CHAPTER 7

# ORDINARY DIFFERENTIAL EQUATIONS

Much of theoretical physics is originally formulated in terms of differential equations in the three-dimensional physical space (and sometimes also time). These variables (e.g., $x$, $y$, $z$, $t$) are usually referred to as **independent variables**, while the function or functions being differentiated are referred to as **dependent variable(s)**. A differential equation involving more than one independent variable is called a **partial differential equation**, often abbreviated **PDE**. The simpler situation considered in the present chapter is that of an equation in a single independent variable, known as an **ordinary differential equation**, abbreviated **ODE**. As we shall see in a later chapter, some of the most frequently used methods for solving PDEs involve their expression in terms of the solutions to ODEs, so it is appropriate to begin our study of differential equations with ODEs.

## 7.1 INTRODUCTION

To start, we note that the taking of a derivative is a **linear operation**, meaning that

$$\frac{d}{dx}\big(a\varphi(x)+b\psi(x)\big)=a\frac{d\varphi}{dx}+b\frac{d\psi}{dx},$$

and the derivative operation can be viewed as defining a linear operator: $\mathcal{L}=d/dx$. Higher derivatives are also linear operators, as for example

$$\frac{d^2}{dx^2}\big(a\varphi(x)+b\psi(x)\big)=a\frac{d^2\varphi}{dx^2}+b\frac{d^2\psi}{dx^2}.$$



**Mathematical Methods for Physicists.** DOI: 10.1016/B978-0-12-384654-9.00007-4

Note that the linearity under discussion is that of the **operator**. For example, if we define

$$\mathcal{L} = p(x)\frac{d}{dx} + q(x),$$

it is identified as **linear** because

$$\begin{aligned}\mathcal{L}\big(a\varphi(x) + b\psi(x)\big) &= a\left(p(x)\frac{d\varphi}{dx} + q(x)\varphi\right) + b\left(p(x)\frac{d\psi}{dx} + q(x)\psi\right)\\ &= a\mathcal{L}\varphi + b\mathcal{L}\psi.\end{aligned}$$

We see that the linearity of $\mathcal{L}$ imposes no requirement that either $p(x)$ or $q(x)$ be a linear function of $x$. Linear differential operators therefore include those of the form

$$\mathcal{L} \equiv \sum_{\nu=0}^{n} p_\nu(x)\left(\frac{d^\nu}{dx^\nu}\right),$$

where the functions $p_\nu(x)$ are arbitrary.

An ODE is termed **homogeneous** if the dependent variable (here $\varphi$) occurs to the same power in all its terms, and **inhomogeneous** otherwise; it is termed **linear** if it can be written in the form

$$\mathcal{L}\varphi(x) = F(x), \tag{7.1}$$

where $\mathcal{L}$ is a linear differential operator and $F(x)$ is an algebraic function of $x$ (i.e., not a differential operator). An important class of ODEs are those that are both linear and homogeneous, and thereby of the form $\mathcal{L}\varphi = 0$.

The solutions to ODEs are in general not unique, and if there are multiple solutions it is useful to identify those that are linearly independent (**linear dependence** is discussed in Section 2.1). Homogeneous linear ODEs have the general property that any multiple of a solution is also a solution, and that if there are multiple linearly independent solutions, any linear combination of those solutions will also solve the ODE. This statement is equivalent to noting that if $\mathcal{L}$ is linear, then, for all $a$ and $b$,

$$\mathcal{L}\varphi = 0 \quad \text{and} \quad \mathcal{L}\psi = 0 \quad \longrightarrow \quad \mathcal{L}(a\varphi + b\psi) = 0.$$

The Schrödinger equation of quantum mechanics is a homogeneous linear ODE (or if in more than one dimension, a homogeneous linear PDE), and the property that any linear combination of its solutions is also a solution is the conceptual basis for the well-known **superposition principle** in electrodynamics, wave optics and quantum theory.

Notationally, it is often convenient to use the symbols $x$ and $y$ to refer, respectively, to independent and dependent variables, and a typical linear ODE then takes the form $\mathcal{L}y = F(x)$. It is also customary to use primes to indicate derivatives: $y' \equiv dy/dx$. In terms of this notation, the superposition property of solutions $y_1$ and $y_2$ of a homogeneous linear ODE tells us that the ODE also has as solutions $c_1y_1$, $c_2y_2$, and $c_1y_1 + c_2y_2$, with the $c_i$ arbitrary constants.

Some physically important problems (particularly in fluid mechanics and in chaos theory) give rise to nonlinear differential equations. A well-studied example is the Bernoulli equation

$$y' = p(x)y + q(x)y^n, \quad n \neq 0,\ 1,$$

which cannot be written in terms of a linear operator applied to $y$.

Further terms used to classify ODEs include their **order** (highest derivative appearing therein), and **degree** (power to which the highest derivative appears after the ODE is rationalized if that is necessary). For many applications, the concept of **linearity** is more relevant than that of **degree**.

## 7.2 FIRST-ORDER EQUATIONS

Physics involves some first-order differential equations. For completeness it seems desirable to touch upon them briefly. We consider the general form

$$\frac{dy}{dx} = f(x, y) = -\frac{P(x, y)}{Q(x, y)}. \tag{7.2}$$

While there is no systematic way to solve the most general first-order ODE, there are a number of techniques that are often useful. After reviewing some of these techniques, we proceed to a more detailed treatment of linear first-order ODEs, for which systematic procedures are available.

### Separable Equations

Frequently Eq. (7.2) will have the special form

$$\frac{dy}{dx} = -\frac{P(x)}{Q(y)}. \tag{7.3}$$

Then it may be rewritten as

$$P(x)dx + Q(y)dy = 0.$$

Integrating from $(x_0, y_0)$ to $(x, y)$ yields

$$\int_{x_0}^{x} P(x)dx + \int_{y_0}^{y} Q(y)dy = 0.$$

Since the lower limits, $x_0$ and $y_0$, contribute constants, we may ignore them and simply add a constant of integration. Note that this separation of variables technique does **not** require that the differential equation be linear.

#### *Example 7.2.1* PARACHUTIST

We want to find the velocity of a falling parachutist as a function of time and are particularly interested in the constant limiting velocity, $v_0$, that comes about by air drag, taken to be quadratic, $-bv^2$, and opposing the force of the gravitational attraction, $mg$, of the Earth on the parachutist. We choose a coordinate system in which the positive direction is downward so that the gravitational force is positive. For simplicity we assume that the parachute opens immediately, that is, at time $t = 0$, where $v(t) = 0$, our initial condition. Newton's law applied to the falling parachutist gives

$$m\dot{v} = mg - bv^2, \tag{7.4}$$

where $m$ includes the mass of the parachute.

The terminal velocity, $v_0$, can be found from the equation of motion as $t \to \infty$; when there is no acceleration, $\dot{v} = 0$, and

$$bv_0^2 = mg, \quad \text{or} \quad v_0 = \sqrt{\frac{mg}{b}}.$$

It simplifies further work to rewrite Eq. (7.4) as

$$\frac{m}{b}\dot{v} = v_0^2 - v^2.$$

This equation is separable, and we write it in the form

$$\frac{dv}{v_0^2 - v^2} = \frac{b}{m}dt. \tag{7.5}$$

Using partial fractions to write

$$\frac{1}{v_0^2 - v^2} = \frac{1}{2v_0}\left(\frac{1}{v + v_0} - \frac{1}{v - v_0}\right),$$

it is straightforward to integrate both sides of Eq. (7.5) (the left-hand side from $v = 0$ to $v$, the right-hand side from $t = 0$ to $t$), yielding

$$\frac{1}{2v_0}\ln\frac{v_0 + v}{v_0 - v} = \frac{b}{m}t.$$

Solving for the velocity, we have

$$v = \frac{e^{2t/T} - 1}{e^{2t/T} + 1}v_0 = v_0\frac{\sinh(t/T)}{\cosh(t/T)} = v_0 \tanh\frac{t}{T},$$

where $T = \sqrt{m/gb}$ is the time constant governing the asymptotic approach of the velocity to its limiting value, $v_0$.

Inserting numerical values, $g = 9.8$ m/s$^2$, and taking $b = 700$ kg/m, $m = 70$ kg, gives $v_0 = \sqrt{9.8/10} \approx 1$ m/s $\approx 3.6$ km/h $\approx 2.234$ mi/h, the walking speed of a pedestrian at landing, and $T = \sqrt{m/bg} = 1/\sqrt{10 \cdot 9.8} \approx 0.1$ s. Thus, the constant speed $v_0$ is reached within a second. Finally, because **it is always important to check the solution**, we verify that our solution satisfies the original differential equation:

$$\dot{v} = \frac{\cosh(t/T)}{\cosh(t/T)}\frac{v_0}{T} - \frac{\sinh^2(t/T)}{\cosh^2(t/T)}\frac{v_0}{T} = \frac{v_0}{T} - \frac{v^2}{Tv_0} = g - \frac{b}{m}v^2.$$

A more realistic case, where the parachutist is in free fall with an initial speed $v(0) > 0$ before the parachute opens, is addressed in Exercise 7.2.16. ■

## Exact Differentials

Again we rewrite Eq. (7.2) as

$$P(x, y)dx + Q(x, y)dy = 0. \tag{7.6}$$

This equation is said to be **exact** if we can match the left-hand side of it to a differential $d\varphi$, and thereby reach

$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy = 0. \tag{7.7}$$

Exactness therefore implies that there exists a function $\varphi(x, y)$ such that

$$\frac{\partial\varphi}{\partial x} = P(x, y) \quad \text{and} \quad \frac{\partial\varphi}{\partial y} = Q(x, y), \tag{7.8}$$

because then our ODE corresponds to an instance of Eq. (7.7), and its solution will be $\varphi(x, y) = \text{constant}$.

Before seeking to find a function $\varphi$ satisfying Eq. (7.8), it is useful to determine whether such a function exists. Taking the two formulas from Eq. (7.8), differentiating the first with respect to $y$ and the second with respect to $x$, we find

$$\frac{\partial^2\varphi}{\partial y\partial x} = \frac{\partial P(x, y)}{\partial y} \quad \text{and} \quad \frac{\partial^2\varphi}{\partial x\partial y} = \frac{\partial Q(x, y)}{\partial x},$$

and these are consistent if and only if

$$\frac{\partial P(x, y)}{\partial y} = \frac{\partial Q(x, y)}{\partial x}. \tag{7.9}$$

We therefore conclude that Eq. (7.6) is exact only if Eq. (7.9) is satisfied. Once exactness has been verified, we can integrate Eqs. (7.8) to obtain $\varphi$ and therewith a solution to the ODE.

The solution takes the form

$$\varphi(x, y) = \int_{x_0}^{x} P(x, y)dx + \int_{y_0}^{y} Q(x_0, y)dy = \text{constant}. \tag{7.10}$$

Proof of Eq. (7.10) is left to Exercise 7.2.7.

We note that separability and exactness are independent attributes. All separable ODEs are automatically exact, but not all exact ODEs are separable.

### *Example 7.2.2* A Nonseparable Exact ODE

Consider the ODE

$$y' + \left(1 + \frac{y}{x}\right) = 0.$$

Multiplying by $x\,dx$, this ODE becomes

$$(x + y)dx + x\,dy = 0,$$

which is of the form

$$P(x, y)dx + Q(x, y)dy = 0,$$

with $P(x, y) = x + y$ and $Q(x, y) = x$. The equation is not separable. To check if it is exact, we compute

$$\frac{\partial P}{\partial y} = \frac{\partial(x + y)}{\partial y} = 1, \quad \frac{\partial Q}{\partial x} = \frac{\partial x}{\partial x} = 1.$$

These partial derivatives are equal; the equation is exact, and can be written in the form

$$d\varphi = P\,dx + Q\,dy = 0.$$

The solution to the ODE will be $\varphi = C$, with $\varphi$ computed according to Eq. (7.10):

$$\varphi = \int_{x_0}^{x} (x + y)dx + \int_{y_0}^{y} x_0 dy = \left(\frac{x^2}{2} + xy - \frac{x_0^2}{2} - x_0 y\right) + (x_o y - x_0 y_0)$$

$$= \frac{x^2}{2} + xy + \text{constant terms.}$$

Thus, the solution is

$$\frac{x^2}{2} + xy = C,$$

which if desired can be solved to give $y$ as a function of $x$. We can also check to make sure that our solution actually solves the ODE. ■

It may well turn out that Eq. (7.6) is not exact and that Eq. (7.9) is not satisfied. However, there always exists at least one and perhaps an infinity of **integrating factors** $\alpha(x, y)$ such that

$$\alpha(x, y)P(x, y)dx + \alpha(x, y)Q(x, y)dy = 0$$

is exact. Unfortunately, an integrating factor is not always obvious or easy to find. A systematic way to develop an integrating factor is known only when a first-order ODE is linear; this will be discussed in the subsection on linear first-order ODEs.

## Equations Homogeneous in *x* and *y*

An ODE is said to be homogeneous (of order $n$) in $x$ and $y$ if the combined powers of $x$ and $y$ add to $n$ in all the terms of $P(x, y)$ and $Q(x, y)$ when the ODE is written as in Eq. (7.6). Note that this use of the term "homogeneous" has a different meaning than when it was used to describe a linear ODE as given in Eq. (7.1) with the term $F(x)$ equal to zero, because it now applies to the combined power of $x$ and $y$.

A first-order ODE, which is homogeneous of order $n$ in the present sense (and not necessarily linear), can be made separable by the substitution $y = xv$, with $dy = x\,dv + v\,dx$. This substitution causes the $x$ dependence of all the terms of the equation containing $dv$ to be $x^{n+1}$, with all the terms containing $dx$ having $x$-dependence $x^n$. The variables $x$ and $v$ can then be separated.

### *Example 7.2.3* AN ODE HOMOGENEOUS IN $x$ AND $y$

Consider the ODE

$$(2x + y)dx + x\,dy = 0,$$

which is homogeneous in $x$ and $y$. Making the substitution $y = xv$, with $dy = x\,dv + v\,dx$, the ODE becomes

$$(2v + 2)dx + x\,dv = 0,$$

which is separable, with solution $\ln x + \frac{1}{2}\ln(v + 1) = C$, which is equivalent to $x^2(v + 1) = C$. Forming $y = xv$, the solution can be rearranged into

$$y = \frac{C}{x} - x.$$

■

## Isobaric Equations

A generalization of the preceding subsection is to modify the definition of homogeneity by assigning different weights to $x$ and $y$ (note that corresponding weights must then also be assigned to $dx$ and $dy$). If assigning unit weight to each instance of $x$ or $dx$ and a weight $m$ to each instance of $y$ or $dy$ makes the ODE homogeneous as defined here, then the substitution $y = x^m v$ will make the equation separable. We illustrate with an example.

### *Example 7.2.4* AN ISOBARIC ODE

Here is an isobaric ODE:

$$(x^2 - y)dx + x\,dy = 0.$$

Assigning $x$ weight 1, and $y$ weight $m$, the term $x^2dx$ has weight 3; the other two terms have weight $1 + m$. Setting $3 = 1 + m$, we find that all terms can be assigned equal weight if we take $m = 2$. This means that we should make the substitution y= $x^2v$. Doing so, we get

$$(1 - v)dx + x\,dv = 0,$$

which separates into

$$\frac{dx}{x} + \frac{dv}{v + 1} = 0 \quad \longrightarrow \quad \ln x + \ln(v + 1) = \ln C, \quad \text{or} \quad x(v + 1) = C.$$

From this, we get $v = \dfrac{C}{x} - 1$. Since $y = x^2v$, the ODE has solution $y = Cx - x^2$. ■

## Linear First-Order ODEs

While nonlinear first-order ODEs can often (but not always) be solved using the strategies already presented, the situation is different for the linear first-order ODE because procedures exist for solving the most general equation of this type, which we write in the form

$$\frac{dy}{dx} + p(x)y = q(x). \tag{7.11}$$

If our linear first-order ODE is exact, its solution is straightforward. If it is not exact, we make it exact by introducing an integrating factor $\alpha(x)$, so that the ODE becomes

$$\alpha(x)\frac{dy}{dx} + \alpha(x)p(x)y = \alpha(x)q(x). \tag{7.12}$$

The reason for multiplication by $\alpha(x)$ is to cause the left-hand side of Eq. (7.12) to become a perfect differential, so we require that $\alpha(x)$ be such that

$$\frac{d}{dx}\left[\alpha(x)y\right] = \alpha(x)\frac{dy}{dx} + \alpha(x)p(x)y. \tag{7.13}$$

Expanding the left-hand side of Eq. (7.13), that equation becomes

$$\alpha(x)\frac{dy}{dx} + \frac{d\alpha}{dx}y = \alpha(x)\frac{dy}{dx} + \alpha(x)p(x)y,$$

so $\alpha$ must satisfy

$$\frac{d\alpha}{dx} = \alpha(x)p(x). \tag{7.14}$$

This is a **separable** equation and therefore soluble. Separating the variables and integrating, we obtain

$$\int^{\alpha} \frac{d\alpha}{\alpha} = \int^{x} p(x)dx.$$

We need not consider the lower limits of these integrals because they combine to yield a constant that does not affect the performance of the integrating factor and can be set to zero. Completing the evaluation, we reach

$$\alpha(x) = \exp\left[\int^{x} p(x)dx\right]. \tag{7.15}$$

With $\alpha$ now known we proceed to integrate Eq. (7.12), which, because of Eq. (7.13), assumes the form

$$\frac{d}{dx}[\alpha(x)y(x)] = \alpha(x)q(x),$$

which can be integrated (and divided through by $\alpha$) to yield

$$y(x) = \frac{1}{\alpha(x)}\left[\int^{x} \alpha(x)q(x)dx + C\right] \equiv y_2(x) + y_1(x). \tag{7.16}$$

The two terms of Eq. (7.16) have an interesting interpretation. The term $y_1 = C/\alpha(x)$ is the general solution of the homogeneous equation obtained by replacing $q(x)$ with zero. To see this, write the homogeneous equation as

$$\frac{dy}{y} = -p(x)dx,$$

which integrates to

$$\ln y = -\int^x p(x)dx + C = -\ln\alpha + C.$$

Taking the exponential of both sides and renaming $e^C$ as $C$, we get just $y = C/\alpha(x)$. The other term of Eq. (7.16),

$$y_2 = \frac{1}{\alpha(x)}\int^x \alpha(x)q(x)dx \tag{7.17}$$

corresponds to the right-hand side (**source**) term $q(x)$, and is a solution of the original inhomogeneous equation (as is obvious because $C$ can be set to zero). We thus have the general solution to the inhomogeneous equation presented as a **particular solution** (or, in ODE parlance, a **particular integral**) plus the general solution to the corresponding homogeneous equation.

The above observations illustrate the following theorem:

*The solution of an inhomogeneous first-order linear ODE is unique except for an arbitrary multiple of the solution of the corresponding homogeneous ODE.*

To show this, suppose $y_1$ and $y_2$ both solve the inhomogeneous ODE, Eq. (7.11). Then, subtracting the equation for $y_2$ from that for $y_1$, we have

$$y_1' - y_2' + p(x)(y_1 - y_2) = 0.$$

This shows that $y_1 - y_2$ is (at some scale) a solution of the homogeneous ODE. Remember that any solution of the homogeneous ODE remains a solution when multiplied by an arbitrary constant.

We also have the theorem:

*A first-order linear homogeneous ODE has only one linearly independent solution.*

Two functions $y_1(x)$ and $y_2(x)$ are linearly dependent if there exist two constants $a$ and $b$, both nonzero, that cause $ay_1 + by_2$ to vanish for all $x$. In the present situation, this is equivalent to the statement that $y_1$ and $y_2$ are linearly dependent if they are proportional to each other.

To prove the theorem, assume that the homogeneous ODE has the linearly independent solutions $y_1$ and $y_2$. Then, from the homogeneous ODE, we have

$$\frac{y_1'}{y_1} = -p(x) = \frac{y_2'}{y_2}.$$

Integrating the first and last members of this equation, we obtain

$$\ln y_1 = \ln y_2 + C, \quad \text{equivalent to} \quad y_1 = Cy_2,$$

contradicting our assumption that $y_1$ and $y_2$ are linearly independent.

### *Example 7.2.5* RL Circuit

For a resistance-inductance circuit Kirchoff's law leads to

$$L\frac{dI(t)}{dt} + RI(t) = V(t),$$

where $I(t)$ is the current, $L$ and $R$ are, respectively, constant values of the inductance and the resistance, and $V(t)$ is the time-dependent input voltage.

From Eq. (7.15), our integrating factor $\alpha(t)$ is

$$\alpha(t) = \exp\int^t \frac{R}{L}dt = e^{Rt/L}.$$

Then, by Eq. (7.16),

$$I(t) = e^{-Rt/L}\left[\int^t e^{Rt/L}\frac{V(t)}{L}dt + C\right],$$

with the constant $C$ to be determined by an initial condition.

For the special case $V(t) = V_0$, a constant,

$$I(t) = e^{-Rt/L}\left[\frac{V_0}{L}\cdot\frac{L}{R}e^{Rt/L} + C\right] = \frac{V_0}{R} + Ce^{-Rt/L}.$$

If the initial condition is $I(0) = 0$, then $C = -V_0/R$ and

$$I(t) = \frac{V_0}{R}\left[1 - e^{-Rt/L}\right].$$

■

We close this section by pointing out that the inhomogeneous linear first-order ODE can also be solved by a method called **variation of the constant**, or alternatively **variation of parameters**, as follows. First, we solve the homogeneous ODE $y' + py = 0$ by separation of variables as before, giving

$$\frac{y'}{y} = -p, \quad \ln y = -\int^x p(X)dX + \ln C, \quad y(x) = C\exp\left(-\int^x p(X)dX\right).$$

Next we allow the integration constant to become $x$-dependent, that is, $C \to C(x)$. This is the reason the method is called "variation of the constant." To prepare for substitution into the inhomogeneous ODE, we calculate $y'$:

$$y' = \exp\left(-\int^x p(X)dX\right)\left[-pC(x) + C'(x)\right] = -py(x) + C'(x)\exp\left(-\int^x p(X)dX\right).$$

Making the substitution for $y'$ into the inhomogeneous ODE $y' + py = q$, some cancellation occurs, and we are left with

$$C'(x)\exp\left(-\int^x p(X)dX\right) = q,$$

which is a separable ODE for $C(x)$ that integrates to yield

$$C(x) = \int^{x} \exp\left(\int^{X} p(Y)dY\right) q(X)dX \quad \text{and} \quad y = C(x)\exp\left(-\int^{x} p(X)dX\right).$$

This particular solution of the inhomogeneous ODE is in agreement with that called $y_2$ in Eq. (7.17).

## Exercises

**7.2.1** From Kirchhoff's law the current $I$ in an $RC$ (resistance-capacitance) circuit (Fig. 7.1) obeys the equation

$$R\frac{dI}{dt} + \frac{1}{C}I = 0.$$

(a) Find $I(t)$.

(b) For a capacitance of 10,000 $\mu$F charged to 100 V and discharging through a resistance of 1 M$\Omega$, find the current $I$ for $t = 0$ and for $t = 100$ seconds.

*Note*. The initial voltage is $I_0 R$ or $Q/C$, where $Q = \int_0^\infty I(t)dt$.

**7.2.2** The Laplace transform of Bessel's equation $(n = 0)$ leads to

$$(s^2 + 1)f'(s) + sf(s) = 0.$$

Solve for $f(s)$.

**7.2.3** The decay of a population by catastrophic two-body collisions is described by

$$\frac{dN}{dt} = -kN^2.$$

This is a first-order, **nonlinear** differential equation. Derive the solution

$$N(t) = N_0\left(1 + \frac{t}{\tau_0}\right)^{-1},$$

where $\tau_0 = (kN_0)^{-1}$. This implies an infinite population at $t = -\tau_0$.

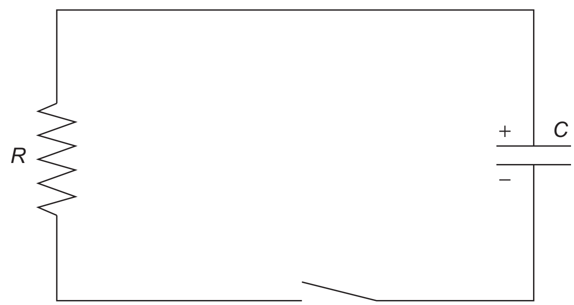


FIGURE 7.1 RC circuit.

**7.2.4** The rate of a particular chemical reaction $A + B \to C$ is proportional to the concentrations of the reactants $A$ and $B$:

$$\frac{dC(t)}{dt} = \alpha[A(0) - C(t)][B(0) - C(t)].$$

(a) Find $C(t)$ for $A(0) \neq B(0)$.

(b) Find $C(t)$ for $A(0) = B(0)$.

The initial condition is that $C(0) = 0$.

**7.2.5** A boat, coasting through the water, experiences a resisting force proportional to $v^n$, $v$ being the boat's instantaneous velocity. Newton's second law leads to

$$m\frac{dv}{dt} = -kv^n.$$

With $v(t = 0) = v_0$, $x(t = 0) = 0$, integrate to find $v$ as a function of time and $v$ as a function of distance.

**7.2.6** In the first-order differential equation $dy/dx = f(x, y)$, the function $f(x, y)$ is a function of the ratio $y/x$:

$$\frac{dy}{dx} = g(y/x).$$

Show that the substitution of $u = y/x$ leads to a separable equation in $u$ and $x$.

**7.2.7** The differential equation

$$P(x, y)dx + Q(x, y)dy = 0$$

is **exact**. Show that its solution is of the form

$$\varphi(x, y) = \int_{x_0}^{x} P(x, y)dx + \int_{y_0}^{y} Q(x_0, y)dy = \text{constant}.$$

**7.2.8** The differential equation

$$P(x, y)dx + Q(x, y)dy = 0$$

is **exact**. If

$$\varphi(x, y) = \int_{x_0}^{x} P(x, y)dx + \int_{y_0}^{y} Q(x_0, y)dy,$$

show that

$$\frac{\partial\varphi}{\partial x} = P(x, y), \qquad \frac{\partial\varphi}{\partial y} = Q(x, y).$$

Hence, $\varphi(x, y) = \text{constant}$ is a solution of the original differential equation.

**7.2.9** Prove that Eq. (7.12) is exact in the sense of Eq. (7.9), provided that $\alpha(x)$ satisfies Eq. (7.14).

**7.2.10** A certain differential equation has the form

$$f(x)dx + g(x)h(y)dy = 0,$$

with none of the functions $f(x)$, $g(x)$, $h(y)$ identically zero. Show that a necessary and sufficient condition for this equation to be exact is that $g(x) = \text{constant}$.

**7.2.11** Show that

$$y(x) = \exp\left[-\int^{x} p(t)dt\right]\left\{\int^{x} \exp\left[\int^{s} p(t)dt\right] q(s)ds + C\right\}$$

is a solution of

$$\frac{dy}{dx} + p(x)y(x) = q(x)$$

by differentiating the expression for $y(x)$ and substituting into the differential equation.

**7.2.12** The motion of a body falling in a resisting medium may be described by

$$m\frac{dv}{dt} = mg - bv$$

when the retarding force is proportional to the velocity, $v$. Find the velocity. Evaluate the constant of integration by demanding that $v(0) = 0$.

**7.2.13** Radioactive nuclei decay according to the law

$$\frac{dN}{dt} = -\lambda N,$$

$N$ being the concentration of a given nuclide and $\lambda$, the particular decay constant. In a radioactive series of two different nuclides, with concentrations $N_1(t)$ and $N_2(t)$, we have

$$\frac{dN_1}{dt} = -\lambda_1 N_1,$$

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2.$$

Find $N_2(t)$ for the conditions $N_1(0) = N_0$ and $N_2(0) = 0$.

**7.2.14** The rate of evaporation from a particular spherical drop of liquid (constant density) is proportional to its surface area. Assuming this to be the sole mechanism of mass loss, find the radius of the drop as a function of time.

**7.2.15** In the linear homogeneous differential equation

$$\frac{dv}{dt} = -av$$

the variables are separable. When the variables are separated, the equation is exact. Solve this differential equation subject to $v(0) = v_0$ by the following three methods:

(a) Separating variables and integrating.

(b) Treating the separated variable equation as exact.

(c) Using the result for a linear homogeneous differential equation.

*ANS.* $v(t) = v_0 e^{-at}$.

**7.2.16** (a) Solve Example 7.2.1, assuming that the parachute opens when the parachutist's velocity has reached $v_i = 60$ mi/h (regard this time as $t = 0$). Find $v(t)$.

(b) For a skydiver in free fall use the friction coefficient $b = 0.25$ kg/m and mass $m = 70$ kg. What is the limiting velocity in this case?

**7.2.17** Solve the ODE

$$(xy^2 - y)dx + x\,dy = 0.$$

**7.2.18** Solve the ODE

$$(x^2 - y^2 e^{y/x})dx + (x^2 + xy)e^{y/x}dy = 0.$$

*Hint.* Note that the quantity $y/x$ in the exponents is of combined degree zero and does not affect the determination of homogeneity.

## 7.3 ODEs with Constant Coefficients

Before addressing second-order ODEs, the main topic of this chapter, we discuss a specialized, but frequently occurring class of ODEs that are not constrained to be of specific order, namely those that are linear and whose homogeneous terms have constant coefficients. The generic equation of this type is

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0 y = F(x). \tag{7.18}$$

The homogeneous equation corresponding to Eq. (7.18) has solutions of the form $y = e^{mx}$, where $m$ is a solution of the algebraic equation

$$m^n + a_{n-1}m^{n-1} + \cdots + a_1 m + a_0 = 0,$$

as may be verified by substitution of the assumed form of the solution.

In the case that the $m$ equation has a multiple root, the above prescription will not yield the full set of $n$ linearly independent solutions for the original $n$th order ODE. If one then considers the limiting process in which two roots approach each other, it is possible to conclude that if $e^{mx}$ is a solution, then so is $d\,e^{mx}/dm = xe^{mx}$. A triple root would have solutions $e^{mx}$, $xe^{mx}$, $x^2e^{mx}$, etc.

### *Example 7.3.1* Hooke's Law Spring

A mass $M$ attached to a Hooke's Law spring (of spring constant $k$) is in oscillatory motion. Letting $y$ be the displacement of the mass from its equilibrium position, Newton's law of motion takes the form

$$M\frac{d^2y}{dt^2} = -ky,$$

which is an ODE of the form $y'' + a_0 y = 0$, with $a_0 = +k/M$. The general solution to this ODE is of the form $C_1 e^{m_1 t} + C_2 e^{m_2 t}$, where $m_1$ and $m_2$ are the solutions of the algebraic equation $m^2 + a_0 = 0$.

The values of $m_1$ and $m_2$ are $\pm i\omega$, where $\omega = \sqrt{k/M}$, so the ODE has solution

$$y(t) = C_1 e^{+i\omega t} + C_2 e^{-i\omega t}.$$

Since the ODE is homogeneous, we may alternatively describe its general solution using arbitrary linear combinations of the above two terms. This permits us to combine them to obtain forms that are real and therefore appropriate to the current problem. Noting that

$$\frac{e^{i\omega t} + e^{-i\omega t}}{2} = \cos\omega t \quad \text{and} \quad \frac{e^{i\omega t} - e^{-i\omega t}}{2i} = \sin\omega t,$$

a convenient alternate form is

$$y(t) = C_1 \cos\omega t + C_2 \sin\omega t.$$

The solution to a specific oscillation problem will now involve fitting the coefficients $C_1$ and $C_2$ to the initial conditions, as for example $y(0)$ and $y'(0)$. ■

## Exercises

Find the general solutions to the following ODEs. Write the solutions in forms that are entirely real (i.e., that contain no complex quantities).

**7.3.1** $y''' - 2y'' - y' + 2y = 0.$

**7.3.2** $y''' - 2y'' + y' - 2y = 0.$

**7.3.3** $y''' - 3y' + 2y = 0.$

**7.3.4** $y'' + 2y' + 2y = 0.$

# 7.4 Second-Order Linear ODEs

We now turn to the main topic of this chapter, second-order linear ODEs. These are of particular importance because they arise in the most frequently used methods for solving PDEs in quantum mechanics, electromagnetic theory, and other areas in physics. Unlike the first-order linear ODE, we do not have a universally applicable closed-form solution, and in general it is found advisable to use methods that produce solutions in the form of power series. As a precursor to the general discussion of series-solution methods, we begin by examining the notion of singularity as applied to ODEs.

## Singular Points

The concept of singularity of an ODE is important to us for two reasons: (1) it is useful for classifying ODEs and identifying those that can be transformed into common forms (discussed later in this subsection), and (2) it bears on the feasibility of finding series solutions to the ODE. This feasibility is the topic of Fuchs' theorem (to be discussed shortly).

When a linear homogeneous second-order ODE is written in the form

$$y'' + P(x)y' + Q(x)y = 0, \tag{7.19}$$

points $x_0$ for which $P(x)$ and $Q(x)$ are finite are termed **ordinary points** of the ODE. However, if either $P(x)$ or $Q(x)$ diverge as $x \to x_0$, the point $x_0$ is called a **singular point**. Singular points are further classified as **regular** or **irregular** (the latter also sometimes called **essential singularities**):

- A singular point $x_0$ is **regular** if either $P(x)$ or $Q(x)$ diverges there, but $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ remain finite.
- A singular point $x_0$ is **irregular** if $P(x)$ diverges faster than $1/(x - x_0)$ so that $(x - x_0)P(x)$ goes to infinity as $x \to x_0$, or if $Q(x)$ diverges faster than $1/(x - x_0)^2$ so that $(x - x_0)^2 Q(x)$ goes to infinity as $x \to x_0$.

These definitions hold for all finite values of $x_0$. To analyze the behavior at $x \to \infty$, we set $x = 1/z$, substitute into the differential equation, and examine the behavior in the limit $z \to 0$. The ODE, originally in the dependent variable $y(x)$, will now be written in terms of $w(z)$, defined as $w(z) = y(z^{-1})$. Converting the derivatives,

$$y' = \frac{dy(x)}{dx} = \frac{dy(z^{-1})}{dz}\frac{dz}{dx} = \frac{dw(z)}{dz}\left(-\frac{1}{x^2}\right) = -z^2 w', \tag{7.20}$$

$$y'' = \frac{dy'}{dz}\frac{dz}{dx} = (-z^2)\frac{d}{dz}\left[-z^2 w'\right] = z^4 w'' + 2z^3 w'. \tag{7.21}$$

Using Eqs. (7.20) and (7.21), we transform Eq. (7.19) into

$$z^4 w'' + \left[2z^3 - z^2 P(z^{-1})\right]w' + Q(z^{-1})w = 0. \tag{7.22}$$

Dividing through by $z^4$ to place the ODE in standard form, we see that the possibility of a singularity at $z = 0$ depends on the behavior of

$$\frac{2z - P(z^{-1})}{z^2} \quad \text{and} \quad \frac{Q(z^{-1})}{z^4}.$$

If these two expressions remain finite at $z = 0$, the point $x = \infty$ is an ordinary point. If they diverge no more rapidly than $1/z$ and $1/z^2$, respectively, $x = \infty$ is a regular singular point; otherwise it is an irregular singular point (an essential singularity).

### *Example 7.4.1* BESSEL'S EQUATION

Bessel's equation is

$$x^2 y'' + xy' + (x^2 - n^2)y = 0.$$

Comparing it with Eq. (7.19), we have

$$P(x) = \frac{1}{x}, \quad Q(x) = 1 - \frac{n^2}{x^2},$$

which shows that the point $x = 0$ is a regular singularity. By inspection we see that there are no other singularities in the finite range. As $x \to \infty$ ($z \to 0$), from Eq. (7.22) we have the coefficients

$$\frac{2z - z}{z^2} \quad \text{and} \quad \frac{1 - n^2 z^2}{z^4}.$$

Since the latter expression diverges as $1/z^4$, the point $x = \infty$ is an irregular, or essential, singularity. ■

Table 7.1 lists the singular points of a number of ODEs of importance in physics. It will be seen that the first three equations in Table 7.1, the hypergeometric, Legendre, and Chebyshev, all have three regular singular points. The hypergeometric equation, with regular singularities at 0, 1, and $\infty$, is taken as the standard, the canonical form. The solutions of the other two may then be expressed in terms of its solutions, the hypergeometric functions. This is done in Chapter 18.

In a similar manner, the confluent hypergeometric equation is taken as the canonical form of a linear second-order differential equation with one regular and one irregular singular point.

**Table 7.1** Singularities of Some Important ODEs.

| Equation | Regular Singularity $x =$ | Irregular Singularity $x =$ |
|---|---|---|
| 1. Hypergeometric<br>$x(x-1)y'' + [(1+a+b)x + c]y' + aby = 0$ | $0, 1, \infty$ | $\cdots$ |
| 2. Legendre[a]<br>$(1-x^2)y'' - 2xy' + l(l+1)y = 0$ | $-1, 1, \infty$ | $\cdots$ |
| 3. Chebyshev<br>$(1-x^2)y'' - xy' + n^2 y = 0$ | $-1, 1, \infty$ | $\cdots$ |
| 4. Confluent hypergeometric<br>$xy'' + (c-x)y' - ay = 0$ | 0 | $\infty$ |
| 5. Bessel<br>$x^2 y'' + xy' + (x^2 - n^2)y = 0$ | 0 | $\infty$ |
| 6. Laguerre[a]<br>$xy'' + (1-x)y' + ay = 0$ | 0 | $\infty$ |
| 7. Simple harmonic oscillator<br>$y'' + \omega^2 y = 0$ | $\cdots$ | $\infty$ |
| 8. Hermite<br>$y'' - 2xy' + 2\alpha y = 0$ | $\cdots$ | $\infty$ |

[a] The associated equations have the same singular points.

## *Exercises*

**7.4.1** Show that Legendre's equation has regular singularities at $x = -1,\ 1$, and $\infty$.

**7.4.2** Show that Laguerre's equation, like the Bessel equation, has a regular singularity at $x = 0$ and an irregular singularity at $x = \infty$.

**7.4.3** Show that Chebyshev's equation, like the Legendre equation, has regular singularities at $x = -1,\ 1$, and $\infty$.

**7.4.4** Show that Hermite's equation has no singularity other than an irregular singularity at $x = \infty$.

**7.4.5** Show that the substitution

$$x \to \frac{1-x}{2}, \quad a = -l, \quad b = l+1, \quad c = 1$$

converts the hypergeometric equation into Legendre's equation.

# 7.5 Series Solutions—Frobenius' Method

In this section we develop a method of obtaining solution(s) of the linear, second-order, homogeneous ODE. For the moment, we develop the mechanics of the method. After studying examples, we return to discuss the conditions under which we can expect these series solutions to exist.

Consider a linear, second-order, homogeneous ODE, in the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0. \tag{7.23}$$

In this section we develop (at least) one solution of Eq. (7.23) by expansion about the point $x = 0$. In the next section we develop the **second, independent solution and prove that no third, independent solution exists**. Therefore the **most general solution** of Eq. (7.23) may be written in terms of the two independent solutions as

$$y(x) = c_1 y_1(x) + c_2 y_2(x). \tag{7.24}$$

Our physical problem may lead to a **nonhomogeneous**, linear, second-order ODE,

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = F(x). \tag{7.25}$$

The function on the right, $F(x)$, typically represents a source (such as electrostatic charge) or a driving force (as in a driven oscillator). Methods for solving this inhomogeneous ODE are also discussed later in this chapter and, using Laplace transform techniques, in Chapter 20. Assuming a single **particular integral** (i.e., specific solution), $y_p$, of the inhomogeneous ODE to be available, we may add to it any solution of the corresponding homogeneous equation, Eq. (7.23), and write the **most general solution** of Eq. (7.25) as

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x). \tag{7.26}$$

In many problems, the constants $c_1$ and $c_2$ will be fixed by boundary conditions.

For the present, we assume that $F(x)=0$, and that therefore our differential equation is homogeneous. We shall attempt to develop a solution of our linear, second-order, homogeneous differential equation, Eq. (7.23), by substituting into it a power series with undetermined coefficients. Also available as a parameter is the power of the lowest nonvanishing term of the series. To illustrate, we apply the method to two important differential equations.

## First Example—Linear Oscillator

Consider the linear (classical) oscillator equation

$$\frac{d^2y}{dx^2}+\omega^2 y=0, \tag{7.27}$$

which we have already solved by another method in Example 7.3.1. The solutions we found there were $y=\sin\omega x$ and $\cos\omega x$.

We try

$$\begin{aligned} y(x) &= x^s(a_0+a_1x+a_2x^2+a_3x^3+\cdots) \\ &= \sum_{j=0}^{\infty} a_j x^{s+j}, \quad a_0\neq 0, \end{aligned} \tag{7.28}$$

with the exponent $s$ and all the coefficients $a_j$ still undetermined. Note that $s$ need not be an integer. By differentiating twice, we obtain

$$\frac{dy}{dx}=\sum_{j=0}^{\infty} a_j(s+j)x^{s+j-1},$$

$$\frac{d^2y}{dx^2}=\sum_{j=0}^{\infty} a_j(s+j)(s+j-1)x^{s+j-2}.$$

By substituting into Eq. (7.27), we have

$$\sum_{j=0}^{\infty} a_j(s+j)(s+j-1)x^{s+j-2}+\omega^2\sum_{j=0}^{\infty} a_j x^{s+j}=0. \tag{7.29}$$

From our analysis of the uniqueness of power series (Chapter 1), we know that the coefficient of each power of $x$ on the left-hand side of Eq. (7.29) must vanish individually, $x^s$ being an overall factor.

The lowest power of $x$ appearing in Eq. (7.29) is $x^{s-2}$, occurring only for $j=0$ in the first summation. The requirement that this coefficient vanish yields

$$a_0 s(s-1)=0.$$

Recall that we chose $a_0$ as the coefficient of the lowest nonvanishing term of the series in Eq. (7.28), so that, by definition, $a_0\neq 0$. Therefore we have

$$s(s-1)=0. \tag{7.30}$$

This equation, coming from the coefficient of the lowest power of $x$, is called the **indicial equation**. The indicial equation and its roots are of critical importance to our analysis. Clearly, in this example it informs us that either $s = 0$ or $s = 1$, so that our series solution must start either with an $x^0$ or an $x^1$ term.

Looking further at Eq. (7.29), we see that the next lowest power of $x$, namely $x^{s-1}$, also occurs uniquely (for $j = 1$ in the first summation). Setting the coefficient of $x^{s-1}$ to zero, we have

$$a_1(s+1)s = 0.$$

This shows that if $s = 1$, we must have $a_1 = 0$. However, if $s = 0$, this equation imposes no requirement on the coefficient set.

Before considering further the two possibilities for $s$, we return to Eq. (7.29) and demand that the remaining net coefficients vanish. The contributions to the coefficient of $x^{s+j}$, $(j \geq 0)$, come from the term containing $a_{j+2}$ in the first summation and from that with $a_j$ in the second. Because we have already dealt with $j = 0$ and $j = 1$ in the first summation, when we have used all $j \geq 0$, we will have used all the terms of both series. For each value of $j$, the vanishing of the net coefficient of $x^{s+j}$ results in

$$a_{j+2}(s+j+2)(s+j+1) + \omega^2 a_j = 0,$$

equivalent to

$$a_{j+2} = -a_j \frac{\omega^2}{(s+j+2)(s+j+1)}. \tag{7.31}$$

This is a two-term **recurrence relation**.[1] In the present problem, given $a_j$, Eq. (7.31) permits us to compute $a_{j+2}$ and then $a_{j+4}, a_{j+6}$, and so on, continuing as far as desired. Thus, if we start with $a_0$, we can make the even coefficients $a_2, a_4, \ldots$, but we obtain no information about the odd coefficients $a_1, a_3, a_5, \ldots$. But because $a_1$ is arbitrary if $s = 0$ and necessarily zero if $s = 1$, let us set it equal to zero, and then, by Eq. (7.31),

$$a_3 = a_5 = a_7 = \cdots = 0;$$

the result is that all the odd-numbered coefficients vanish.

Returning now to Eq. (7.30), our indicial equation, we first try the solution $s = 0$. The recurrence relation, Eq. (7.31), becomes

$$a_{j+2} = -a_j \frac{\omega^2}{(j+2)(j+1)}, \tag{7.32}$$

[1] In some problems, the recurrence relation may involve more than two terms; its exact form will depend on the functions $P(x)$ and $Q(x)$ of the ODE.

which leads to

$$a_2 = -a_0 \frac{\omega^2}{1 \cdot 2} = -\frac{\omega^2}{2!} a_0,$$

$$a_4 = -a_2 \frac{\omega^2}{3 \cdot 4} = +\frac{\omega^4}{4!} a_0,$$

$$a_6 = -a_4 \frac{\omega^2}{5 \cdot 6} = -\frac{\omega^6}{6!} a_0, \quad \text{and so on.}$$

By inspection (and mathematical induction, see Section 1.4),

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n)!} a_0, \tag{7.33}$$

and our solution is

$$y(x)_{s=0} = a_0 \left[ 1 - \frac{(\omega x)^2}{2!} + \frac{(\omega x)^4}{4!} - \frac{(\omega x)^6}{6!} + \cdots \right] = a_0 \cos \omega x. \tag{7.34}$$

If we choose the indicial equation root $s = 1$ from Eq. (7.30), the recurrence relation of Eq. (7.31) becomes

$$a_{j+2} = -a_j \frac{\omega^2}{(j+3)(j+2)}. \tag{7.35}$$

Evaluating this successively for $j = 0, 2, 4, \ldots$, we obtain

$$a_2 = -a_0 \frac{\omega^2}{2 \cdot 3} = -\frac{\omega^2}{3!} a_0,$$

$$a_4 = -a_2 \frac{\omega^2}{4 \cdot 5} = +\frac{\omega^4}{5!} a_0,$$

$$a_6 = -a_4 \frac{\omega^2}{6 \cdot 7} = -\frac{\omega^6}{7!} a_0, \quad \text{and so on.}$$

Again, by inspection and mathematical induction,

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n+1)!} a_0. \tag{7.36}$$

For this choice, $s = 1$, we obtain

$$\begin{aligned} y(x)_{s=1} &= a_0 x \left[ 1 - \frac{(\omega x)^2}{3!} + \frac{(\omega x)^4}{5!} - \frac{(\omega x)^6}{7!} + \cdots \right] \\ &= \frac{a_0}{\omega} \left[ (\omega x) - \frac{(\omega x)^3}{3!} + \frac{(\omega x)^5}{5!} - \frac{(\omega x)^7}{7!} + \cdots \right] \\ &= \frac{a_0}{\omega} \sin \omega x. \end{aligned} \tag{7.37}$$

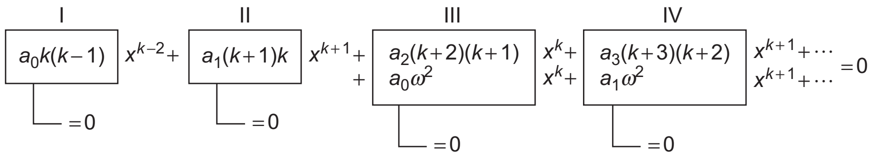


FIGURE 7.2 Schematics of series solution.

For future reference we note that the ODE solution from the indicial equation root $s = 0$ consisted only of even powers of $x$, while the solution from the root $s = 1$ contained only odd powers.

*To summarize this approach, we may write Eq. (7.29) schematically as shown in Fig. 7.2. From the uniqueness of power series (Section 1.2), the total coefficient of each power of* x *must vanish—all by itself. The requirement that the first coefficient vanish (I) leads to the indicial equation, Eq. (7.30). The second coefficient is handled by setting* $a_1 = 0$ *(II). The vanishing of the coefficients of* $x^s$ *(and higher powers, taken one at a time) is ensured by imposing the recurrence relation, Eq. (7.31), (III), (IV).*

This expansion in power series, known as Frobenius' method, has given us two series solutions of the linear oscillator equation. However, there are two points about such series solutions that must be strongly emphasized:

1. The series solution should always be substituted back into the differential equation, to see if it works, as a precaution against algebraic and logical errors. If it works, it is a solution.
2. The acceptability of a series solution depends on its convergence (including asymptotic convergence). It is quite possible for Frobenius' method to give a series solution that satisfies the original differential equation when substituted in the equation but that does **not** converge over the region of interest. Legendre's differential equation (examined in Section 8.3) illustrates this situation.

## Expansion about $x_0$

Equation (7.28) is an expansion about the origin, $x_0 = 0$. It is perfectly possible to replace Eq. (7.28) with

$$y(x) = \sum_{j=0}^{\infty} a_j (x - x_0)^{s+j}, \quad a_0 \neq 0. \tag{7.38}$$

Indeed, for the Legendre, Chebyshev, and hypergeometric equations, the choice $x_0 = 1$ has some advantages. The point $x_0$ should not be chosen at an essential singularity, or Frobenius' method will probably fail. The resultant series ($x_0$ an ordinary point or regular singular point) will be valid where it converges. You can expect a divergence of some sort when $|x - x_0| = |z_1 - x_0|$, where $z_1$ is the ODE's closest singularity to $x_0$ (in the complex plane).

## Symmetry of Solutions

Let us note that for the classical oscillator problem we obtained one solution of even symmetry, $y_1(x) = y_1(-x)$, and one of odd symmetry, $y_2(x) = -y_2(-x)$. This is not just an accident but a direct consequence of the form of the ODE. Writing a general homogeneous ODE as

$$\mathcal{L}(x)y(x) = 0, \tag{7.39}$$

in which $\mathcal{L}(x)$ is the differential operator, we see that for the linear oscillator equation, Eq. (7.27), $\mathcal{L}(x)$ is even under parity; that is,

$$\mathcal{L}(x) = \mathcal{L}(-x).$$

Whenever the differential operator has a specific parity or symmetry, either even or odd, we may interchange $+x$ and $-x$, and Eq. (7.39) becomes

$$\pm\mathcal{L}(x)y(-x) = 0.$$

Clearly, if $y(x)$ is a solution of the differential equation, $y(-x)$ is also a solution. Then, either $y(x)$ and $y(-x)$ are linearly dependent (i.e., proportional), meaning that $y$ is either even or odd, or they are linearly independent solutions that can be combined into a pair of solutions, one even, and one odd, by forming

$$y_{\text{even}} = y(x) + y(-x), \quad y_{\text{odd}} = y(x) - y(-x).$$

For the classical oscillator example, we obtained two solutions; our method for finding them caused one to be even, the other odd.

If we refer back to Section 7.4 we can see that Legendre, Chebyshev, Bessel, simple harmonic oscillator, and Hermite equations are all based on differential operators with even parity; that is, their $P(x)$ in Eq. (7.19) is odd and $Q(x)$ even. Solutions of all of them may be presented as series of even powers of $x$ or separate series of odd powers of $x$. The Laguerre differential operator has neither even nor odd symmetry; hence its solutions cannot be expected to exhibit even or odd parity. Our emphasis on parity stems primarily from the importance of parity in quantum mechanics. We find that in many problems wave functions are either even or odd, meaning that they have a definite parity. Most interactions (beta decay is the big exception) are also even or odd, and the result is that parity is conserved.

## A Second Example—Bessel's Equation

This attack on the linear oscillator was perhaps a bit too easy. By substituting the power series, Eq. (7.28), into the differential equation, Eq. (7.27), we obtained two independent solutions with no trouble at all.

To get some idea of other things that can happen, we try to solve Bessel's equation,

$$x^2y'' + xy' + (x^2 - n^2)y = 0. \tag{7.40}$$

Again, assuming a solution of the form

$$y(x) = \sum_{j=0}^{\infty} a_j x^{s+j},$$

we differentiate and substitute into Eq. (7.40). The result is

$$\sum_{j=0}^{\infty} a_j (s+j)(s+j-1)x^{s+j} + \sum_{j=0}^{\infty} a_j (s+j)x^{s+j}$$
$$+ \sum_{j=0}^{\infty} a_j x^{s+j+2} - \sum_{j=0}^{\infty} a_j n^2 x^{s+j} = 0. \tag{7.41}$$

By setting $j = 0$, we get the coefficient of $x^s$, the lowest power of $x$ appearing on the left-hand side,

$$a_0\left[s(s-1) + s - n^2\right] = 0, \tag{7.42}$$

and again $a_0 \neq 0$ by definition. Equation (7.42) therefore yields the **indicial** equation

$$s^2 - n^2 = 0, \tag{7.43}$$

with solutions $s = \pm n$.

We need also to examine the coefficient of $x^{s+1}$. Here we obtain

$$a_1[(s+1)s + s + 1 - n^2] = 0,$$

or

$$a_1(s+1-n)(s+1+n) = 0. \tag{7.44}$$

For $s = \pm n$, neither $s+1-n$ nor $s+1+n$ vanishes and we **must** require $a_1 = 0$.

Proceeding to the coefficient of $x^{s+j}$ for $s = n$, we see that it is the term containing $a_j$ in the first, second, and fourth terms of Eq. (7.41), but is that containing $a_{j-2}$ in the third term. By requiring the overall coefficient of $x^{s+j}$ to vanish, we obtain

$$a_j[(n+j)(n+j-1) + (n+j) - n^2] + a_{j-2} = 0.$$

When $j$ is replaced by $j+2$, this can be rewritten for $j \geq 0$ as

$$a_{j+2} = -a_j \frac{1}{(j+2)(2n+j+2)}, \tag{7.45}$$

which is the desired recurrence relation. Repeated application of this recurrence relation leads to

$$a_2 = -a_0 \frac{1}{2(2n+2)} = -\frac{a_0 n!}{2^2 1!(n+1)!},$$

$$a_4 = -a_2 \frac{1}{4(2n+4)} = \frac{a_0 n!}{2^4 2!(n+2)!},$$

$$a_6 = -a_4 \frac{1}{6(2n+6)} = -\frac{a_0 n!}{2^6 3!(n+3)!}, \quad \text{and so on,}$$

and in general,

$$a_{2p} = (-1)^p \frac{a_0 n!}{2^{2p} p!(n+p)!}. \tag{7.46}$$

Inserting these coefficients in our assumed series solution, we have

$$y(x) = a_0 x^n \left[ 1 - \frac{n!\,x^2}{2^2 1!(n+1)!} + \frac{n!\,x^4}{2^4 2!(n+2)!} - \cdots \right]. \tag{7.47}$$

In summation form,

$$\begin{aligned} y(x) &= a_0 \sum_{j=0}^{\infty} (-1)^j \frac{n!\,x^{n+2j}}{2^{2j} j!(n+j)!} \\ &= a_0\, 2^n n! \sum_{j=0}^{\infty} (-1)^j \frac{1}{j!(n+j)!} \left(\frac{x}{2}\right)^{n+2j}. \end{aligned} \tag{7.48}$$

In Chapter 14 the final summation (with $a_0 = 1/2^n n!$) is identified as the Bessel function $J_n(x)$:

$$J_n(x) = \sum_{j=0}^{\infty} (-1)^j \frac{1}{j!(n+j)!} \left(\frac{x}{2}\right)^{n+2j}. \tag{7.49}$$

Note that this solution, $J_n(x)$, has either even or odd symmetry,[2] as might be expected from the form of Bessel's equation.

When $s = -n$ and $n$ is not an integer, we may generate a second distinct series, to be labeled $J_{-n}(x)$. However, when $-n$ is a negative integer, trouble develops. The recurrence relation for the coefficients $a_j$ is still given by Eq. (7.45), but with $2n$ replaced by $-2n$. Then, when $j + 2 = 2n$ or $j = 2(n-1)$, the coefficient $a_{j+2}$ blows up and Frobenius' method does not produce a solution consistent with our assumption that the series starts with $x^{-n}$.

By substituting in an infinite series, we have obtained two solutions for the linear oscillator equation and one for Bessel's equation (two if $n$ is not an integer). To the questions "Can we always do this? Will this method always work?" the answer is "No, we cannot always do this. This method of series solution will not always work."

## Regular and Irregular Singularities

The success of the series substitution method depends on the roots of the indicial equation and the degree of singularity of the coefficients in the differential equation. To understand better the effect of the equation coefficients on this naive series substitution approach,

[2] $J_n(x)$ is an even function if $n$ is an even integer, and an odd function if $n$ is an odd integer. For nonintegral $n$, $J_n$ has no such simple symmetry.

consider four simple equations:

$$y'' - \frac{6}{x^2} y = 0, \tag{7.50}$$

$$y'' - \frac{6}{x^3} y = 0, \tag{7.51}$$

$$y'' + \frac{1}{x} y' - \frac{b^2}{x^2} y = 0, \tag{7.52}$$

$$y'' + \frac{1}{x^2} y' - \frac{b^2}{x^2} y = 0. \tag{7.53}$$

The reader may show easily that for Eq. (7.50) the indicial equation is

$$s^2 - s - 6 = 0,$$

giving $s = 3$ and $s = -2$. Since the equation is homogeneous in $x$ (counting $d^2/dx^2$ as $x^{-2}$), there is no recurrence relation. However, we are left with two perfectly good solutions, $x^3$ and $x^{-2}$.

Equation (7.51) differs from Eq. (7.50) by only one power of $x$, but this sends the indicial equation to

$$-6a_0 = 0,$$

with no solution at all, for we have agreed that $a_0 \neq 0$. Our series substitution worked for Eq. (7.50), which had only a regular singularity, but broke down at Eq. (7.51), which has an irregular singular point at the origin.

Continuing with Eq. (7.52), we have added a term $y'/x$. The indicial equation is

$$s^2 - b^2 = 0,$$

but again, there is no recurrence relation. The solutions are $y = x^b$ and $x^{-b}$, both perfectly acceptable one-term series.

When we change the power of $x$ in the coefficient of $y'$ from $-1$ to $-2$, in Eq. (7.53), there is a drastic change in the solution. The indicial equation (with only the $y'$ term contributing) becomes

$$s = 0.$$

There is a recurrence relation,

$$a_{j+1} = +a_j \frac{b^2 - j(j-1)}{j+1}.$$

Unless the parameter $b$ is selected to make the series terminate, we have

$$\lim_{j\to\infty} \left| \frac{a_{j+1}}{a_j} \right| = \lim_{j\to\infty} \frac{j(j+1)}{j+1}$$

$$= \lim_{j\to\infty} \frac{j^2}{j} = \infty.$$

Hence our series solution diverges for all $x \neq 0$. Again, our method worked for Eq. (7.52) with a regular singularity but failed when we had the irregular singularity of Eq. (7.53).

## Fuchs' Theorem

The answer to the basic question as to when the method of series substitution can be expected to work is given by Fuchs' theorem, which asserts that we can always obtain at least one power-series solution, provided that we are expanding about a point which is an ordinary point or at worst a regular singular point.

If we attempt an expansion about an irregular or essential singularity, our method may fail as it did for Eqs. (7.51) and (7.53). Fortunately, the more important equations of mathematical physics, listed in Section 7.4, have no irregular singularities in the finite plane. Further discussion of Fuchs' theorem appears in Section 7.6.

From Table 7.1, Section 7.4, infinity is seen to be a singular point for all the equations considered. As a further illustration of Fuchs' theorem, Legendre's equation (with infinity as a regular singularity) has a convergent series solution in negative powers of the argument (Section 15.6). In contrast, Bessel's equation (with an irregular singularity at infinity) yields asymptotic series (Sections 12.6 and 14.6). Although only asymptotic, these solutions are nevertheless extremely useful.

## Summary

If we are expanding about an ordinary point or at worst about a regular singularity, the series substitution approach will yield at least one solution (Fuchs' theorem).

Whether we get one or two distinct solutions depends on the roots of the indicial equation.

1. If the two roots of the indicial equation are equal, we can obtain only one solution by this series substitution method.
2. If the two roots differ by a nonintegral number, two independent solutions may be obtained.
3. If the two roots differ by an integer, the larger of the two will yield a solution, while the smaller may or may not give a solution, depending on the behavior of the coefficients.

The usefulness of a series solution for numerical work depends on the rapidity of convergence of the series and the availability of the coefficients. Many ODEs will not yield nice, simple recurrence relations for the coefficients. In general, the available series will probably be useful for very small $|x|$ (or $|x - x_0|$). Computers can be used to determine additional series coefficients using a symbolic language, such as Mathematica[3] or Maple.[4] Often, however, for numerical work a direct numerical integration will be preferred.

[3] S. Wolfram, *Mathematica: A System for Doing Mathematics by Computer*. Reading, MA. Addison Wesley (1991).
[4] A. Heck, *Introduction to Maple*. New York: Springer (1993).

## Exercises

**7.5.1** Uniqueness theorem. The function $y(x)$ satisfies a second-order, linear, homogeneous differential equation. At $x = x_0,\ y(x) = y_0$ and $dy/dx = y'_0$. Show that $y(x)$ is unique, in that no other solution of this differential equation passes through the points $(x_0, y_0)$ with a slope of $y'_0$.

*Hint*. Assume a second solution satisfying these conditions and compare the Taylor series expansions.

**7.5.2** A series solution of Eq. (7.23) is attempted, expanding about the point $x = x_0$. If $x_0$ is an ordinary point, show that the indicial equation has roots $s = 0, 1$.

**7.5.3** In the development of a series solution of the simple harmonic oscillator (SHO) equation, the second series coefficient $a_1$ was neglected except to set it equal to zero. From the coefficient of the next-to-the-lowest power of $x, x^{s-1}$, develop a second-indicial type equation.

(a) (SHO equation with $s = 0$). Show that $a_1$, may be assigned any finite value (including zero).

(b) (SHO equation with $s = 1$). Show that $a_1$ must be set equal to zero.

**7.5.4** Analyze the series solutions of the following differential equations to see when $a_1$ **may** be set equal to zero without irrevocably losing anything and when $a_1$ **must** be set equal to zero.

(a) Legendre, (b) Chebyshev, (c) Bessel, (d) Hermite.

*ANS.* (a) Legendre, (b) Chebyshev, and (d) Hermite: For $s = 0$, $a_1$ **may** be set equal to zero; for $s = 1$, $a_1$ **must** be set equal to zero.
(c) Bessel: $a_1$ **must** be set equal to zero (except for $s = \pm n = -\frac{1}{2}$).

**7.5.5** Obtain a series solution of the hypergeometric equation

$$x(x-1)y'' + [(1+a+b)x - c]y' + aby = 0.$$

Test your solution for convergence.

**7.5.6** Obtain two series solutions of the confluent hypergeometric equation

$$xy'' + (c-x)y' - ay = 0.$$

Test your solutions for convergence.

**7.5.7** A quantum mechanical analysis of the Stark effect (parabolic coordinates) leads to the differential equation

$$\frac{d}{d\xi}\left(\xi\frac{du}{d\xi}\right) + \left(\frac{1}{2}E\xi + \alpha - \frac{m^2}{4\xi} - \frac{1}{4}F\xi^2\right)u = 0.$$

Here $\alpha$ is a constant, $E$ is the total energy, and $F$ is a constant such that $Fz$ is the potential energy added to the system by the introduction of an electric field.

Using the larger root of the indicial equation, develop a power-series solution about $\xi = 0$. Evaluate the first three coefficients in terms of $a_o$.

*ANS.* Indicial equation $s^2 - \frac{m^2}{4} = 0$,

$$u(\xi) = a_0 \xi^{m/2} \left\{ 1 - \frac{\alpha}{m+1}\xi + \left[ \frac{\alpha^2}{2(m+1)(m+2)} - \frac{E}{4(m+2)} \right] \xi^2 + \cdots \right\}.$$

Note that the perturbation $F$ does not appear until $a_3$ is included.

**7.5.8** For the special case of no azimuthal dependence, the quantum mechanical analysis of the hydrogen molecular ion leads to the equation

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{du}{d\eta}\right] + \alpha u + \beta \eta^2 u = 0.$$

Develop a power-series solution for $u(\eta)$. Evaluate the first three nonvanishing coefficients in terms of $a_0$.

*ANS.* Indicial equation $s(s-1) = 0$,

$$u_{k=1} = a_0 \eta \left\{ 1 + \frac{2-\alpha}{6}\eta^2 + \left[ \frac{(2-\alpha)(12-\alpha)}{120} - \frac{\beta}{20} \right] \eta^4 + \cdots \right\}.$$

**7.5.9** To a good approximation, the interaction of two nucleons may be described by a mesonic potential

$$V = \frac{Ae^{-ax}}{x},$$

attractive for $A$ negative. Show that the resultant Schrödinger wave equation

$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (E - V)\psi = 0$$

has the following series solution through the first three nonvanishing coefficients:

$$\psi = a_0 \left\{ x + \frac{1}{2}A' x^2 + \frac{1}{6}\left[ \frac{1}{2}A'^2 - E' - aA' \right] x^3 + \cdots \right\},$$

where the prime indicates multiplication by $2m/\hbar^2$.

**7.5.10** If the parameter $b^2$ in Eq. (7.53) is equal to 2, Eq. (7.53) becomes

$$y'' + \frac{1}{x^2}y' - \frac{2}{x^2}y = 0.$$

From the indicial equation and the recurrence relation, **derive** a solution $y = 1 + 2x + 2x^2$. Verify that this is indeed a solution by substituting back into the differential equation.

**7.5.11** The modified Bessel function $I_0(x)$ satisfies the differential equation

$$x^2 \frac{d^2}{dx^2} I_0(x) + x \frac{d}{dx} I_0(x) - x^2 I_0(x) = 0.$$

Given that the leading term in an asymptotic expansion is known to be

$$I_0(x) \sim \frac{e^x}{\sqrt{2\pi x}},$$

assume a series of the form

$$I_0(x) \sim \frac{e^x}{\sqrt{2\pi x}} \left\{ 1 + b_1 x^{-1} + b_2 x^{-2} + \cdots \right\}.$$

Determine the coefficients $b_1$ and $b_2$.

*ANS.* $b_1 = \frac{1}{8}, \quad b_2 = \frac{9}{128}.$

**7.5.12** The even power-series solution of Legendre's equation is given by Exercise 8.3.1. Take $a_0 = 1$ and $n$ not an even integer, say $n = 0.5$. Calculate the partial sums of the series through $x^{200}, x^{400}, x^{600}, \ldots, x^{2000}$ for $x = 0.95(0.01)1.00$. Also, write out the individual term corresponding to each of these powers.

*Note*. This calculation does **not** constitute proof of convergence at $x = 0.99$ or divergence at $x = 1.00$, but perhaps you can see the difference in the behavior of the sequence of partial sums for these two values of $x$.

**7.5.13** (a) The odd power-series solution of Hermite's equation is given by Exercise 8.3.3. Take $a_0 = 1$. Evaluate this series for $\alpha = 0, x = 1, 2, 3$. Cut off your calculation after the last term calculated has dropped below the maximum term by a factor of $10^6$ or more. Set an upper bound to the error made in ignoring the remaining terms in the infinite series.

(b) As a check on the calculation of part (a), show that the Hermite series $y_{\text{odd}}(\alpha = 0)$ corresponds to $\int_0^x \exp(x^2)dx$.

(c) Calculate this integral for $x = 1, 2, 3$.

# 7.6 OTHER SOLUTIONS

In Section 7.5 a solution of a second-order homogeneous ODE was developed by substituting in a power series. By Fuchs' theorem this is possible, provided the power series is an expansion about an ordinary point or a nonessential singularity.[5] There is no guarantee that this approach will yield the two independent solutions we expect from a linear second-order ODE. In fact, we shall prove that such an ODE has at most two linearly independent solutions. Indeed, the technique gave only one solution for Bessel's equation ($n$ an integer). In this section we also develop two methods of obtaining a second independent solution: an integral method and a power series containing a logarithmic term. First, however, we consider the question of independence of a set of functions.

[5]This is why the classification of singularities in Section 7.4 is of vital importance.

## Linear Independence of Solutions

In Chapter 2 we introduced the concept of linear dependence for forms of the type $a_1 x_1 + a_2 x_2 + \dots$, and identified a set of such forms as linearly dependent if any one of the forms could be written as a linear combination of others. We need now to extend the concept to a set of functions $\varphi_\lambda$. The criterion for linear dependence of a set of functions of a variable $x$ is the existence of a relation of the form

$$\sum_\lambda k_\lambda \varphi_\lambda(x) = 0, \tag{7.54}$$

in which not all the coefficients $k_\lambda$ are zero. The interpretation we attach to Eq. (7.54) is that it indicates linear dependence if it is satisfied for all relevant values of $x$. Isolated points or partial ranges of satisfaction of Eq. (7.54) do not suffice to indicate linear dependence. The essential idea being conveyed here is that if there is linear dependence, the function space spanned by the $\varphi_\lambda(x)$ can be spanned using less than all of them. On the other hand, if the only global solution of Eq. (7.54) is $k_\lambda = 0$ for all $\lambda$, the set of functions $\varphi_\lambda(x)$ is said to be linearly **independent**.

If the members of a set of functions are mutually orthogonal, then they are automatically linearly independent. To establish this, consider the evaluation of

$$S = \left\langle \sum_\lambda k_\lambda \varphi_\lambda \middle| \sum_\mu k_\mu \varphi_\mu \right\rangle$$

for a set of orthonormal $\varphi_\lambda$ and with arbitrary values of the coefficients $k_\lambda$. Because of the orthonormality, $S$ evaluates to $\sum_\lambda |k_\lambda|^2$, and will be nonzero (showing that $\sum_\lambda k_\lambda \varphi_\lambda \neq 0$) unless all the $k_\lambda$ vanish.

We now proceed to consider the ramifications of linear dependence for solutions of ODEs, and for that purpose it is appropriate to assume that the functions $\varphi_\lambda(x)$ are differentiable as needed. Then, differentiating Eq. (7.54) repeatedly, with the assumption that it is valid for all $x$, we generate a set of equations

$$\sum_\lambda k_\lambda \varphi'_\lambda(x) = 0,$$

$$\sum_\lambda k_\lambda \varphi''_\lambda(x) = 0,$$

continuing until we have generated as many equations as the number of $\lambda$ values. This gives us a set of homogeneous linear equations in which $k_\lambda$ are the unknown quantities. By Section 2.1 there is a solution other than all $k_\lambda = 0$ only if the determinant of the coefficients of the $k_\lambda$ vanishes. This means that the linear dependence we have assumed by accepting Eq. (7.54) implies that

$$\begin{vmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_n \\ \varphi'_1 & \varphi'_2 & \dots & \varphi'_n \\ \dots & \dots & \dots & \dots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \dots & \varphi_n^{(n-1)} \end{vmatrix} = 0. \tag{7.55}$$

This determinant is called the **Wronskian**, and the analysis leading to Eq. (7.55) shows that:

1. If the Wronskian is not equal to zero, then Eq. (7.54) has no solution other than $k_\lambda = 0$. The set of functions $\varphi_\lambda$ is therefore linearly independent.
2. If the Wronskian vanishes at isolated values of the argument, this does not prove linear dependence. However, if the Wronskian is zero over the entire range of the variable, the functions $\varphi_\lambda$ are linearly dependent over this range.[6]

## *Example 7.6.1* LINEAR INDEPENDENCE

The solutions of the linear oscillator equation, Eq. (7.27), are $\varphi_1 = \sin\omega x$, $\varphi_2 = \cos\omega x$. The Wronskian becomes

$$\begin{vmatrix} \sin\omega x & \cos\omega x \\ \omega\cos\omega x & -\omega\sin\omega x \end{vmatrix} = -\omega \neq 0.$$

These two solutions, $\varphi_1$ and $\varphi_2$, are therefore linearly independent. For just two functions this means that one is not a multiple of the other, which is obviously true here.

Incidentally, you know that

$$\sin\omega x = \pm(1 - \cos^2\omega x)^{1/2},$$

but this is **not** a **linear** relation, of the form of Eq. (7.54). ■

## *Example 7.6.2* LINEAR DEPENDENCE

For an illustration of linear dependence, consider the solutions of the ODE

$$\frac{d^2\varphi(x)}{dx^2} = \varphi(x).$$

This equation has solutions $\varphi_1 = e^x$ and $\varphi_2 = e^{-x}$, and we add $\varphi_3 = \cosh x$, also a solution. The Wronskian is

$$\begin{vmatrix} e^x & e^{-x} & \cosh x \\ e^x & -e^{-x} & \sinh x \\ e^x & e^{-x} & \cosh x \end{vmatrix} = 0.$$

The determinant vanishes for all $x$ because the first and third rows are identical. Hence $e^x$, $e^{-x}$, and $\cosh x$ are linearly dependent, and, indeed, we have a relation of the form of Eq. (7.54):

$$e^x + e^{-x} - 2\cosh x = 0 \quad \text{with} \quad k_\lambda \neq 0.$$

■

[6]Compare H. Lass, *Elements of Pure and Applied Mathematics*, New York: McGraw-Hill (1957), p. 187, for proof of this assertion. It is assumed that the functions have continuous derivatives and that at least one of the minors of the bottom row of Eq. (7.55) (Laplace expansion) does not vanish in $[a, b]$, the interval under consideration.

## Number of Solutions

Now we are ready to prove the theorem that a second-order homogeneous ODE has two linearly independent solutions.

Suppose $y_1$, $y_2$, $y_3$ are three solutions of the homogeneous ODE, Eq. (7.23). Then we form the Wronskian $W_{jk} = y_j y_k' - y_j' y_k$ of any pair $y_j$, $y_k$ of them and note also that

$$\begin{aligned} W_{jk}' &= (y_j' y_k' + y_j y_k'') - (y_j'' y_k + y_j' y_k') \\ &= y_j y_k'' - y_j'' y_k. \end{aligned} \tag{7.56}$$

Next we divide the ODE by $y$ and move $Q(x)$ to its right-hand side (where it becomes $-Q(x)$), so, for solutions $y_j$ and $y_k$:

$$\frac{y_j''}{y_j} + P(x)\frac{y_j'}{y_j} = -Q(x) = \frac{y_k''}{y_k} + P(x)\frac{y'_k}{y_k}.$$

Taking now the first and third members of this equation, multiplying by $y_j y_k$ and rearranging, we find that

$$(y_j y_k'' - y_j'' y_k) + P(x)(y_j y_k' - y_j' y_k) = 0,$$

which simplifies for any pair of solutions to

$$W_{jk}' = -P(x) W_{jk}. \tag{7.57}$$

Finally, we evaluate the Wronskian of all three solutions, expanding it by minors along the second row and identifying each term as containing a $W_{ij}'$ as given by Eq. (7.56):

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = -y_1' W_{23}' + y_2' W_{13}' - y_3' W_{12}'.$$

We now use Eq. (7.57) to replace each $W_{ij}'$ by $-P(x)W_{ij}$ and then reassemble the minors into a $3 \times 3$ determinant, which vanishes because it contains two identical rows:

$$W = P(x)\left(y_1' W_{23} - y_2' W_{13} + y_3' W_{12}\right) = -P(x) \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \end{vmatrix} = 0.$$

We therefore have $W = 0$, which is just the condition for linear dependence of the solutions $y_j$. Thus, we have proved the following:

> *A linear second-order homogeneous ODE has at most two linearly independent solutions. Generalizing, a linear homogeneous* n*th-order ODE has at most* n *linearly independent solutions* $y_j$*, and its general solution will be of the form* $y(x) = \sum_{j=1}^{n} c_j y_j(x)$.

## Finding a Second Solution

Returning to our linear, second-order, homogeneous ODE of the general form

$$y'' + P(x)y' + Q(x)y = 0, \tag{7.58}$$

let $y_1$ and $y_2$ be two independent solutions. Then the Wronskian, by definition, is

$$W = y_1 y_2' - y_1' y_2. \tag{7.59}$$

By differentiating the Wronskian, we obtain, as already demonstrated in Eq. (7.57),

$$W' = -P(x)W. \tag{7.60}$$

In the special case that $P(x) = 0$, that is,

$$y'' + Q(x)y = 0, \tag{7.61}$$

the Wronskian

$$W = y_1 y_2' - y_1' y_2 = \text{constant}. \tag{7.62}$$

Since our original differential equation is homogeneous, we may multiply the solutions $y_1$ and $y_2$ by whatever constants we wish and arrange to have the Wronskian equal to unity (or $-1$). This case, $P(x) = 0$, appears more frequently than might be expected. Recall that $\nabla^2(\psi/r)$ in spherical polar coordinates contains no first radial derivative. Finally, every linear second-order differential equation can be transformed into an equation of the form of Eq. (7.61) (compare Exercise 7.6.12).

For the general case, let us now assume that we have one solution of Eq. (7.58) by a series substitution (or by guessing). We now proceed to develop a second, independent solution for which $W \neq 0$. Rewriting Eq. (7.60) as

$$\frac{dW}{W} = -P\,dx,$$

we integrate over the variable $x$, from $a$ to $x$, to obtain

$$\ln \frac{W(x)}{W(a)} = -\int_a^x P(x_1)dx_1,$$

or[7]

$$W(x) = W(a) \exp\left[-\int_a^x P(x_1)dx_1\right]. \tag{7.63}$$

[7]If $P(x)$ remains finite in the domain of interest, $W(x) \neq 0$ unless $W(a) = 0$. That is, the Wronskian of our two solutions is either identically zero or never zero. However, if $P(x)$ does not remain finite in our interval, then $W(x)$ can have isolated zeros in that domain and one must be careful to choose $a$ so that $W(a) \neq 0$.

Now we make the observation that

$$W(x) = y_1 y_2' - y_1' y_2 = y_1^2 \frac{d}{dx}\left(\frac{y_2}{y_1}\right), \tag{7.64}$$

and, by combining Eqs. (7.63) and (7.64), we have

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = W(a) \frac{\exp[-\int_a^x P(x_1)dx_1]}{y_1^2}. \tag{7.65}$$

Finally, by integrating Eq. (7.65) from $x_2 = b$ to $x_2 = x$ we get

$$y_2(x) = y_1(x) W(a) \int_b^x \frac{\exp\left[-\int_a^{x_2} P(x_1)dx_1\right]}{[y_1(x_2)]^2} dx_2. \tag{7.66}$$

Here $a$ and $b$ are arbitrary constants and a term $y_1(x)y_2(b)/y_1(b)$ has been dropped, because it is a multiple of the previously found first solution $y_1$. Since $W(a)$, the Wronskian evaluated at $x = a$, is a constant and our solutions for the homogeneous differential equation always contain an arbitrary normalizing factor, we set $W(a) = 1$ and write

$$y_2(x) = y_1(x) \int^x \frac{\exp[-\int^{x_2} P(x_1)dx_1]}{[y_1(x_2)]^2} dx_2. \tag{7.67}$$

Note that the lower limits $x_1 = a$ and $x_2 = b$ have been omitted. If they are retained, they simply make a contribution equal to a constant times the known first solution, $y_1(x)$, and hence add nothing new. If we have the important special case $P(x) = 0$, Eq. (7.67) reduces to

$$y_2(x) = y_1(x) \int^x \frac{dx_2}{[y_1(x_2)]^2}. \tag{7.68}$$

This means that by using either Eq. (7.67) or Eq. (7.68) we can take one known solution and by integrating can generate a second, independent solution of Eq. (7.58). This technique is used in Section 15.6 to generate a second solution of Legendre's differential equation.

## *Example 7.6.3* A Second Solution for the Linear Oscillator Equation

From $d^2y/dx^2 + y = 0$ with $P(x) = 0$ let one solution be $y_1 = \sin x$. By applying Eq. (7.68), we obtain

$$y_2(x) = \sin x \int^x \frac{dx_2}{\sin^2 x_2} = \sin x(-\cot x) = -\cos x,$$

which is clearly independent (not a linear multiple) of $\sin x$. ■

## Series Form of the Second Solution

Further insight into the nature of the second solution of our differential equation may be obtained by the following sequence of operations.

1. Express $P(x)$ and $Q(x)$ in Eq. (7.58) as

$$P(x) = \sum_{i=-1}^{\infty} p_i x^i, \quad Q(x) = \sum_{j=-2}^{\infty} q_j x^j. \tag{7.69}$$

   The leading terms of the summations are selected to create the strongest possible **regular** singularity (at the origin). These conditions just satisfy Fuchs' theorem and thus help us gain a better understanding of that theorem.
2. Develop the first few terms of a power-series solution, as in Section 7.5.
3. Using this solution as $y_1$, obtain a second series-type solution, $y_2$, from Eq. (7.67), by integrating it term by term.

Proceeding with Step 1, we have

$$y'' + (p_{-1}x^{-1} + p_0 + p_1 x + \cdots)y' + (q_{-2}x^{-2} + q_{-1}x^{-1} + \cdots)y = 0, \tag{7.70}$$

where $x = 0$ is at worst a regular singular point. If $p_{-1} = q_{-1} = q_{-2} = 0$, it reduces to an ordinary point. Substituting

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{s+\lambda}$$

(Step 2), we obtain

$$\sum_{\lambda=0}^{\infty}(s+\lambda)(s+\lambda-1)a_\lambda x^{s+\lambda-2} + \sum_{i=-1}^{\infty} p_i x^i \sum_{\lambda=0}^{\infty}(s+\lambda)a_\lambda x^{s+\lambda-1}$$

$$+ \sum_{j=-2}^{\infty} q_j x^j \sum_{\lambda=0}^{\infty} a_\lambda x^{s+\lambda} = 0. \tag{7.71}$$

Assuming that $p_{-1} \neq 0$, our indicial equation is

$$s(s-1) + p_{-1}k + q_{-2} = 0,$$

which sets the net coefficient of $x^{s-2}$ equal to zero. This reduces to

$$s^2 + (p_{-1} - 1)s + q_{-2} = 0. \tag{7.72}$$

We denote the two roots of this indicial equation by $s = \alpha$ and $s = \alpha - n$, where $n$ is zero or a positive integer. (If $n$ is not an integer, we expect two independent series solutions by the methods of Section 7.5 and we are done.) Then

$$(s-\alpha)(s-\alpha+n) = 0, \tag{7.73}$$

or

$$s^2 + (n-2\alpha)s + \alpha(\alpha-n) = 0,$$

and equating coefficients of $s$ in Eqs. (7.72) and (7.73), we have

$$p_{-1} - 1 = n - 2\alpha. \tag{7.74}$$

The known series solution corresponding to the larger root $s = \alpha$ may be written as

$$y_1 = x^\alpha \sum_{\lambda=0}^{\infty} a_\lambda x^\lambda.$$

Substituting this series solution into Eq. (7.67) (Step 3), we are faced with

$$y_2(x) = y_1(x) \int^x \left( \frac{\exp\left(-\int_a^{x_2} \sum_{i=-1}^{\infty} p_i x_1^i \, dx_1\right)}{x_2^{2\alpha} \left(\sum_{\lambda=0}^{\infty} a_\lambda x_2^\lambda\right)^2} \right) dx_2, \tag{7.75}$$

where the solutions $y_1$ and $y_2$ have been normalized so that the Wronskian $W(a) = 1$. Tackling the exponential factor first, we have

$$\int_a^{x_2} \sum_{i=-1}^{\infty} p_i x_1^i dx_1 = p_{-1} \ln x_2 + \sum_{k=0}^{\infty} \frac{p_k}{k+1} x_2^{k+1} + f(a), \tag{7.76}$$

with $f(a)$ an integration constant that may depend on $a$. Hence,

$$\begin{aligned} \exp\left(-\int_a^{x_2} \sum_i p_i x_1^i dx_1\right) &= \exp[-f(a)] x_2^{-p_{-1}} \exp\left(-\sum_{k=0}^{\infty} \frac{p_k}{k+1} x_2^{k+1}\right) \\ &= \exp[-f(a)] x_2^{-p_{-1}} \left[ 1 - \sum_{k=0}^{\infty} \frac{p_k}{k+1} x_2^{k+1} + \frac{1}{2!} \left( -\sum_{k=0}^{\infty} \frac{p_k}{k+1} x_2^{k+1} \right)^2 + \cdots \right]. \end{aligned} \tag{7.77}$$

This final series expansion of the exponential is certainly convergent if the original expansion of the coefficient $P(x)$ was uniformly convergent.

The denominator in Eq. (7.75) may be handled by writing

$$\left[ x_2^{2\alpha} \left( \sum_{\lambda=0}^{\infty} a_\lambda x_2^\lambda \right)^2 \right]^{-1} = x_2^{-2\alpha} \left( \sum_{\lambda=0}^{\infty} a_\lambda x_2^\lambda \right)^{-2} = x_2^{-2\alpha} \sum_{\lambda=0}^{\infty} b_\lambda x_2^\lambda. \tag{7.78}$$

Neglecting constant factors, which will be picked up anyway by the requirement that $W(a) = 1$, we obtain

$$y_2(x) = y_1(x) \int^x x_2^{-p_{-1}-2\alpha} \left( \sum_{\lambda=0}^{\infty} c_\lambda x_2^\lambda \right) dx_2. \tag{7.79}$$

Applying Eq. (7.74),

$$x_2^{-p_{-1}-2\alpha} = x_2^{-n-1}, \tag{7.80}$$

and we have assumed here that $n$ is an integer. Substituting this result into Eq. (7.79), we obtain

$$y_2(x) = y_1(x) \int^x \left( c_0 x_2^{-n-1} + c_1 x_2^{-n} + c_2 x_2^{-n+1} + \cdots + c_n x_2^{-1} + \cdots \right) dx_2. \qquad (7.81)$$

The integration indicated in Eq. (7.81) leads to a coefficient of $y_1(x)$ consisting of two parts:

1. A power series starting with $x^{-n}$.
2. A logarithm term from the integration of $x^{-1}$ (when $\lambda = n$). This term always appears when $n$ is an integer, **unless** $c_n$ fortuitously happens to vanish.[8]

If we choose to combine $y_1$ and the power series starting with $x^{-n}$, our second solution will assume the form

$$y_2(x) = y_1(x) \ln |x| + \sum_{j=-n}^{\infty} d_j x^{j+\alpha}. \qquad (7.82)$$

## *Example 7.6.4* A Second Solution of Bessel's Equation

From Bessel's equation, Eq. (7.40), divided by $x^2$ to agree with Eq. (7.59), we have

$$P(x) = x^{-1} \quad Q(x) = 1 \quad \text{for the case} \quad n = 0.$$

Hence $p_{-1} = 1$, $q_0 = 1$; all other $p_i$ and $q_j$ vanish. The Bessel indicial equation, Eq. (7.43) with $n = 0$, is

$$s^2 = 0.$$

Hence we verify Eqs. (7.72) to (7.74) with $n$ and $\alpha$ set to zero.

Our first solution is available from Eq. (7.49). It is[9]

$$y_1(x) = J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - O(x^6). \qquad (7.83)$$

Now, substituting all this into Eq. (7.67), we have the specific case corresponding to Eq. (7.75):

$$y_2(x) = J_0(x) \int^x \left( \frac{\exp\left[ -\int^{x_2} x_1^{-1} dx_1 \right]}{\left[ 1 - \frac{x_2^2}{4} + \frac{x_2^4}{64} - \cdots \right]^2} \right) dx_2. \qquad (7.84)$$

[8] For parity considerations, ln $x$ is taken to be ln $|x|$, even.

[9] The capital $O$ (order of) as written here means terms proportional to $x^6$ and possibly higher powers of $x$.

From the numerator of the integrand,

$$\exp\left[-\int^{x_2}\frac{dx_1}{x_1}\right]=\exp[-\ln x_2]=\frac{1}{x_2}.$$

This corresponds to the $x_2^{-p-1}$ in Eq. (7.77). From the denominator of the integrand, using a binomial expansion, we obtain

$$\left[1-\frac{x_2^2}{4}+\frac{x_2^4}{64}\right]^{-2}=1+\frac{x_2^2}{2}+\frac{5x_2^4}{32}+\cdots.$$

Corresponding to Eq. (7.79), we have

$$y_2(x)=J_0(x)\int^{x}\frac{1}{x_2}\left[1+\frac{x_2^2}{2}+\frac{5x_2^4}{32}+\cdots\right]dx_2$$

$$=J_0(x)\left\{\ln x+\frac{x^2}{4}+\frac{5x^4}{128}+\cdots\right\}. \tag{7.85}$$

Let us check this result. From Eq. (14.62), which gives the standard form of the second solution, which is called a **Neumann function** and designated $Y_0$,

$$Y_0(x)=\frac{2}{\pi}\left[\ln x-\ln 2+\gamma\right]J_0(x)+\frac{2}{\pi}\left\{\frac{x^2}{4}-\frac{3x^4}{128}+\cdots\right\}. \tag{7.86}$$

Two points arise: (1) Since Bessel's equation is homogeneous, we may multiply $y_2(x)$ by any constant. To match $Y_0(x)$, we multiply our $y_2(x)$ by $2/\pi$. (2) To our second solution, $(2/\pi)y_2(x)$, we may add any constant multiple of the first solution. Again, to match $Y_0(x)$ we add

$$\frac{2}{\pi}\left[-\ln 2+\gamma\right]J_0(x),$$

where $\gamma$ is the Euler-Mascheroni constant, defined in Eq. (1.13).[10] Our new, modified second solution is

$$y_2(x)=\frac{2}{\pi}\left[\ln x-\ln 2+\gamma\right]J_0(x)+\frac{2}{\pi}J_0(x)\left\{\frac{x^2}{4}+\frac{5x^4}{128}+\cdots\right\}. \tag{7.87}$$

Now the comparison with $Y_0(x)$ requires only a simple multiplication of the series for $J_0(x)$ from Eq. (7.83) and the curly bracket of Eq. (7.87). The multiplication checks, through terms of order $x^2$ and $x^4$, which is all we carried. Our second solution from Eqs. (7.67) and (7.75) agrees with the standard second solution, the Neumann function $Y_0(x)$. ■

The analysis that indicated the second solution of Eq. (7.58) to have the form given in Eq. (7.82) suggests the possibility of just substituting Eq. (7.82) into the original differential equation and determining the coefficients $d_j$. However, the process has some features different from that of Section 7.5, and is illustrated by the following example.

[10] The Neumann function $Y_0$ is defined as it is in order to achieve convenient asymptotic properties; see Sections 14.3 and 14.6.

### *Example 7.6.5* More Neumann Functions

We consider here second solutions to Bessel's ODE of integer orders $n > 0$, using the expansion given in Eq. (7.82). The first solution, designated $J_n$ and presented in Eq. (7.49), arises from the value $\alpha = n$ from the indicial equation, while the quantity called $n$ in Eq. (7.82), the separation of the two roots of the indicial equation, has in the current context the value $2n$. Thus, Eq. (7.82) takes the form

$$y_2(x) = J_n(x)\ln|x| + \sum_{j=-2n}^{\infty} d_j x^{j+n}, \tag{7.88}$$

where $y_2$ must, apart from scale and a possible multiple of $J_n$, be the second solution $Y_n$ of the Bessel equation. Substituting this form into Bessel's equation, carrying out the indicated differentiations and using the fact that $J_n(x)$ is a solution of our ODE, we get after combining similar terms

$$x^2 y_2'' + x y_2' + (x^2 - n^2) y_2 =$$
$$2x J_n'(x) + \sum_{j\ge -2n} j(j+2n) d_j x^{j+n} + \sum_{j\ge -2n} d_j x^{j+n+2} = 0. \tag{7.89}$$

We next insert the power-series expansion

$$2x J_n'(x) = \sum_{j\ge 0} a_j x^{j+n}, \tag{7.90}$$

where the coefficients can be obtained by differentiation of the expansion of $J_n$, see Eq. (7.49), and have the values (for $j \ge 0$)

$$a_{2j} = \frac{(-1)^j (n+2j)}{j!(n+j)!2^{n+2j-1}},$$
$$a_{2j+1} = 0. \tag{7.91}$$

This, and a redefinition of the index $j$ in the last term, bring Eq. (7.89) to the form

$$\sum_{j\ge 0} a_j x^{j+n} + \sum_{j\ge -2n} j(j+2n) d_j x^{j+n} + \sum_{j\ge -2n+2} d_{j-2} x^{j+n} = 0. \tag{7.92}$$

Considering first the coefficient of $x^{-n+1}$ (corresponding to $j = -2n+1$), we note that its vanishing requires that $d_{-2n+1}$ vanish, as the only contribution comes from the middle summation. Since all $a_j$ of odd $j$ vanish, the vanishing of $d_{-2n+1}$ implies that all other $d_j$ of odd $j$ must also vanish. We therefore only need to give further consideration to even $j$.

We next note that the coefficient $d_0$ is arbitrary, and may without loss of generality be set to zero. This is true because we may bring $d_0$ to any value by adding to $y_2$ an appropriate multiple of the solution $J_n$, whose expansion has an $x^n$ leading term. We have then exhausted all freedom in specifying $y_2$; its scale is determined by our choice of its logarithmic term.

Now, taking the coefficient of $x^n$ (terms with $j = 0$), and remembering that $d_0 = 0$, we have

$$d_{-2} = -a_0,$$

and we may recur **downward** in steps of 2, using formulas based on the coefficients of $x^{n-2}, x^{n-4}, \ldots$, corresponding to

$$d_{j-2} = -j(2n+j)d_j, \quad j = -2, -4, \ldots, -2n+2.$$

To obtain $d_j$ with positive $j$, we recur upward, obtaining from the coefficient of $x^{n+j}$

$$d_j = \frac{-a_j - d_{j-2}}{j(2n+j)}, \quad j = 2, 4, \ldots,$$

again remembering that $d_0 = 0$.

Proceeding to $n = 1$ as a specific example, we have from Eq. (7.91) $a_0 = 1$, $a_2 = -3/8$, and $a_4 = 5/192$, so

$$d_{-2} = -1, \quad d_2 = -\frac{a_2}{8} = \frac{3}{64}, \quad d_4 = \frac{-a_4 - d_2}{24} = -\frac{7}{2304};$$

thus

$$y_2(x) = J_1(x)\ln|x| - \frac{1}{x} + \frac{3}{64}x^3 - \frac{7}{2304}x^5 + \cdots,$$

in agreement (except for a multiple of $J_1$ and a scale factor) with the standard form of the Neumann function $Y_1$:

$$Y_1(x) = \frac{2}{\pi}\left[\ln\left|\frac{x}{2}\right| + \gamma - \frac{1}{2}\right]J_1(x) + \frac{2}{\pi}\left[-\frac{1}{x} + \frac{3}{64}x^3 - \frac{7}{2304}x^5 + \cdots\right]. \tag{7.93}$$

■

As shown in the examples, the second solution will usually diverge at the origin because of the logarithmic factor and the negative powers of $x$ in the series. For this reason $y_2(x)$ is often referred to as the **irregular solution**. The first series solution, $y_1(x)$, which usually converges at the origin, is called the **regular solution**. The question of behavior at the origin is discussed in more detail in Chapters 14 and 15, in which we take up Bessel functions, modified Bessel functions, and Legendre functions.

# Summary

The two solutions of both sections (together with the exercises) provide a **complete solution** of our linear, homogeneous, second-order ODE, assuming that the point of expansion is no worse than a regular singularity. At least one solution can always be obtained by series substitution (Section 7.5). A **second, linearly independent solution** can be constructed by the **Wronskian** double integral, Eq. (7.67). This is all there are: **No third, linearly independent solution exists** (compare Exercise 7.6.10).

The **inhomogeneous**, linear, second-order ODE will have a general solution formed by adding a **particular solution** to the complete inhomogeneous equation to the general solution of the corresponding homogeneous ODE. Techniques for finding particular solutions of linear but inhomogeneous ODEs are the topic of the next section.

## Exercises

**7.6.1** You know that the three unit vectors $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, and $\hat{\mathbf{e}}_z$ are mutually perpendicular (orthogonal). Show that $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, and $\hat{\mathbf{e}}_z$ are linearly independent. Specifically, show that no relation of the form of Eq. (7.54) exists for $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, and $\hat{\mathbf{e}}_z$.

**7.6.2** The criterion for the linear **independence** of three vectors **A, B**, and **C** is that the equation

$$a\mathbf{A} + b\mathbf{B} + c\mathbf{C} = 0,$$

analogous to Eq. (7.54), has no solution other than the trivial $a = b = c = 0$. Using components $\mathbf{A} = (A_1, A_2, A_3)$, and so on, set up the determinant criterion for the existence or nonexistence of a nontrivial solution for the coefficients $a, b$, and $c$. Show that your criterion is equivalent to the scalar triple product $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} \neq 0$.

**7.6.3** Using the Wronskian determinant, show that the set of functions

$$\left\{1, \frac{x^n}{n!}(n = 1, 2, \ldots, N)\right\}$$

is linearly independent.

**7.6.4** If the Wronskian of two functions $y_1$ and $y_2$ is identically zero, show by direct integration that

$$y_1 = cy_2,$$

that is, that $y_1$ and $y_2$ are linearly dependent. Assume the functions have continuous derivatives and that at least one of the functions does not vanish in the interval under consideration.

**7.6.5** The Wronskian of two functions is found to be zero at $x_0 - \varepsilon \leq x \leq x_0 + \varepsilon$ for arbitrarily small $\varepsilon > 0$. Show that this Wronskian vanishes for all $x$ and that the functions are linearly dependent.

**7.6.6** The three functions $\sin x$, $e^x$, and $e^{-x}$ are linearly independent. No one function can be written as a linear combination of the other two. Show that the Wronskian of $\sin x$, $e^x$, and $e^{-x}$ vanishes but only at isolated points.

*ANS.* $W = 4\sin x$,
$W = 0$ for $x = \pm n\pi, \quad n = 0, 1, 2, \ldots.$

**7.6.7** Consider two functions $\varphi_1 = x$ and $\varphi_2 = |x|$. Since $\varphi_1' = 1$ and $\varphi_2' = x/|x|$, $W(\varphi_1, \varphi_2) = 0$ for any interval, including $[-1, +1]$. Does the vanishing of the Wronskian over $[-1, +1]$ prove that $\varphi_1$ and $\varphi_2$ are linearly dependent? Clearly, they are not. What is wrong?

**7.6.8** Explain that **linear independence** does not mean the absence of any dependence. Illustrate your argument with $\cosh x$ and $e^x$.

**7.6.9** Legendre's differential equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

has a regular solution $P_n(x)$ and an irregular solution $Q_n(x)$. Show that the Wronskian of $P_n$ and $Q_n$ is given by

$$P_n(x)Q_n'(x) - P_n'(x)Q_n(x) = \frac{A_n}{1-x^2},$$

with $A_n$ **independent** of $x$.

**7.6.10** Show, by means of the Wronskian, that a linear, second-order, homogeneous ODE of the form

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0$$

**cannot have three independent solutions**.

*Hint.* Assume a third solution and show that the Wronskian vanishes for all $x$.

**7.6.11** Show the following when the linear second-order differential equation $py'' + qy' + ry = 0$ is expressed in self-adjoint form:

(a) The Wronskian is equal to a constant divided by $p$:

$$W(x) = \frac{C}{p(x)}.$$

(b) A second solution $y_2(x)$ is obtained from a first solution $y_1(x)$ as

$$y_2(x) = Cy_1(x)\int^x \frac{dt}{p(t)[y_1(t)]^2}.$$

**7.6.12** Transform our linear, second-order ODE

$$y'' + P(x)y' + Q(x)y = 0$$

by the substitution

$$y = z\exp\left[-\frac{1}{2}\int^x P(t)dt\right]$$

and show that the resulting differential equation for $z$ is

$$z'' + q(x)z = 0,$$

where

$$q(x) = Q(x) - \frac{1}{2}P'(x) - \frac{1}{4}P^2(x).$$

*Note*. This substitution can be derived by the technique of Exercise 7.6.25.

**7.6.13** Use the result of Exercise 7.6.12 to show that the replacement of $\varphi(r)$ by $r\varphi(r)$ may be expected to eliminate the first derivative from the Laplacian in spherical polar coordinates. See also Exercise 3.10.34.

**7.6.14** By direct differentiation and substitution show that

$$y_2(x) = y_1(x) \int^x \frac{\exp[-\int^s P(t)dt]}{[y_1(s)]^2} ds$$

satisfies, like $y_1(x)$, the ODE

$$y_2''(x) + P(x)y_2'(x) + Q(x)y_2(x) = 0.$$

*Note*. The Leibniz formula for the derivative of an integral is

$$\frac{d}{d\alpha} \int_{g(\alpha)}^{h(\alpha)} f(x,\alpha)dx = \int_{g(\alpha)}^{h(\alpha)} \frac{\partial f(x,\alpha)}{\partial\alpha} dx + f[h(\alpha),\alpha]\frac{dh(\alpha)}{d\alpha} - f[g(\alpha),\alpha]\frac{dg(\alpha)}{d\alpha}.$$

**7.6.15** In the equation

$$y_2(x) = y_1(x) \int^x \frac{\exp[-\int^s P(t)dt]}{[y_1(s)]^2} ds,$$

$y_1(x)$ satisfies

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0.$$

The function $y_2(x)$ is a linearly **independent** second solution of the same equation. Show that the inclusion of lower limits on the two integrals leads to nothing new, that is, that it generates only an overall constant factor and a constant multiple of the known solution $y_1(x)$.

**7.6.16** Given that one solution of

$$R'' + \frac{1}{r}R' - \frac{m^2}{r^2}R = 0$$

is $R = r^m$, show that Eq. (7.67) predicts a second solution, $R = r^{-m}$.

**7.6.17** Using

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

as a solution of the linear oscillator equation, follow the analysis that proceeds through Eq. (7.81) and show that in that equation $c_n = 0$, so that in this case the second solution does not contain a logarithmic term.

**7.6.18** Show that when $n$ is **not** an integer in Bessel's ODE, Eq. (7.40), the second solution of Bessel's equation, obtained from Eq. (7.67), does **not** contain a logarithmic term.

**7.6.19** (a) One solution of Hermite's differential equation

$$y'' - 2xy' + 2\alpha y = 0$$

for $\alpha = 0$ is $y_1(x) = 1$. Find a second solution, $y_2(x)$, using Eq. (7.67). Show that your second solution is equivalent to $y_{\text{odd}}$ (Exercise 8.3.3).

(b) Find a second solution for $\alpha = 1$, where $y_1(x) = x$, using Eq. (7.67). Show that your second solution is equivalent to $y_{\text{even}}$ (Exercise 8.3.3).

**7.6.20** One solution of Laguerre's differential equation

$$xy'' + (1 - x)y' + ny = 0$$

for $n = 0$ is $y_1(x) = 1$. Using Eq. (7.67), develop a second, linearly independent solution. Exhibit the logarithmic term explicitly.

**7.6.21** For Laguerre's equation with $n = 0$,

$$y_2(x) = \int^x \frac{e^s}{s} ds.$$

(a) Write $y_2(x)$ as a logarithm plus a power series.

(b) Verify that the integral form of $y_2(x)$, previously given, is a solution of Laguerre's equation ($n = 0$) by direct differentiation of the integral and substitution into the differential equation.

(c) Verify that the series form of $y_2(x)$, part (a), is a solution by differentiating the series and substituting back into Laguerre's equation.

**7.6.22** One solution of the Chebyshev equation

$$(1 - x^2)y'' - xy' + n^2 y = 0$$

for $n = 0$ is $y_1 = 1$.

(a) Using Eq. (7.67), develop a second, linearly independent solution.

(b) Find a second solution by direct integration of the Chebyshev equation.

*Hint*. Let $v = y'$ and integrate. Compare your result with the second solution given in Section 18.4.

*ANS.* (a) $y_2 = \sin^{-1} x$.
(b) The second solution, $V_n(x)$, is not defined for $n = 0$.

**7.6.23** One solution of the Chebyshev equation

$$(1 - x^2)y'' - xy' + n^2 y = 0$$

for $n = 1$ is $y_1(x) = x$. Set up the Wronskian double integral solution and derive a second solution, $y_2(x)$.

*ANS.* $y_2 = -(1 - x^2)^{1/2}$.

**7.6.24** The radial Schrödinger wave equation for a spherically symmetric potential can be written in the form

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + l(l+1)\frac{\hbar^2}{2mr^2} + V(r)\right] y(r) = E y(r).$$

The potential energy $V(r)$ may be expanded about the origin as

$$V(r) = \frac{b_{-1}}{r} + b_0 + b_1 r + \cdots .$$

(a) Show that there is one (regular) solution $y_1(r)$ starting with $r^{l+1}$.

(b) From Eq. (7.69) show that the irregular solution $y_2(r)$ diverges at the origin as $r^{-l}$.

**7.6.25** Show that if a second solution, $y_2$, is assumed to be related to the first solution, $y_1$, according to $y_2(x) = y_1(x) f(x)$, substitution back into the original equation

$$y_2'' + P(x) y_2' + Q(x) y_2 = 0$$

leads to

$$f(x) = \int^x \frac{\exp[-\int^s P(t)dt]}{[y_1(s)]^2} ds,$$

in agreement with Eq. (7.67).

**7.6.26** (a) Show that

$$y'' + \frac{1-\alpha^2}{4x^2} y = 0$$

has two solutions:

$$y_1(x) = a_0 x^{(1+\alpha)/2},$$
$$y_2(x) = a_0 x^{(1-\alpha)/2}.$$

(b) For $\alpha = 0$ the two linearly independent solutions of part (a) reduce to the single solution $y_{1'} = a_0 x^{1/2}$. Using Eq. (7.68) derive a second solution,

$$y_{2'}(x) = a_0 x^{1/2} \ln x.$$

Verify that $y_{2'}$ is indeed a solution.

(c) Show that the second solution from part (b) may be obtained as a limiting case from the two solutions of part (a):

$$y_{2'}(x) = \lim_{\alpha \to 0} \left(\frac{y_1 - y_2}{\alpha}\right).$$

# 7.7 INHOMOGENEOUS LINEAR ODES

We frame the discussion in terms of second-order ODEs, although the methods can be extended to equations of higher order. We thus consider ODEs of the general form

$$y'' + P(x)y' + Q(x)y = F(x), \tag{7.94}$$

and proceed under the assumption that the corresponding homogeneous equation, with $F(x) = 0$, has been solved, thereby obtaining two independent solutions designated $y_1(x)$ and $y_2(x)$.

## Variation of Parameters

The method of variation of parameters (variation of the constant) starts by writing a particular solution of the inhomogeneous ODE, Eq. (7.94), in the form

$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x). \tag{7.95}$$

We have specifically written $u_1(x)$ and $u_2(x)$ to emphasize that these are functions of the independent variable, and **not** constant coefficients. This, of course, means that Eq. (7.95) does not constitute a restriction to the functional form of $y(x)$. For clarity and compactness, we will usually write these functions just as $u_1$ and $u_2$.

In preparation for inserting $y(x)$, from Eq. (7.95), into the inhomogeneous ODE, we compute its derivative:

$$y' = u_1 y_1' + u_2 y_2' + (y_1 u_1' + y_2 u_2'),$$

and take advantage of the redundancy in the form assumed for $y$ by choosing $u_1$ and $u_2$ in such a way that

$$y_1 u_1' + y_2 u_2' = 0, \tag{7.96}$$

where Eq. (7.96) is assumed to be an identity (i.e., to apply for all $x$). We will shortly show that requiring Eq. (7.96) does not lead to an inconsistency.

After applying Eq. (7.96), $y'$, and its derivative $y''$, are found to be

$$y' = u_1 y_1' + u_2 y_2',$$
$$y'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2',$$

and substitution into Eq. (7.94) yields

$$(u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2') + P(x)(u_1 y_1' + u_2 y_2') + Q(x)(u_1 y_1 + u_2 y_2) = F(x),$$

which, because $y_1$ and $y_2$ are solutions of the homogeneous equation, reduces to

$$u_1' y_1' + u_2' y_2' = F(x). \tag{7.97}$$

Equations (7.96) and (7.97) are, for each value of $x$, a set of two simultaneous **algebraic** equations in the variables $u_1'$ and $u_2'$; to emphasize this point we repeat them here:

$$\begin{aligned} y_1 u_1' + y_2 u_2' &= 0, \\ y_1' u_1' + y_2' u_2' &= F(x). \end{aligned} \tag{7.98}$$

The determinant of the coefficients of these equations is

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix},$$

which we recognize as the Wronskian of the linearly independent solutions to the homogeneous equation. That means this determinant is nonzero, so there will, for each $x$, be a unique solution to Eqs. (7.98), i.e., unique functions $u_1'$ and $u_2'$. We conclude that the restriction implied by Eq. (7.96) is permissible.

Once $u_1'$ and $u_2'$ have been identified, each can be integrated, respectively yielding $u_1$ and $u_2$, and, via Eq. (7.95), a particular solution of our inhomogeneous ODE.

## *Example 7.7.1* An Inhomogeneous ODE

Consider the ODE

$$(1-x)y'' + xy' - y = (1-x)^2. \tag{7.99}$$

The corresponding homogeneous ODE has solutions $y_1 = x$ and $y_2 = e^x$. Thus, $y_1' = 1$, $y_2' = e^x$, and the simultaneous equations for $u_1'$ and $u_2'$ are

$$\begin{aligned} x\,u_1' + e^x u_2' &= 0, \\ u_1' + e^x u_2' &= F(x). \end{aligned} \tag{7.100}$$

Here $F(x)$ is the inhomogeneous term when the ODE has been written in the standard form, Eq. (7.94). This means that we must divide Eq. (7.99) through by $1-x$ (the coefficient of $y''$), after which we see that $F(x) = 1 - x$.

With the above choice of $F(x)$, we solve Eqs. (7.100), obtaining

$$u_1' = 1, \quad u_2' = -xe^{-x},$$

which integrate to

$$u_1 = x, \quad u_2 = (x+1)e^{-x}.$$

Now forming a particular solution to the inhomogeneous ODE, we have

$$y_p(x) = u_1 y_1 + u_2 y_2 = x(x) + \left((x+1)e^{-x}\right)e^x = x^2 + x + 1.$$

Because $x$ is a solution to the homogeneous equation, we may remove it from the above expression, leaving the more compact formula $y_p = x^2 + 1$.

The general solution to our ODE therefore takes the final form

$$y(x) = C_1 x + C_2 e^x + x^2 + 1.$$

■

## Exercises

**7.7.1** If our linear, second-order ODE is inhomogeneous, that is, of the form of Eq. (7.94), the **most general solution is**

$$y(x) = y_1(x) + y_2(x) + y_p(x),$$

where $y_1$ and $y_2$ are independent solutions of the homogeneous equation.
Show that

$$y_p(x) = y_2(x) \int^x \frac{y_1(s)F(s)ds}{W\{y_1(s), y_2(s)\}} - y_1(x) \int^x \frac{y_2(s)F(s)ds}{W\{y_1(s), y_2(s)\}},$$

with $W\{y_1(x), y_2(x)\}$ the Wronskian of $y_1(s)$ and $y_2(s)$.

Find the general solutions to the following inhomogeneous ODEs:

**7.7.2** $y'' + y = 1.$

**7.7.3** $y'' + 4y = e^x.$

**7.7.4** $y'' - 3y' + 2y = \sin x.$

**7.7.5** $xy'' - (1 + x)y' + y = x^2.$

# 7.8 Nonlinear Differential Equations

The main outlines of large parts of physical theory have been developed using mathematics in which the objects of concern possessed some sort of linearity property. As a result, linear algebra (matrix theory) and solution methods for linear differential equations were appropriate mathematical tools, and the development of these mathematical topics has progressed in the directions illustrated by most of this book. However, there is some physics that requires the use of nonlinear differential equations (NDEs). The hydrodynamics of viscous, compressible media is described by the Navier-Stokes equations, which are nonlinear. The nonlinearity evidences itself in phenomena such as turbulent flow, which cannot be described using linear equations. Nonlinear equations are also at the heart of the description of behavior known as **chaotic**, in which the evolution of a system is so sensitive to its initial conditions that it effectively becomes unpredictable.

The mathematics of nonlinear ODEs is both more difficult and less developed than that of linear ODEs, and accordingly we provide here only an extremely brief survey. Much of the recent progress in this area has been in the development of computational methods for nonlinear problems; that is also outside the scope of this text.

In this final section of the present chapter we discuss briefly some specific NDEs, the classical Bernoulli and Riccati equations.

## Bernoulli and Riccati Equations

Bernoulli equations are nonlinear, having the form

$$y'(x) = p(x)y(x) + q(x)[y(x)]^n, \tag{7.101}$$

where $p$ and $q$ are real functions and $n \neq 0, 1$ to exclude first-order linear ODEs. However, if we substitute

$$u(x) = [y(x)]^{1-n},$$

then Eq. (7.101) becomes a first-order linear ODE,

$$u' = (1-n)y^{-n}y' = (1-n)\big[p(x)u(x) + q(x)\big], \tag{7.102}$$

which we can solve (using an integrating factor) as described in Section 7.2.

Riccati equations are quadratic in $y(x)$:

$$y' = p(x)y^2 + q(x)y + r(x), \tag{7.103}$$

where we require $p \neq 0$ to exclude linear ODEs and $r \neq 0$ to exclude Bernoulli equations. There is no known general method for solving Riccati equations. However, when a special solution $y_0(x)$ of Eq. (7.103) is known by a guess or inspection, then one can write the general solution in the form $y = y_0 + u$, with $u$ satisfying the Bernoulli equation

$$u' = pu^2 + (2py_0 + q)u, \tag{7.104}$$

because substitution of $y = y_0 + u$ into Eq. (7.103) removes $r(x)$ from the resulting equation.

There are no general methods for obtaining exact solutions of most nonlinear ODEs. This fact makes it more important to develop methods for finding the qualitative behavior of solutions. In Section 7.5 of this chapter we mentioned that power-series solutions of ODEs exist except (possibly) at essential singularities of the ODE. The coefficients in the power-series expansions provide us with the asymptotic behavior of the solutions. By making expansions of solutions to NDEs and retaining only the linear terms, it will often be possible to understand the qualitative behavior of the solutions in the neighborhood of the expansion point.

## Fixed and Movable Singularities, Special Solutions

A first step in analyzing the solutions of NDEs is to identify their singularity structures. Solutions of NDEs may have singular points that are independent of the initial or boundary conditions; these are called **fixed singularities**. But in addition they may have **spontaneous**, or **movable**, singularities that vary with the initial or boundary conditions. This feature complicates the asymptotic analysis of NDEs.

## *Example 7.8.1* MOVEABLE SINGULARITY

Compare the linear ODE

$$y' + \frac{y}{x-1} = 0,$$

(which has an obvious regular singularity at $x = 1$), with the NDE $y' = y^2$. Both have the same solution with initial condition $y(0) = 1$, namely $y(x) = 1/(1-x)$. But for $y(0) = 2$, the linear ODE has solution $y = 1 + 1/(1-x)$, while the NDE now has solution $y(x) = 2/(1-2x)$. The singularity in the solution of the NDE has moved to $x = 1/2$. ■

For a linear second-order ODE we have a complete description of its solutions and their asymptotic behavior when two linearly independent solutions are known. But for NDEs there may still be **special solutions** whose asymptotic behavior is not obtainable from two independent solutions. This is another characteristic property of NDEs, which we illustrate again by an example.

## *Example 7.8.2* SPECIAL SOLUTION

The NDE $y'' = yy'/x$ has two linearly independent solutions that define the two-parameter family of curves

$$y(x) = 2c_1 \tan(c_1 \ln x + c_2) - 1, \tag{7.105}$$

where the $c_i$ are integration constants. However, this NDE also has the special solution $y = c_3 =$ constant, which cannot be obtained from Eq. (7.105) by any choice of the parameters $c_1, c_2$.

The "general solution" in Eq. (7.105) can be obtained by making the substitution $x = e^t$, and then defining $Y(t) \equiv y(e^t)$ so that $x(dy/dx) = dY/dt$, thereby obtaining the ODE $Y'' = Y'(Y + 1)$. This ODE can be integrated once to give $Y' = \frac{1}{2}Y^2 + Y + c$ with $c = 2(c_1^2 + 1/4)$ an integration constant. The equation for $Y'$ is separable and can be integrated again to yield Eq. (7.105). ■

## *Exercises*

**7.8.1** Consider the Riccati equation $y' = y^2 - y - 2$. A particular solution to this equation is $y = 2$. Find a more general solution.

**7.8.2** A particular solution to $y' = y^2/x^3 - y/x + 2x$ is $y = x^2$. Find a more general solution.

**7.8.3** Solve the Bernoulli equation $y' + xy = xy^3$.

**7.8.4** ODEs of the form $y = xy' + f(y')$ are known as Clairaut equations. The first step in solving an equation of this type is to differentiate it, yielding

$$y' = y' + xy'' + f'(y')y'', \quad \text{or} \quad y''\big(x + f'(y')\big) = 0.$$

Solutions may therefore be obtained both from $y'' = 0$ and from $f'(y') = -x$. The so-called general solution comes from $y'' = 0$.

For $f(y') = (y')^2$,

(a) Obtain the general solution (note that it contains a single constant).
(b) Obtain the so-called singular solution from $f'(y') = -x$. By substituting back into the original ODE show that this singular solution contains no adjustable constants.

*Note.* The singular solution is the envelope of the general solutions.

## Additional Readings


Cohen, H., *Mathematics for Scientists and Engineers*. Englewood Cliffs, NJ: Prentice-Hall (1992).

Golomb, M., and M. Shanks, *Elements of Ordinary Differential Equations*. New York: McGraw-Hill (1965).

Hubbard, J., and B. H. West, *Differential Equations*. Berlin: Springer (1995).

Ince, E. L., *Ordinary Differential Equations*. New York: Dover (1956). The classic work in the theory of ordinary differential equations.

Jackson, E. A., *Perspectives of Nonlinear Dynamics*. Cambridge: Cambridge University Press (1989).

Jordan, D. W., and P. Smith, *Nonlinear Ordinary Differential Equations*, 2nd ed. Oxford: Oxford University Press (1987).

Margenau, H., and G. M. Murphy, *The Mathematics of Physics and Chemistry*, 2nd ed. Princeton, NJ: Van Nostrand (1956).

Miller, R. K., and A. N. Michel, *Ordinary Differential Equations*. New York: Academic Press (1982).

Murphy, G. M., *Ordinary Differential Equations and Their Solutions*. Princeton, NJ: Van Nostrand (1960). A thorough, relatively readable treatment of ordinary differential equations, both linear and nonlinear.

Ritger, P. D., and N. J. Rose, *Differential Equations with Applications*. New York: McGraw-Hill (1968).

Sachdev, P. L., *Nonlinear Differential Equations and their Applications*. New York: Marcel Dekker (1991).

Tenenbaum, M., and H. Pollard, *Ordinary Differential Equations*. New York: Dover (1985). Detailed and readable (over 800 pages). This is a reprint of a work originally published in 1963, and stresses formal manipulations. Its references to numerical methods are somewhat dated.