Chapter 11
Linear Algebra

Linear algebra involves the systematic solving of linear algebraic or differential equations. These equations arise in a wide variety of situations. They usually involve some system, either electrical, mechanical, or even human, where two or more components are interacting with each other. In this chapter we present efficient techniques for expressing these systems and their solution.

11.1 FUNDAMENTALS OF LINEAR ALGEBRA

In this chapter we shall study the solution of m simultaneous linear equations in n unknowns $x_1, x_2, x_3, \ldots, x_n$ of the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned} \tag{11.1.1}$$

where the a's and b's are known real or complex numbers. *Matrix algebra* allows us to solve these systems. First, succinct notation is introduced so that we can replace (11.1.1) with rather simple expressions. Then a set of rules is used to manipulate these simple expressions. In this section we focus on developing these simple expressions.

The fundamental quantity in linear algebra is the *matrix*. A matrix is an ordered rectangular array of numbers or mathematical expressions. We shall use upper case letters to denote them. The $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{ij} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & a_{m3} & \cdot & \cdot & \cdot & a_{mn} \end{pmatrix} \tag{11.1.2}$$

has m *rows* and n *columns*. The *order* (or size) of a matrix is determined by the number of rows and columns; (11.1.2) is of order m by n. If $m = n$, the matrix is a *square* matrix; otherwise, A is *rectangular*. The numbers or expressions in the array a_{ij} are the *elements* of A and may be either real or complex. When all of the elements are real, A is a *real matrix*. If some or all of the elements are complex, then A is a *complex matrix*. For a square matrix, the diagonal from the top left corner to the bottom right corner is the *principal diagonal*.

From the limitless number of possible matrices, certain ones appear with sufficient regularity that they are given special names. A *zero* matrix (sometimes called a *null* matrix) has all of its elements equal to zero. It fulfills the role in matrix algebra that is analogous to that of zero in scalar algebra. The *unit* or *identity* matrix is a $n \times n$ matrix having 1's along its principal diagonal and zero everywhere else. The unit matrix serves essentially the same purpose in matrix algebra as does the number one in scalar algebra. A *symmetric* matrix is one where $a_{ij} = a_{ji}$ for all i and j.

• Example 11.1.1

Examples of zero, identity, and symmetric matrices are

$$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 1 & 0 \\ 4 & 0 & 5 \end{pmatrix}, \tag{11.1.3}$$

respectively.

A special class of matrices are *column vectors* and *row vectors*:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}. \tag{11.1.4}$$

We denote row and column vectors by lower case, boldface letters. The length or *norm* of the vector $\mathbf{x}$ of n elements is

$$||\mathbf{x}|| = \left(\sum_{k=1}^{n} x_k^2 \right)^{1/2} . \tag{11.1.5}$$

Two matrices A and B are equal if and only if $a_{ij} = b_{ij}$ for all possible i and j and they have the same dimensions.

Having defined a matrix, let us explore some of its arithmetic properties. For two matrices A and B with the same dimensions (conformable for addition), the matrix $C = A + B$ contains the elements $c_{ij} = a_{ij} + b_{ij}$. Similarly, $C = A - B$ contains the elements $c_{ij} = a_{ij} - b_{ij}$. Because the order of addition does not matter, addition is *commutative*: $A + B = B + A$.

Consider now a scalar constant k. The product kA is formed by multiplying every element of A by k. Thus the matrix kA has elements ka_{ij}.

So far the rules for matrix arithmetic have conformed to their scalar counterparts. However, there are several possible ways of multiplying two matrices together. For example, we might simply multiply together the corresponding elements from each matrix. As we will see, the multiplication rule is designed to facilitate the solution of linear equations.

We begin by requiring that the dimensions of A be $m \times n$ while for B they are $n \times p$. That is, the number of columns in A must equal the number of rows in B. The matrices A and B are then said to be *conformable* for multiplication. If this is true, then $C = AB$ will be a matrix $m \times p$, where its elements equal

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}. \tag{11.1.6}$$

The right side of (11.1.6) is referred to as an *inner product* of the ith row of A and the jth column of B. Although (11.1.6) is the method used with a computer, an easier method for human computation is as a running sum of the products given by successive elements of the ith row of A and the corresponding elements of the jth column of B.

The product AA is usually written A^2; the product AAA, A^3, and so forth.

• Example 11.1.2

If

$$A = \begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \tag{11.1.7}$$

then

$$AB = \begin{pmatrix} [(-1)(1) + (4)(3)] & [(-1)(2) + (4)(4)] \\ [(2)(1) + (-3)(3)] & [(2)(2) + (-3)(4)] \end{pmatrix} \tag{11.1.8}$$

$$= \begin{pmatrix} 11 & 14 \\ -7 & -8 \end{pmatrix}. \tag{11.1.9}$$

Matrix multiplication is associative and distributive with respect to addition:

$$(kA)B = k(AB) = A(kB), \tag{11.1.10}$$

$$A(BC) = (AB)C, \tag{11.1.11}$$

$$(A + B)C = AC + BC \tag{11.1.12}$$

and

$$C(A + B) = CA + CB. \tag{11.1.13}$$

On the other hand, matrix multiplication is *not commutative.* In general, $AB \neq BA$.

• Example 11.1.3

Does $AB = BA$ if

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}? \tag{11.1.14}$$

Because

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{11.1.15}$$

and

$$BA = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \tag{11.1.16}$$

$$AB \neq BA. \tag{11.1.17}$$

• Example 11.1.4

Given

$$A = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad (11.1.18)$$

find the product AB.

Performing the calculation, we find that

$$AB = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \qquad (11.1.19)$$

The point here is that just because $AB = 0$, this does *not* imply that either A or B equals the zero matrix.

We cannot properly speak of division when we are dealing with matrices. Nevertheless, a matrix A is said to be *nonsingular* or *invertible* if there exists a matrix B such that $AB = BA = I$. This matrix B is the multiplicative inverse of A or simply the *inverse* of A, written A^{-1}. A $n \times n$ matrix is *singular* if it does not have a multiplicative inverse.

• Example 11.1.5

If

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix}, \qquad (11.1.20)$$

let us verify that its inverse is

$$A^{-1} = \begin{pmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{pmatrix}. \qquad (11.1.21)$$

We perform the check by finding AA^{-1} or $A^{-1}A$,

$$AA^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (11.1.22)$$

In a later section we will show how to compute the inverse, given A.

Another matrix operation is transposition. The *transpose* of a matrix A with dimensions $m \times n$ is another matrix, written A^T, where we have interchanged the rows and columns from A. Clearly, $(A^T)^T = A$ as well as $(A + B)^T = A^T + B^T$ and $(kA)^T = kA^T$. If A and B are conformable for multiplication, then $(AB)^T = B^T A^T$. Note the reversal of order between the two sides. To prove this last result, we first show that the results are true for two 3×3 matrices A and B and then generalize to larger matrices.

Having introduced some of the basic concepts of linear algebra, we are ready to rewrite (11.1.1) in a canonical form so that we may present techniques for its solution. We begin by writing (11.1.1) as a single column vector:

$$\begin{pmatrix} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n \\ \vdots & & \vdots & & \vdots & & \vdots \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_m \end{pmatrix}. \tag{11.1.23}$$

On the left side of (11.1.23) we can use the multiplication rule to write

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_m \end{pmatrix} \tag{11.1.24}$$

or

$$A\mathbf{x} = \mathbf{b}, \tag{11.1.25}$$

where $\mathbf{x}$ is the solution vector. If $\mathbf{b} = \mathbf{0}$, we have a *homogeneous* set of equations; otherwise, we have a *nonhomogeneous* set. In the next few sections, we will give a number of methods for finding $\mathbf{x}$.

• Example 11.1.6: Solution of a tridiagonal system

A common problem in linear algebra involves solving systems such as

$$b_1y_1 + c_1y_2 = d_1 \tag{11.1.26}$$

$$a_2y_1 + b_2y_2 + c_2y_3 = d_2 \tag{11.1.27}$$

$$\vdots$$

$$a_{N-1}y_{N-2} + b_{N-1}y_{N-1} + c_{N-1}y_N = d_{N-1} \tag{11.1.28}$$

$$b_Ny_{N-1} + c_Ny_N = d_N. \tag{11.1.29}$$

Such systems arise in the numerical solution of ordinary and partial differential equations.

We begin our analysis by rewriting (11.1.26)–(11.1.29) in the matrix notation:

$$\begin{pmatrix} b_1 & c_1 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & \cdots & 0 & 0 & 0 \\ 0 & a_3 & b_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{N-1} & b_{N-1} & c_{N-1} \\ 0 & 0 & 0 & \cdots & 0 & a_N & b_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-1} \\ y_N \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{N-1} \\ d_N \end{pmatrix}. \qquad \textbf{(11.1.30)}$$

The matrix in (11.1.30) is an example of a *banded matrix*: a matrix where all of the elements in each row are zero except for the diagonal element and a limited number on either side of it. In our particular case, we have a *tridiagonal* matrix in which only the diagonal element and the elements immediately to its left and right in each row are nonzero.

Consider the nth equation. We can eliminate a_n by multiplying the $(n-1)$th equation by a_n/b_{n-1} and subtracting this new equation from the nth equation. The values of b_n and d_n become

$$b'_n = b_n - a_n c_{n-1}/b_{n-1} \qquad \textbf{(11.1.31)}$$

and

$$d'_n = d_n - a_n d_{n-1}/b_{n-1} \qquad \textbf{(11.1.32)}$$

for $n = 2, 3, \ldots, N$. The coefficient c_n is unaffected. Because elements a_1 and c_N are never involved, their values can be anything or they can be left undefined. The new system of equations may be written

$$\begin{pmatrix} b'_1 & c_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & b'_2 & c_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & b'_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & b'_{N-1} & c_{N-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & b'_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-1} \\ y_N \end{pmatrix} = \begin{pmatrix} d'_1 \\ d'_2 \\ d'_3 \\ \vdots \\ d'_{N-1} \\ d'_N \end{pmatrix}. \qquad \textbf{(11.1.33)}$$

The matrix in (11.1.33) is in *upper triangular* form because all of the elements below the principal diagonal are zero. This is particularly useful because y_n may be computed by *back substitution*. That is, we first compute y_N. Next, we calculate y_{N-1} in terms of y_N. The solution y_{N-2} may then be computed in terms of y_N and y_{N-1}. We continue this process until we find y_1 in terms of $y_N, y_{N-1}, \ldots, y_2$. In the present case, we have the rather simple:

$$y_N = d'_N/b'_N \qquad \textbf{(11.1.34)}$$

and

$$y_n = (d'_n - c_n d'_{n+1})/b'_n \qquad \textbf{(11.1.35)}$$

for $n = N-1, N-2, \ldots, 2, 1$.

As we shall show shortly, this is an example of solving a system of linear equations by Gaussian elimination. For a tridiagonal case, we have the advantage that the solution can be expressed in terms of a recurrence relationship, a very convenient feature from a computational point of view. This algorithm is very robust, being stable[1] as long as $|a_i + c_i| < |b_i|$. By stability, we mean that if we change $\mathbf{b}$ by $\Delta\mathbf{b}$ so that $\mathbf{x}$ changes by $\Delta\mathbf{x}$, then $||\Delta\mathbf{x}|| < M\epsilon$, where $\epsilon \geq ||\Delta\mathbf{b}||$, $0 < M < \infty$, for *any* N.

Problems

Given $A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, find

1. $A+B$, $B+A$
2. $A-B$, $B-A$
3. $3A-2B$, $3(2A-B)$
4. A^T, B^T, $(B^T)^T$
5. $(A+B)^T$, A^T+B^T
6. $B+B^T$, $B-B^T$
7. AB, A^TB, BA, B^TA
8. A^2, B^2
9. BB^T, B^TB
10. A^2-3A+I
11. A^3+2A
12. A^4-4A^2+2I

Can multiplication occur between the following matrices? If so, compute it.

13. $\begin{pmatrix} 3 & 5 & 1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 4 & 1 \\ 1 & 3 \end{pmatrix}$

14. $\begin{pmatrix} -2 & 4 \\ -4 & 6 \\ -6 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

15. $\begin{pmatrix} 1 & 4 & 2 \\ 0 & 0 & 4 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$

16. $\begin{pmatrix} 4 & 6 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \end{pmatrix}$

17. $\begin{pmatrix} 6 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 & 4 \\ 2 & 0 & 6 \end{pmatrix}$

If $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 3 & 1 \end{pmatrix}$ verify that

18. $7A = 4A + 3A$,
19. $10A = 5(2A)$,
20. $(A^T)^T = A$.

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[1] Torii, T., 1966: Inversion of tridiagonal matrices and the stability of tridiagonal systems of linear systems. *Tech. Rep. Osaka Univ.*, **16**, 403–414.

If $A = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -2 \\ 4 & 0 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, verify that

21. $(A+B)+C = A+(B+C)$,
22. $(AB)C = A(BC)$,
23. $A(B+C) = AB+AC$,
24. $(A+B)C = AC+BC$.

Verify that the following A^{-1} are indeed the inverse of A:

25. $A = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$ $\qquad A^{-1} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$

26. $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\qquad A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Write the following linear systems of equations in matrix form: $A\mathbf{x} = \mathbf{b}$.

27.

$$x_1 - 2x_2 = 5$$
$$3x_1 + x_2 = 1$$

28.

$$2x_1 + x_2 + 4x_3 = 2$$
$$4x_1 + 2x_2 + 5x_3 = 6$$
$$6x_1 - 3x_2 + 5x_3 = 2$$

29.

$$x_2 + 2x_3 + 3x_4 = 2$$
$$3x_1 - 4x_3 - 4x_4 = 5$$
$$x_1 + x_2 + x_3 + x_4 = -3$$
$$2x_1 - 3x_2 + x_3 - 3x_4 = 7.$$

11.2 DETERMINANTS

Determinants appear naturally during the solution of simultaneous equations. Consider, for example, two simultaneous equations with two unknowns x_1 and x_2,

$$a_{11}x_1 + a_{12}x_2 = b_1 \tag{11.2.1}$$

and

$$a_{21}x_1 + a_{22}x_2 = b_2. \tag{11.2.2}$$

The solution to these equations for the value of x_1 and x_2 is

$$x_1 = \frac{b_1 a_{22} - a_{12} b_2}{a_{11}a_{22} - a_{12}a_{21}} \tag{11.2.3}$$

and

$$x_2 = \frac{b_2 a_{11} - a_{21} b_1}{a_{11} a_{22} - a_{12} a_{21}}. \qquad \textbf{(11.2.4)}$$

Note that the denominator of (11.2.3) and (11.2.4) are the same. This term, which will always appear in the solution of 2×2 systems, is formally given the name of *determinant* and written

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \qquad \textbf{(11.2.5)}$$

Although determinants have their origin in the solution of systems of equations, any square array of numbers or expressions possesses a unique determinant, independent of whether it is involved in a system of equations or not. This determinant is evaluated (or expanded) according to a formal rule known as *Laplace's expansion of cofactors.*[2] The process revolves around expanding the determinant using any arbitrary column *or* row of A. If the ith row or jth column is chosen, the determinant is given by

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \qquad \textbf{(11.2.6)}$$

$$= a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}, \qquad \textbf{(11.2.7)}$$

where A_{ij}, the *cofactor* of a_{ij}, equals $(-1)^{i+j}M_{ij}$. The minor M_{ij} is the determinant of the $(n-1) \times (n-1)$ submatrix obtained by deleting row i, column j of A. This rule, of course, was chosen so that determinants are still useful in solving systems of equations.

• **Example 11.2.1**

Let us evaluate

$$\begin{vmatrix} 2 & -1 & 2 \\ 1 & 3 & 2 \\ 5 & 1 & 6 \end{vmatrix}$$

by an expansion in cofactors.

Using the first column,

$$\begin{vmatrix} 2 & -1 & 2 \\ 1 & 3 & 2 \\ 5 & 1 & 6 \end{vmatrix} = 2(-1)^2 \begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} + 1(-1)^3 \begin{vmatrix} -1 & 2 \\ 1 & 6 \end{vmatrix} + 5(-1)^4 \begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} \qquad \textbf{(11.2.8)}$$

$$= 2(16) - 1(-8) + 5(-8) = 0. \qquad \textbf{(11.2.9)}$$

[2] Laplace, P. S., 1772: Recherches sur le calcul intégral et sur le système du monde. *Hist. Acad. R. Sci., II^e Partie*, 267–376. *Œuvres*, **8**, pp. 369–501. See Muir, T., 1960: *The Theory of Determinants in the Historical Order of Development, Vol. I, Part 1, General Determinants Up to 1841*, Dover Publishers, Mineola, NY, pp. 24–33.

The greatest source of error is forgetting to take the factor $(-1)^{i+j}$ into account during the expansion.

Although Laplace's expansion does provide a method for calculating $\det(A)$, the number of calculations equals $(n!)$. Consequently, for hand calculations, an obvious strategy is to select the column or row that has the greatest number of zeros. An even better strategy would be to manipulate a determinant with the goal of introducing zeros into a particular column or row. In the remaining portion of section, we show some operations that may be performed on a determinant to introduce the desired zeros. Most of the properties follow from the expansion of determinants by cofactors.

- *Rule 1* : For every square matrix A, $\det(A^T) = \det(A)$.

The proof is left as an exercise.

- *Rule 2* : If any two rows or columns of A are identical, $\det(A) = 0$.

To see that this is true, consider the following 3×3 matrix:

$$\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} = c_1(b_2b_3 - b_3b_2) - c_2(b_1b_3 - b_3b_1) + c_3(b_1b_2 - b_2b_1) = 0. \tag{11.2.10}$$

- *Rule 3* : The determinant of a triangular matrix is equal to the product of its diagonal elements.

If A is lower triangular, successive expansions by elements in the first column give

$$\det(A) = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} \tag{11.2.11}$$

$$= \cdots = a_{11}a_{22}\cdots a_{nn}. \tag{11.2.12}$$

If A is upper triangular, successive expansions by elements of the first row proves the property.

- *Rule 4* : If a square matrix A has either a row or a column of all zeros, then $\det(A) = 0$.

The proof is left as an exercise.

• *Rule 5* : If each element in one row (column) of a determinant is multiplied by a number c, the value of the determinant is multiplied by c.

Suppose $|B|$ has been obtained from $|A|$ by multiplying row i (column j) of $|A|$ by c. Upon expanding $|B|$ in terms of row i (column j) each term in the expansion contains c as a factor. Factor out the common c, the result is just c times the expansion $|A|$ by the same row (column).

• *Rule 6* : If each element of a row (or a column) of a determinant can be expressed as a binomial, the determinant can be written as the sum of two determinants.

To understand this property, consider the following 3×3 determinant:

$$\begin{vmatrix} a_1+d_1 & b_1 & c_1 \\ a_2+d_2 & b_2 & c_2 \\ a_3+d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}. \quad \textbf{(11.2.13)}$$

The proof follows by expanding the determinant by the row (or column) that contains the binomials.

• *Rule 7* : If B is a matrix obtained by interchanging any two rows (columns) of a square matrix A, then $\det(B) = -\det(A)$.

The proof is by induction. It is easily shown for any 2×2 matrix. Assume that this rule holds of any $(n-1) \times (n-1)$ matrix. If A is $n \times n$, then let B be a matrix formed by interchanging rows i and j. Expanding $|B|$ and $|A|$ by a different row, say k, we have that

$$|B| = \sum_{s=1}^{n} (-1)^{k+s} b_{ks} M_{ks} \quad \text{and} \quad |A| = \sum_{s=1}^{n} (-1)^{k+s} a_{ks} N_{ks}, \quad \textbf{(11.2.14)}$$

where M_{ks} and N_{ks} are the minors formed by deleting row k, column s from $|B|$ and $|A|$, respectively. For $s = 1, 2, \ldots, n$, we obtain N_{ks} and M_{ks} by interchanging rows i and j. By the induction hypothesis and recalling that N_{ks} and M_{ks} are $(n-1) \times (n-1)$ determinants, $N_{ks} = -M_{ks}$ for $s = 1, 2, \ldots, n$. Hence, $|B| = -|A|$. Similar arguments hold if two columns are interchanged.

• *Rule 8* : If one row (column) of a square matrix A equals to a number c times some other row (column), then $\det(A) = 0$.

Suppose one row of a square matrix A is equal to c times some other row. If $c = 0$, then $|A| = 0$. If $c \neq 0$, then $|A| = c|B|$, where $|B| = 0$ because $|B|$ has two identical rows. A similar argument holds for two columns.

• *Rule 9* : The value of det(A) is unchanged if any arbitrary multiple of any line (row or column) is added to any other line.

To see that this is true, consider the simple example:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} cb_1 & b_1 & c_1 \\ cb_2 & b_2 & c_2 \\ cb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + cb_1 & b_1 & c_1 \\ a_2 + cb_2 & b_2 & c_2 \\ a_3 + cb_3 & b_3 & c_3 \end{vmatrix}, \qquad (11.2.15)$$

where $c \neq 0$. The first determinant on the left side is our original determinant. In the second determinant, we can again expand the first column and find that

$$\begin{vmatrix} cb_1 & b_1 & c_1 \\ cb_2 & b_2 & c_2 \\ cb_3 & b_3 & c_3 \end{vmatrix} = c \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} = 0. \qquad (11.2.16)$$

• **Example 11.2.2**

Let us evaluate

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 2 & 3 \\ 1 & -1 & 1 & 2 \\ -1 & 1 & -1 & 5 \end{vmatrix}$$

using a combination of the properties stated above and expansion by cofactors.

By adding or subtracting the first row to the other rows, we have that

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 2 & 3 \\ 1 & -1 & 1 & 2 \\ -1 & 1 & -1 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 5 & 7 \\ 0 & -3 & -2 & -2 \\ 0 & 3 & 2 & 9 \end{vmatrix} \qquad (11.2.17)$$

$$= \begin{vmatrix} 3 & 5 & 7 \\ -3 & -2 & -2 \\ 3 & 2 & 9 \end{vmatrix} = \begin{vmatrix} 3 & 5 & 7 \\ 0 & 3 & 5 \\ 0 & -3 & 2 \end{vmatrix} \qquad (11.2.18)$$

$$= 3 \begin{vmatrix} 3 & 5 \\ -3 & 2 \end{vmatrix} = 3 \begin{vmatrix} 3 & 5 \\ 0 & 7 \end{vmatrix} = 63. \qquad (11.2.19)$$

Problems

Evaluate the following determinants:

1. $\begin{vmatrix} 3 & 5 \\ -2 & -1 \end{vmatrix}$

2. $\begin{vmatrix} 5 & -1 \\ -8 & 4 \end{vmatrix}$

3. $\begin{vmatrix} 3 & 1 & 2 \\ 2 & 4 & 5 \\ 1 & 4 & 5 \end{vmatrix}$

4. $\begin{vmatrix} 4 & 3 & 0 \\ 3 & 2 & 2 \\ 5 & -2 & -4 \end{vmatrix}$

5. $\begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix}$

6. $\begin{vmatrix} 2 & -1 & 2 \\ 1 & 3 & 3 \\ 5 & 1 & 6 \end{vmatrix}$

7. $\begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 1 & 0 \\ 1 & 1 & -2 & 3 \end{vmatrix}$

8. $\begin{vmatrix} 2 & 1 & 2 & 1 \\ 3 & 0 & 2 & 2 \\ -1 & 2 & -1 & 1 \\ -3 & 2 & 3 & 1 \end{vmatrix}$

9. Using the properties of determinants, show that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = (b-a)(c-a)(d-a)(c-b)(d-b)(d-c).$$

This determinant is called *Vandermonde's determinant.*

10. Show that

$$\begin{vmatrix} a & b+c & 1 \\ b & a+c & 1 \\ c & a+b & 1 \end{vmatrix} = 0.$$

11. Show that if all of the elements of a row or column are zero, then $\det(A) = 0$.

12. Prove that $\det(A^T) = \det(A)$.

11.3 CRAMER'S RULE

One of the most popular methods for solving simple systems of linear equations is Cramer's rule.[3] It is very useful for 2×2 systems, acceptable for 3×3 systems, and of doubtful use for 4×4 or larger systems.

Let us have n equations with n unknowns, $A\mathbf{x} = \mathbf{b}$. Cramer's rule states that

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}, \tag{11.3.1}$$

where A_i is a matrix obtained from A by replacing the ith column with $\mathbf{b}$ and $n = 1, 2, 3, \ldots$. Obviously, $\det(A) \neq 0$ if Cramer's rule is to work.

To prove Cramer's rule, consider

$$x_1 \det(A) = \begin{vmatrix} a_{11}x_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21}x_1 & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31}x_1 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1}x_1 & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \tag{11.3.2}$$

by Rule 5 from the previous section. By adding x_2 times the second column to the first column,

$$x_1 \det(A) = \begin{vmatrix} a_{11}x_1 + a_{12}x_2 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21}x_1 + a_{22}x_2 & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31}x_1 + a_{32}x_2 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}. \tag{11.3.3}$$

Multiplying each of the columns by the corresponding x_i and adding it to the first column yields,

$$x_1 \det(A) = \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}. \tag{11.3.4}$$

[3] Cramer, G., 1750: *Introduction à l'analyse des lignes courbes algébriques*, Geneva, p. 657.

The first column of (11.3.4) equals $A\mathbf{x}$ and we replace it with $\mathbf{b}$. Thus,

$$x_1 \det(A) = \begin{vmatrix} b_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ b_2 & a_{22} & a_{23} & \cdots & a_{2n} \\ b_3 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_n & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = \det(A_1) \tag{11.3.5}$$

or

$$x_1 = \frac{\det(A_1)}{\det(A)} \tag{11.3.6}$$

provided $\det(A) \neq 0$. To complete the proof we do exactly the same procedure to the jth column. □

• Example 11.3.1

Let us solve the following system of equations by Cramer's rule:

$$2x_1 + x_2 + 2x_3 = -1, \tag{11.3.7}$$

$$x_1 + x_3 = -1 \tag{11.3.8}$$

and

$$-x_1 + 3x_2 - 2x_3 = 7. \tag{11.3.9}$$

From the matrix form of the equations,

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ -1 & 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 7 \end{pmatrix}, \tag{11.3.10}$$

we have that

$$\det(A) = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ -1 & 3 & -2 \end{vmatrix} = 1, \tag{11.3.11}$$

$$\det(A_1) = \begin{vmatrix} -1 & 1 & 2 \\ -1 & 0 & 1 \\ 7 & 3 & -2 \end{vmatrix} = 2, \tag{11.3.12}$$

$$\det(A_2) = \begin{vmatrix} 2 & -1 & 2 \\ 1 & -1 & 1 \\ -1 & 7 & -2 \end{vmatrix} = 1 \tag{11.3.13}$$

and

$$\det(A_3) = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & 3 & 7 \end{vmatrix} = -3. \tag{11.3.14}$$

Finally,

$$x_1 = \frac{2}{1} = 2, \quad x_2 = \frac{1}{1} = 1 \quad \text{and} \quad x_3 = \frac{-3}{1} = -3. \qquad \textbf{(11.3.15)}$$

Problems

Solve the following systems of equations by Cramer's rule:

1. $x_1 + 2x_2 = 3, \quad 3x_1 + x_2 = 6$
2. $2x_1 + x_2 = -3, \;\; x_1 - x_2 = 1$
3. $x_1 + 2x_2 - 2x_3 = 4, \quad 2x_1 + x_2 + x_3 = -2, \;\; -x_1 + x_2 - x_3 = 2$
4. $2x_1 + 3x_2 - x_3 = -1, \;\; -x_1 - 2x_2 + x_3 = 5, \;\; 3x_1 - x_2 = -2.$

11.4 ROW ECHELON FORM AND GAUSSIAN ELIMINATION

So far, we have assumed that every system of equations has a unique solution. This is not necessary true as the following examples show.

• Example 11.4.1

Consider the system

$$x_1 + x_2 = 2 \qquad \textbf{(11.4.1)}$$

and

$$2x_1 + 2x_2 = -1. \qquad \textbf{(11.4.2)}$$

This system is inconsistent because the second equation does not follow after multiplying the first by 2. Geometrically (11.4.1) and (11.4.2) are parallel lines; they never intersect to give a unique x_1 and x_2.

• Example 11.4.2

Even if a system is consistent, it still may not have a unique solution. For example, the system

$$x_1 + x_2 = 2 \qquad \textbf{(11.4.3)}$$

and

$$2x_1 + 2x_2 = 4 \qquad \textbf{(11.4.4)}$$

is consistent, the second equation formed by multiplying the first by 2. However, there are an infinite number of solutions.

Our examples suggest the following:

Theorem: *A system of m linear equation in n unknowns may: (1) have no solution, in which case it is called an* inconsistent *system, or (2) have exactly one solution (called a unique solution), or (3) have an infinite number of solutions. In the latter two cases, the system is said to be* consistent.

Before we can prove this theorem at the end of this section, we need to introduce some new concepts.

The first one is equivalent systems. Two systems of equations involving the same variables are *equivalent* if they have the same solution set. Of course, the only reason for introducing equivalent systems is the possibility of transforming one system of linear systems into another which is easier to solve. But what operations are permissible? Also what is the ultimate goal of our transformation?

From a complete study of possible operations, there are only three operations for transforming one system of linear equations into another. These three *elementary row operations* are

(1) interchanging any two rows in the matrix,

(2) multiplying any row by a nonzero scalar, and

(3) adding any arbitrary multiple of any row to any other row.

Armed with our elementary row operations, let us now solve the following set of linear equations:

$$x_1 - 3x_2 + 7x_3 = 2, \tag{11.4.5}$$

$$2x_1 + 4x_2 - 3x_3 = -1 \tag{11.4.6}$$

and

$$-x_1 + 13x_2 - 21x_3 = 2. \tag{11.4.7}$$

We begin by writing (11.4.5)–(11.4.7) in matrix notation:

$$\begin{pmatrix} 1 & -3 & 7 \\ 2 & 4 & -3 \\ -1 & 13 & -21 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}. \tag{11.4.8}$$

The matrix in (11.4.8) is called the *coefficient matrix* of the system.

We now introduce the concept of the *augmented matrix*: a matrix B composed of A plus the column vector $\mathbf{b}$ or

$$B = \left(\begin{array}{ccc|c} 1 & -3 & 7 & 2 \\ 2 & 4 & -3 & -1 \\ -1 & 13 & -21 & 2 \end{array}\right). \tag{11.4.9}$$

We can solve our original system by performing elementary row operations on the augmented matrix. Because the x_i's function essentially as placeholders, we can omit them until the end of the computation.

Returning to the problem, the first row may be used to eliminate the elements in the first column of the remaining rows. For this reason the first row is called the *pivotal* row and the element a_{11} is the *pivot*. By using the third elementary row operation twice (to eliminate the 2 and -1 in the first column), we finally have the equivalent system

$$B = \left(\begin{array}{ccc|c} 1 & -3 & 7 & 2 \\ 0 & 10 & -17 & -5 \\ 0 & 10 & -14 & 4 \end{array}\right). \qquad \textbf{(11.4.10)}$$

At this point we choose the second row as our new pivotal row and again apply the third row operation to eliminate the last element in the second column. This yields

$$B = \left(\begin{array}{ccc|c} 1 & -3 & 7 & 2 \\ 0 & 10 & -17 & -5 \\ 0 & 0 & 3 & 9 \end{array}\right). \qquad \textbf{(11.4.11)}$$

Thus, elementary row operations have transformed (11.4.5)–(11.4.7) into the triangular system:

$$x_1 - 3x_2 + 7x_3 = 2, \qquad \textbf{(11.4.12)}$$

$$10x_2 - 17x_3 = -5, \qquad \textbf{(11.4.13)}$$

$$3x_3 = 9, \qquad \textbf{(11.4.14)}$$

which is *equivalent* to the original system. The final solution is obtained by *back substitution*, solving from (11.4.14) back to (11.4.12). In the present case, $x_3 = 3$. Then, $10x_2 = 17(3) - 5$ or $x_2 = 4.6$. Finally, $x_1 = 3x_2 - 7x_3 + 2 = -5.2$.

In general, if an $n \times n$ linear system can be reduced to triangular form, then it will have a unique solution that we can obtain by performing back substitution. This reduction involves $n - 1$ steps. In the first step, a pivot element, and thus the pivotal row, is chosen from the nonzero entries in the first column of the matrix. We interchange rows (if necessary) so that the pivotal row is the first row. Multiples of the pivotal row are then subtracted from each of the remaining $n - 1$ rows so that there are 0's in the $(2, 1), ..., (n, 1)$ positions. In the second step, a pivot element is chosen from the nonzero entries in column 2, rows 2 through n, of the matrix. The row containing the pivot is then interchanged with the second row (if necessary) of the matrix and is used as the pivotal row. Multiples of the pivotal row are then subtracted from the remaining $n - 2$ rows, eliminating all entries below the diagonal in the second column. The same procedure is repeated for columns 3 through $n-1$. Note that in the second step, row 1 and column 1 remain unchanged, in the third step the first two rows and first two columns remain unchanged, and so on.

If elimination is carried out as described, we will arrive at an equivalent upper triangular system after $n-1$ steps. However, the procedure will fail if, at any step, all possible choices for a pivot element equal zero. Let us now examine such cases.

Consider now the system

$$x_1 + 2x_2 + x_3 = -1, \tag{11.4.15}$$
$$2x_1 + 4x_2 + 2x_3 = -2, \tag{11.4.16}$$
$$x_1 + 4x_2 + x_3 = 2. \tag{11.4.17}$$

Its augmented matrix is

$$B = \left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 2 & 4 & 2 & -2 \\ 1 & 4 & 2 & 2 \end{array}\right). \tag{11.4.18}$$

Choosing the first row as our pivotal row, we find that

$$B = \left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \end{array}\right) \tag{11.4.19}$$

or

$$B = \left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right). \tag{11.4.20}$$

The difficulty here is the presence of the zeros in the third row. Clearly any finite numbers will satisfy the equation $0x_1 + 0x_2 + 0x_3 = 0$ and we have an infinite number of solutions. Closer examination of the original system shows a underdetermined system; (11.4.15) and (11.4.16) differ by a factor of 2. An important aspect of this problem is the fact that the final augmented matrix is of the form of a staircase or *echelon form* rather than of triangular form.

Let us modify (11.4.15)–(11.4.17) to read

$$x_1 + 2x_2 + x_3 = -1, \tag{11.4.21}$$
$$2x_1 + 4x_2 + 2x_3 = 3, \tag{11.4.22}$$
$$x_1 + 4x_2 + x_3 = 2, \tag{11.4.23}$$

then the final augmented matrix is

$$B = \left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 5 \end{array}\right). \tag{11.4.24}$$

We again have a problem with the third row because $0x_1+0x_2+0x_3 = 5$, which is impossible. There is no solution in this case and we have an *overdetermined system*. Note, once again, that our augmented matrix has a row echelon form rather than a triangular form.

In summary, to include all possible situations in our procedure, we must rewrite the augmented matrix in row echelon form. *Row echelon form* consists of:

(1) The first nonzero entry in each row is 1.

(2) If row k does not consist entirely of zeros, the number of leading zero entries in row $k+1$ is greater than the number of leading zero entries in row k.

(3) If there are rows whose entries are all zero, they are below the rows having nonzero entries.

The number of nonzero rows in the row echelon form of a matrix is known as its *rank*. *Gaussian elimination* is the process of using elementary row operations to transform a linear system into one whose augmented matrix is in row echelon form.

• Example 11.4.3

Each of the following matrices is *not* of row echelon form because they violate one of the conditions for row echelon form:

$$\begin{pmatrix} 2 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{11.4.25}$$

• Example 11.4.4

The following matrices are in row echelon form:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 4 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{11.4.26}$$

• Example 11.4.5

Gaussian elimination may also be used to solve the general problem $AX = B$. One of the most common applications is in finding the inverse. For example, let us find the inverse of the matrix

$$A = \begin{pmatrix} 4 & -2 & 2 \\ -2 & -4 & 4 \\ -4 & 2 & 8 \end{pmatrix} \tag{11.4.27}$$

by Gaussian elimination.

Because the inverse is defined by $AA^{-1} = I$, our augmented matrix is

$$\left(\begin{array}{ccc|ccc} 4 & -2 & 2 & 1 & 0 & 0 \\ -2 & -4 & 4 & 0 & 1 & 0 \\ -4 & 2 & 8 & 0 & 0 & 1 \end{array}\right). \tag{11.4.28}$$

Then, by elementary row operations,

$$\left(\begin{array}{ccc|ccc} 4 & -2 & 2 & 1 & 0 & 0 \\ -2 & -4 & 4 & 0 & 1 & 0 \\ -4 & 2 & 8 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} -2 & -4 & 4 & 0 & 1 & 0 \\ 4 & -2 & 2 & 1 & 0 & 0 \\ -4 & 2 & 8 & 0 & 0 & 1 \end{array}\right) \tag{11.4.29}$$

$$= \left(\begin{array}{ccc|ccc} -2 & -4 & 4 & 0 & 1 & 0 \\ 4 & -2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 10 & 1 & 0 & 1 \end{array}\right) \tag{11.4.30}$$

$$= \left(\begin{array}{ccc|ccc} -2 & -4 & 4 & 0 & 1 & 0 \\ 0 & -10 & 10 & 1 & 2 & 0 \\ 0 & 0 & 10 & 1 & 0 & 1 \end{array}\right) \tag{11.4.31}$$

$$= \left(\begin{array}{ccc|ccc} -2 & -4 & 4 & 0 & 1 & 0 \\ 0 & -10 & 0 & 0 & 2 & -1 \\ 0 & 0 & 10 & 1 & 0 & 1 \end{array}\right) \tag{11.4.32}$$

$$= \left(\begin{array}{ccc|ccc} -2 & -4 & 0 & -2/5 & 1 & -2/5 \\ 0 & -10 & 0 & 0 & 2 & -1 \\ 0 & 0 & 10 & 1 & 0 & 1 \end{array}\right) \tag{11.4.33}$$

$$= \left(\begin{array}{ccc|ccc} -2 & 0 & 0 & -2/5 & 1/5 & 0 \\ 0 & -10 & 0 & 0 & 2 & -1 \\ 0 & 0 & 10 & 1 & 0 & 1 \end{array}\right) \tag{11.4.34}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & -1/10 & 0 \\ 0 & 1 & 0 & 0 & -1/5 & 1/10 \\ 0 & 0 & 1 & 1/10 & 0 & 1/10 \end{array}\right). \tag{11.4.35}$$

Thus, the right half of the augmented matrix yields the inverse and it equals

$$A^{-1} = \begin{pmatrix} 1/5 & -1/10 & 0 \\ 0 & -1/5 & 1/10 \\ 1/10 & 0 & 1/10 \end{pmatrix}. \tag{11.4.36}$$

Of course, we can always check our answer by multiplying A^{-1} by A.

Gaussian elimination may be used with overdetermined systems. *Overdetermined systems* are linear systems where there are more equations than unknowns ($m > n$). These systems are usually (but not always) inconsistent.

• Example 11.4.6

Consider the linear system

$$x_1 + x_2 = 1, \tag{11.4.37}$$

$$-x_1 + 2x_2 = -2, \tag{11.4.38}$$

$$x_1 - x_2 = 4. \tag{11.4.39}$$

After several row operations, the augmented matrix

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 2 & -2 \\ 1 & -1 & 4 \end{array}\right) \tag{11.4.40}$$

becomes

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -7 \end{array}\right). \tag{11.4.41}$$

From the last row of the augmented matrix (11.4.41) we see that the system is inconsistent. However, if we change the system to

$$x_1 + x_2 = 1, \tag{11.4.42}$$

$$-x_1 + 2x_2 = 5, \tag{11.4.43}$$

$$x_1 = -1, \tag{11.4.44}$$

the final form of the augmented matrix is

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right). \tag{11.4.45}$$

which has the unique solution $x_1 = -1$ and $x_2 = 2$.

Finally, by introducing the set:

$$x_1 + x_2 = 1, \tag{11.4.46}$$
$$2x_1 + 2x_2 = 2, \tag{11.4.47}$$
$$3x_1 + 3x_3 = 3, \tag{11.4.48}$$

the final form of the augmented matrix is

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right). \tag{11.4.49}$$

There are an infinite number of solutions: $x_1 = 1 - \alpha$ and $x_2 = \alpha$.

Gaussian elimination can also be employed with underdetermined systems. An *underdetermined linear system* is one where there are fewer equations than unknowns ($m < n$). These systems usually have an infinite number of solutions although they can be inconsistent.

• Example 11.4.7

Consider the underdetermined system:

$$2x_1 + 2x_2 + x_3 = -1, \tag{11.4.50}$$
$$4x_1 + 4x_2 + 2x_3 = 3. \tag{11.4.51}$$

Its augmented matrix may be transformed into the form:

$$\left(\begin{array}{ccc|c} 2 & 2 & 1 & -1 \\ 0 & 0 & 0 & 4 \end{array}\right). \tag{11.4.52}$$

Clearly this case corresponds to an inconsistent set of equations. On the other hand, if (11.4.51) is changed to

$$4x_1 + 4x_2 + 2x_3 = -2, \tag{11.4.53}$$

then the final form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{11.4.54}$$

and we have an infinite number of solutions, namely $x_3 = \alpha$, $x_2 = \beta$, and $2x_1 = -1 - \alpha - 2\beta$.

Consider now one of most important classes of linear equations: the homogeneous equations $A\mathbf{x} = \mathbf{0}$. If $\det(A) \neq 0$, then by Cramer's rule

$x_1 = x_2 = x_3 = \cdots = x_n = 0$. Thus, the only possibility for a nontrivial solution is $\det(A) = 0$. In this case, A is singular, no inverse exists, and nontrivial solutions exist but they are not unique.

• Example 11.4.8

Consider the two homogeneous equations:

$$x_1 + x_2 = 0 \tag{11.4.55}$$

$$x_1 - x_2 = 0. \tag{11.4.56}$$

Note that $\det(A) = -2$. Solving this system yields $x_1 = x_2 = 0$.

However, if we change the system to

$$x_1 + x_2 = 0 \tag{11.4.57}$$

$$x_1 + x_2 = 0 \tag{11.4.58}$$

which has the $\det(A) = 0$ so that A is singular. Both equations yield $x_1 = -x_2 = \alpha$, any constant. Thus, there is an infinite number of solutions for this set of homogeneous equations.

We close this section by outlining the proof of the theorem which we introduced at the beginning.

Consider the system $A\mathbf{x} = \mathbf{b}$. By elementary row operations, the first equation in this system can be reduced to

$$x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n = \beta_1. \tag{11.4.59}$$

The second equation has the form

$$x_p + \alpha_{2,p+1}x_{p+1} + \cdots + \alpha_{2n}x_n = \beta_2, \tag{11.4.60}$$

where $p > 1$. The third equation has the form

$$x_q + \alpha_{3,q+1}x_{q+1} + \cdots + \alpha_{3n}x_n = \beta_3, \tag{11.4.61}$$

where $q > p$, and so on. To simplify the notation, we introduce z_i where we choose the first k values so that $z_1 = x_1$, $z_2 = x_p$, $z_3 = x_q$, ... Thus, the question of the existence of solutions depends upon the three integers: m, n, and k. The resulting set of equations have the form:

$$\begin{pmatrix} 1 & \gamma_{12} & \cdots & \gamma_{1,k} & \gamma_{1,k+1} & \cdots & \gamma_{1n} \\ 0 & 1 & \cdots & \gamma_{2,k} & \gamma_{2,k+1} & \cdots & \gamma_{2n} \\ & & & \vdots & & & \\ 0 & 0 & \cdots & 1 & \gamma_{k,k+1} & \cdots & \gamma_{kn} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ & & & \vdots & & & \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \\ \beta_{k+1} \\ \vdots \\ \beta_m \end{pmatrix}. \tag{11.4.62}$$

Note that $\beta_{k+1}, \ldots, \beta_m$ need not be all zero.

There are three possibilities:

(a) $k < m$ and at least one of the elements $\beta_{k+1}, \ldots, \beta_m$ is nonzero. Suppose that an element β_p is nonzero ($p > k$). Then the pth equation is

$$0z_1 + 0z_2 + \cdots + 0z_n = \beta_p \neq 0. \tag{11.4.63}$$

However, this is a contradiction and the equations are inconsistent.

(b) $k = n$ and either (i) $k < m$ and all of the elements $\beta_{k+1}, \ldots, \beta_m$ are zero, or (ii) $k = m$. Then the equations have a unique solution which can be obtained by back-substitution.

(c) $k < n$ and either (i) $k < m$ and all of the elements $\beta_{k+1}, \ldots, \beta_m$ are zero, or (ii) $k = m$. Then, arbitrary values can be assigned to the $n-k$ variables $z_{k+1}, \ldots, z_n$. The equations can be solved for $z_1, z_2, \ldots, z_k$ and there is an infinity of solutions.

For homogeneous equations $\mathbf{b} = \mathbf{0}$, all of the β_i are zero. In this case, we have only two cases:

(b′) $k = n$, then (11.4.62) has the solution $\mathbf{z} = \mathbf{0}$ which leads to the trivial solution for the original system $A\mathbf{x} = \mathbf{0}$.

(c′) $k < n$, the equations possess an infinity of solutions given by assigning arbitrary values to $z_{k+1}, \ldots, z_n$. □

Problems

Solve the following systems of linear equations by Gaussian elimination:

1. $2x_1 + x_2 = 4, \quad 5x_1 - 2x_2 = 1$

2. $x_1 + x_2 = 0, \quad 3x_1 - 4x_2 = 1$

3. $-x_1 + x_2 + 2x_3 = 0, \quad 3x_1 + 4x_2 + x_3 = 0, \quad -x_1 + x_2 + 2x_3 = 0$

4. $4x_1 + 6x_2 + x_3 = 2, \quad 2x_1 + x_2 - 4x_3 = 3, \quad 3x_1 - 2x_2 + 5x_3 = 8$

5. $3x_1 + x_2 - 2x_3 = -3, \quad x_1 - x_2 + 2x_3 = -1, \quad -4x_1 + 3x_2 - 6x_3 = 4$

6. $x_1 - 3x_2 + 7x_3 = 2, \quad 2x_1 + 4x_2 - 3x_3 = -1, \quad -3x_1 + 7x_2 + 2x_3 = 3$

7. $x_1 - x_2 + 3x_3 = 5, \quad 2x_1 - 4x_2 + 7x_3 = 7, \quad 4x_1 - 9x_2 + 2x_3 = -15$

8. $x_1 + x_2 + x_3 + x_4 = -1, \quad 2x_1 - x_2 + 3x_3 = 1, \quad 2x_2 + 3x_4 = 15, \quad -x_1 + 2x_2 + x_4 = -2$

Find the inverse of each of the following matrices by Gaussian elimination:

9. $\begin{pmatrix} -3 & 5 \\ 2 & 1 \end{pmatrix}$

10. $\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$

11. $\begin{pmatrix} 19 & 2 & -9 \\ -4 & -1 & 2 \\ -2 & 0 & 1 \end{pmatrix}$ 12. $\begin{pmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 2 & 4 & 11 \end{pmatrix}$

13. Does $(A^2)^{-1} = (A^{-1})^2$? Justify your answer.

11.5 EIGENVALUES AND EIGENVECTORS

One of the classic problems of linear algebra[4] is finding all of the λ's which satisfy the $n \times n$ system

$$A\mathbf{x} = \lambda\mathbf{x}. \tag{11.5.1}$$

The nonzero quantity λ is the *eigenvalue* or *characteristic value* of A. The vector $\mathbf{x}$ is the *eigenvector* or *characteristic vector* belonging to λ. The set of the eigenvalues of A is called the *spectrum* of A. The largest of the absolute values of the eigenvalues of A is called the *spectral radius* of A.

To find λ and $\mathbf{x}$, we first rewrite (11.5.1) as a set of homogeneous equations:

$$(A - \lambda I)\mathbf{x} = \mathbf{0}. \tag{11.5.2}$$

From the theory of linear equations, (11.5.2) has trivial solutions unless its determinant equals zero. On the other hand, if

$$\det(A - \lambda I) = 0, \tag{11.5.3}$$

there are an infinity of solutions.

The expansion of the determinant (11.5.3) yields an nth-degree polynomial in λ, the *characteristic polynomial.* The roots of the characteristic polynomial are the eigenvalues of A. Because the characteristic polynomial has exactly n roots, A will have n eigenvalues, some of which may be repeated (with multiplicity $k \leq n$) and some of which may be complex numbers. For each eigenvalue λ_i, there will be a corresponding eigenvector $\mathbf{x}_i$. This eigenvector is the solution of the homogeneous equations $(A - \lambda_i I)\mathbf{x}_i = \mathbf{0}$.

An important property of eigenvectors is their *linear independence* if there are n distinct eigenvalues. Vectors are linearly independent if the equation

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_n\mathbf{x}_n = 0 \tag{11.5.4}$$

can be satisfied only by taking *all* of the α's equal to zero.

---

[4] The standard reference is Wilkinson, J. H., 1965: *The Algebraic Eigenvalue Problem,* Clarendon Press, Oxford.

To show that this is true in the case of n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, each eigenvalue λ_i having a corresponding eigenvector $\mathbf{x}_i$, we first write down the linear dependence condition

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_n\mathbf{x}_n = \mathbf{0}. \tag{11.5.5}$$

Premultiplying (11.5.5) by A,

$$\alpha_1 A\mathbf{x}_1 + \alpha_2 A\mathbf{x}_2 + \cdots + \alpha_n A\mathbf{x}_n = \alpha_1\lambda_1\mathbf{x}_1 + \alpha_2\lambda_2\mathbf{x}_2 + \cdots + \alpha_n\lambda_n\mathbf{x}_n = \mathbf{0}. \tag{11.5.6}$$

Premultiplying (11.5.5) by A^2,

$$\alpha_1 A^2\mathbf{x}_1 + \alpha_2 A^2\mathbf{x}_2 + \cdots + \alpha_n A^2\mathbf{x}_n = \alpha_1\lambda_1^2\mathbf{x}_1 + \alpha_2\lambda_2^2\mathbf{x}_2 + \cdots + \alpha_n\lambda_n^2\mathbf{x}_n = \mathbf{0}. \tag{11.5.7}$$

In similar manner, we obtain the system of equations:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix} \begin{pmatrix} \alpha_1\mathbf{x}_1 \\ \alpha_2\mathbf{x}_2 \\ \alpha_3\mathbf{x}_3 \\ \vdots \\ \alpha_n\mathbf{x}_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{11.5.8}$$

Because

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix} = \begin{matrix} (\lambda_2-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)(\lambda_4-\lambda_3) \\ (\lambda_4-\lambda_2)\cdots(\lambda_n-\lambda_1) \neq 0, \end{matrix} \tag{11.5.9}$$

since it is a Vandermonde determinant, $\alpha_1\mathbf{x}_1 = \alpha_2\mathbf{x}_2 = \alpha_3\mathbf{x}_3 = \cdots = \alpha_n\mathbf{x}_n = 0$. Because the eigenvectors are nonzero, $\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$ and the eigenvectors are linearly independent. □

This property of eigenvectors allows us to express any arbitrary vector $\mathbf{x}$ as a linear sum of the eigenvectors $\mathbf{x}_i$ or

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n. \tag{11.5.10}$$

We will make good use of this property in Example 11.5.3.

• Example 11.5.1

Let us find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} -4 & 2 \\ -1 & -1 \end{pmatrix}. \tag{11.5.11}$$

We begin by setting up the characteristic equation:

$$\det(A-\lambda I)=\begin{vmatrix}-4-\lambda & 2\\ -1 & -1-\lambda\end{vmatrix}=0. \tag{11.5.12}$$

Expanding the determinant,

$$(-4-\lambda)(-1-\lambda)+2=\lambda^2+5\lambda+6=(\lambda+3)(\lambda+2)=0. \tag{11.5.13}$$

Thus, the eigenvalues of the matrix A are $\lambda_1=-3$ and $\lambda_2=-2$.

To find the corresponding eigenvectors, we must solve the linear system:

$$\begin{pmatrix}-4-\lambda & 2\\ -1 & -1-\lambda\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}. \tag{11.5.14}$$

For example, for $\lambda_1=-3$,

$$\begin{pmatrix}-1 & 2\\ -1 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \tag{11.5.15}$$

or

$$x_1=2x_2. \tag{11.5.16}$$

Thus, any nonzero multiple of the vector $\begin{pmatrix}2\\ 1\end{pmatrix}$ is an eigenvector belonging to $\lambda_1=-3$. Similarly, for $\lambda_2=-2$, the eigenvector is any nonzero multiple of the vector $\begin{pmatrix}1\\ 1\end{pmatrix}$.

- **Example 11.5.2**

Let us now find the eigenvalues and corresponding eigenvectors of the matrix

$$A=\begin{pmatrix}-4 & 5 & 5\\ -5 & 6 & 5\\ -5 & 5 & 6\end{pmatrix}. \tag{11.5.17}$$

Setting up the characteristic equation:

$$\det(A-\lambda I)$$
$$=\begin{vmatrix}-4-\lambda & 5 & 5\\ -5 & 6-\lambda & 5\\ -5 & 5 & 6-\lambda\end{vmatrix}=\begin{vmatrix}-4-\lambda & 5 & 5\\ -5 & 6-\lambda & 5\\ 0 & \lambda-1 & 1-\lambda\end{vmatrix} \tag{11.5.18}$$

$$=(\lambda-1)\begin{vmatrix}-4-\lambda & 5 & 5\\ -5 & 6-\lambda & 5\\ 0 & 1 & -1\end{vmatrix}=(\lambda-1)^2\begin{vmatrix}-1 & 1 & 0\\ -5 & 6-\lambda & 5\\ 0 & 1 & -1\end{vmatrix} \tag{11.5.19}$$

$$=(\lambda-1)^2\begin{vmatrix}-1 & 0 & 0\\ -5 & 6-\lambda & 0\\ 0 & 1 & -1\end{vmatrix}=(\lambda-1)^2(6-\lambda)=0. \tag{11.5.20}$$

Thus, the eigenvalues of the matrix A are $\lambda_{1,2} = 1$ (twice) and $\lambda_3 = 6$.

To find the corresponding eigenvectors, we must solve the linear system:

$$(-4-\lambda)x_1 + 5x_2 + 5x_3 = 0, \tag{11.5.21}$$

$$-5x_1 + (6-\lambda)x_2 + 5x_3 = 0 \tag{11.5.22}$$

and

$$-5x_1 + 5x_2 + (6-\lambda)x_3 = 0. \tag{11.5.23}$$

For $\lambda_3 = 6$, (11.5.21)–(11.5.23) become

$$-10x_1 + 5x_2 + 5x_3 = 0, \tag{11.5.24}$$

$$-5x_1 + 5x_3 = 0 \tag{11.5.25}$$

and

$$-5x_1 + 5x_2 = 0. \tag{11.5.26}$$

Thus, $x_1 = x_2 = x_3$ and the eigenvector is any nonzero multiple of the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

The interesting aspect of this example involves finding the eigenvector for the eigenvalue $\lambda_{1,2} = 1$. If $\lambda_{1,2} = 1$, then (11.5.21)–(11.5.23) collapses into one equation

$$-x_1 + x_2 + x_3 = 0 \tag{11.5.27}$$

and we have *two* free parameters at our disposal. Let us take $x_2 = \alpha$ and $x_3 = \beta$. Then the eigenvector equals $\alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ for $\lambda_{1,2} = 1$.

In this example our 3×3 matrix has three *linearly independent* eigenvectors: $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ associated with $\lambda_1 = 1$, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ associated with $\lambda_2 = 1$, and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ associated with $\lambda_3 = 6$. However, with repeated eigenvalues this is not always true. For example,

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \tag{11.5.28}$$

has the repeated eigenvalues $\lambda_{1,2} = 1$. However, there is only a single eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for *both* λ_1 and λ_2.

- **Example 11.5.3**

When we discussed the stability of numerical schemes for the wave equation in Section 7.6, we examined the behavior of a prototypical Fourier harmonic to variation in the parameter $c\Delta t/\Delta x$. In this example we shall show another approach to determining the stability of a numerical scheme via matrices.

Consider the explicit scheme for the numerical integration of the wave equation (7.6.11). We can rewrite that single equation as the coupled difference equations:

$$u_m^{n+1} = 2(1-r^2)u_m^n + r^2(u_{m+1}^n + u_{m-1}^n) - v_m^n \tag{11.5.29}$$

and

$$v_m^{n+1} = u_m^n, \tag{11.5.30}$$

where $r = c\Delta t/\Delta x$. Let $u_{m+1}^n = e^{i\beta\Delta x}u_m^n$ and $u_{m-1}^n = e^{-i\beta\Delta x}u_m^n$, where β is real. Then (11.5.29)–(11.5.30) becomes

$$u_m^{n+1} = 2\left[1 - 2r^2\sin^2\left(\frac{\beta\Delta x}{2}\right)\right]u_m^n - v_m^n \tag{11.5.31}$$

and

$$v_m^{n+1} = u_m^n \tag{11.5.32}$$

or in the matrix form

$$\mathbf{u}_m^{n+1} = \begin{pmatrix} 2\left[1-2r^2\sin^2\left(\frac{\beta\Delta x}{2}\right)\right] & -1 \\ 1 & 0 \end{pmatrix}\mathbf{u}_m^n, \tag{11.5.33}$$

where $\mathbf{u}_m^n = \begin{pmatrix} u_m^n \\ v_m^n \end{pmatrix}$. The eigenvalues λ of this *amplification matrix* are given by

$$\lambda^2 - 2\left[1 - 2r^2\sin^2\left(\frac{\beta\Delta x}{2}\right)\right]\lambda + 1 = 0 \tag{11.5.34}$$

or

$$\lambda_{1,2} = 1 - 2r^2\sin^2\left(\frac{\beta\Delta x}{2}\right) \pm 2r\sin\left(\frac{\beta\Delta x}{2}\right)\sqrt{r^2\sin^2\left(\frac{\beta\Delta x}{2}\right) - 1}. \tag{11.5.35}$$

Because each successive time step consists of multiplying the solution from the previous time step by the amplification matrix, the solution will be stable only if $\mathbf{u}_m^n$ remains bounded. This will occur only if all of the eigenvalues have a magnitude less or equal to one because

$$\mathbf{u}_m^n = \sum_k c_k A^n \mathbf{x}_k = \sum_k c_k \lambda_k^n \mathbf{x}_k, \tag{11.5.36}$$

where A denotes the amplification matrix and $\mathbf{x}_k$ denotes the eigenvectors corresponding to the eigenvalues λ_k. Equation (11.5.36) follows from our ability to express any initial condition in terms of an eigenvector expansion:

$$\mathbf{u}_m^0 = \sum_k c_k \mathbf{x}_k. \tag{11.5.37}$$

In our particular example, two cases arise. If $r^2 \sin^2(\beta\Delta x/2) \leq 1$,

$$\lambda_{1,2} = 1 - 2r^2 \sin^2\left(\frac{\beta\Delta x}{2}\right) \pm 2ri \sin\left(\frac{\beta\Delta x}{2}\right)\sqrt{1 - r^2 \sin^2\left(\frac{\beta\Delta x}{2}\right)} \tag{11.5.38}$$

and $|\lambda_{1,2}| = 1$. On the other hand, if $r^2 \sin^2(\beta\Delta x/2) > 1$, $|\lambda_{1,2}| > 1$. Thus, we will have stability only if $c\Delta t/\Delta x \leq 1$.

Problems

Find the eigenvalues and corresponding eigenvectors for the following matrices:

1. $A = \begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix}$

2. $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$

3. $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$

4. $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

5. $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

6. $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{pmatrix}$

7. $A = \begin{pmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$

8. $A = \begin{pmatrix} -2 & 0 & 1 \\ 3 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

Project: Numerical Solution of the Sturm-Liouville Problem

You may have been struck by the similarity of the algebraic eigenvalue problem to the Sturm-Liouville problem. In both cases nontrivial solutions exist only for characteristic values of λ. The purpose of this project is to further deepen your insight into these similarities.

Consider the Sturm-Liouville problem:

$$y'' + \lambda y = 0, \qquad y(0) = y(\pi) = 0. \tag{11.5.39}$$

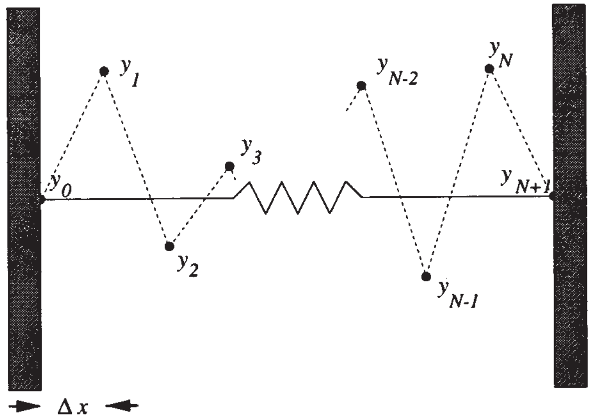

Figure 11.5.1: Schematic for finite-differencing a Sturm-Liouville problem into a set of difference equations.

We know that it has the nontrivial solutions $\lambda_m = m^2$, $y_m(x) = \sin(mx)$, where $m = 1, 2, 3, \ldots$

Step 1: Let us solve this problem numerically. Introducing centered finite differencing and the grid shown in Figure 11.5.1, show that

$$y'' \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta x}, \qquad n = 1, 2, \ldots, N, \tag{11.5.40}$$

where $\Delta x = \pi/(N+1)$. Show that the finite-differenced form of (11.5.39) is

$$-h^2 y_{n+1} + 2h^2 y_n - h^2 y_{n-1} = \lambda y_n \tag{11.5.41}$$

with $y_0 = y_{N+1} = 0$ and $h = 1/(\Delta x)$.

Step 2: Solve (11.5.41) as an algebraic eigenvalue problem using $N = 1, 2, \ldots$. Show that (11.5.41) can be written in the matrix form of

$$\begin{pmatrix} 2h^2 & -h^2 & 0 & \cdots & 0 & 0 & 0 \\ -h^2 & 2h^2 & -h^2 & \cdots & 0 & 0 & 0 \\ 0 & -h^2 & 2h^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -h^2 & 2h^2 & -h^2 \\ 0 & 0 & 0 & \cdots & 0 & -h^2 & 2h^2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-1} \\ y_N \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-1} \\ y_N \end{pmatrix}. \tag{11.5.42}$$

Note that the coefficient matrix is symmetric. Except for very small N, computing the values of λ using determinants is very difficult. Consequently you must use one of the numerical schemes that have been

Table 11.5.1: Eigenvalues computed from (11.5.42) as a numerical approximation of the Sturm-Liouville problem (11.5.39).

N	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
1	0.81057						
2	0.91189	2.73567					
3	0.94964	3.24228	5.53491				
4	0.96753	3.50056	6.63156	9.16459			
5	0.97736	3.64756	7.29513	10.94269	13.61289		
6	0.98333	3.73855	7.71996	12.13899	16.12040	18.87563	
7	0.98721	3.79857	8.00605	12.96911	17.93217	22.13966	24.95100
8	0.98989	3.84016	8.20702	13.56377	19.26430	24.62105	28.98791
20	0.99813	3.97023	8.84993	15.52822	23.85591	33.64694	44.68265
50	0.99972	3.99498	8.97438	15.91922	24.80297	35.59203	48.24538

developed for the efficient solution of the algebraic eigenvalue problem.[5] Packages for numerically solving the algebraic eigenvalue problem may already exist on your system or you may find code in a numerical methods book.

In Table 11.5.1 I have given the computed values of λ as a function of N using the IMSL routine EVLSF so that you may check your answers. How do your computed eigenvalues compare to the eigenvalues given by the Sturm-Liouville problem? What happens as you increase N? Which computed eigenvalues agree best with those given by the Sturm-Liouville problem? Which ones compare the worst?

Step 3: Let us examine the eigenfunctions now. First, reorder (if necessary) your eigenvectors so that each consecutive eigenvalue increases in magnitude. Starting with the smallest eigenvalue, construct an xy plot for each consecutive eigenvectors where $x_i = i\Delta x$, $i = 1, 2, \ldots, N$, and y_i are the corresponding element from the eigenvector. On the same plot, graph $y_m(x) = \sin(mx)$. Which eigenvectors and eigenfunctions agree the best? Which eigenvectors and eigenfunctions agree the worst? Why? Why are there N eigenvectors and an infinite number of eigenfunctions?

Step 4: The most important property of eigenfunctions is orthogonality. But what do we mean by orthogonality in the case of eigenvectors? Recall from three-dimensional vectors we had the scalar dot product:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3. \qquad \textbf{(11.5.43)}$$

[5] See Press, W. H., Flannery, B. F., Teukolsky, S. A., and Vetterling, W. T., 1986: *Numerical Recipes: The Art of Scientific Computing*, Cambridge University Press, Cambridge, chap. 11.

For n-dimensional vectors, this dot product is generalized to the inner product

$$\mathbf{x} \cdot \mathbf{y} = \sum_{k=1}^{n} x_k y_k. \tag{11.5.44}$$

Orthogonality implies that $\mathbf{x} \cdot \mathbf{y} = 0$ if $\mathbf{x} \neq \mathbf{y}$. Are your eigenvectors orthogonal? How might you use this property with eigenvectors?

11.6 SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

In this section we show how we may apply the classic algebraic eigenvalue problem to solve a system of ordinary differential equations.

Let us solve the following system:

$$x_1' = x_1 + 3x_2 \tag{11.6.1}$$

and

$$x_2' = 3x_1 + x_2, \tag{11.6.2}$$

where the primes denote the time derivative.

We begin by rewriting (11.6.1)–(11.6.2) in linear algebra notation:

$$\mathbf{x}' = A\mathbf{x}, \tag{11.6.3}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}. \tag{11.6.4}$$

Note that

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{x}'. \tag{11.6.5}$$

Assuming a solution of the form

$$\mathbf{x} = \mathbf{x}_0 e^{\lambda t}, \quad \text{where} \quad \mathbf{x}_0 = \begin{pmatrix} a \\ b \end{pmatrix} \tag{11.6.6}$$

is a constant vector, we substitute (11.6.6) into (11.6.3) and find that

$$\lambda e^{\lambda t}\mathbf{x}_0 = A e^{\lambda t}\mathbf{x}_0. \tag{11.6.7}$$

Because $e^{\lambda t}$ does not generally equal zero, we have that

$$(A - \lambda I)\mathbf{x}_0 = \mathbf{0}, \tag{11.6.8}$$

which we solved in the previous section. This set of homogeneous equations is the *classic eigenvalue problem*. In order for this set not to have trivial solutions,

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 0. \tag{11.6.9}$$

Expanding the determinant,

$$(1-\lambda)^2 - 9 = 0 \quad \text{or} \quad \lambda = -2, 4. \tag{11.6.10}$$

Thus, we have two real and distinct eigenvalues: $\lambda = -2$ and 4.

We must now find the corresponding $\mathbf{x}_0$ or *eigenvector* for each eigenvalue. From (11.6.8),

$$(1-\lambda)a + 3b = 0 \tag{11.6.11}$$

and

$$3a + (1-\lambda)b = 0. \tag{11.6.12}$$

If $\lambda = 4$, these equations are consistent and yield $a = b = c_1$. If $\lambda = -2$, we have that $a = -b = c_2$. Therefore, the general solution in matrix notation is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}. \tag{11.6.13}$$

To evaluate c_1 and c_2, we must have initial conditions. For example, if $x_1(0) = x_2(0) = 1$, then

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{11.6.14}$$

Solving for c_1 and c_2, $c_1 = 1$ and $c_2 = 0$ and the solution with this particular set of initial conditions is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}. \tag{11.6.15}$$

• Example 11.6.1

Let us solve the following set of linear ordinary differential equations:

$$x_1' = -x_2 + x_3, \tag{11.6.16}$$

$$x_2' = 4x_1 - x_2 - 4x_3 \tag{11.6.17}$$

and

$$x_3' = -3x_1 - x_2 + 4x_3; \tag{11.6.18}$$

or in matrix form,

$$\mathbf{x}' = \begin{pmatrix} 0 & -1 & 1 \\ 4 & -1 & -4 \\ -3 & -1 & 4 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{11.6.19}$$

Assuming the solution $\mathbf{x} = \mathbf{x}_0 e^{\lambda t}$,

$$\begin{pmatrix} 0 & -1 & 1 \\ 4 & -1 & -4 \\ -3 & -1 & 4 \end{pmatrix} \mathbf{x}_0 = \lambda \mathbf{x}_0 \tag{11.6.20}$$

or

$$\begin{pmatrix} -\lambda & -1 & 1 \\ 4 & -1-\lambda & -4 \\ -3 & -1 & 4-\lambda \end{pmatrix} \mathbf{x}_0 = \mathbf{0}. \tag{11.6.21}$$

For nontrivial solutions,

$$\begin{vmatrix} -\lambda & -1 & 1 \\ 4 & -1-\lambda & -4 \\ -3 & -1 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 4-4\lambda & -5-\lambda & -4 \\ -3+4\lambda-\lambda^2 & 3-\lambda & 4-\lambda \end{vmatrix} = 0 \tag{11.6.22}$$

and

$$(\lambda-1)(\lambda-3)(\lambda+1) = 0 \quad \text{or} \quad \lambda = -1, 1, 3. \tag{11.6.23}$$

To determine the eigenvectors, we rewrite (11.6.21) as

$$-\lambda a - b + c = 0, \tag{11.6.24}$$

$$4a - (1+\lambda)b - 4c = 0 \tag{11.6.25}$$

and

$$-3a - b + (4-\lambda)c = 0. \tag{11.6.26}$$

For example, if $\lambda = 1$,

$$-a - b + c = 0, \tag{11.6.27}$$

$$4a - 2b - 4c = 0 \tag{11.6.28}$$

and

$$-3a - b + 3c = 0; \tag{11.6.29}$$

or $a = c$ and $b = 0$. Thus, the eigenfunction for $\lambda = 1$ is $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Similarly, for $\lambda = -1$, $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and for $\lambda = 3$, $\mathbf{x}_0 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. Thus, the most general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} e^{3t}. \tag{11.6.30}$$

• **Example 11.6.2**

Let us solve the following set of linear ordinary differential equations:

$$x_1' = x_1 - 2x_2 \tag{11.6.31}$$

and

$$x_2' = 2x_1 - 3x_2; \tag{11.6.32}$$

or in matrix form,

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{11.6.33}$$

Assuming the solution $\mathbf{x} = \mathbf{x}_0 e^{\lambda t}$,

$$\begin{pmatrix} 1-\lambda & -2 \\ 2 & -3-\lambda \end{pmatrix} \mathbf{x}_0 = \mathbf{0}. \tag{11.6.34}$$

For nontrivial solutions,

$$\begin{vmatrix} 1-\lambda & -2 \\ 2 & -3-\lambda \end{vmatrix} = (\lambda+1)^2 = 0. \tag{11.6.35}$$

Thus, we have the solution

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}. \tag{11.6.36}$$

The interesting aspect of this example is the single solution that the traditional approach yields because we have repeated roots. To find the second solution, we try a solution of the form

$$\mathbf{x} = \begin{pmatrix} a + ct \\ b + dt \end{pmatrix} e^{-t}. \tag{11.6.37}$$

Equation (11.6.37) was guessed based upon our knowledge of solutions to differential equations when the characteristic polynomial has repeated roots. Substituting (11.6.37) into (11.6.33), we find that $c = d = 2c_2$ and $a - b = c_2$. Thus, we have one free parameter, which we will choose to be b, and set it equal to zero. This is permissible because (11.6.37) can be broken into two terms: $b \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$ and $c_2 \begin{pmatrix} 1+2t \\ 2t \end{pmatrix} e^{-t}$. The first term may be incorporated into the $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$ term. Thus, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + 2c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t}. \tag{11.6.38}$$

• **Example 11.6.3**

Let us solve the system of linear differential equations:

$$x_1' = 2x_1 - 3x_2 \tag{11.6.39}$$

and

$$x_2' = 3x_1 + 2x_2; \tag{11.6.40}$$

or in matrix form,

$$\mathbf{x}' = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{11.6.41}$$

Assuming the solution $\mathbf{x} = \mathbf{x}_0 e^{\lambda t}$,

$$\begin{pmatrix} 2-\lambda & -3 \\ 3 & 2-\lambda \end{pmatrix} \mathbf{x}_0 = \mathbf{0}. \tag{11.6.42}$$

For nontrivial solutions,

$$\begin{vmatrix} 2-\lambda & -3 \\ 3 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 + 9 = 0 \tag{11.6.43}$$

and $\lambda = 2 \pm 3i$. If $\mathbf{x}_0 = \begin{pmatrix} a \\ b \end{pmatrix}$, then $b = -ai$ if $\lambda = 2 + 3i$ and $b = ai$ if $\lambda = 2 - 3i$. Thus, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{2t+3it} + c_2 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{2t-3it}, \tag{11.6.44}$$

where c_1 and c_2 are arbitrary complex constants. Using Euler relationships, we can rewrite (11.6.44) as

$$\mathbf{x} = c_3 \begin{bmatrix} \cos(3t) \\ \sin(3t) \end{bmatrix} e^{2t} + c_4 \begin{bmatrix} \sin(3t) \\ -\cos(3t) \end{bmatrix} e^{2t}, \tag{11.6.45}$$

where $c_3 = c_1 + c_2$ and $c_4 = i(c_1 - c_2)$.

Problems

Find the general solution of the following sets of ordinary differential equations using matrix techniques:

1. $x_1' = x_1 + 2x_2 \qquad x_2' = 2x_1 + x_2.$
2. $x_1' = x_1 - 4x_2 \qquad x_2' = 3x_1 - 6x_2.$

3. $x_1' = x_1 + x_2$ $\quad x_2' = 4x_1 + x_2.$
4. $x_1' = x_1 + 5x_2$ $\quad x_2' = -2x_1 - 6x_2.$
5. $x_1' = -\frac{3}{2}x_1 - 2x_2$ $\quad x_2' = 2x_1 + \frac{5}{2}x_2.$
6. $x_1' = -3x_1 - 2x_2$ $\quad x_2' = 2x_1 + x_2.$
7. $x_1' = x_1 - x_2$ $\quad x_2' = x_1 + 3x_2.$
8. $x_1' = 3x_1 + 2x_2$ $\quad x_2' = -2x_1 - x_2.$
9. $x_1' = -2x_1 - 13x_2$ $\quad x_2' = x_1 + 4x_2.$
10. $x_1' = 3x_1 - 2x_2$ $\quad x_2' = 5x_1 - 3x_2.$
11. $x_1' = 4x_1 - 2x_2$ $\quad x_2' = 25x_1 - 10x_2.$
12. $x_1' = -3x_1 - 4x_2$ $\quad x_2' = 2x_1 + x_2.$
13. $x_1' = 3x_1 + 4x_2$ $\quad x_2' = -2x_1 - x_2.$
14. $x_1' + 5x_1 + x_2' + 3x_2 = 0$ $\quad 2x_1' + x_1 + x_2' + x_2 = 0.$
15. $x_1' - x_1 + x_2' - 2x_2 = 0$ $\quad x_1' - 5x_1 + 2x_2' - 7x_2 = 0.$
16. $x_1' = x_1 - 2x_2$ $\quad x_2' = 0$ $\quad x_3' = -5x_1 + 7x_3.$
17. $x_1' = 2x_1$ $\quad x_2' = x_1 + 2x_3$ $\quad x_3' = x_3.$
18. $x_1' = 3x_1 - 2x_3$ $\quad x_2' = -x_1 + 2x_2 + x_3$ $\quad x_3' = 4x_1 - 3x_3.$
19. $x_1' = 3x_1 - x_3$ $\quad x_2' = -2x_1 + 2x_2 + x_3$ $\quad x_3' = 8x_1 - 3x_3.$