Discrete Mathematics 255 (2002) 249–258
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Linear spaces with projective lines

Alexander Kreuzer
Mathematisches Seminar, Universität Hamburg, Bundesstr. 55, 20146 Hamburg, Germany

Received 5 April 1999; received in revised form 13 November 2000; accepted 22 December 2000

Abstract

A line L of a linear space $(P,\mathfrak{L})$ is a projective line, if L intersects every line G of the plane $\overline{L\cup\{x\}}$ for every $x\in P\setminus L$. In this paper a linear space $(P,\mathfrak{L})$ with projective lines is considered. We assume that for any two planes E_1,E_2 which intersect in a line G, there are two projective lines $L_i,K_i\subset E_i$ with distinct intersection points $p=L_1\cap L_2$, $q=K_1\cap K_2\in G$. Furthermore, it is assumed that for two intersecting lines H_1,H_2 of a plane F and a point $x\in F$ there exists a line G through x with $\emptyset\neq G\cap H_1\neq G\cap H_2\neq\emptyset$. Then the Bundle Theorem holds and $(P,\mathfrak{L})$ is locally projective. Therefore $(P,\mathfrak{L})$ is embeddable in a projective space (cf. Theorem 4.1).

1. Introduction

In order to embed absolute planes in pappian projective planes, Karzel introduced in [2] the kinematic space which points are defined by motions of the absolute plane. The aim is to embed the kinematic space in a projective space (cf. [3,4,6]). A characteristic property is that the kinematic space contains projective lines. This situation suggests to consider linear spaces with projective lines. But we remark that the assumptions of this paper differs from that of [3,4]. We assume only a few projective lines, but in every plane, while in [3,4] there are many projective lines, but not necessarily in every plane.

Now we give some definitions and recall some notations. A *linear space* $(P,\mathfrak{L},I)$ will be defined as a set P of elements, called *points*, a distinct set $\mathfrak{L}$ of elements, called *lines*, and an incidence relation I such that any two distinct points are incident with exactly one line and every line is incident with at least two points. Usually one identifies every line $L\in\mathfrak{L}$ with the set of points which are incident with L, hence the lines of $(P,\mathfrak{L},I)=(P,\mathfrak{L})$ are subsets of P.

E-mail address: kreuzer@math.uni-hamburg.de (A. Kreuzer).

0012-365X/02/$ - see front matter
PII: S0012-365X(01)00402-2

A *subspace* is a subset $U \subset P$ such that for all distinct points $x, y \in U$ the unique line passing through x and y is contained in U. Let $\mathfrak{U}$ denote the set of all subspaces. For every subset $X \subset P$ we define the following *closure operation*:

$$\bar{\ }: \mathfrak{P}(P) \to \mathfrak{U} : X \mapsto \bar{X} := \bigcap_{\substack{U \in \mathfrak{U} \\ X \subset U}} U. \tag{1.1}$$

For $U \in \mathfrak{U}$ we call $\dim U := \inf\{|X| - 1 : X \subset U \text{ and } \bar{X} = U\}$ the *dimension* of U. A subspace of dimension two is a *plane*.

A linear space $(P, \mathfrak{L})$ satisfies the *exchange condition* if for $S \subset P$ and $x, y \in P$ with $x \in \overline{S \cup \{y\}} \backslash \bar{S}$, it follows that $y \in \overline{S \cup \{x\}}$.

A line $L \in \mathfrak{L}$ is a *projective line* if $|L| \geqslant 3$ and if for every point $x \in P \backslash L$ every line $G \subset \overline{L \cup \{x\}}$ has a non-empty intersection with L.

A linear space $(P, \mathfrak{L})$ is called *locally projective*, if for every point $x \in P$ the lines and planes of $(P, \mathfrak{L})$ containing x form the points and lines of a projective space.

The Bundle Theorem states that for four lines $A, B, C, D \in \mathfrak{L}$, no three in a common plane, the coplanarities of $\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}$ imply the coplanarity of $\{C, D\}$.

The *Theorem of Kahn* states that every locally projective linear space $(P, \mathfrak{L})$ of $\dim P \geqslant 3$ satisfying the Bundle Theorem is embeddable into a projective space [1,5,9].

In this paper we consider the following properties for a linear space $(P, \mathfrak{L})$:

(E) For any two intersecting lines H_1, H_2 and every point $x \in \overline{H_1 \cup H_2}$ there exists a line G through x with $\emptyset \neq G \cap H_1 \neq G \cap H_2 \neq \emptyset$.

(P_1) Let E_1, E_2 be planes which intersect in a non-projective line $G = E_1 \cap E_2$. Then there exists a point $p \in G$ and projective lines $L_1 \subset E_1$, $L_2 \subset E_2$ with $p = L_1 \cap L_2$.

(P_2) Let E_1, E_2 be planes which intersect in a line $G = E_1 \cap E_2$. Then there are two distinct points $p, q \in G$ and projective lines $L_i, K_i \subset E_i$ with $p = L_1 \cap L_2$, $q = K_1 \cap K_2$ and $G \neq L_i, K_i$ for $i = 1, 2$.

Clearly (P_2) implies (P_1). We will show that (E) and (P_1) imply the Bundle Theorem (Section 2), and, if $|L| \geqslant 3$ for every line $L \in \mathfrak{L}$, the Bundle Theorem and (P_2) imply that $(P, \mathfrak{L})$ is locally projective (Section 3). Hence we can use the Theorem of Kahn and we summarize that (E) and (P_2) imply that $(P, \mathfrak{L})$ is embeddable in a projective space $(P', \mathfrak{L}')$ with $\dim P = \dim P'$.

2. Bundle Theorem

In this section we assume that (E) and (P_1) are satisfied. By (P_1) we have:

Lemma 2.1. *Every plane contains a projective line.*

Lemma 2.2. *Every plane satisfies the exchange condition.*

Proof. By Lemma 2.1 every plane E contains a projective line $L \subset E$. Hence $E = \bigcup_{x \in L} \overline{x, p} = \overline{L \cup \{p\}}$ for a point $p \in E \backslash L$. For non-collinear points $a, b, c \in E$ there are $x_a, x_b, x_c \in L$ with $a \in \overline{x_a, p}$, $b \in \overline{x_b, p}$, $c \in \overline{x_c, p}$. Since $L \cap \overline{a, b} \neq \emptyset$, $L \cap \overline{a, c} \neq \emptyset$ and $L \cap \overline{b, c} \neq \emptyset$, it follows that $L \cup \{p\} \subset \overline{a, b, c}$, hence $E = \overline{L \cup \{p\}} = \overline{a, b, c}$. $\square$

Remark. One can find this result in [4, (1.1)] where it is shown that for any three non-collinear points a, b, c of a plane E we have $E = \overline{a, b, c}$.

Lemma 2.3. *Let A, B, C be three lines, not in a common plane, which are pairwise coplanar. If $x = A \cap C$, then $x = A \cap B = B \cap C$.*

Proof. It holds $x \in \overline{A \cup B}$, $\overline{B \cup C}$. Since by Lemma 2.2 the planes satisfy the exchange condition, we have $x \in \overline{A \cup B} \cap \overline{B \cup C} = B$. Since the lines are pairwise distinct the assertion follows. $\square$

Lemma 2.4. *Let X, Y, Z, L be four lines, no three in a common plane, let X, Y, Z be pairwise coplanar and let X, Y, L be pairwise coplanar. If L is a projective line, then Z and L are coplanar.*

Proof. Since L is a projective line, the point $x := Y \cap L$ exists. By Lemma 2.3, $x = X \cap Y$ and again by Lemma 2.3, $x = X \cap Z$. Therefore $x = Z \cap L$ and Z, L are coplanar. $\square$

Lemma 2.5. *Let A, B, C, D be four lines, no three in a common plane, let A, B, C be pairwise coplanar and let A, B, D be pairwise coplanar. If there exists a projective line L with $c = L \cap C$, $d = L \cap D$, then C and D are coplanar.*

Proof. If any two of the lines A, B, C, D have a non-empty intersection, Lemma 2.3 implies the assertion. Hence we may assume that A, B, C, D have pairwise an empty intersection.

Case I. Assume that there exists a projective line $M \subset \overline{A \cup C}$ through c. Let $a := M \cap A$. By Lemma 2.1 there is a projective line $\tilde{N} \subset \overline{B \cup D}$.

Case Ia. If $d \in \tilde{N}$, let denote $N := \tilde{N}$ and $b := N \cap B$. Choose a point $c' \in C \backslash \{c\}$. By (E) there is a line $G' \subset \overline{B \cup C}$ through c' with $x := \overline{c, b} \cap G' \neq b' := B \cap G'$. Again by (E) a line $H' \subset \overline{A \cup B}$ through b' exists with $u := H' \cap \overline{a, b} \neq a' := H' \cap A$, and a line $K' \subset \overline{A \cup D}$ through a' with $y' := \overline{a, d} \cap K' \neq d' := K' \cap D$. Since $M = \overline{a, c}$, $N = \overline{b, d}$ are projective lines, the points $y := M \cap \overline{a', c'}$ and $x' := N \cap \overline{b', d'}$ exist (see Fig. 1).

Case Ib. If $d \notin \tilde{N}$, let denote $N' := \tilde{N}$, $d' := N' \cap D$ and $b' := N' \cap B$. By (E) there is a line $K' \subset \overline{A \cup D}$ through d' with $y' := K' \cap \overline{a, d} \neq a' := K' \cap A$. Again by (E) a line $H \subset \overline{A \cup B}$ through a exists with $u := H \cap \overline{a', b'}$ and $b := H \cap B$, and a line $G' \subset \overline{B \cup C}$ through b' with $x := G' \cap \overline{b, c} \neq c' := G' \cap C$. Since $M = \overline{a, c}$, $N' = \overline{b', d'}$ are projective lines, $y := M \cap \overline{a', c'}$ and $x' := N' \cap \overline{b, d}$ exist.

Case II. If there is no projective line in $\overline{A \cup C}$ passing through c, by (P_1) projective lines $M' \subset \overline{A \cup C}$ and $H' \subset \overline{A \cup B}$ exist with $a' := A \cap M' = A \cap H'$. Let $b' := B \cap H'$ and $c' := C \cap M'$. By the assumption of this case, we have $c' \neq c$. By (E) there is a line $G \subset \overline{B \cup C}$ through c with $x := \overline{c', b'} \cap G \neq b := G \cap B$. Again by (E) a line $N' \subset \overline{B \cup D}$

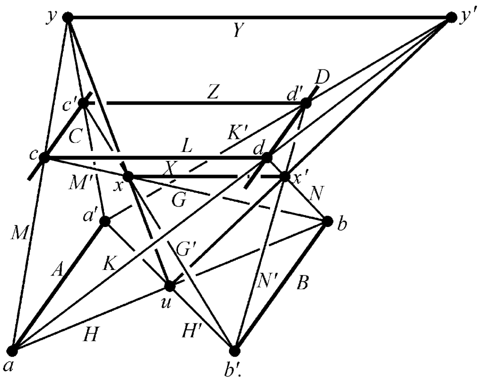

Fig. 1.

through b' exists with $x' := N' \cap \overline{b,d} \neq d' := N' \cap D$, and a line $K \subset \overline{A \cup D}$ through d with $y' := K \cap \overline{a',d'} \neq a := K \cap A$. Since $M' = \overline{a',c'}$ and $H' = \overline{a',b'}$ are projective lines, $y := M' \cap \overline{a,c}$ and $u := H' \cap \overline{a,b}$ exist.

In all cases we have the same figure. Since the lines A,B,C,D have pairwise an empty intersection and since no three of these lines are contained in a plane we have $x \neq x'$, $y \neq y'$, $c \neq d$ and $c' \neq d'$. Because the planes satisfy the exchange condition by Lemma 2.2, the points $u,x,y \in \overline{a,b,c}$, $\overline{a',b',c'}$ are contained in the intersection of two distinct planes, hence u,x,y are collinear. Also $u,x',y' \in \overline{a,b,d} \cap \overline{a',b',d'}$ are collinear. Therefore the lines $X := \overline{x,x'}$, $Y := \overline{y,y'}$ are coplanar. Let $L = \overline{c,d}$ and $Z := \overline{c',d'}$. Since $X,L \subset \overline{b,c,d}$ and $X,Z \subset \overline{b',c',d'}$ as well as $Y,L \subset \overline{a,c,d}$ and $Y,Z \subset \overline{a',c',d'}$, the three lines X,Y,Z and also X,Y,L, respectively, are pairwise coplanar. Because L is a projective line by Lemma 2.4 the coplanarity of L,Z follows and therefore also the coplanarity of C and D. $\square$

Theorem 2.6. *Let* $(P,\mathfrak{L})$ *be a linear space satisfying* (E) *and* (P_1). *Then the Bundle Theorem holds.*

Proof. Let A,B,C,D be four lines, no three in a common plane and let A,B,C be pairwise coplanar and also A,B,D. We have to show that C,D are coplanar. By Lemma 2.1 there is a projective line $M \subset \overline{A \cup C}$ with $a := A \cap M$, $c := C \cap M$. Also in the plane $\overline{\{c\} \cup D}$ there exists a projective line $\tilde{L}$. Let $d' := \tilde{L} \cap D$. If $c \in \tilde{L}$, it follows by Lemma 2.5 that C,D are coplanar. Hence we may assume $c \notin \tilde{L}$.

Since $\tilde{L}$ is a projective line, for every $d \in D \setminus \{d'\}$ we have $\tilde{L} \cap \overline{c,d} \neq \emptyset$. For every $c' \in C \setminus \{c\}$ it holds that $d' = \tilde{L} \cap \overline{c',d'}$, hence the projective line $\tilde{L}$ intersects $\overline{c,d}$ and $\overline{c',d'}$. We will show by Lemma 2.5 that $\overline{c,d}$ and $\overline{c',d'}$ are coplanar. Then it will follow that also C,D are coplanar (see Fig. 2).

Let $\tilde{N} \subset \overline{B \cup D}$ be a projective line.

Case I. If $d' \in \tilde{N}$, denote $N' := \tilde{N}$ and $b' := N' \cap B$. Choose any point $d \in D \setminus \{d'\}$.

Case II. If $d' \notin \tilde{N}$, denote $N := \tilde{N}$, $d := N \cap D$ and $b := N \cap B$.

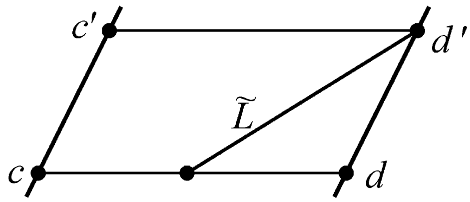

Fig. 2.

In both cases by (E), there is a line $K' \subset \overline{A \cup D}$ through d' with $y' := K' \cap \overline{a,d} \neq a' := K' \cap A$.

For case I by (E) a line $H \subset \overline{A \cup B}$ through a with $u := H \cap \overline{a',b'} \neq b := H \cap B$ exists. For case II we have a line $H' \subset \overline{A \cup B}$ through a' with $u := H' \cap \overline{a,b} \neq b' := H' \cap B$. In both cases there is a line $G' \subset \overline{B \cup C}$ through b' with $x := G' \cap \overline{b,c} \neq c' := G' \cap C$. Since $M = \overline{a,c}$ and $\tilde{N}$ are projective lines, the points $y := M \cap \overline{a',c'}$ and $x' := \overline{b,d} \cap \overline{b',d'}$ exist (cf. Fig. 1).

Again the points $u, x, y \in \overline{a,b,c} \cap \overline{a',b',c'}$ and $u, x', y' \in \overline{a,b,d} \cap \overline{a',b',d'}$ are collinear, hence $\overline{x,x'}, \overline{y,y'}$ are coplanar. It follows that $\overline{c,d}, \overline{x,x'}, \overline{y,y'}$ are pairwise coplanar and also $\overline{c',d'}, \overline{x,y'}, \overline{y,y'}$ are pairwise coplanar. Since $\tilde{L}$ is a projective line meeting $\overline{c,d}$ and $\overline{c',d'}$, by Lemma 2.5 $\overline{c,d}, \overline{c',d'}$ are coplanar. $\square$

3. Intersections of planes

Now let $(P, \mathfrak{L})$ be a linear space with $\dim P \geqslant 3$ and $|L| \geqslant 3$ for every line $L \in \mathfrak{L}$. In this section we assume the Bundle Theorem and the property (P_2).

Lemma 3.1. *Every plane E contains three projective lines. For every point $x \in E$ there is a projective line $L \subset E$ with $x \notin L$.*

Proof. By Lemma 2.1, E contains a projective line H. Since $\dim P \geqslant 3$, by (P_2) there are two distinct points $p, q \in H$ which are incident with projective lines $L, K \subset E$. The lines H, L, K do not contain a common point. $\square$

Theorem 3.2. *Let $(P, \mathfrak{L})$ be a linear space with $\dim P \geqslant 3$ and $|L| \geqslant 3$ for every line $L \in \mathfrak{L}$ which satisfies (P_2) and the Bundle Theorem. Then for two coplanar lines G, H and $z \in P \setminus \overline{G \cup H}$, the intersection of the planes $\overline{G \cup \{z\}}$, $\overline{H \cup \{z\}}$ is a line.*

Proof. If G or H is a projective line, then for $x := G \cap H$ we have $\overline{G \cup \{z\}} \cap \overline{H \cup \{z\}} = \overline{x,z}$. Hence we assume in what follows that G and H are not projective lines.

By (P_2) there are projective lines $L_1 \subset \overline{G \cup \{z\}}$, $L_2 \subset \overline{G \cup H}$ with $a := L_1 \cap L_2 \cap G$. Let $b := L_2 \cap H$. We give the proof in six steps. In what follows we denote the projective lines which we need in the proof by L_i or L_i', $i = 1, 2, \ldots, 5$. We assume the existence of intersecting points (for example the points $a, b, c, d, \ldots$) of a projective line L_i with a coplanar line without further explanation.

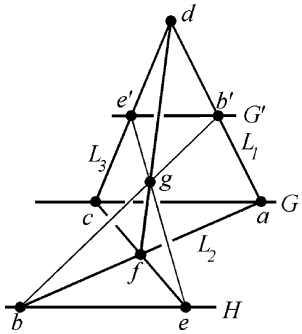

Fig. 3.

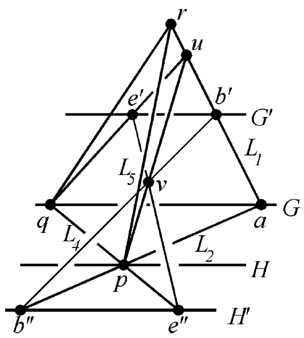

Fig. 4.

Step 1. There exists a line $G' \subset \overline{G \cup \{z\}} \backslash G$ which is coplanar to H:

By Lemma 3.1 there is a projective line $L_3 \subset \overline{G \cup \{z\}}$ with $c := L_3 \cap G \neq a$. Let $d := L_3 \cap L_1$. For a point $e \in H \backslash \{b\}$, the point $f := \overline{c, e} \cap L_2$ exists, since L_2 is a projective line. Let $g \in \overline{d, f} \backslash \{d, f\}$, then the points $b' := L_1 \cap \overline{b, g}$ and $e' := L_3 \cap \overline{e, g}$ exist and $G' := \overline{b', e'}$ is coplanar to H (see Fig. 3).

By (P_2) there are projective lines $L_4 \subset \overline{G \cup H}$ and $L_5 \subset \overline{L_1 \cup L_2}$, and a point $p \in L_2 \backslash \{a\}$ with $p = L_2 \cap L_4 \cap L_5$. Let $q := L_4 \cap G$ and $r := L_5 \cap L_1$. If $b \neq p$, i.e. if $p \notin H$, we continue with step 3. If $p \in H$ we need step 2.

Step 2. There exists a line $H' \subset \overline{G \cup H}$ with $p \notin H'$ which is coplanar to G':

For $e' \in G'$ the point $u := \overline{e', q} \cap L_1 \neq e'$ exists. For $v \in \overline{u, p} \backslash \{u, p\}$ also $e'' := L_4 \cap \overline{v, e'}$ and $b'' := L_2 \cap \overline{v, b'}$ exist and $H' := \overline{b'', e''}$ is coplanar to G' (see Fig. 4).

Since H, G', H' are pairwise coplanar, any line $K \subset \overline{G \cup \{z\}} = \overline{G \cup G'}$ which is coplanar to H' is also coplanar to H by the Bundle Theorem. Hence we could use H' instead of H. Because of this we may assume in the following that $p \notin H$.

Let $s := L_2 \cap H$ and $t := L_4 \cap H$.

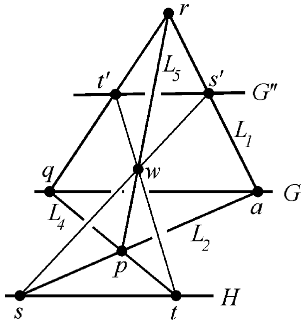

Fig. 5.

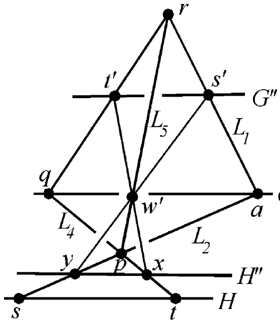

Fig. 6.

Step 3. There is a line $G'' \subset \overline{G \cup \{z\}}$ with $s' = G'' \cap L_1 \neq t' = G'' \cap \overline{q,r}$ which is coplanar to H:

Choose any $t' \in \overline{q,r} \setminus \{q,r\}$. Then the points $w := \overline{t,t'} \cap L_5$ and $s' := \overline{s,w} \cap L_1$ exist and $G'' := \overline{s',t'}$ is coplanar to H (see Fig. 5).

Step 4. Through every $x \in L_4 \setminus \{p\}$ there exists a line $H'' \subset \overline{G \cup H}$ which is coplanar to G'':

For $x \in L_4 \setminus \{p\}$ the points $w' := \overline{x,t'} \cap L_5$ and $y := \overline{s',w'} \cap L_2$ exist. Define $H'' := \overline{x,y}$ (see Fig. 6).

Step 5. If $z \notin L_1$, then the line $K := \overline{G \cup \{z\}} \cap \overline{H \cup \{z\}}$ exists:

We have $z \notin G$, hence the point $h := \overline{q,z} \cap L_1$ exists. For $h' \in \overline{h,p} \setminus \{h,p\}$ let $x := \overline{z,h'} \cap L_4$. By step 4 a line H'' with $x = H'' \cap L_4$ and $y := H'' \cap L_2$ exists which is coplanar to G''. Let $y' := \overline{h',y} \cap L_1$. Then $K := \overline{z,y'} \subset \overline{G \cup \{z\}}$ is coplanar to G'' and H''. Since also H is coplanar to G'' and H'', by the Bundle Theorem it follows that H and K are coplanar, hence $K \subset \overline{H \cup \{z\}}$. Because $z \notin \overline{G \cup H}$ it follows that $K = \overline{G \cup \{z\}} \cap \overline{H \cup \{z\}}$ (see Fig. 7).

Step 6. The case $z \in L_1$:

By (P_2) there is a second point $a' \in G \setminus \{a\}$ incident with projective lines $L_2' \subset \overline{G \cup H}$ and $L_1' \subset \overline{G \cup \{z\}}$. For every point $y \in \overline{G \cup \{z\}}$ distinct from $\tilde{z} := L_1 \cap L_1'$ the line

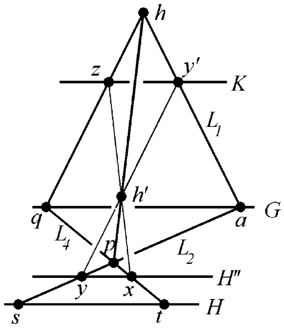

Fig. 7.

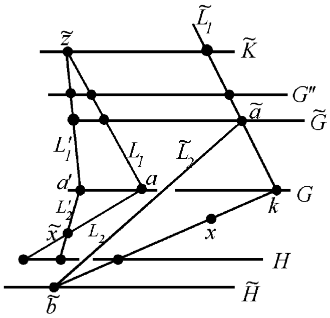

Fig. 8.

$\overline{G\cup\{y\}}\cap\overline{H\cup\{y\}}$ exists by step 5. Also for every point $x\in\overline{G\cup H}$ distinct from $\tilde{x}:=L_2\cap L_2'$ the line $\overline{G''\cup\{x\}}\cap\overline{G\cup\{x\}}$ exists (see Fig. 8).

By Lemma 3.1 there is a projective line $\tilde{L}_1\subset\overline{G\cup\{z\}}$ with $\tilde{z}\notin\tilde{L}_1$ and $k:=G\cap\tilde{L}_1$. Let $x\in\overline{G\cup H}$ with $\tilde{x}\notin\overline{x,k}$. By Lemma 3.1 we have a projective line $\tilde{L}_2\subset\overline{\tilde{L}_1\cup\{x\}}$ with $\tilde{a}:=\tilde{L}_2\cap\tilde{L}_1\neq\tilde{b}:=\tilde{L}_2\cap\overline{x,k}$. By step 2 we may assume that $\tilde{b}\notin H$. Since $\tilde{z}\notin\tilde{L}_1$ and $\tilde{x}\notin\overline{x,k}$ the lines $\tilde{G}:=\overline{G\cup\{\tilde{a}\}}\cap\overline{H\cup\{\tilde{a}\}}$ and $\tilde{H}:=\overline{G''\cup\{\tilde{b}\}}\cap\overline{H\cup\{\tilde{b}\}}$ exist. Since G'', $G\subset\overline{G\cup\{\tilde{a}\}}=\overline{G\cup\{c\}}$, by the Bundle Theorem applied on the lines $G'',H,\tilde{G},\tilde{H}$, the lines $\tilde{G}$ and $\tilde{H}$ are coplanar.

Now step 5 for $\tilde{G},\tilde{H},\tilde{z}$ implies the existence of the line $\tilde{K}:=\overline{\tilde{G}\cup\{\tilde{z}\}}\cap\overline{\tilde{H}\cup\{\tilde{z}\}}$. The line $\tilde{K}$ and H, respectively, are coplanar to $\tilde{H},\tilde{G}$. By the Bundle Theorem $\tilde{K}$ is coplanar to H, hence $\tilde{K}=\overline{G\cup\{\tilde{z}\}}\cap\overline{H\cup\{\tilde{z}\}}$. □

We remark that the hypotheses IG1, IG2 and IG3 made by Wyler [9] imply that the *incidence geometry* of Wyler is a linear space. The Theorem 3.2 coincide with IG4 and for $\dim P\geqslant 3$ there are four distinct points not on a plane, i.e., the last hypothesis IG5 made by Wyler is also satisfied. Therefore by [9, Theorem 2.5], we get the following theorem which is also proved by Teirlinck in [8, Proposition 1]:

Theorem 3.3. *Let $(P, \mathfrak{L})$ be a linear space with $|L| \geqslant 3$ for every line $L \in \mathfrak{L}$ which satisfies* (P_2) *and the Bundle Theorem. Then $(P, \mathfrak{L})$ is locally projective.*

We remark that for $\dim P \leqslant 2$ the assertion is trivial.

Corollary 3.4. *Let $(P, \mathfrak{L})$ be a linear space with $|L| \geqslant 3$ for every line $L \in \mathfrak{L}$ which satisfies* (E) *and* (P_2). *Then $(P, \mathfrak{L})$ is locally projective.*

Proof. By Theorem 2.6, $(P, \mathfrak{L})$ satisfies the Bundle Theorem, since (P_2) implies (P_1). Hence Theorem 3.3 proves the assertion. □

For Theorem 4.1 we do not need to know how the closure of a linear space $(P, \mathfrak{L})$ satisfying (P_2) and (E) works, but for completness we will show it for a three-dimensional subspace. (Then by induction one get the closure of any subspace.)

Lemma 3.5. *Let E be a plane and $a \in P \backslash E$. Then for the set*

$$T := \bigcup \{\overline{a,z}\colon\ z \in P \backslash \{a\} \text{ for which } \overline{a,z} \text{ is coplanar to a line } G \subset E\}$$

we have $T = \overline{E \cup \{a\}}$.

Proof. We show that T is a subspace of P. Then $E \cup \{a\} \subset T$ implies $\overline{E \cup \{a\}} \subset T$. By definition of T and by Lemma 2.2 clearly $T \subset \overline{E \cup \{a\}}$.

Let $x, y \in T \backslash \{a\}$ with $y \notin \overline{a,x}$ and let $z \in \overline{x,y}$. We will show that $z \in T$. For this we have to prove the existence of a line $Z \subset E$ which is coplanar to $\overline{a,z}$.

By definition there are lines $G, H \subset E$ such that $\overline{a,x}, G$ are coplanar and $\overline{a,y}, H$ are coplanar. We may assume $b := G \cap H \neq \emptyset$, since for $b \in G \backslash H$ the line $H' := E \cap \overline{a,y,b} = \overline{H \cup \{b\}} \cap \overline{\overline{a,y} \cup \{b\}}$ exists by Theorem 3.2, and then $H', \overline{a,y}$ are coplanar with $b \in H'$ (see Fig. 9).

Let $g \in G \backslash \{b\}$. Then by Theorem 3.2 the lines $K := \overline{x,g,y} \cap \overline{a,b,y}$ and $L := \overline{x,g,z} \cap \overline{a,b,z}$ exist, since $\overline{x,g}$ and $\overline{a,b}$ are coplanar. Because $K, H \subset \overline{a,b,y}$ the lines K, H are coplanar and by Theorem 3.2 the line $M := \overline{K \cup \{g\}} \cap \overline{H \cup \{g\}}$ exists.

Notice that $M \subset E = \overline{H \cup \{g\}} = \overline{M \cup \{b\}}$ and that $L, M \subset \overline{x,y,g}$ are coplanar. Let denote $Z := \overline{L \cup \{b\}} \cap \overline{M \cup \{b\}} = \overline{a,b,z} \cap E$. Then $\overline{a,z}$ and $Z \subset E$ are coplanar, hence $z \in T$. □

4. Embedding Theorem

Theorem 4.1. *Let $(P, \mathfrak{L})$ be a linear space with $|L| \geqslant 3$ for every line $L \in \mathfrak{L}$* and $\dim P \geqslant 3$. *Assume that $(P, \mathfrak{L})$ satisfies the properties* (E) *and* (P_2). *Then $(P, \mathfrak{L})$ is embeddable in a projective space $(P', \mathfrak{L}')$ with* $\dim P = \dim P'$.

Proof. By Theorem 2.6 and Corollary 3.4, $(P, \mathfrak{L})$ satisfies the Bundle Theorem and is locally projective. Hence $(P, \mathfrak{L})$ is embeddable by the Theorem of Kahn (cf. [1,5]). □

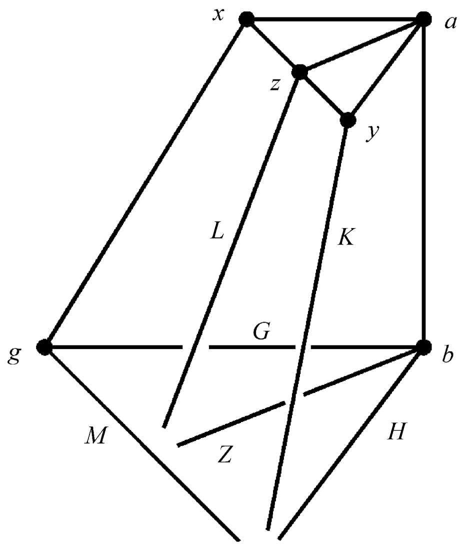

Fig. 9.

References

[1] J. Kahn, Locally projective-planar lattices which satisfy the Bundle Theorem, Math. Z. 175 (1980) 219–247.

[2] H. Karzel, Gruppentheoretische Begründung metrischer Geometrien. Vorlesungsausarbeitung von G. Graumann, Universität Hamburg, 1963.

[3] H. Karzel, H.-J. Kroll, Einbettung von Inzidenzräumen mit Eigentlichkeitsbereich, Abh. Math. Sem. Univ. Hamburg 49 (1979) 82–94.

[4] H. Karzel, H.-J. Kroll, Zur Inzidenzstruktur kinematischer Räume und absoluter Ebenen. Beiträge zur Geometrie und Algebra (TUM-Bericht M 8010, TU München) 6 (1980) 24–61.

[5] A. Kreuzer, Locally projective spaces which satisfy the Bundle Theorem, J. Geom. 56 (1996) 87–98.

[6] E.M. Schröder, Aufbau metrischer Geometrie aus der Hexagrammbedingung, Atti Sem. Mat. Fis. Univ. Modena 33 (1984) 183–217.

[7] K. Sörensen, Projektive Einbettung angeordneter Räume, Beiträge zur Geometrie und Algebra (TUM-Bericht M 8612, TU München) 15 (1986) 8–15.

[8] L. Teirlinck, Combinatorical properties of planar spaces and embeddability, J. Combin. Theory Ser. A 43 (1986) 291–302.

[9] O. Wyler, Incidence geometry, Duke Math. J. 20 (1953) 601–610.