fulfilled, then $b \in L$ is the right and left inverse element of an element $a \in L$, and we write $\ominus a := b$. **(K5)** is only efficient, if **(I)** is satisfied and means that the map

$$\ominus : L \to L;\ x \to \ominus x$$

is an automorphism. Furthermore we have $\ominus \circ \alpha = \alpha \circ \ominus$ for any $\alpha \in Aut(L, \oplus)$ (cf. **(2.5.c)**). We remark that for a right loop, **(K4)**, or **(K6)** and **(K1ℓ)**, or **(KB)**, respectively, imply **(I)** (cf. (2.5),(**2.10**), (2.13)). Then we write $x \ominus y$ for $x \oplus (\ominus y)$.

A loop $(L, \oplus)$ is called a **WK-loop** , if **(K3)** is fulfilled and a **K-loop** , if **(K3)**,**(K5)**, and **(K6)** are fulfilled. We will show that a K-loop satisfies all of the above axioms, i.e. a K-loop fulfils also **(I)**, **(K4)** and **(KB)**.

A loop $(L, \oplus)$ is called a **Bol loop**, if **(KB)** is fulfilled and a **Bruck loop**, if **(KB)** and **(K5)** is fulfilled (cf. [2, 6, 26]).

M. Kikkawa called a loop **homogeneous**, if **(K3)** and **(K4)** are fulfilled, and homogeneous with the **symmetric property**, if **(K3)**, **(K4)**, and **(K5)** are fulfilled (cf [18]).

(**2.4**) Let $(L, \oplus)$ be a right loop with **(K3)**. Then for $a, b, c \in L$:

$$\delta_{a \oplus b, \delta_{a,b}(c)} \circ \delta_{a,b} = \delta_{a, b \oplus c} \circ \delta_{b,c}$$

The **proof** follows by (2.3).

(2.5) Let $(L, \oplus)$ be a right loop with **(K4)**. Then for $a, b \in L$:

(a) **(I)** is satisfied, i.e. $a \oplus b = 0$ implies $b \oplus a = 0$.

(b) $a \oplus b = \ominus \delta_{a,b}(\ominus b \ominus a)$.

(c) $\alpha(\ominus x) = \ominus \alpha(x)$ for every $x \in L$ and every $\alpha \in Aut(L, \oplus)$.

Proof. (a). For $a,b \in L$ with $a \oplus b = 0$ we get $a \oplus (b \ominus a) = (a \oplus b) \ominus \delta_{a,b}(a) \overset{(K4)}{=} 0 \ominus a = a$, i.e., $b \oplus a = 0$ by **(K1r)**.
(b). $(a \oplus b) \oplus \delta_{a,b}(\ominus b \ominus a) = a \oplus \big(b \oplus (\ominus b \ominus a)\big) = a \oplus \big((b \oplus (\ominus b)) \oplus \delta_{b,\ominus b}(\ominus a)\big) \overset{(K4)}{=}$
$a \oplus \big(0 \oplus (\ominus a)\big) = 0$, i.e. by (i), $a \oplus b = \ominus \delta_{a,b}(\ominus b \ominus a)$.
(c). $(\ominus x) + x = 0 = \alpha(0) = \alpha((\ominus x) + x) = \alpha(\ominus x) + \alpha(x)$, hence $\alpha(\ominus x) = \ominus \alpha(x)$ by **(K1ℓ)**.

A right loop $(L,\oplus)$ is said to have the **left inverse property**, if for any $a \in L$ there exists an element $\ominus a \in L$ with $(\ominus a) \oplus (a \oplus x) = x$ for every $x \in L$.

(2.6) For a right loop $(L,\oplus)$ the following assertions are equivalent:
(α) **(K4)** .
(β) The left inverse property.
(γ) **(I)** and $a \oplus b = \ominus \delta_{a,b}(\ominus b \ominus a)$ for $a,b \in L$.

Proof. (α)⟺(β). $(\ominus a) \oplus (a \oplus x) = \big((\ominus a) \oplus a\big) \oplus \delta_{\ominus a,a}(x) \overset{!}{=} x$ for every $x \in L$ iff $\delta_{b,a} = id$ for $b \in L$ with $b \oplus a = 0$, i.e., iff **(K4)** is fulfilled.
(α) ⇒ (γ) follows by **(2.5)** .
(γ) ⇒ (β). $0 = (a \oplus b) \oplus \delta_{a,b}(\ominus b \ominus a) = a \oplus \big(b \oplus (\ominus b \ominus a)\big)$. Hence by the definition of $\ominus a$ it follows that $\ominus a = b \oplus (\ominus b \ominus a)$

A right loop $(L,\oplus)$ is called **left alternative**, if $a \oplus (a \oplus b) = (a \oplus a) \oplus b$ for all $a,b \in L$.

(2.7) A right loop $(L,\oplus)$ is left alternative, if and only if **(K4')** is fulfilled

Proof. $a \oplus (a \oplus b) = (a \oplus a) \oplus \delta_{a,a}(b) \overset{!}{=} (a \oplus a) \oplus b$, iff $\delta_{a,a}(b) = b$ for all $b \in L$, i.e., $\delta_{a,a} = id$.

(2.8) Let $(L,\oplus)$ be a right loop with **(K3)** and **(K4)**. Then for $a,b,c \in L$:
(a) $\delta_{a,b}^{-1} = \delta_{\ominus a, a \oplus b}$.
(b) $\delta_{a,b}^{-1} = \delta_{\ominus b, \ominus a}$,

Proof. (a). We set in **(2.4)** $b = \ominus a$, hence $\delta_{a,\ominus a \oplus c} \circ \delta_{\ominus a,c} = \delta_{0,\delta_{a,b}(c)} \circ \delta_{a,\ominus a} \overset{(K4)}{=} id$.
(b). We set $c = \ominus b$ in **(2.4)** , hence $id \overset{(K4)}{=} \delta_{a,0} \circ \delta_{b,\ominus b} = \delta_{a \oplus b, \delta_{a,b}(\ominus b)} \circ \delta_{a,b}$. By **(a)** the inverse of the right side is $\delta_{\ominus a, a \oplus b} \circ \delta_{\ominus(a \oplus b), (a \oplus b) \oplus \delta_{a,b}(\ominus b)} =$
$\delta_{\ominus a, a \oplus b} \circ \delta_{\ominus(a \oplus b), a \oplus (b \ominus b)} = \delta_{\ominus a, a \oplus b} \circ \delta_{\ominus(a \oplus b), a} = id$. By **(K1r)** for every $y \in L$, there is an element $b \in L$ with $a \oplus b = y$. Hence with $x := \ominus a$, $\delta_{x,y} \circ \delta_{\ominus y, \ominus x} = id$.

(**2.9**) Let $(L,\oplus)$ be a right loop which satisfies **(K3)**, **(K4)** and **(K5)**. Then for $a,b,c \in L$:

(a) $\delta_{a,b}(b\oplus a) = a\oplus b$.

(b) $\delta_{a,b} = \delta_{\ominus a,\ominus b}$

(c) $\delta_{a,b}^{-1} = \delta_{b,a}$.

(d) $a\oplus(b\oplus c) = \delta_{a,b}((b\oplus a)\oplus c)$

(e) $\delta_{a,b} = \delta_{a\oplus b,\ominus a}$

(f) $\delta_{a,b} = \delta_{\ominus b,b\oplus a}$

(g) $\delta_{a,b}(b) = (a\oplus b)\ominus a$

(h) $\delta_{a,b\oplus c}\circ\delta_{b,c} = \delta_{a,b}\circ\delta_{b\oplus a,c}$

(i) $\delta_{a,b} = \delta_{a,z}\circ\delta_{z\oplus a,\ominus z\oplus b}\circ\delta_{\ominus z,b}$

(j) $\ominus a\oplus\delta_{a,b}(b) = b\ominus\delta_{b,a}(a)$

Proof. (a). By **(2.5.b)** $a\oplus b = \ominus\delta_{a,b}(\ominus b\oplus\ominus a) \overset{(K5)}{=} \ominus\delta_{a,b}(\ominus(b\oplus a)) \overset{(K3)}{=} \delta_{a,b}(b\oplus a)$.
(**b**) follows from **(K5)**, (2.2c), and (2.5.c). (**c**) follows from (**b**) and (**2.8.b**).
(**d**). $a\oplus(b\oplus c) = (a\oplus b)\oplus\delta_{a,b}(c) \overset{(a)}{=} \delta_{a,b}(b\oplus a)\oplus\delta_{a,b}(c) \overset{(K3)}{=} \delta_{a,b}((b\oplus a)\oplus c)$.
(**e**) follows from (**2.8.a**) and (**c**). (**f**). Put in (**2.4**) $a:=\ominus b$, $b:=b$ and $c:=a$ and use (**c**).
(**g**). We have $(a\oplus b)\ominus a \overset{(a)}{=} \delta_{a\oplus b,\ominus a}(\ominus a\oplus(a\oplus b)) \overset{(e)}{=} \delta_{a,b}(b)$, using (**2.6**) .
(**h**). By (**2.4**), $\delta_{a,b\oplus c}\circ\delta_{b,c} = \delta_{a\oplus b,\delta_{a,b}(c)}\circ\delta_{a,b} \overset{(a)}{=} \delta_{\delta_{a,b}(b\oplus a),\delta_{a,b}(c)}\circ\delta_{a,b} \overset{(2.2.c)}{=\!=} \delta_{a,b}\circ\delta_{b\oplus a,c}$.
(**i**). Using (**c**), we write (**h**) in the form $\delta_{b\oplus a,c} = \delta_{b,a}\circ\delta_{a,b\oplus c}\circ\delta_{b,c}$ and replace b by $\ominus z$, c by b and a by $z\oplus a$. Since $\delta_{\ominus z\oplus(z\oplus a),b} = \delta_{a,b}$ and $\delta_{\ominus z,z\oplus a} = \delta_{a,z}$ by (**f**), (**i**) follows.
(**j**). $\ominus a\oplus\delta_{a,b}(b) \overset{(g)}{=} \ominus a\oplus((a\oplus b)\ominus a) \overset{(2.2.a)(c)}{=\!=\!=} \ominus a\oplus(a\oplus(b\ominus\delta_{b,a}(a)) \overset{(2.6)}{=} b\ominus\delta_{b,a}(a)$.

(**2.10**) Let $(L,\oplus)$ be a right loop. Then:

(a) **(K6)** implies **(K4')**.

(b) **(K6)** and **(K1ℓ)** imply **(I)**.

(c) **(K6)** and **(I)** imply **(K4)**.

Proof. (**a**). By **(2.1)** $id = \delta_{a,0} \overset{(K6)}{=} \delta_{a,a}$. (**b**). Let $a\oplus b = 0$, then $\delta_{b,a} \overset{(K6)}{=} \delta_{b,a\oplus b} = \delta_{b,0} \overset{(2.1)}{=} id$, hence $b = b\oplus(a\oplus b) = (b\oplus a)\oplus\delta_{b,a}(b) = (b\oplus a)\oplus b$, i.e., by **(K1ℓ)** $b\oplus a = 0$. (**c**). Let $a\oplus b = 0$. It follows that, $\delta_{a,b} \overset{(K6)}{=} \delta_{a,b\oplus a} = \delta_{a,0} \overset{(2.1)}{=} id$.

Remark 1. The inversions of the assertions of (**2.10**) are not true. Since there are examples of loops satisfying **(K3)**,**(K4)** , **(K4')** and **(K5)**, but not **(K6)**, i.e., there exists homogeneous loops with the symmetric property which are not K-loops (cf. (**5.3**)) Furthermore there are examples of right loops, fulfilling **(I)** and **(K5)**, but not **(K1ℓ)** (cf. (**5.1**)). But we remark that in a right loop **(K3)**, **(K4)**, **(K5)**, and **(K6)** imply **(K1ℓ)** and **(I)** (cf. (**2.15**)).

(2.11) Let $(L,\oplus)$ be a right loop. Then the Bol identity **(KB)** and $\delta_{a,b\oplus a}=\delta_{b,a}^{-1}$ are equivalent.

Proof. $a\oplus(b\oplus(a\oplus c)) = a\oplus\big((b\oplus a)\oplus\delta_{b,a}(c)\big)=\big(a\oplus(b\oplus a)\big)\oplus\delta_{a,b\oplus a}\circ\delta_{b,a}(c) = \big(a\oplus(b\oplus a)\big)\oplus c$ iff $\delta_{a,b\oplus a}\circ\delta_{b,a} = id$, i.e. $\delta_{a,b\oplus a}=\delta_{b,a}^{-1}$.

By **(2.9.c)**, it follows from **(2.11)**:

(2.12) If for a right loop $(L,\oplus)$ the properties **(K3) (K4)** and **(K5)** are valid or if $\delta_{a,b}^{-1}=\delta_{b,a}$, then **(K6)** and the Bol identity **(KB)** are equivalent.

(2.13) Let for a right loop $(L,\oplus)$ the Bol identity **(KB)** be valid. Then:

(a) **(I)** is fulfilled.

(b) **(K4)** is fulfilled.

(c) **(K1ℓ)** is fulfilled.

Proof. By **(2.11)**, **(KB)** implies $\delta_{a,b\oplus a}=\delta_{b,a}^{-1}$. Now let $a\oplus b=0$. Hence by **(2.1)** $\delta_{a,b}^{-1} \overset{((KB)}{=} \delta_{b,a\oplus b} = \delta_{b,0}=id$ and therefore $\delta_{a,b}=id$, i.e. **(b)** is valid. By **(2.5.a)**, (a) follows.

(c). By **(b)** and **(2.6)** the left inverse property is fulfilled. Let $a,b\in L$. For $y:=(\ominus a)\oplus\big((a\oplus b)\ominus a\big)$, we get $y\oplus a=\big((\ominus a)\oplus\big((a\oplus b)\ominus a\big)\big)\oplus a \overset{(KB)}{=} (\ominus a)\oplus\big((a\oplus b)\oplus((\ominus a)\oplus a)\big)= (\ominus a)\oplus(a\oplus b) \overset{(2.6)}{=} b$. If for $y'\in L$ it holds $y'\oplus a=b$, then $y'\oplus a=y\oplus a$, hence $a\oplus(y'\oplus a)=a\oplus(y\oplus a)$ and $\big(a\oplus(y'\oplus a)\big)\ominus a=\big(a\oplus(y\oplus a)\big)\ominus a$. By **(KB)** we get $a\oplus y' = a\oplus\big(y'\oplus(a\ominus a)\big) \overset{(KB)}{=} \big(a\oplus(y'\oplus a)\big)\ominus a = \big(a\oplus(y\oplus a)\big)\ominus a \overset{(KB)}{=} a\oplus y$ and by **(K1r)** it follows $y=y'$.

This proof together with **(2.9.g,j)** gives also an explicit solution y of $y\oplus a=b$ for $a,b\in L$.

(2.14) In a right loop $(L,\oplus)$ with the Bol identity **(KB)** the solution of $y\oplus a=b$ can be written as: $y=(\ominus a)\oplus\big((a\oplus b)\ominus a\big) = (\ominus a)\oplus\delta_{a,b}(b) = b\ominus\delta_{b,a}(a)$

By **(2.12)** we get:

(2.15) A right loop with the properties **(K3)**, **(K4)**, **(K5)**, and **(K6)** fulfills **(K1ℓ)**.

Remark 2. There are examples of right loops with the properties **(I)** and **(K5)** which do not fulfill **(K1ℓ)** (cf (5.1)). A simple proof shows that this examples satisfy also **(K4)**.

Remark 3. By (2.12), every K-loop is a Bol loop and a Bruck loop.

By [21, Theorem 1] (cf. also [6, 19]):

(2.16) Every Bruck loop $(L,\oplus)$ with $x\oplus x\neq 0$ for every $x\in L\setminus\{0\}$ is a K-loop.
It means that for a right loop $(L,\oplus)$ with $x\oplus x\neq 0$ for every $x\in L\setminus\{0\}$, **(KB)** and **(K5)** imply **(K1ℓ)**, **(K3)**, and **(K6)**.

(2.17) Let $(L,\oplus)$ be a K-loop. Then for $a,b,x\in L$:

(a) $\delta_{a,b}=\delta_{a\oplus b,b}$.

(b) $x\ominus a=b$ if and only if $x\ominus b=a$.

Proof. (a). By **(K6)**, $\delta_{a,b}\overset{(2.9.c)}{=}\delta_{b,a}^{-1}\overset{(K6)}{=}\delta_{b,a\oplus b}^{-1}\overset{(2.9.c)}{=}\delta_{a\oplus b,b}$.
(b). Let $x\ominus a=b$, then $x\ominus b=x\ominus(x\ominus a)\overset{(K5)}{=}x\oplus((\ominus x)\oplus a)\overset{(2.6)}{=}a$. On the other hand $x\ominus b=a$ implies $x\ominus a=x\ominus(x\ominus b)=b$.

(2.18) Let $(L,\oplus)$ be a K-loop. With the new operation

$$a\boxdot b:=a\oplus\delta_{\ominus a,b}(b)$$

for $a,b\in L$, $(L,\boxdot)$ is a commutative loop.

Proof. We replace in **(2.9.j)** a by $\ominus a$ and use $\delta_{\ominus b,a}=\delta_{b,\ominus a}$ by **(2.9.b)**. This implies $a\boxdot b=a\oplus\delta_{\ominus a,b}(b)=b\oplus\delta_{b,\ominus a}(a)=b\oplus\delta_{\ominus b,a}(a)=b\boxdot a$.
By **(2.14)** for $x:=b\ominus a$ it follows $b=a\oplus\delta_{\ominus a,x}(x)=a\boxdot b$, hence $x=b\ominus a$ is a solution of $a\boxdot x=b$. Let for $y\in L$, $a\boxdot y=a\boxdot x=b$, Then $b=a\oplus\delta_{\ominus a,y}(y)=a\oplus\delta_{\ominus a,x}(x)$. Hence by **(K1r)** $\delta_{\ominus a,y}(y)=\delta_{\ominus a,x}(x)$. By **(2.9.g)**, $\big((\ominus a)\oplus y\big)\oplus a=\big((\ominus a)\oplus x\big)\oplus a$ and by **(K1r)** and **(K1ℓ)**, $y=x$. Thus there exists an unique solution of $a\boxdot x=b$ and $(L,\boxdot)$ is a loop.

3. LOOP OPERATIONS ON SUBSETS OF GROUPS

Let $(G,\cdot)$ be a group with the neutral element e, and let $A<G$ be a subgroup and $K\subset G$ be a subset of G with the following properties.

(L1r) $G=K\cdot A$ is an exact decomposition, i.e. for every $\mathfrak{a}\in G$, there are unique elements $a\in K$ and $\alpha\in A$ with $\mathfrak{a}=a\cdot\alpha$.

(L2) $e\in K$.

Remark 4. **(L1r)** is obviously equivalent to

(L1r') $\quad G = \dot{\bigcup}_{x \in K} x \cdot A.$

We remark that for any $z \in G$, $G = z \cdot G = \dot{\bigcup}_{x \in K} z \cdot x \cdot A = \dot{\bigcup}_{y \in K'} y \cdot A$ for $K' := z \cdot K$. Hence if **(L1r)** is valid, we can always find a set K' with $G = \dot{\bigcup}_{y \in K'} y \cdot A$ and $e \in K'$, i.e. **(L2)** is fulfilled.

Remark 5. By **(L1r)** it follows that, $|K \cap A| = 1$, since $e = y \cdot \alpha \in K \cdot A$ for an element $y \in K$, hence $y = \alpha^{-1} \in A$. Let $z \in K \cap A$, then $z = y \cdot (y^{-1} \cdot z) = z \cdot e \in K \cdot A$ and **(L1r)** implies $z = y$. If we assume **(L1r)** and **(L2)**, then $K \cap A = \{e\}$.

(3.1) Example of an exact decomposition $G = K \cdot A$ for subsets $K, A \subset G$ with $K \cap A = \emptyset$. (This example was found by J. Gräter [7] after a discussion in Luminy.)

Let G be a finite group with a subgroup U which is not normal, i.e. there exists an element $a \in G$ with $aU \neq Ua$. Then $gU \neq Ua$ for any $g \in U$, because the existence of an element $g_0 \in G$ with $g_0 U = Ua$ would imply $g_0 u_0 = a$ with an appropriate $u_0 \in U$ and $Ua = g_0 U = a u_0^{-1} U = aU$ in contradiction to the assumption.

Let $G = g_1 U \dot{\cup} g_2 U \dot{\cup} \ldots \dot{\cup} g_n U$ be the union of disjoint cosets. Then there exists $v_i \in U$ such that $g_i v_i \notin Ua$, since otherwise $g_i U \subset Ua$ and our finiteness condition would imply $|g_i U| = |Ua|$, hence $g_i U = Ua$, contradicting $g_i U \neq Ua$. Therefore we may assume $g_i \notin Ua$, and

$$G = Ga = g_1 Ua \dot{\cup} g_2 Ua \dot{\cup} \ldots \dot{\cup} g_n Ua$$

is a disjoint decomposition of G with $K = \{g_1, g_2, \ldots, g_n\}$ and $A = Ua$. But clearly, in this example A is not a subgroup of G.

In the following let **(L1r)** and **(L2)** be always satisfied. Then obviously it holds $K \cap A = \{e\}$, since A is a subgroup. By **(L1r)** for $a, b \in K$ there are unique elements $z \in K$ and $d_{a,b} \in A$ with $a \cdot b = z \cdot d_{a,b}$. Therefore

$$\oplus : K \times K \to K, \ (a,b) \to a \oplus b := z = a \cdot b \cdot d_{a,b}^{-1}$$

is a binary operation on K. We note: $a \cdot b = (a \oplus b) \cdot d_{a,b}$

(3.2) $(K, \oplus)$ is a right loop.

Proof. For $a, b \in K$ by **(L1r)** there exists $x \in K$ and $\alpha \in A$ with $a^{-1} \cdot b = x \cdot \alpha$, hence $a \cdot x = b \cdot \alpha^{-1}$, i.e., $a \oplus x = b$. Let $y \in K$ with $a \oplus y = b$, i.e., $a \cdot y = b \cdot \beta^{-1}$ for $b \in A$. It follows $a \cdot y \cdot \beta = b = a \cdot x \cdot \alpha$, hence $y\cdot\beta = x\cdot\alpha$. This implies by **(L1r)**, $x=y$ and $\alpha=\beta$. By **(L2)** $e\in K$ and $a=a\cdot e=a\oplus e=e\oplus a=e\cdot a$, i.e., e is the neutral element of $(K,\oplus)$.

Now we consider the following properties:

(L1ℓ) $|K\cdot a \cap b\cdot A| = 1$ for all $a,b\in K$.

(L3) $\alpha\cdot K\cdot\alpha^{-1}\subset K$ for every $\alpha\in A$.

(L4) $K^{-1}\subset K$.

(L5) If $a,b\in K$ and $\alpha\in A$ with $a\cdot b\cdot\alpha\in K$, then there exists $\beta\in A$ with $\beta\cdot b\cdot a=a\cdot b\cdot\alpha$.

(LB) $a\cdot K\cdot a\subset K$ for every $a\in K$.

(3.3) **(LB)** implies **(L1ℓ)** and **(L4)**.

Proof. For $a\in K$, by **(L1r)** there are $a'\in K$ and $\alpha\in A$ with $a^{-1}=a'\cdot\alpha$. By **(LB)**, $a'\cdot a\cdot a' = a'\cdot a\cdot a^{-1}\cdot\alpha^{-1}=a'\cdot\alpha^{-1}\in K$ and by **(L1r)** we get $\alpha^{-1}=\alpha=e$, i.e., $a^{-1}=a'\in K$. For $a,b\in K$ let $d_{a,b}\in A$ with $a\cdot b\cdot d_{a,b}^{-1}\in K$. By **(LB)** $a^{-1}\in K$ and $y:=a^{-1}\cdot a\cdot b\cdot d_{a,b}^{-1}\cdot a^{-1}\in K$, hence $b\cdot d_{a,b}^{-1} = y\cdot a=(y\oplus a)\cdot d_{y,a}$ and therefore $b=y\oplus a$ by **(L1r)**. Now let $x\in K$ with $x\oplus a=b$, thus $x\cdot a=b\cdot\alpha$ with $\alpha\in K$. It follows by **(LB)** $a\cdot x\cdot a=a\cdot b\cdot\alpha\in K$, i.e. by **(L1r)**, $\alpha=d_{a,b}^{-1}$ and $x\cdot a=b\cdot d_{a,b}^{-1}$, hence $x=y$.

(3.4) For $a,b\in K$, $(a\oplus b)^2=a\cdot b^2\cdot a$ implies **(L5)**.

Proof. $(a\oplus b)^2 = (a\cdot b\cdot d_{a,b}^{-1})^2 = a\cdot b\cdot d_{a,b}^{-1}\cdot a\cdot b\cdot d_{a,b}^{-1} \overset{!}{=} a\cdot b^2\cdot a$ if and only if $a\cdot b\cdot d_{a,b}^{-1} = d_{a,b}\cdot b.a$, i.e., **(L5)** is fulfilled with $\beta=d_{a,b}$.

(3.5) Example. Let $(L,\oplus)$ be a right loop with **(K3)** and **(K4)**, and let D be a subgroup of the automorphismgroup of $(L,\oplus)$ containing $\{\delta_{a,b}: a,b\in L\}$. For example let $D:=\langle\{\delta_{a,b}: a,b\in L\}\rangle$ be the subgroup which is generated by the automorphisms $\delta_{a,b}$, called the **structure group of L**. Then for $G:=L\times D$ and the operation

$$(a,\alpha)\cdot(b,\beta):=(a\oplus\alpha(b),\delta_{a,\alpha(b)}\circ\alpha\circ\beta) \qquad \text{(cf. [19, page 28])},$$

$(G,\cdot)$ is a group with the properties **(L1r)**, **(L2)**, **(L3)**, called the **quasidirect product** of L and D. For the centralizer

$$C_G(\overline{L})=\{(x,\alpha)\in G: (x,\alpha)\cdot(a,id)=(a,id)\cdot(x,\alpha) \text{ for every } a\in L\}$$

of $\overline{L}:=L\times\{id\}$ in G, it holds $C_G(\overline{L})\cap\{(0,\delta_{a,b}): a,b\in L\} = \{(0,id)\}$.

$(G,\cdot)$ fulfilles **(L1ℓ)**, if $(L,\oplus)$ is a loop, i.e. if additionally **(K1ℓ)** is valid.

Proof. $(G,\cdot)$ is associative: Let $(a,\alpha),(b,\beta),(c,\gamma)\in G$. We compute $\big((a,\alpha)\cdot(b,\beta)\big)\cdot(c,\gamma) = \big((a\oplus\alpha(b))\oplus\delta_{a,\alpha(b)}\circ\alpha\circ\beta(c),\ \delta_{a\oplus\alpha(b),\,\delta_{a,\alpha(b)}\circ\alpha\circ\beta(c)}\circ\delta_{a,\alpha(b)}\circ\alpha\circ\beta\circ\gamma\big)$ and $(a,\alpha)\cdot\big((b,\beta)\cdot(c,\gamma)\big)= \big(a\oplus(\alpha(b)\oplus\alpha\circ\beta(c)),\ \delta_{a,\,\alpha(b\oplus\beta(c))}\circ\alpha\circ\delta_{b,\beta(c)}\circ\beta\circ\gamma\big)$. By definition of $\delta_{a,\alpha(b)}$ we get $\big(a\oplus\alpha(b)\big)\oplus\delta_{a,\alpha(b)}(\alpha(\beta(c)))= a\oplus\big(\alpha(b)\oplus\alpha(\beta(c))\big)$. By **(2.4)** with for $a:=a$, $b:=\alpha(b)$ and $x:=\alpha(\beta(c))$ it follows $\delta_{a\oplus\alpha(b),\,\delta_{a,\alpha(b)}\circ\alpha\circ\beta(c)}\circ\delta_{a,\alpha(b)}\circ\alpha = \delta_{a,\alpha(b)\oplus\alpha(\beta(c))}\circ\delta_{\alpha(b),\alpha(\beta(c))}\circ\alpha \overset{(2.2.c)}{=} \delta_{a,\alpha(b\oplus\beta(c))}\circ\alpha\circ\delta_{b,\beta(c)}$, which proves the assertion.

$e=(0,id)$ is the neutral element and $(\alpha^{-1}(\ominus a),\alpha^{-1})$ is the inverse element of (a,α), since $(a,\alpha)\circ(\alpha^{-1}(\ominus a),\alpha^{-1}) = (a\oplus(\ominus a),\delta_{a,\ominus a}\circ\alpha\circ\alpha^{-1}) = (0,id)$ by **(K4)**, hence $(G,\cdot)$ is a group.

(L1r): For $(0,\alpha),(0,\beta)\in G$, $(0,\alpha)\circ(0,\beta) = (0\oplus\alpha(0),\delta_{0,0}\circ\alpha\circ\beta) = (0,\alpha\circ\beta)$ by **(2.1)**, i.e., $A:=\{(0,\alpha):\alpha\in D\}$ is a subgroup of $(G,\cdot)$ which is isomorphic to $(D,\circ)$. For $K:=\{(a,id):a\in L\}\subset G$ we obtain $(a,\alpha)=(a,id)\cdot(0,\alpha)=(a\oplus id(0),\delta_{a,0}\circ id\circ\alpha)=(a,\alpha)\in K\cdot A$ (cf.**(2.1)**) and $(a,id),(0,\alpha)$ are unique, i.e. **(L1r)** is fulfilled.

Obviously **(L2)** is valid, i.e., $(0,id)\in K$.

(L3): Let $(0,\alpha)\in A$ and $(a,id)\in K$. Then $(0,\alpha)\cdot(a,id)\cdot(0,\alpha)^{-1}=(\alpha(a),\alpha)\cdot(0,\alpha^{-1})=(\alpha(a),\alpha\circ\alpha^{-1}) = (\alpha(a),id)\in K$, since $\alpha(0)=0$ and $\delta_{\alpha(a),0} = id$ by **(2.1)**. Hence **(L3)** is fulfilled.

(L1ℓ): For $(a,id),(b,id)\in K$, $K\cdot(a,id)=\{(x\oplus a,\delta_{x,a}) : x\in L\}$ and $(b,id)\cdot A=\{(b,\alpha):\alpha\in D\}$. By **(K1ℓ)** there is a unique element $x\in K$ with $x\oplus a=b$, hence also $\alpha=\delta_{x,a}$ is uniquely determined, i.e. $|K\cdot(a,id)\cap(b,id)\cdot A|=1$ and **(L1ℓ)** is valid.

For $(0,\alpha)\in A$, we compute $(0,\alpha)\cdot(a,id)=(\alpha(a),\delta_{0,\alpha(a)}\circ\alpha) =(\alpha(a),\alpha)$ and $(a,id)\cdot(0,\alpha)=(a,\alpha)$. We have $(0,\alpha)\cdot(a,id) = (a,id)\cdot(0,\alpha)$ for every $a\in K$ if and only if $\alpha(a)=a$ for every $a\in K$, i.e. if $\alpha=id$. Hence $C_G(K)\cap\{(0,\delta_{a,b}):a,b\in L\} = \{(0,id)\}$.

(3.6) $(K,\oplus)$ satisfies **(K1ℓ)** if and only if **(L1ℓ)** is satisfied. Hence $(K,\oplus)$ is a loop, if **(L1r)**, **(L1ℓ)** and **(L2)** are fulfilled.

Proof. For $a,b\in K$, there exists an unique element $y\in K$ with $y\oplus a=b$, i.e., $y\cdot a=b\cdot\alpha$ with $\alpha\in A$ if and only if $|K\cdot a\cap b\cdot A|=1$.

If **(L3)** is valid, for $a,b\in K$ and $d_{a,b}\in A$ with $a\cdot b=(a\oplus b)\cdot d_{a,b}$,

$$\delta_{a,b}: K\to K,\ x\to\delta_{a,b}(x):=d_{a,b}\cdot x\cdot d_{a,b}^{-1},$$

is a bijective map with the property $a\oplus(b\oplus x) = (a\oplus b)\oplus\delta_{a,b}(x)$, since $a\oplus(b\oplus x)= a\oplus(b\cdot x\cdot d_{b,x}^{-1})=a\cdot b\cdot x\cdot d_{b,x}^{-1}\cdot d_{a,b\oplus x}^{-1}$ and $(a\oplus b)\oplus\delta_{a,b}(x)=a\cdot b\cdot d_{a,b}^{-1}\cdot d_{a,b}\cdot x\cdot d_{a,b}^{-1}\cdot d_{a\oplus b,\delta_{a,b}(x)}^{-1}\cdot$

We get $(a\oplus(b\oplus x))\cdot d_{a,b\oplus x}\cdot d_{b,x} = a\cdot b\cdot x = ((a\oplus b)\oplus \delta_{a,b}(x))\cdot d_{a\oplus b,\delta_{a,b}(x)}\cdot d_{a,b} \in K\cdot A$. Since by **(L1r)** the decomposition is exact, it follows $d_{a,b\oplus x}\cdot d_{b,x} = d_{a\oplus b,\delta_{a,b}(x)}\cdot d_{a,b}$. Hence:

(3.7) If **(L3)** is satisfied, then for every $a,b\in K$, $\delta_{a,b}$ is an automorphism of $(K,\oplus)$ with the properties $a\oplus(b\oplus x) = (a\oplus b)\oplus\delta_{a,b}(x)$ and $d_{a,b\oplus x}\cdot d_{b,x} = d_{a\oplus b,\delta_{a,b}(x)}\cdot d_{a,b}$.

Proof. It remains only to show that $\delta_{a,b}$ is an automorphism. Obviously $\delta_{a,b}(x)\oplus\delta_{a,b}(y) = d_{a,b}\cdot x\cdot y\cdot d_{a,b}^{-1}\cdot d^{-1}_{\delta_{a,b}(x),\delta_{a,b}(y)}$ and $\delta_{a,b}(x\oplus y) = d_{a,b}\cdot x\cdot y\cdot d_{x,y}^{-1}\cdot d_{a,b}^{-1}$. We get by **(L3)**, $(\delta_{a,b}(x)\oplus\delta_{a,b}(y))\cdot d_{\delta_{a,b}(x),\delta_{a,b}(y)} = d_{a,b}\cdot x\cdot y\cdot d_{a,b}^{-1} = \delta_{a,b}(x\oplus y)\cdot d_{a,b}\cdot d_{x,y}\cdot d_{a,b}^{-1} \in K\cdot A$ and by **(L1r)** it follows $\delta_{a,b}(x)\oplus\delta_{a,b}(y) = \delta_{a,b}(x\oplus y)$.

Remark 6. If $d_{a,b}=e$, obviously $\delta_{a,b}=id$. But for $\delta_{a,b}=id$, it need not be $d_{a,b}=e$. For $\delta_{a,b}=id$ we get only $d_{a,b}\in C_G(K)=\{\xi\in G : \xi\cdot a=a\cdot\xi$ for every $a\in K\}$.

Remark 7. Let **(L1ℓ)**, **(L2)**, and **(L3)** be fulfilled. Then $(K,\oplus)$ is not associative if and only if $\{d_{a,b} : a,b\in K\} \not\subset C_G(K)=\{\xi\in G : \xi\cdot a=a\cdot\xi$ for every $a\in K\}$. Since then there exist $a,b\in K$ with $\delta_{a,b}\neq id$ (cf. Remark 6) and by **(3.7)**, $(K,\oplus)$ is non associative.

(3.8) Theorem. Let $A<G$ be a subgroup and $K\subset G$ be a subset of a group $(G,\cdot)$, fulfilling **(L1r)** and **(L2)**. Then $(K,\oplus)$ with $a\oplus b := a\cdot b\cdot d_{a,b}$ for $a,b\in K$ and $d_{a,b}\in A$ is a right loop. **(Lx)** imply **(Kx)** for $x\in\{$ **1ℓ, 3, 4, B**$\}$.
If **(L4)** is valid, then **(L5)** and **(K5)** are equivalent.
Hence $(K,\oplus)$ is a Bruck loop if **(L5)** and **(LB)** are fulfilled, and a K-loop if **(L3)**, **(L5)** and **(LB)** are valid.

Proof. The first part follows by **(3.4)**, **(3.6)**, **(3.7)**, **(3.9)**, **(3.12)**, and **(3.13)**. By **(2.12)** and **(2.13)** we obtain the second part.

Remark 8. If

(*) $\{d_{a,b} : a,b\in K\} \cap C_G(K)=\{\xi\in G : \xi\cdot a=a\cdot\xi$ for every $a\in K\} = \{e\}$

then $\delta_{a,b}=id$ if and only if $d_{a,b}=e$, and **(Lx)** and **(Kx)** are equivalent for $x\in\{$**3, 4, B**$\}$.

(3.9) **(L4)** implies **(K4)**. If **(*)**, then **(L4)** and **(K4)** are equivalent.

Proof. For $a \in K$ let $b \in K$ and $\alpha \in A$ with $a^{-1} = b \cdot \alpha$ (cf. **(L1r)**). Then $e = a \cdot a^{-1} = a \cdot b \cdot \alpha$, hence $(a \oplus b) = e = a \cdot b \cdot \alpha^{-1} = a \cdot b \cdot d_{a,b}^{-1}$. Therefore $a^{-1} \in K$ if and only if $\alpha = d_{a,b} = e$. It follows $\delta_{a,b} = id$. If **(*)**, then $\delta_{a,b} = id$ and $d_{a,b} = e$ are equivalent (cf. Remark 6).

(3.10) Let **(L4)** be valid. Then for $a,b \in K$:

(i) $\ominus a = a^{-1}$. (ii) $d_{a,b}^{-1} = d_{a^{-1}, a \oplus b}$.

Let **(L3)** and **(L4)** be valid, then for $a,b \in K$:

(iii) $d_{a,b}^{-1} = d_{b^{-1},a^{-1}}$ (iv) $(a \oplus b)^{-1} = \delta_{a,b}(b^{-1} \oplus a^{-1})$

Proof. (i). Since $a^{-1} \in K$, $a \oplus a^{-1} = a \cdot a^{-1} = e$. (ii). Since **(L4)** implies **(K4)**, by **(2.6)** $b = a^{-1} \oplus (a \oplus b) = a^{-1} \cdot a \cdot b \cdot d_{a,b}^{-1} \cdot d_{a^{-1}, a \oplus b}^{-1}$ hence by **(L1r)** $d_{a,b}^{-1} \cdot d_{a^{-1}, a \oplus b}^{-1} = e$.
(iii). Since $d_{a,e} = e$ and $d_{b,b^{-1}} = e$ by **(L4)**, **(3.7)** implies $d_{a \oplus b, \delta_{a,b}(b^{-1})} \cdot d_{a,b} = d_{a,e} \cdot d_{b,b^{-1}} = e$.
Hence $d_{a^{-1}, a \oplus b} \overset{(ii)}{=} d_{a,b}^{-1} = d_{a \oplus b, \delta_{a,b}(b^{-1})} \overset{(ii)}{=} d^{-1}_{(a \oplus b)^{-1}, (a \oplus b) \oplus \delta_{a,b}(b^{-1})} = d^{-1}_{(a \oplus b)^{-1}, a}$.
(iv). $(a \oplus b)^{-1} = d_{a,b} \cdot b^{-1} \cdot a^{-1} \overset{(iii)}{=} d_{a,b} \cdot b^{-1} \cdot a^{-1} \cdot d_{b^{-1}, a^{-1}}^{-1} \cdot d_{a,b}^{-1} = \delta_{a,b}(b^{-1} \oplus a^{-1})$.

(3.11) If $a \cdot a \in K$ for $a \in K$, then $(K, \oplus)$ fulfilles **(K4')** and $a \oplus a = a \cdot a$.

Proof. By **(L1r)** for $a \in K$, $a \cdot a = (a \oplus a) \cdot d_{a,a}$. Hence $a \cdot a \in K$ implies $d_{a,a} = e$ and $\delta_{a,a} = id$.

(3.12) Let **(L4)** be valid and for $a \in K$ let $\ominus a = a^{-1} \in K$ be the right and left inverse of a with respect to $\oplus$, according to (3.10). Then **(L5)** and **(K5)** are equivalent.

Proof. For $a,b \in K$, let $d_{a,b} \in A$ with $a \cdot b \cdot d_{a,b}^{-1} \in K$. Let **(K5)** be valid. Hence $(a^{-1} \oplus b^{-1})^{-1} = (a^{-1} \cdot b^{-1} \cdot d_{a^{-1},b^{-1}}^{-1})^{-1} = d_{a^{-1},b^{-1}} \cdot b \cdot a \overset{(K5)}{=} a \oplus b = a \cdot b \cdot d_{a,b}^{-1} \in K$ implies the existence of $\beta \in A$ with $\beta \cdot b \cdot a = a \cdot b \cdot d_{a,b}^{-1} \in K$. Now we assume **(L5)**. Let $\beta \in A$ with $\beta \cdot b \cdot a = a \cdot b \cdot d_{a,b}^{-1} = a \oplus b$. Then $a \oplus b = \left((\beta \cdot b \cdot a)^{-1}\right)^{-1} = \left(a^{-1} \cdot b^{-1} \cdot \beta^{-1}\right)^{-1} \in K$, and by **(L4)** $a^{-1} \cdot b^{-1} \cdot \beta^{-1} \in K$, and by **(L1r)** $a^{-1} \cdot b^{-1} \cdot \beta^{-1} = a^{-1} \oplus b^{-1}$. Hence $a \oplus b = \left(a^{-1} \oplus b^{-1}\right)^{-1}$.

(3.13) **(LB)** implies **(KB)**. If **(*)**, then **(LB)** and **(KB)** are equivalent.

Proof. For $a,b \in K$, $a \oplus (b \oplus a) = a \cdot b \cdot a \cdot d_{b,a}^{-1} \cdot d_{a,b \oplus a}^{-1} \in K$, hence $a \cdot b \cdot a \in K$ if and only if $d_{b,a} \cdot d_{a,b \oplus a} = e$, i.e., $d_{a,b \oplus a} = d_{b,a}^{-1}$. It follows $\delta_{a,b \oplus a} = \delta_{b,a}^{-1}$, which is equivalent to **(KB)** by **(2.11)**. If **(*)**, then $d_{a,b \oplus a} = d_{b,a}^{-1}$ and $\delta_{a,b \oplus a} = \delta_{b,a}^{-1}$ are equivalent.

4. CONSTRUCTION

Let $(G,+)$ and $(H,+)$ be commutative groups with the neutral elements $0 \in G$ and $0 \in H$, and let $U \leq G$ be a subgroup of G. We consider the following maps

$$\mu : G \times G \times H \to U \quad ; \quad (a,c,d) \to \mu(a,c,d) = \mu_{a,c,d}$$

$$\circ : H \times U \to U \quad ; \quad (b,\mu_{a,c,d}) \to b \circ \mu_{a,c,d}$$

with the properties for $a,c,x \in G$ and $b,d \in H$:

(M1r) For every $t \in U$, $b \circ \mu_{a,c+t,d} = b \circ \mu_{a,c,d}$

(M1ℓ) For every $t \in U$, $b \circ \mu_{a+t,c,d} = b \circ \mu_{a,c,d}$

(M2) $0 \circ \mu_{a,c,d} = 0 = b \circ \mu_{a,0,0} = b \circ \mu_{0,0,d}$.

(MI) $b \circ \mu_{a,-a,-b} = 0$.

(M3) $b \circ \mu_{a,c+x,d+y} = b \circ \mu_{a,c,d} + b \circ \mu_{a,x,y}$.

(M4) $-b \circ \mu_{a,c,d} = (-b) \circ \mu_{-a,c,d}$

(M5) $-b \circ \mu_{a,c,d} = (-b) \circ \mu_{-a,-c,-d}$

(MB) $(b+d+b) \circ \mu_{a+c+a,x,y} = b \circ \mu_{a,x,y} + d \circ \mu_{c,x,y} + b \circ \mu_{a,x,y}$.

(M) There are $u,v \in G$ and $h \in H$ with $h \circ \mu_{u,v,-h} \neq 0$.

(4.1) i) (MB) implies **(M4)**.
ii) (M2), **(M3)**, **(M4)** and **(M5)** imply $\mathrm{ord}(b \circ \mu_{a,c,d}) \leq 2$ for all $a,c \in G$ and all $b,d \in H$.

Proof. i. We set $d=-b$ and $c=-a$ in **(MB)**, then $b \circ \mu_{a,x,y} = b \circ \mu_{a,x,y} + (-b) \circ \mu_{-a,x,y} + b \circ \mu_{a,x,y}$, hence $-b \circ \mu_{a,x,y} = (-b) \circ \mu_{-a,x,y}$. **ii.** By **(M2)**, $0 = b \circ \mu_{a,0,0} = b \circ \mu_{a,c-c,d-d} \overset{(M3)}{=} b \circ \mu_{a,c,d} + b \circ \mu_{a,-c,-d} \overset{(M4)}{=} b \circ \mu_{a,c,d} - (-b) \circ \mu_{-a,-c,-d} \overset{(M5)}{=} b \circ \mu_{a,c,d} + b \circ \mu_{a,c,d}$.

We set $L := G \times H$ and define for $(a,b),(c,d) \in L$:

$$\oplus : L \times L \to L; \; (a,b) \oplus (c,d) := (a+c+b \circ \mu_{a,c,d},\; b+d).$$

(4.2) Theorem. Let **(M1r)** and **(M2)** be fulfilled. Then $(L,\oplus)$ is a right loop with the neutral element $(0,0)$.

Proof. Let $(a,b),(c,d) \in L$. Then $(a,b) \oplus (x,y) = (c,d)$ has the solution $y=-b+d$ and $x=-a+c-b \circ \mu_{a,-a+c,y}$, since $b \circ \mu_{a,-a+c,y} \in U$ and $\mu_{a,-a+c-b \circ \mu_{a,-a+c,y},\, y} \overset{(M1r)}{=} \mu_{a,-a+c,y}$. We compute $(a,b) \oplus (x,y) = (a-a+c-b \circ \mu_{a,-a+c,y} + b \circ \mu_{a,-a+c,y},\; b-b+d) = (c,d)$. We assume that also (x',y) with $y=-b+d$ is a solution, hence $(a,b)\oplus(x',-b+d)=(a+x'+b\circ\mu_{a,x',y},d)=$ $=(c,d)$, i.e. $x'=-a+c-b\circ\mu_{a,x',y}$. We compute $-x'+x=b\circ\mu_{a,x',y}-b\circ\mu_{a,-a+c,y}\in U$, since $b\circ\mu_{a,x',y}$, $b\circ\mu_{a,-a+c,y}\in U$ and U is a group. Therefore there is a $t\in U$ with $x'=x+t$ and with **(M1r)** we get $b\circ\mu_{a,x',y}=b\circ\mu_{a,x,y}=b\circ\mu_{a,-a+c,y}$, hence $x=x'$ and the solution is unique. By **(M2)**, $0\circ\mu_{0,a,b}=0=b\circ\mu_{a,0,0}$, hence $(0,0)\oplus(a,b)=(a+0\circ\mu_{0,a,b},b)=(a,b)=$ $=(a+b\circ\mu_{a,0,0},b)=(a,b)\oplus(0,0)$.

In the following let **(M1r)** and **(M2)** be always fulfilled, i.e. that $(L,\oplus)$ is a right loop.

(4.3) If **(M1ℓ)** is fulfilled, $(L,\oplus)$ is a loop.

Proof. For $(a,b),(c,d)\in L$, the equation $(x,y)\oplus(a,b)=(c,d)$ has the unique solution $y=d-b$ and $x=c-a-y\circ\mu_{c-a,a,b}$, since $y\circ\mu_{c-a-y\circ\mu_{c-a,a,b},a,b}=y\circ\mu_{c-a,a,b}$ by **(M1ℓ)** and $(x,y)\oplus(a,b)=(c-a-y\circ\mu_{c-a,a,b}+a+y\circ\mu_{c-a,a,b},d-b+b)=(c,d)$.

(4.4) Let **(MI)** be fulfilled.

i) Then $(L,\oplus)$ satisfies **(I)**, and $\ominus(a,b)=(-a,-b)$ is the inverese of $(a,b)\in L$.

ii) If **(M)** is valid, $(L,\oplus)$ is not assosiative.

Proof. i. Since **(MI)**, $b\circ\mu_{a,-a,-b}=0=(-b)\circ\mu_{-a,a,b}$, thus $(-a,-b)\oplus(a,b)=((-b)\circ\mu_{-a,a,b},0)=$ $(0,0)$ and $(a,b)\oplus(-a,-b)=(b\circ\mu_{a,-a,-b},0)=(0,0)$. **ii.** By **(M)**, there exist $u,v\in G$ and $h\in H$ with $h\circ\mu_{u,v,-h}\neq 0$. Hence we get $(u,h)\oplus\big((v,-h)\oplus(-v,h)\big)\overset{(i)}{=}(u,h)\oplus(0,0)=(u,h)\neq$ $\big((u,h)\oplus(v,-h)\big)\oplus(-v,h)=(u+v+h\circ\mu_{u,v,-h},0)\oplus(-v,h)\overset{(M2)}{=}(u+h\circ\mu_{u,v,-h},h)$.

(4.5) For $G'=G\times\{0\}$ and $H'=\{0\}\times H$, $(G',\oplus)$ and $(H',\oplus)$ are commutative subgroups of $(L,\oplus)$ with $L=G'\oplus H'$. $(G',\oplus)$ is isomorphic to $(G,+)$ and $(H',\oplus)$ is isomorphic to $(H,+)$.

Proof. Let $(a,0),(c,0)\in G\times\{0\}$. Then $(a,0)\oplus(c,0)=(a+c+0\circ\mu_{a,c,0},0)\overset{(M2)}{=}(a+c,0)$. Hence $G\times\{0\}$ is closed under $\oplus$ and $f\colon G\times\{0\}\to G$ is an isomorphism of $(G\times\{0\},\oplus)$ onto $(G,+)$ and $(G\times\{0\},\oplus)$ is a subgroup of $(L,\oplus)$. For $(0,b),(0,d)\in\{0\}\times H$, it holds $(0,b)\oplus(0,d)=(b\circ\mu_{0,0,d},b+d)\overset{(M2)}{=}(0,b+d)\in\{0\}\times H$. Just as $(\{0\}\times H,\oplus)$ is isomorphic to $(H,+)$. Furthermore for $(a,b)\in L$ it holds $(a,0)\oplus(0,b)=(a+0\circ\mu_{a,0,b},b)\overset{(M2)}{=}(a,b)\in G'\oplus H'$.

(4.6) Let **(MI)** be valid. Then $(L,\oplus)$ fulfills **(K5)** if and only if **(M5)** is fulfilled.

Proof. By **(4.4.i)**, $\ominus\big((a,b)\oplus(c,d)\big)=\big(-a-c-b\circ\mu_{a,c,d},-(b+d)\big)\overset{!}{=}\big(-a-c+(-b)\circ\mu_{-a,-c,-d},-b-d\big)=$ $(-a,-b)\oplus(-c,-d)=\big(\ominus(a,b)\big)\oplus\big(\ominus(c,d)\big)$, if and only if **(M5)** $-b\circ\mu_{a,c,d}=(-b)\circ\mu_{-a,-c,-d}$ is valid.

Now we assume additionaly the properties **(M1ℓ)** and **(M3)**. For $\mathfrak{a}=(a,b), \mathfrak{b}=(c,d)\in L$ let

$$\delta_{\mathfrak{a},\mathfrak{b}}: L \to L\ ;\ (x,y) \to (x+b\circ\mu_{a,x,y}+d\circ\mu_{c,x,y}-(b+d)\circ\mu_{a+c,x,y}, y)$$

(4.7) For all $\mathfrak{a},\mathfrak{b},\mathfrak{x}\in L$, $\mathfrak{a}\oplus(\mathfrak{b}\oplus\mathfrak{x})=(\mathfrak{a}\oplus\mathfrak{b})\oplus\delta_{\mathfrak{a},\mathfrak{b}}(\mathfrak{x})$

Proof. For $\mathfrak{a}=(a,b)$, $\mathfrak{b}=(c,d)$, $\mathfrak{x}=(x,y)$, $\mathfrak{a}\oplus(\mathfrak{b}\oplus\mathfrak{x})\overset{(M1r)}{=}(a+c+x+d\circ\mu_{c,x,y}+b\circ\mu_{a,c+x,d+y}, b+d+y)$ $\overset{(M3)}{=}\big(a+c+x+d\circ\mu_{c,x,y}+b\circ\mu_{a,c,d}+b\circ\mu_{a,x,y}-(b+d)\circ\mu_{a+c,x,y}+(b+d)\circ\mu_{a+c,x,y}, b+d+y\big)\overset{(M1\ell)}{=}$ $=(\mathfrak{a}\oplus\mathfrak{b})\oplus\delta_{\mathfrak{a},\mathfrak{b}}(\mathfrak{x})$

(4.8) Theorem. Let **(M1r)**,**(M1ℓ)**, **(M2)** and **(M3)** be fulfilled.

i) Then $(L,\oplus)$ is a WK-loop.

ii) $(L,\oplus)$ fulfills **(KB)** if and only if **(MB)** is fulfilled.

iii) Let **(MI)** be valid. Then $(L,\oplus)$ fulfills **(K4)** if and only if **(M4)** is fulfilled.

iv) Let **(MI)** be valid. $(L,\oplus)$ is a Bruck loop if and only if **(M5)** and **(MB)** are fulfilled.

Proof. i. By **(4.3)**, $(L,\oplus)$ is a loop. For all $\mathfrak{a},\mathfrak{b}\in L$, the map $\delta_{\mathfrak{a},\mathfrak{b}}$ is an automorphism of $(L,\oplus)$, since for $(x,y),(z,w)\in L$, $\delta_{\mathfrak{a},\mathfrak{b}}\big((x,y)\oplus(z,w)\big) = \delta_{\mathfrak{a},\mathfrak{b}}\big((x+z+y\circ\mu_{x,z,w}, y+w)\big)\overset{(M1r)}{=}$ $\big(x+z+y\circ\mu_{x,z,w}+b\circ\mu_{a,x+z,y+w}+d\circ\mu_{c,x+z,y+w}-(b+d)\circ\mu_{a+c,x+z,y+w}, y+w\big)\overset{(M3)}{=}$ $\big(x+z+y\circ\mu_{x,z,w}+b\circ\mu_{a,x,y}+b\circ\mu_{a,z,w}+d\circ\mu_{c,x,y}+d\circ\mu_{c,z,w}-(b+d)\circ\mu_{a+c,x,y}-(b+d)\circ\mu_{a+c,z,w}, y+w\big)$ $\overset{(M1r,\ell)}{=}\delta_{\mathfrak{a},\mathfrak{b}}\big((x,y)\big)\oplus\delta_{\mathfrak{a},\mathfrak{b}}\big((z,w)\big)$.

ii. We compute $(a,b)\oplus[(c,d)\oplus((a,b)\oplus(x,y))]\overset{(M1r)}{=}\big(a+c+a+x+b\circ\mu_{a,x,y}+d\circ\mu_{c,a+x,b+y}+$ $= b\circ\mu_{a,c+a+x,d+b+y}, b+d+b+y\big)\overset{!}{=}[(a,b)\oplus((c,d)\oplus(a,b))]\oplus(x,y)\overset{(M1r,\ell)}{=}\big(a+c+a+x+d\circ\mu_{c,a,b}+$ $b\circ\mu_{a,c+a,d+b}+(b+d+b)\circ\mu_{a+c+a,x,y}, b+d+b+y\big)$ if and only if $b\circ\mu_{a,x,y}+d\circ\mu_{c,a+x,b+y}+$ $b\circ\mu_{a,c+a+x,d+b+y}\overset{(M3)}{=}b\circ\mu_{a,x,y}+d\circ\mu_{c,a,b}+d\circ\mu_{c,x,y}+b\circ\mu_{a,c+a,d+b}+b\circ\mu_{a,x,y}\overset{!}{=}$ $= d\circ\mu_{c,a,b}+b\circ\mu_{a,c+a,d+b}+(b+d+b)\circ\mu_{a+c+a,x,y}$. This is correct, i.e. **(KB)** is correct, if and only if **(MB)** $(b+d+b)\circ\mu_{a+c+a,x,y}=b\circ\mu_{a,x,y}+d\circ\mu_{c,x,y}+b\circ\mu_{a,x,y}$ is fulfilled.

iii. Since $\ominus\mathfrak{a}=\ominus(a,b)=(-a,-b)$ by **(4.4.i)**, we get $\delta_{\mathfrak{a},\ominus\mathfrak{a}}\big((x,y)\big)=\big(x+b\circ\mu_{a,x,y}+(-b)\circ\mu_{-a,x,y}-$ $(b-b)\circ\mu_{a-a,x,y}, y\big)\overset{(M2)}{=}\big(x+b\circ\mu_{a,x,y}+(-b)\circ\mu_{-a,x,y}, y\big)\overset{!}{=}(x,y)$, if and only if $-b\circ\mu_{a,x,y}=$ $(-b)\circ\mu_{-a,x,y}$.

iv) follows by **(4.6)** and **ii)**.

(4.9) Let **(M1r)**, **(M1ℓ)**, **(MI)**, **(M2)**, **(M3)**, **(M5)** and **(MB)** be fulfilled. Then $(L,\oplus)$ is a K-loop.

Proof. Since **(4.6)** and **(4.7.i)**, we have only to prove that **(K6)**, $\delta_{\mathfrak{a},\mathfrak{b}\oplus\mathfrak{a}}=\delta_{\mathfrak{a},\mathfrak{b}}$ is fulfilled. We compute $\delta_{\mathfrak{a},\mathfrak{b}\oplus\mathfrak{a}}\big((x,y)\big) \overset{(M1\ell)}{=} (x+b\circ\mu_{a,x,y}+(d+b)\circ\mu_{c+a,x,y} - (b+d+b)\circ\mu_{a+c+a,x,y}, y) \overset{!}{=} \delta_{\mathfrak{a},\mathfrak{b}}\big((x,y)\big) = (x+b\circ\mu_{a,x,y}+d\circ\mu_{c,x,y}-(b+d)\circ\mu_{a+c,x,y}, y)$, if $(d+b)\circ\mu_{c+a,x,y}+(b+d)\circ\mu_{a+c,x,y}= 2\cdot\big((b+d)\circ\mu_{a+c,x,y}\big) \overset{!}{=} d\circ\mu_{c,x,y} + (b+d+b)\circ\mu_{a+c+a,x,y} \overset{(MB)}{=} 2\cdot(b\circ\mu_{a,x,y}) + 2\cdot(d\circ\mu_{c,x,y})$. Since **(4.1.ii)**, both sides of the equation are 0, hence **(K6)** is fulfilled.

5. EXAMPLES

Using the Theorems **(4.2)** and **(4.8)**, we give examples of loops with additionally properties. If the map $\mu: G\times G\times H \to U$, $(a,c,d) \to \mu_{a,c,d}$ of section 4 depends only on $a,c\in G$ and not on $d\in H$, i.e. $\mu_{a,c,d}=\mu_{a,c,y}$ for every $y\in H$, then we denote these map with

$$\lambda:\ G\times G \to U;\ (a,c) \to \lambda_{a,c}.$$

Let $(G,+)$ and $(H,+)$ be commutative groups and let always T be a subgroup of G with index 2, i.e. for every $a\in G$ it follows $a+a\in G$. Let $m\in T$ be an element of order 2. Then $U:=\{0,m\}$ is a subgroup of G of order 2 with $U\subset T$.

Let always $L:=G\times H$ and for $(a,b),(c,d)\in L$: $(a,b)\oplus(c,d) := (a+c+b\circ\lambda_{a,c}, b+d)$.

(5.1) Example of a right loop with **(I)** and **(K5)**

We define

$$\lambda:\ G\times G \to \{0,m\}\ ;\ (a,c)\to\lambda_{a,c} := \begin{cases} m & \text{for } a=0 \text{ and } c\notin T \\ 0 & \text{otherwise} \end{cases}$$

Let V be a subgroup of H with index 2, i.e. for every $b\in H$ it follows $b+b\in V$.

$$\circ:\ H\times\{0,m\} \to \{0,m\}\ ;\ (b,\lambda_{a,c})\to b\circ\lambda_{a,c} := \begin{cases} \lambda_{a,c} & \text{for } b\notin V \\ 0 & \text{for } b\in V \end{cases}$$

Then **(M)**,**(M1r)**, **(M2)**,**(MI)**, and **(M5)** are fulfilled, but not **(M1ℓ)**, i.e. $(L,\oplus)$ is a non-associative right loop with **(I)**, and **(K5)** which is not a loop..

Proof. (M): Let $h\in H\setminus V$ and $u\in G\setminus T$, then $h\circ\lambda_{0,u}=m\neq 0$. **(M1r)** : Since U is a group, we get for $t\in U\subset T$, $c+t\in T$ iff $c\in T$ and $\lambda_{a,c}=\lambda_{a,c+t}$. Not **(M1ℓ)**: For $c\in G\setminus T$ and $b\in H\setminus V$, $b\circ\lambda_{0,c}=m\neq 0=b\circ\lambda_{m,c}$. **(M2)**: Since $0\in T$ and $0\in V$, $b\circ\lambda_{a,0}=b\circ 0=0=0\circ\lambda_{a,c}$. **(MI)**: If $a=0\in G\setminus T$, $\lambda_{0,0}=0$, and if $a\neq 0$, also $\lambda_{a,-a}=0$, hence $b\circ\lambda_{a,-a}=0$. **(M5)**: Since T and $\{0\}$ are subgroups, obviously $\lambda_{a,c}=\lambda_{-a,-c}$. Because $b\in V$ iff $-b\in V$ and $-m=m$, it follows $-b\cdot\lambda_{a,c}=b\circ\lambda_{a,c}=(-b)\circ\lambda_{a,c}=(-b)\circ\lambda_{-a,-c}$.

Hence by **(4.2)** and **(4.4.ii)**, $(L,\oplus)$ is a right loop with **(I)** and **(K5)**. Now for $a \in G \setminus T$ and $b \in H \setminus V$ we consider $(x,y)\oplus(a,0)=(a,b)$, hence necessarily $y=b$ and $x+a+b\circ\lambda_{x,a}=a$. If $x=0$, then $0+a+b\circ\lambda_{0,a}=a+m\neq a$, and if $x\neq 0$, then $\lambda_{x,a}=0$, hence $x+a+b\circ\lambda_{x,a}=x+a\neq a$. Therefore $(x,y)\oplus(a,0)=(a,b)$ has no solution.

(5.2) Example of a WK-loop with **(I)**, **(K4)**, and **(K5)**.

We define

$$\lambda : G \times G \to \{0,m\} \; ; \; (a,c) \to \lambda_{a,c} := \begin{cases} m & \text{for } a\in T \text{ and } c \notin T \\ 0 & \text{otherwise} \end{cases}$$

Let $V \subsetneqq H$ be an arbitrary subset with $0 \in V$ and $-V = V$. We define

$$\circ : H \times \{0,m\} \to \{0,m\} \; ; \; (b,\lambda_{a,c}) \to b\circ\lambda_{a,c} := \begin{cases} \lambda_{a,c} & \text{for } b \notin V \\ 0 & \text{for } b \in V \end{cases}$$

Then **(M)**,**(M1r)**,**(M1ℓ)**,**(MI)**,**(M2)**,**(M3)**,**(M4)** and **(M5)** are fulfilled, i.e. $(L,\oplus)$ is a non-associative WK-loop with **(I)**, **(K4)**, and **(K5)** .

i) If there is an element $b\in H$ with $b+b\notin V$, then **(MB)** is not fulfilled.

ii) If V is subgroup of H with index 2, i.e. for every $b \in H$ it holds $b+b \in V$, then **(MB)** is fulfilled.

Proof. (M): Let $b\in H\setminus V$ and $c \in G\setminus T$, then $b\circ\lambda_{0,c}=m\neq 0$. **(M1r)** and **(M1ℓ)** : Since U is a group, we get for $t \in U \subset T$ $a+t\in T$ iff $a\in T$, and $c\notin T$ iff $c+t\notin T$, hence $\lambda_{a+t,c}=\lambda_{a,c}=\lambda_{a,c+t}$. **(M2)**: Since $0\in T$ and $0\in V$, $b\circ\lambda_{a,0}=b\circ 0=0=0\circ\lambda_{a,c}$. **(MI)**: Clearly $a\in T$ iff $-a\in T$, hence $\lambda_{a,-a}=0$ and $b\circ\lambda_{a,-a}=0$. **(M3)**: We have to show $\lambda_{a,c+x}=\lambda_{a,c}+\lambda_{a,x}$. Let $a\in T$. If $c,x\in T$, then $a+c\in T$, hence $0=\lambda_{a,c+x}=\lambda_{a,c}+\lambda_{a,x}=0+0$. If $c\in T$ and $x\notin T$ or $c\notin T$ and $x\in T$, respectively, then $c+x\notin T$, hence $m=\lambda_{a,c+x}=\lambda_{a,c}+\lambda_{a,x}=m$. If $c,x\notin T$, then $c+x\in T$, since T has index 2. Thus $0=\lambda_{a,c+x}=\lambda_{a,c}+\lambda_{a,x}=m+m=0$. **(M4)** and **(M5)**: Since T is a group, obviously $\lambda_{a,c}= =\lambda_{-a,-c}=\lambda_{-a,c}$. Because $b\in V$ iff $-b\in V$ and $-m=m$, it follows $-b\cdot\lambda_{a,c}=b\circ\lambda_{a,c}=(-b)\circ\lambda_{a,c}=(-b)\circ\lambda_{-a,c}=(-b)\circ\lambda_{-a,-c}$. Hence by **(4.4.ii)** and **(4.8)**, $(L,\oplus)$ is a WK-loop with **(I)** **(K4)**, and **(K5)**.

(MB): Since T is a subgroup with index 2, $a+c+a\in T$ iff $c\in T$. Hence **(MB)** is equivalent to $(b+d+b)\circ\lambda_{c,x}=d\circ\lambda_{c,x}$, since $\operatorname{ord} m=2$. Let $b\in H$ with $b+b\notin V$ and set $d=0$, $c=0$, $x\in G\setminus T$, then $(b+d+b)\circ\lambda_{c,x}=(b+b)\circ\lambda_{0,x}=m\neq 0=0\circ\lambda_{c,x}$, i.e., **(MB)** is not valid. If V is a subgroup with $b+b\in V$ for every $b\in H$, then $b+d+b\in V$ iff $d\in V$, hence $(b+d+b)\circ\lambda_{c,x}=d\circ\lambda_{c,x}$ and **(MB)** is fulfilled.

(5.3) Corollary. There exist WK-loops fulfilling **(K4)** and **(K5)**, but not **(K6)**.
For $n,k\in\mathbb{N}$ with $k\geq 3$ set for example $G:=\mathbb{Z}_{4n}$, $T:=2\mathbb{Z}_{4n}$, $U:=\{0,2n\}$ and set $H:=\mathbb{Z}_k$, $V=\{0\}$.

Proof. Obviously T is a subgroup of G with index 2, ord U = 2 and V = -V. For $1\in H$, clearly $1+1=2\notin V$. Hence by **(5.2.i)** the assertion follows.

(5.4) Corollary. Let $n,k \in \mathbb{N}$ and let $G:=\mathbb{Z}_{4n}$, $T:=2\mathbb{Z}_{4n}$, $U:=\{0,2n\}$, $H:=\mathbb{Z}_{2k}$, $V=2\mathbb{Z}_{2k}$. Then $(L,\oplus)$ is a K-loop and a Bruck loop of order 8nk.

Proof. Obviously T is a subgroup of G with index 2, ord U = 2 and V is a subgroup with index 2. Hence by **(5.2.ii)** the assertion follows.

Now we consider the situation that the map $\mu: G\times G\times H \to U;\ (a,c,d)\to \mu_{a,c,d}$ of section 4 depends on $d\in H$.

(5.5) Example of a K- loop and a Bruck loop.

Let V be a subgroup of H with index 2. We define the following maps

$$\mu: G\times G\times H \to \{0,m\}\ ;\ (a,c,d) \to \mu_{a,c,d} := \begin{cases} m & \text{for } d\in V \text{ and } c\notin T \\ m & \text{for } d\notin V \text{ and } a+c\notin T \\ 0 & \text{otherwise} \end{cases}$$

$$\circ: H\times \{0,m\} \to \{0,m\}\ ;\ (b,\mu_{a,c,d}) \to b\circ\mu_{a,c,d} := \begin{cases} \mu_{a,c,d} & \text{for } b\notin V \\ 0 & \text{for } b\in V \end{cases}$$

Then $(L,\oplus)$ is a non-associative Bruck and K-loop.

Proof. (M): Let $h\in H\setminus V$ and $c\in G\setminus T$, then $h\circ\mu_{0,c,-h}=m\neq 0$. **(M1r)** and **(M1ℓ)**: For $t\in U\subset T$, $a\in T$ iff $a+t\in T$, $c\in T$ iff $c+t\in T$ and $a+c\in T$ iff $a+c+t\in T$. Hence $\mu_{a+t,c,d}=\mu_{a,c,d}=\mu_{a,c+t,d}$. **(M2)**: Since $0\in T$ and $0\in V$, $b\circ\mu_{a,0,0}=b\circ 0=0=b\circ\mu_{0,0,d}=0\circ\mu_{a,c,d}$. **(MI)**: For $a\in G$ we get $0=a-a\in T$, hence $\mu_{a,-a,-b}=0$ if $b\notin V$ and $b\circ\mu_{a,-a,-b}=0$, if $b\in V$.
(M3): Let $d,y\in V$. Since ord $m=2$, $\mu_{a,c,d}+\mu_{a,x,y}=m$, iff $c\in T$ and $x\notin T$ or $c\notin T$ and $x\in T$, i.e. iff $c+x\notin T$ and $\mu_{a,c+x,d+y}=m$. Now let $d\in V$, $y\notin V$. Then $\mu_{a,c+x,d+y}=m$, iff $a+c+x\notin T$, i.e. either $c\notin T$ or $a+x\notin T$. On the other hand $\mu_{a,c,d}=m$ iff $c\notin T$, and $\mu_{a,x,y}=m$ iff $a+x\notin T$, thus $\mu_{a,c,d}+\mu_{a,x,y}=m$, iff $a+c+x\notin T$. In the same way we get $\mu_{a,c,d}+\mu_{a,x,y}=\mu_{a,c+x,d+y}$ for $d\notin V$ and $y\in V$. The last case is $d,y\notin V$. Then $\mu_{a,c+x,d+y}=m$, iff $c+x\notin T$, i.e. iff $2a+c+x\notin T$. On the other hand $\mu_{a,c,d}+\mu_{a,x,y}=m$, iff either $a+c\notin T$ or $a+x\notin T$, i.e. $2a+c+x\notin T$. Hence **(M3)** is valid.
(M4) and **(M5)**: Since T and V are groups, $\mu_{a,c,d}=\mu_{-a,c,d}=\mu_{-a,-c,-d}$. Because ord $m=2$ and V is a group, it follows $-b\circ\mu_{a,c,d}=(-b)\circ\mu_{-a,c,d}=(-b)\circ\mu_{-a,-c,-d}$. **(MB)**: Since T and V are subgroups with index 2, $(b+d+b)\in V$ iff $d\in V$, and $a+c+a\in T$ iff $c\in T$, hence $(b+d+b)\circ\mu_{a+c+a,x,y}=d\circ\mu_{c,x,y}$. On the other hand $d\circ\mu_{c,x,y}+b\circ\mu_{a,x,y}+d\circ\mu_{c,x,y}=b\circ\mu_{a,x,y}$, because ord $m=2$.

(5.6) Corollary. Let $n, K \in \mathbb{N}$ and let $G := \mathbb{Z}_{4n}$, $T := 2\mathbb{Z}_{4n}$, $U := \{0, 2n\}$, $H := \mathbb{Z}_{2k}$, $V = 2\mathbb{Z}_{2k}$. Then $(L, \oplus)$ is a K-loop and a Bruck loop of order 8nk.

Remark 9. In [24, **(4.7)**] it is shown that for $k, n \in \mathbb{N}$ with $k \neq 2n$ and for $G = \mathbb{Z}_{4n}$, $H = \mathbb{Z}_{2k}$, the K-loops of example **(5.2.ii)** and **(5.5)** are non-isomorphic.

6. K-LOOPS OF ORDER 8

The idea of the following example **(6.1)** is due to H. Zassenhaus and can be found by G. Bol in [1, p. 430]. For the case $p=2$, this example is mentioned by D. A. Robinson [26, Example 2.2, p. 346] (cf. [21, **(1.7)**]). Also T. Kepka proves in [12] in a general context that the following construction defines a Bruck loop.

(6.1) Example. For a prime p we consider the finite ring $(\mathbb{Z}_p, +, \cdot)$. Let $L := \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. Then for

$$\oplus : L \times L \to L,\ (a_1, a_2, a_3) \oplus (b_1, b_2, b_3) := \big(a_1 + b_1, a_2 + b_2, a_3 + b_3 + a_1 b_2 (a_2 + b_2)\big).$$

$(L, \oplus)$ is a Bruck and K-loop of order p^3 in which every element has order p.

For $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$, $x = (x_1, x_2, x_3) \in L$, $\delta_{a,b}(x) = \big(x_1, x_2, x_3 + (a_1 b_2 - a_2 b_1) x_2\big)$.

Proof. Since by [12], $(L, \oplus)$ is a Bruck loop, i.e. in particular the Bol identity is valid, by **(2.12)** we have only to prove **(K3)** and **(K4)**. The third coordinate of $(a \oplus b) \oplus \delta_{a,b}(x)$ is $a_3 + b_3 + x_3 + a_1 b_2 (a_2 + b_2) + (a_1 + b_1) x_2 (a_2 + b_2 + x_2) + (a_1 b_2 - a_2 b_1) x_2 = a_3 + b_3 + x_3 + a_1 b_2 (a_2 + b_2) + a_1 x_2 (a_2 + b_2 + x_2) + b_1 x_2 (b_2 + x_2) + a_1 b_2 x_2$, and the third coordinate of $a \oplus (b \oplus x)$ is $a_3 + b_3 + x_3 + b_1 x_2 (b_2 + x_2) + a_1 (b_2 + x_2)(a_2 + b_2 + x_2) = a_3 + b_3 + x_3 + b_1 x_2 (b_2 + x_2) + a_1 x_2 (a_2 + b_2 + x_2) + a_1 b_2 (a_2 + b_2) + a_1 b_2 x_2 = (a \oplus b) \oplus \delta_{a,b}(x)$. Obviously we have $\delta_{a,b}(x \oplus y) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 y_2 (x_2 + y_2) + (a_1 b_2 - a_2 b_1)(x_2 + y_2) = \delta_{a,b}(x) \oplus \delta_{a,b}(y)$. For $b = (-a_1, -a_2, -a_3)$ we have $a \oplus b = (a_1, a_2, a_3) \oplus (-a_1, -a_2, -a_3) = 0$ and $\delta_{a,b}(x) = (x_1, x_2, x_3)$, i.e. **(K4)** holds.

(6.2) There are at most three non- isomorphic non-associative K-loops of order 8.

Proof. Since every K-loop is a Bol loop, we can use a result of R. P. Burn, who shows in [4] that there are exactly six non-isomorphic non-associative Bol loops of order 8. (Burn uses the dual Bol-identity $\big((a+b)+c\big)+b = a+\big((b+c)+b\big)$.) In [4, p. 382] he gives representations $\Pi_1, \ldots, \Pi_6$ of these Bol loops. We show now that Π_1, Π_4, and Π_5 do not fulfil **(K5)**, hence there remains at most three non-associative K-loops of order 8. Burn gives a representation of a Bool loop $B = \{R_1, R_2, \ldots, R_8\}$ as a subset of the symmetric group $(S_8, \circ)$. With R_i he denotes the permutation mapping 1 to i and

$R_i \circ R_j = R_{R_j(i)} \in B$. In all his representations Π_1, Π_4, Π_5 it holds $R_2 = (1234)(5678)$ and $R_2^{-1} = R_4 = (1432)(5876)$.

In Π_1: $R_5 = (1537)(2648)$, $R_5^{-1} = R_7 = (1735)(2846)$ $R_6 = (1638)(2547)$ and $R_6^{-1} = R_8 = (1836)(2745)$. Hence $(R_2 \circ R_5)^{-1} = R_6^{-1} = R_8 \neq R_2^{-1} \circ R_5^{-1} = R_4 \circ R_7 = R_6$.

In Π_4: $R_5 = (15)(26)(37)(48) = R_5^{-1}$, $R_6 = (16)(27)(38)(45) = R_6^{-1}$ and $R_8 = (18)(25)(36)(47)$. Hence $(R_2 \circ R_5)^{-1} = R_6^{-1} = R_6 \neq R_2^{-1} \circ R_5^{-1} = R_4 \circ R_5 = R_8$.

In Π_5: $R_5 = (15)(26)(37)(48) = R_5^{-1}$, $R_6 = (16)(25)(38)(47) = R_6^{-1}$ and $R_8 = (18)(27)(36)(45)$. Hence $(R_2 \circ R_5)^{-1} = R_6^{-1} = R_6 \neq R_2^{-1} \circ R_5^{-1} = R_4 \circ R_5 = R_8$.

(6.3) Theorem. There exists exactly three non-associative non-isomorphic K-loops of order 8.

Proof. By **(6.2)** there are at most three non-associative K-loops of order 8. Now we give three non-isomorphic examples. For $p=2$ in **(6.1)** a K-loop L_1 of order 8 is given in which every element has order 2. For $n=k=1$, in **(5.4)** and in **(5.6)** K-loops L_2 and L_3 are constructed, both having elements with order 4, hence L_2, L_3 are not isomorphic to L_1. Further in [24, **(4.7)**] it is shown, that L_2 and L_3 are not isomorphic (cf. Remark **9** in section 5.).

Remark 10. For $p=2$, the example **(6.1)** is isomorphic to the Bruck loop Π_6 of Burn [4, p. 382], for $n=k=1$, example **(5.4)** is isomorphic to Π_2, and example **(5.6)** is isomorphic to Π_3 of Burn.

7. AUTOMORPHISMS OF K-LOOPS

Let $(L,\oplus)$ be a right loop fulfilling **(K3)** and **(K4)**, and let $D := \langle\{\delta_{a,b} \in \mathrm{Aut}(L,\oplus) : a,b \in L\}\rangle$ its so called structure group. With the operation $\cdot$ of example **(3.5)**, the quasidirect product $G := L \times D$ is a group and the connection between $\mathrm{Aut}(L,\oplus)$ and $\mathrm{Aut}(G,\cdot)$ is as follows:

(7.1) Let $(L,\oplus)$ be a right loop with **(K3)** and **(K4)**. Then each automorphism $\alpha \in \mathrm{Aut}(L,\oplus)$ can be extended to an automorphism $\overline{\alpha}$ of the involved quasidirect product $G = L \times D$ with $\overline{\alpha}((k,\tau)) = (\alpha(k), \alpha\tau\alpha^{-1})$ for $(k,\tau) \in G$, $\overline{\alpha}(L \times \{id\}) = L \times \{id\}$ and $\overline{\alpha}(\{0\} \times D) = \{0\} \times D$.

Vice versa any automorphism $\overline{\gamma} \in \mathrm{Aut}(G,\cdot)$ with the properties $\overline{\gamma}(L \times \{id\}) = L \times \{id\}$ and $\overline{\gamma}(\{0\} \times D) = \{0\} \times D$ implies an automorphism γ of $(L,\oplus)$ by simply setting $\gamma(a) = b$ for $\overline{\gamma}((a \times \{id\})) = b \times \{id\}$.

Proof. For $\alpha \in \mathrm{Aut}(L,\oplus)$ we have $\alpha\,\delta_{a,b}\,\alpha^{-1} = \delta_{\alpha(a),\alpha(b)}$ by **(2.2)**, hence $\alpha D \alpha^{-1} = D$.

Obviously $\bar\alpha$ is a bijection. Now for $g_i=(k_i,\tau_i)\in G$, $i=1,2$ we compute $\bar\alpha(g_1\cdot g_2)=\bar\alpha\left(\left(k_1\oplus\tau_1(k_2),\delta_{k_1,\tau_1(k_2)}\tau_1\tau_2\right)\right)=\left(\alpha(k_1)\oplus\alpha\tau_1(k_2),\alpha\,\delta_{k_1,\tau_1(k_2)}\alpha^{-1}\alpha\tau_1\alpha^{-1}\alpha\tau_2\alpha^{-1}\right)=\left(\alpha(k_1)\oplus(\alpha\tau_1\alpha^{-1})\alpha(k_2),\delta_{\alpha(k_2),\alpha\tau_1\alpha^{-1}\alpha(k_2)}\alpha\tau_1\alpha^{-1}\alpha\tau_2\alpha^{-1}\right)=\bar\alpha(g_1)\cdot\bar\alpha(g_2)$, i.e. $\bar\alpha$ is an automorphism. Furthermore $\bar\alpha(L\oplus\{id\})=(\alpha(L)\oplus\{id\})=(L\times\{id\})$ and $\bar\alpha(\{0\}\times D)=\{0\}\oplus\alpha D\alpha^{-1}=\{0\}\times D$. Now let $\bar\gamma\in Aut(G,\cdot)$ with the above properties. Then $\bar\gamma((0,\tau))=(0,\tau')$ for a $\tau'\in D$ and $\bar\gamma((k,\tau))=\bar\gamma((k,id)\cdot(0,\tau))=\bar\gamma((k,id))\cdot\bar\gamma((0,\tau))=(\gamma(k),id)\cdot(0,\tau')=(\gamma(k),\tau')$. Hence for $k_1,k_2\in L$, $\left(\gamma(k_1\oplus k_2),\delta'_{k_1,k_2}\right)=\bar\gamma\left(\left(k_1\oplus k_2,\delta_{k_1,k_2}\right)\right)=\bar\gamma((k_1,id)\cdot(k_2,id))=\bar\gamma\left((k_1,id)\right)\cdot\bar\gamma\left((k_2,id)\right)=\left(\gamma(k_1),id\right)\cdot\left(\gamma(k_2),id\right)=\left(\gamma(k_1)\oplus\gamma(k_2),\delta_{\gamma(k_1),\gamma(k_2)}\right)=\left(\gamma(k_1)\oplus\gamma(k_2),\gamma\delta_{k_1,k_2}\gamma^{-1}\right)$, i.e. $\gamma\in Aut(L,\oplus)$.

The group $G=L\times D$ can be considered as a transformation group operating on L in the following way:

$$g=(a,\tau):\begin{cases} L\to L \\ x\to a\oplus\tau(x)\end{cases}\quad\text{for } g\in G.$$

If $(L,\oplus)$ is also a loop, i.e. besides **(K1r)**, **(K2)**, **(K3)** and **(K4)** also **(K1ℓ)** is valid, then:

(7.2) If each $\tau\in D\setminus\{id\}$ has the only one fixed point $0\in L$, then G is a **Frobeniusgroup** , i.e.

-- G acts transitively on L ,

-- $G_x:=\{g\in G:\ g(x)=x\}\neq\{id\}$ and

-- $G_x\cap G_y=\{id\}$ for $x\neq y$ and $x,y\in L$.

Proof. $(G,\circ)$ is a group: Let $(a,\tau),(b,\sigma)\in T$, then $(a,\tau)\circ(b,\sigma)=(a\oplus\tau(b),\delta_{a,\tau(b)}\tau\sigma)\in G$, since $(a,\tau)\circ(b,\sigma): x\to a\oplus\tau(b\oplus\sigma(x))=(a\oplus\tau(b))\oplus\delta_{a,\tau(b)}\tau\sigma(x)$. Notice $(a,\tau)^{-1}=(\tau^{-1}(\ominus a),\tau^{-1})\in G$. G acts transitively on L, since for given $x,y\in L$ and $\tau\in D$ by **(K1ℓ)** there exists an element $a\in L$ with $a\oplus\tau(x)=y$, i.e. $(a,\tau)\in G$ maps x onto y. Hence for $\tau\neq id$ and $x=y$, there is an element $(a,\tau)\in G_x\setminus\{id\}$. Since G acts transitively, G_x and G_0 are isomorphic. Therefore an automorphism $g\in G_x\setminus\{id\}$ possess a fixed point $y\neq x$ if and only if there is an $h=(0,\tau)\in G_0\setminus\{id\}$ with a fixed point $z\neq 0$, i.e. $\tau(z)=z$. Hence if $\tau\in D\setminus\{id\}$ has no fixed point except 0, then $G_x\cap G_y=\{id\}$ for $x\neq y$.

(7.3) Theorem. Let $(L,\oplus)$ be a finite loop with **(K3)** and **(K4)** and let $D=\langle\{\delta_{a,b}:a,b\in L\}\rangle$ be its stucture group. If $D\setminus\{id\}$ operates fixed point free on $L\setminus\{0\}$, then $(L,\oplus)$ is a group. Or with other words: If $(L,\oplus)$ is a proper loop then there exists a $\tau\in D$ with at least two fixed points.

Proof. Assume that D operates fixed point free on $L\setminus\{0\}$, i.e. $\tau(x)=x$ implies $x=0\in L$ for any $\tau\in D\setminus\{id\}$. Then $G=L\times D$ is a Frobeniusgroup operating on L.
Since $(L,\oplus)$ is a loop, the subset $\bar L:=L\times\{0\}\subset G$ except (0,id) operates transitively and fixed point free on L. Now the Theorem of Frobenius says that in a finite Frobenius-

group G the set T of fixed point free transformations together with {id} form a normal subgroup of G which operates regularly. Let $T \trianglelefteq G$ be this normal subgroup. Then $\overline{L} \subset T$. Since both sets $\overline{L}$ and T operate regularly on L, the finiteness of G implies $\overline{L} = T$. Thus $\overline{L}$ is a group, i.e. $(a,id)\cdot(b,id)=(a\oplus b,\delta_{a,b})\in \overline{L} = L\times\{id\}$, hence $\delta_{a,b}=id$ for any $a,b\in L$ and $(L,\oplus)$ is a group.

Remark 11. To our knowledge it seems rather difficult to prove even partial results in the infinite case (cf. [13 to 17]).

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