CHAPTER 15

LEGENDRE FUNCTIONS

Legendre functions are important in physics because they arise when the Laplace or Helmholtz equations (or their generalizations) for central force problems are separated in spherical coordinates. They therefore appear in the descriptions of wave functions for atoms, in a variety of electrostatics problems, and in many other contexts. In addition, the Legendre polynomials provide a convenient set of functions that is orthogonal (with unit weight) on the interval $(-1, +1)$ that is the range of the sine and cosine functions. And from a pedagogical viewpoint, they provide a set of functions that are easy to work with and form an excellent illustration of the general properties of orthogonal polynomials. Several of these properties were discussed in a general way in Chapter 12. We collect here those results, expanding them with additional material that is of great utility and importance.

As indicated above, Legendre functions are encountered when an equation written in spherical polar coordinates (r, θ, φ), such as

$$-\nabla^2\psi + V(r)\psi = \lambda\psi,$$

is solved by the method of separation of variables. Note that we are assuming that this equation is to be solved for a spherically symmetric region and that $V(r)$ is a function of the distance from the origin of the coordinate system (and therefore not a function of the three-component position vector $\mathbf{r}$). As in Eqs. (9.77) and (9.78), we write $\psi = R(r)\Theta(\theta)\Phi(\varphi)$ and decompose our original partial differential equation (PDE) into the three one-dimensional ordinary differential equations (ODEs):

$$\frac{d^2\Phi}{d\varphi^2} = -m^2\Phi, \tag{15.1}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) - \frac{m^2\Theta}{\sin^2\theta} + l(l+1)\Theta = 0, \tag{15.2}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left[\lambda - V(r)\right]R - \frac{l(l+1)R}{r^2} = 0. \tag{15.3}$$

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The quantities m^2 and $l(l+1)$ are constants that occur when the variables are separated; the ODE in φ is easy to solve and has natural boundary conditions (cf. Section 9.4), which dictate that m must be an integer and that the functions Φ can be written as $e^{\pm im\varphi}$ or as $\sin(m\varphi)$, $\cos(m\varphi)$.

The Θ equation can now be transformed by the substitution $x = \cos\theta$, cf. Eq. (9.79), reaching

$$(1-x^2)P''(x) - 2xP'(x) - \frac{m^2}{1-x^2}P(x) + l(l+1)P(x) = 0. \tag{15.4}$$

This is the **associated Legendre equation**; the special case with $m = 0$, which we will treat first, is the **Legendre** ODE.

15.1 LEGENDRE POLYNOMIALS

The Legendre equation,

$$(1-x^2)P''(x) - 2xP'(x) + \lambda P(x) = 0, \tag{15.5}$$

has regular singular points at $x = \pm 1$ and $x = \infty$ (see Table 7.1), and therefore has a series solution about $x = 0$ that has a unit radius of convergence, i.e., the series solution will (for all values of the parameter λ) converge for $|x| < 1$. In Section 8.3 we found that for most values of λ, the series solutions will diverge at $x = \pm 1$ (corresponding to $\theta = 0$ and $\theta = \pi$), making the solutions inappropriate for use in central force problems. However, if λ has the value $l(l+1)$, with l an integer, the series become truncated after x^l, leaving a polynomial of degree l.

Now that we have identified the desired solutions to the Legendre equations as polynomials of successive degrees, called **Legendre polynomials** and designated P_l, let us use the machinery of Chapter 12 to develop them from a generating-function approach. This course of action will set a scale for the P_l and provide a good starting point for deriving recurrence relations and related formulas.

We found in Example 12.1.3 that the generating function for the polynomial solutions of the Legendre ODE is given by Eq. (12.27):

$$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n. \tag{15.6}$$

To identify the scale that is given to P_n by Eq. (15.6), we simply set $x = 1$ in that equation, bringing its left-hand side to the form

$$g(1,t) = \frac{1}{\sqrt{1-2t+t^2}} = \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n, \tag{15.7}$$

where the last step in Eq. (15.7) was to expand $1/(1-t)$ using the binomial theorem. Comparing with Eq. (15.6), we see that the scaling it predicts is $P_n(1) = 1$.

Next, consider what happens if we replace x by $-x$ and t by $-t$. The value of $g(x,t)$ in Eq. (15.6) is unaffected by this substitution, but the right-hand side takes a different form:

$$\sum_{n=0}^{\infty} P_n(x)t^n = g(x,t) = g(-x,-t) = \sum_{n=0}^{\infty} P_n(-x)(-t)^n, \tag{15.8}$$

showing that

$$P_n(-x) = (-1)^n P_n(x). \tag{15.9}$$

From this result it is obvious that $P_n(-1) = (-1)^n$, and that $P_n(x)$ will have the same parity as x^n.

Another useful special value is $P_n(0)$. Writing P_{2n} and P_{2n+1} to distinguish even and odd index values, we note first that because P_{2n+1} is odd under parity, i.e., $x \to -x$, we must have $P_{2n+1}(0) = 0$. To obtain $P_{2n}(0)$, we again resort to the binomial expansion:

$$g(0,t) = (1+t^2)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} t^{2n} = \sum_{n=0}^{\infty} P_{2n}(0)\, t^{2n}. \tag{15.10}$$

Then, using Eq. (1.74) to evaluate the binomial coefficient, we get

$$P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}. \tag{15.11}$$

It is also useful to characterize the leading terms of the Legendre polynomials. Applying the binomial theorem to the generating function,

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-2xt+t^2)^n, \tag{15.12}$$

from which we see that the maximum power of x that can multiply t^n will be x^n, and is obtained from the term $(-2xt)^n$ in the expansion of the final factor. Thus, the

$$\text{coefficient of } x^n \text{ in } P_n(x) \text{ is } \binom{-1/2}{n}(-2)^n = \frac{(2n-1)!!}{n!}. \tag{15.13}$$

These results are important, so we summarize:

$P_n(x)$ has sign and scaling such that $P_n(1) = 1$ and $P_n(-1) = (-1)^n$. $P_{2n}(x)$ is an even function of x; $P_{2n+1}(x)$ is odd. $P_{2n+1}(0) = 0$, and $P_{2n}(0)$ is given by Eq. (15.11). $P_n(x)$ is a polynomial of degree n in x, with the coefficient of x^n given by Eq. (15.13); $P_n(x)$ contains alternate powers of x: x^n, $x^{n-2}, \cdots, (x^0$ or $x^1)$.

From the fact that P_n is of degree n with alternate powers, it is clear that $P_0(x) =$ constant and that $P_1(x) =$ (constant)$\,x$. From the scaling requirements these must reduce to $P_0(x) = 1$ and $P_1(x) = x$.

Returning to Eq. (15.12), we can get explicit closed expressions for the Legendre polynomials. All we need to do is expand the quantity $(-2xt+t^2)^n$ and rearrange the summations to identify the x dependence associated with each power of t. The result, which is in general less useful than the recurrence formulas to be developed in the next subsection, is

$$P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^n k!\,(n-k)!\,(n-2k)!} x^{n-2k}. \tag{15.14}$$

Here $[n/2]$ stands for the largest integer $\le n/2$. This formula is consistent with the requirement that for n even, $P_n(x)$ has only even powers of x and even parity, while for n odd, it has only odd powers of x and odd parity. Proof of Eq. (15.14) is the topic of Exercise 15.1.2.

Recurrence Formulas

From the generating function equation we can generate recurrence formulas by differentiating $g(x,t)$ with respect to x or t. We start from

$$\frac{\partial g(x,t)}{\partial t} = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} n P_n(x) t^{n-1}, \tag{15.15}$$

which we rearrange to

$$(1-2xt+t^2)\sum_{n=0}^{\infty} n P_n(x) t^{n-1} + (t-x)\sum_{n=0}^{\infty} P_n(x) t^n = 0, \tag{15.16}$$

and then expand, reaching

$$\sum_{n=0}^{\infty} n P_n(x) t^{n-1} - 2\sum_{n=0}^{\infty} nx P_n(x) t^n + \sum_{n=0}^{\infty} n P_n(x) t^{n+1} + \sum_{n=0}^{\infty} P_n(x) t^{n+1} - \sum_{n=0}^{\infty} x P_n(x) t^n = 0. \tag{15.17}$$

Collecting the coefficients of t^n from the various terms and setting the result to zero, Eq. (15.17) is seen to be equivalent to

$$(2n+1)x P_n(x) = (n+1)P_{n+1}(x) + n P_{n-1}(x), \quad n = 1, 2, 3, \dots. \tag{15.18}$$

Equation (15.18) permits us to generate successive P_n from the starting values P_0 and P_1 that we have previously identified. For example,

$$2P_2(x) = 3x P_1(x) - P_0(x) \quad \longrightarrow \quad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right). \tag{15.19}$$

Continuing this process, we can build the list of Legendre polynomials given in Table 15.1.

We can also obtain a recurrence formula involving P_n' by differentiating $g(x,t)$ with respect to x. This gives

$$\frac{\partial g(x,t)}{\partial x} = \frac{t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x) t^n,$$

Table 15.1 Legendre Polynomials

$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2 - 1)$
$P_3(x) = \frac{1}{2}(5x^3 - 3x)$
$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$
$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$
$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$
$P_7(x) = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$
$P_8(x) = \frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)$

or

$$(1 - 2xt + t^2)\sum_{n=0}^{\infty} P_n'(x)t^n - t\sum_{n=0}^{\infty} P_n(x)t^n = 0. \tag{15.20}$$

As before, the coefficient of each power of t is set to zero and we obtain

$$P_{n+1}'(x) + P_{n-1}'(x) = 2x P_n'(x) + P_n(x). \tag{15.21}$$

A more useful relation may be found by differentiating Eq. (15.18) with respect to x and multiplying by 2. To this we add $(2n + 1)$ times Eq. (15.21), canceling the P_n' term. The result is

$$P_{n+1}'(x) - P_{n-1}'(x) = (2n + 1)P_n(x). \tag{15.22}$$

Starting from Eqs. (15.21) and (15.22), numerous additional relations can be developed,[1] including

$$P_{n+1}'(x) = (n + 1)P_n(x) + x P_n'(x), \tag{15.23}$$

$$P_{n-1}'(x) = -n P_n(x) + x P_n'(x), \tag{15.24}$$

$$(1 - x^2)P_n'(x) = n P_{n-1}(x) - nx P_n(x), \tag{15.25}$$

$$(1 - x^2)P_n'(x) = (n + 1)x P_n(x) - (n + 1)P_{n+1}(x). \tag{15.26}$$

Because we derived the generating function $g(x, t)$ from the Legendre ODE and then obtained the recurrence formulas using $g(x, t)$, that ODE will automatically be consistent with these recurrence relations. It is nevertheless of interest to verify this consistency, because then we can conclude that **any** set of functions satisfying the recurrence formulas will be a set of solutions to the Legendre ODE, and that observation will be relevant to

[1] Using the equation numbers in parentheses to indicate how they are to be combined, we may obtain some of these derivative formulas as follows:

$$2 \cdot \frac{d}{dx}(15.18) + (2n + 1) \cdot (15.21) \Rightarrow (15.22), \qquad \frac{1}{2}\{(15.21) + (15.22)\} \Rightarrow (15.23),$$

$$\frac{1}{2}\{(15.21) - (15.22)\} \Rightarrow (15.24), \qquad (15.23)_{n \to n-1} + x\,(15.24) \Rightarrow (15.25).$$

the Legendre functions of the second kind (solutions linearly independent of the polynomials P_l). A demonstration that functions satisfying the recurrence formulas also satisfy the Legendre ODE is the topic of Exercise 15.1.1.

Upper and Lower Bounds for $P_n(\cos\theta)$

Our generating function can be used to set an upper limit on $|P_n(\cos\theta)|$. We have

$$\begin{aligned}(1-2t\cos\theta+t^2)^{-1/2} &= (1-te^{i\theta})^{-1/2}(1-te^{-i\theta})^{-1/2}\\ &= \left(1+\frac{1}{2}te^{i\theta}+\frac{3}{8}t^2e^{2i\theta}+\cdots\right)\left(1+\frac{1}{2}te^{-i\theta}+\frac{3}{8}t^2e^{-2i\theta}+\cdots\right). \end{aligned} \quad (15.27)$$

We may make two immediate observations from Eq. (15.27). First, when any term within the first set of parentheses is multiplied by any term from the second set of parentheses, the power of t in the product will be even if and only if m in the net exponential $e^{im\theta}$ is even. Second, for every term of the form $t^n e^{im\theta}$, there will be another term of the form $t^n e^{-im\theta}$, and the two terms will occur with the same coefficient, which must be positive (since all the terms in both summations are individually positive). These two observations mean that:

(1) Taking the terms of the expansion two at a time, we can write the coefficient of t^n as a linear combination of forms

$$\frac{1}{2}a_{nm}(e^{im\theta}+e^{-im\theta}) = a_{nm}\cos m\theta$$

with all the a_{nm} **positive**, and

(2) The parity of n and m must be the same (either they are both even, or both odd).

This, in turn, means that

$$P_n(\cos\theta) = \sum_{m=0 \text{ or } 1}^{n} a_{nm}\cos m\theta. \quad (15.28)$$

This expression is clearly a maximum when $\theta = 0$, where we already know, from the Summary following Eq. (15.11), that $P_n(1) = 1$. Thus,

The Legendre polynomial $P_n(x)$ has a global maximum on the interval $(-1,+1)$ at $x = 1$, with value $P_n(1) = 1$, and if n is even, also at $x = -1$. If n is odd, $x = -1$ will be a global minimum on this interval with $P_n(-1) = -1$.

The maxima and minima of the Legendre polynomials can be seen from the graphs of P_2 through P_5, in which are plotted in Fig. 15.1.

Rodrigues Formula

In Section 12.1 we showed that orthogonal polynomials could be described by **Rodrigues formulas**, and that the repeated differentiations occurring therein were good

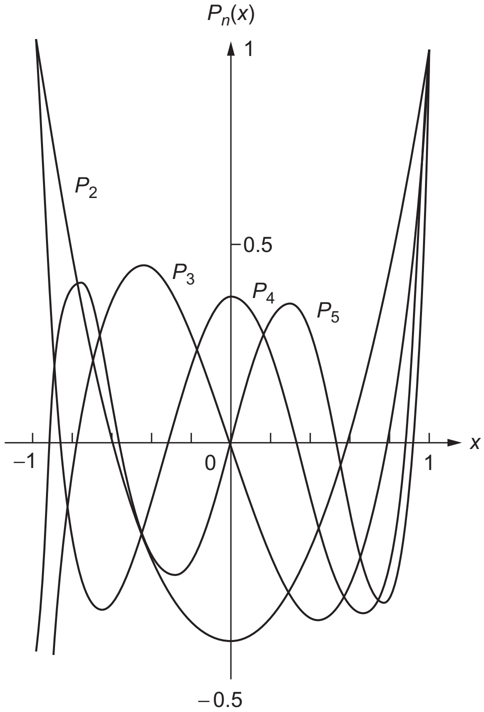

FIGURE 15.1 Legendre polynomials $P_2(x)$ through $P_5(x)$.

starting points for developing properties of these functions. Applying Eq. (12.9), we find that the Rodrigues formula for the Legendre polynomials must be proportional to

$$\left(\frac{d}{dx}\right)^n (1-x^2)^n. \tag{15.29}$$

Equation (12.9) is not sufficient to set the scale of the orthogonal polynomials, and to bring Eq. (15.29) to the scaling already adopted via Eq. (15.6) we multiply Eq. (15.29) by $(-1)^n/2^n\,n!$, so

$$P_n(x) = \frac{1}{2^n\,n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n. \tag{15.30}$$

To establish that Eq. (15.30) has a scaling in agreement with our earlier analyses, it suffices to check the coefficient of a single power of x; we choose x^n. From the Rodrigues formula, this power of x can only arise from the term x^{2n} in the expansion of $(x^2-1)^n$, and the

$$\text{coefficient of } x^n \text{ in } P_n(x) \text{ (Rodrigues) is } \frac{1}{2^n\,n!}\,\frac{(2n)!}{n!} = \frac{(2n-1)!!}{n!},$$

in agreement with Eq. (15.13). This confirms the scale of Eq. (15.30).

Exercises

15.1.1 Derive the Legendre ODE by manipulation of the Legendre polynomial recurrence relations. Suggested starting point: Eqs. (15.24) and (15.25).

15.1.2 Derive the following closed formula for the Legendre polynomials $P_n(x)$.

$$P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^n k!\,(n-k)!\,(n-2k)!}\, x^{n-2k},$$

where $[n/2]$ stands for the integer part of $n/2$.

Hint. Further expand Eq. (15.12) and rearrange the resulting double sum.

15.1.3 By differentiation and direct substitution of the series form given in Exercise 15.1.2, show that $P_n(x)$ satisfies the Legendre ODE. Note that there is no restriction on x. We may have any x, $-\infty < x < \infty$, and indeed any z in the entire finite complex plane.

15.1.4 The **shifted Legendre polynomials**, designated by the symbol $P_n^*(x)$ (where the asterisk does **not** mean complex conjugate) are orthogonal with unit weight on [0, 1], with normalization integral $\langle P_n^* | P_n^* \rangle = 1/(2n+1)$. The P_n^* through $n = 6$ are shown in Table 15.2.

(a) Find the recurrence relation satisfied by the P_n^*.
(b) Show that all the coefficients of the P_n^* are integers.

Hint. Look at the closed formula in Exercise 15.1.2.

15.1.5 Given the series

$$\alpha_0 + \alpha_2 \cos^2\theta + \alpha_4 \cos^4\theta + \alpha_6 \cos^6\theta = a_0 P_0 + a_2 P_2 + a_4 P_4 + a_6 P_6,$$

where the arguments of the P_n are $\cos\theta$, express the coefficients α_i as a column vector $\boldsymbol{\alpha}$ and the coefficients a_i as a column vector $\mathbf{a}$ and determine the matrices A and B such that

$$\mathsf{A}\boldsymbol{\alpha} = \mathbf{a} \quad \text{and} \quad \mathsf{B}\mathbf{a} = \boldsymbol{\alpha}.$$

Table 15.2 Shifted Legendre Polynomials

$P_0^*(x) = 1$
$P_1^*(x) = 2x - 1$
$P_2^*(x) = 6x^2 - 6x + 1$
$P_3^*(x) = 20x^3 - 30x^2 + 12x - 1$
$P_4^*(x) = 70x^4 - 140x^3 + 90x^2 - 20x + 1$
$P_5^*(x) = 252x^5 - 630x^4 + 560x^3 - 210x^2 + 30x - 1$
$P_6^*(x) = 924x^6 - 2772x^5 + 3150x^4 - 1680x^3 + 420x^2 - 42x + 1$

Check your computation by showing that $\mathsf{AB} = \mathbf{1}$ (unit matrix). Repeat for the odd case

$$\alpha_1 \cos\theta + \alpha_3 \cos^3\theta + \alpha_5 \cos^5\theta + \alpha_7 \cos^7\theta = a_1 P_1 + a_3 P_3 + a_5 P_5 + a_7 P_7.$$

Note. $P_n(\cos\theta)$ and $\cos^n\theta$ are tabulated in terms of each other in AMS-55 (see Additional Readings for the complete reference).

15.1.6 By differentiating the generating function $g(x,t)$ with respect to t, multiplying by $2t$, and then adding $g(x,t)$, show that

$$\frac{1-t^2}{(1-2tx+t^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)P_n(x)t^n.$$

This result is useful in calculating the charge induced on a grounded metal sphere by a nearby point charge.

15.1.7 (a) Derive Eq. (15.26),

$$(1-x^2)P_n'(x) = (n+1)xP_n(x) - (n+1)P_{n+1}(x).$$

(b) Write out the relation of Eq. (15.26) to preceding equations in symbolic form analogous to the symbolic forms for Eqs. (15.22) to (15.25).

15.1.8 Prove that

$$P_n'(1) = \frac{d}{dx}P_n(x)\,|_{x=1} = \frac{1}{2}n(n+1).$$

15.1.9 Show that $P_n(\cos\theta) = (-1)^n P_n(-\cos\theta)$ by use of the recurrence relation relating P_n, P_{n+1}, and P_{n-1} and your knowledge of P_0 and P_1.

15.1.10 From Eq. (15.27) write out the coefficient of t^2 in terms of $\cos n\theta$, $n \le 2$. This coefficient is $P_2(\cos\theta)$.

15.1.11 Derive the recurrence relation

$$(1-x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x)$$

from the Legendre polynomial generating function.

15.1.12 Evaluate $\displaystyle\int_0^1 P_n(x)\,dx$.

ANS. $n = 2s$, $\quad 1$ for $s = 0$, 0 for $s > 0$;

$n = 2s+1$, $\quad P_{2s}(0)/(2s+2) = (-1)^s(2s-1)!!/1(2s+2)!!$.

Hint. Use a recurrence relation to replace $P_n(x)$ by derivatives and then integrate by inspection. Alternatively, you can integrate the generating function.

15.1.13 Show that **each** term in the summation

$$\sum_{r=[n/2]+1}^{n}\left(\frac{d}{dx}\right)^n \frac{(-1)^r n!}{r!\,(n-r)!}x^{2n-2r}$$

vanishes (r and n integral). Here $[n/2]$ is the largest integer $\le n/2$.

15.1.14 Show that $\int_{-1}^{1} x^m P_n(x)\, dx = 0$ when $m < n$.

Hint. Use Rodrigues formula or expand x^m in Legendre polynomials.

15.1.15 Show that

$$\int_{-1}^{1} x^n P_n(x) dx = \frac{2\, n!}{(2n+1)!!}.$$

Note. You are expected to use the Rodrigues formula and integrate by parts, but also see if you can get the result from Eq. (15.14) by inspection.

15.1.16 Show that

$$\int_{-1}^{1} x^{2r} P_{2n}(x) dx = \frac{2^{2n+1}(2r)!\,(r+n)!}{(2r+2n+1)!\,(r-n)!}, \quad r \geq n.$$

15.1.17 As a generalization of Exercises 15.1.15 and 15.1.16, show that the Legendre expansions of x^s are

(a) $$x^{2r} = \sum_{n=0}^{r} \frac{2^{2n}(4n+1)(2r)!\,(r+n)!}{(2r+2n+1)!\,(r-n)!} P_{2n}(x), \quad s = 2r,$$

(b) $$x^{2r+1} = \sum_{n=0}^{r} \frac{2^{2n+1}(4n+3)(2r+1)!\,(r+n+1)!}{(2r+2n+3)!\,(r-n)!} P_{2n+1}(x), \; s = 2r+1.$$

15.1.18 In numerical work (for e.g., the Gauss-Legendre quadrature), it is useful to establish that $P_n(x)$ has n real zeros in the interior of $[-1, 1]$. Show that this is so.

Hint. Rolle's theorem shows that the first derivative of $(x^2 - 1)^{2n}$ has one zero in the interior of $[-1, 1]$. Extend this argument to the second, third, and ultimately the nth derivative.

15.2 ORTHOGONALITY

Because the Legendre ODE is self-adjoint and the coefficient of $P''(x)$, namely $(1 - x^2)$, vanishes at $x = \pm 1$, its solutions of different n will automatically be orthogonal with unit weight on the interval $(-1, 1)$,

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = 0, \quad (n \neq m). \tag{15.31}$$

Because the P_n are real, no complex conjugation needs to be indicated in the orthogonality integral. Since P_n is often used with argument $\cos\theta$, we note that Eq. (15.31) is equivalent to

$$\int_0^{\pi} P_n(\cos\theta) P_m(\cos\theta) \sin\theta \, d\theta = 0, \quad (n \neq m). \tag{15.32}$$

The definition of the P_n does not guarantee that they are normalized, and in fact they are not. One way to establish the normalization starts by squaring the generating-function formula, yielding initially

$$(1 - 2xt + t^2)^{-1} = \left[\sum_{n=0}^{\infty} P_n(x) t^n \right]^2. \tag{15.33}$$

Integrating from $x = -1$ to $x = 1$ and dropping the cross terms because they vanish due to orthogonality, Eq. (15.31), we have

$$\int_{-1}^{1} \frac{dx}{1 - 2tx + t^2} = \sum_{n=0}^{\infty} t^{2n} \int_{-1}^{1} \left[P_n(x) \right]^2 dx. \tag{15.34}$$

Making now the substitution $y = 1 - 2tx + t^2$, with $dy = -2t\,dx$, we obtain

$$\int_{-1}^{1} \frac{dx}{1 - 2tx + t^2} = \frac{1}{2t} \int_{(1-t)^2}^{(1+t)^2} \frac{dy}{y} = \frac{1}{t} \ln\left(\frac{1+t}{1-t} \right). \tag{15.35}$$

Expanding this result in a power series (Exercise 1.6.1),

$$\frac{1}{t} \ln\left(\frac{1+t}{1-t} \right) = 2 \sum_{n=0}^{\infty} \frac{t^{2n}}{2n+1}, \tag{15.36}$$

and equating the coefficients of powers of t in Eqs. (15.34) and (15.36), we must have

$$\int_{-1}^{1} \left[P_n(x) \right]^2 dx = \frac{2}{2n+1}. \tag{15.37}$$

Combining Eqs. (15.31) and (15.37), we have the orthonormality condition

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \frac{2\delta_{nm}}{2n+1}. \tag{15.38}$$

This result can also be obtained using the Rodrigues formulas for P_n and P_m. See Exercise 15.2.1.

Legendre Series

The orthogonality of the Legendre polynomials makes it natural to use them as a basis for expansions. Given a function $f(x)$ defined on the range $(-1, 1)$, the coefficients in the expansion

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \tag{15.39}$$

are given by the formula

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx. \tag{15.40}$$

The orthogonality property guarantees that this expansion is unique. Since we can (but perhaps will not wish to) convert our expansion into a power series by inserting the expansion of Eq. (15.14) and collecting the coefficients of each power of x, we can also obtain a power series, which we thereby know must be unique.

An important application of Legendre series is to solutions of the Laplace equation. We saw in Section 9.4 that when the Laplace equation is separated in spherical polar coordinates, its general solution (for spherical symmetry) takes the form

$$\psi(r, \theta, \varphi) = \sum_{l,m} (A_{lm} r^l + B_{lm} r^{-l-1}) P_l^m(\cos\theta)(A'_{lm} \sin m\varphi + B'_{lm} \cos m\varphi), \tag{15.41}$$

with l required to be an integer to avoid a solution that diverges in the polar directions. Here we consider solutions with no azimuthal dependence (i.e., with $m = 0$), so Eq. (15.41) reduces to

$$\psi(r, \theta) = \sum_{l=0}^{\infty} (a_l r^l + b_l r^{-l-1}) P_l(\cos\theta). \tag{15.42}$$

Often our problem is further restricted to a region either within or external to a boundary sphere, and if the problem is such that ψ must remain finite, the solution will have one of the two following forms:

$$\psi(r, \theta) = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta) \quad (r \le r_0), \tag{15.43}$$

$$\psi(r, \theta) = \sum_{l=0}^{\infty} a_l r^{-l-1} P_l(\cos\theta) \quad (r \ge r_0). \tag{15.44}$$

Note that this simplification is not always appropriate; see Example 15.2.2. Sometimes the coefficients (a_l) are determined from the boundary conditions of a problem rather than from the expansion of a known function. See the examples to follow.

Example 15.2.1 EARTH'S GRAVITATIONAL FIELD

An example of a Legendre series is provided by the description of the Earth's gravitational potential U at points exterior to the Earth's surface. Because gravitation is an inverse-square force, its potential in mass-free regions satisfies the Laplace equation, and therefore (if we neglect azimuthal effects, i.e., those dependent on longitude) it has the form given in Eq. (15.44).

To specialize to the current example, we define R to be the Earth's radius at the equator, and take as the expansion variable the dimensionless quantity R/r. In terms of the total mass of the Earth M and the gravitational constant G, we have

$$R = 6378.1 \pm 0.1\,\text{km},$$

$$\frac{GM}{R} = 62.494 \pm 0.001\,\text{km}^2/\text{s}^2,$$

and we write

$$U(r,\theta) = \frac{GM}{R}\left[\frac{R}{r} - \sum_{l=2}^{\infty} a_l \left(\frac{R}{r}\right)^{l+1} P_l(\cos\theta)\right]. \qquad (15.45)$$

The leading term of this expansion describes the result that would be obtained if the Earth were spherically symmetric; the higher terms describe distortions. The P_1 term is absent because the origin from which r is measured is the Earth's center of mass.

Artificial satellite motions have shown that

$$a_2 = (1,082,635 \pm 11) \times 10^{-9},$$

$$a_3 = (-2,531 \pm 7) \times 10^{-9},$$

$$a_4 = (-1,600 \pm 12) \times 10^{-9}.$$

This is the famous pear-shaped deformation of the Earth. Other coefficients have been computed through a_{20}.

More recent satellite data permit a determination of the longitudinal dependence of the Earth's gravitational field. Such dependence may be described by a Laplace series (see Section 15.5). ■

Example 15.2.2 SPHERE IN A UNIFORM FIELD

Another illustration of the use of a Legendre series is provided by the problem of a neutral conducting sphere (radius r_0) placed in a (previously) uniform electric field of magnitude E_0 (Fig. 15.2). The problem is to find the new, perturbed electrostatic potential ψ that satisfies Laplace's equation,

$$\nabla^2\psi = 0.$$

We select spherical polar coordinates with origin at the center of the conducting sphere and the polar (z) axis oriented parallel to the original uniform field, a choice that will simplify

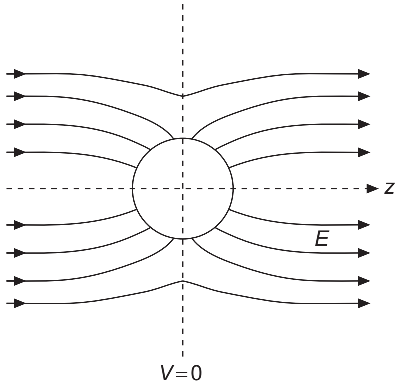

FIGURE 15.2 Conducting sphere in a uniform field.

the application of the boundary condition at the surface of the conductor. Separating variables, we note that because we require a solution to Laplace's equation, the potential for $r \geq r_0$ will be of the form of Eq. (15.42). Our solution will be independent of φ because of the axial symmetry of the problem.

Because the insertion of the conducting sphere will have an effect that is local, the asymptotic behavior of ψ must be of the form

$$\psi(r \to \infty) = -E_0 z = -E_0 r \cos\theta = -E_0 r P_1(\cos\theta), \tag{15.46}$$

equivalent to

$$a_n = 0, \quad n > 1, \quad a_1 = -E_0. \tag{15.47}$$

Note that if $a_n \neq 0$ for any $n > 1$, that term would dominate at large r and the boundary condition, Eq. (15.46), could not be satisfied. In addition, the neutrality of the conducting sphere requires that ψ not contain a contribution proportional to $1/r$, so we also must have $b_0 = 0$.

As a second boundary condition, the conducting sphere must be an equipotential, and without loss of generality we can set its potential to zero. Then, on the sphere $r = r_0$ we have

$$\psi(r_0, \theta) = a_0 + \left(\frac{b_1}{r_0^2} - E_0 r_0\right) P_1(\cos\theta) + \sum_{n=2}^{\infty} b_n \frac{P_n(\cos\theta)}{r_0^{n+1}} = 0. \tag{15.48}$$

In order that Eq. (15.48) may hold for all values of θ, we set

$$a_0 = 0, \quad b_1 = E_0 r_0^3 \quad b_n = 0, \quad n \geq 2. \tag{15.49}$$

The electrostatic potential (outside the sphere) is then completely determined:

$$\psi(r,\theta) = -E_0 r P_1(\cos\theta) + \frac{E_0 r_0^3}{r^2} P_1(\cos\theta)$$
$$= -E_0 r P_1(\cos\theta)\left(1 - \frac{r_0^3}{r^3}\right) = -E_0 z \left(1 - \frac{r_0^3}{r^3}\right). \tag{15.50}$$

In Section 9.5 we showed that Laplace's equation with Dirichlet boundary conditions on a closed boundary (parts of which may be at infinity) had a unique solution. Since we have now found a solution to our current problem, it must (apart from an additive constant) be the only solution.

It may further be shown that there is an induced surface charge density

$$\sigma = -\varepsilon_0 \frac{\partial\psi}{\partial r}\bigg|_{r=r_0} = 3\varepsilon_0 E_0 \cos\theta \tag{15.51}$$

on the surface of the sphere and an induced electric dipole moment of magnitude

$$P = 4\pi r_0^3 \varepsilon_0 E_0. \tag{15.52}$$

See Exercise 15.2.11. ■

Example 15.2.3 Electrostatic Potential for a Ring of Charge

As a further example, consider the electrostatic potential produced by a thin conducting ring of radius a placed symmetrically in the equatorial plane of a spherical polar coordinate system and carrying a total electric charge q (Fig. 15.3). Again we rely on the fact that the potential ψ satisfies Laplace's equation. Separating the variables and recognizing that a solution for the region $r > a$ must go to zero as $r \to \infty$, we use the form given by

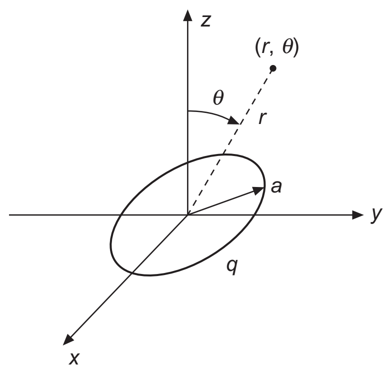

FIGURE 15.3 Charged, conducting ring.

Eq. (15.44), obtaining

$$\psi(r,\theta)=\sum_{n=0}^{\infty}c_n\frac{a^n}{r^{n+1}}P_n(\cos\theta),\qquad r>a. \tag{15.53}$$

There is no φ (azimuthal) dependence because of the cylindrical symmetry of the system. Note also that by including an explicit factor a^n we cause all the coefficients c_n to have the same dimensionality; this choice simply modifies the definition of c_n and was, of course, not required.

Our problem is to determine the coefficients c_n in Eq. (15.53). This may be done by evaluating $\psi(r,\theta)$ at $\theta=0$, $r=z$, and comparing with an independent calculation of the potential from Coulomb's law. In effect, we are using a boundary condition along the z-axis. From Coulomb's law (using the fact that all the charge is equidistant from any point on the z axis),

$$\psi(z,0)=\frac{q}{4\pi\varepsilon_0}\,\frac{1}{(z^2+a^2)^{1/2}}=\frac{q}{4\pi\varepsilon_0 z}\sum_{s=0}^{\infty}\binom{-1/2}{s}\left(\frac{a^2}{z^2}\right)^s$$

$$=\frac{q}{4\pi\varepsilon_0 z}\sum_{s=0}^{\infty}(-1)^s\frac{(2s-1)!!}{(2s)!!}\left(\frac{a}{z}\right)^{2s},\qquad z>a, \tag{15.54}$$

where we have evaluated the binomial coefficient using Eq. (1.74).

Now, evaluating $\psi(z,0)$ from Eq. (15.53), remembering that $P_n(1)=1$ for all n, we have

$$\psi(z,0)=\sum_{n=0}^{\infty}c_n\frac{a^n}{z^{n+1}}. \tag{15.55}$$

Since the power series expansion in z is unique, we may equate the coefficients of corresponding powers of z from Eqs. (15.54) and (15.55), reaching the conclusion that $c_n=0$ for n odd, while for n even and equal to $2s$,

$$c_{2s}=\frac{q}{4\pi\varepsilon_0 z}\,(-1)^s\frac{(2s-1)!!}{(2s)!!}, \tag{15.56}$$

and our electrostatic potential $\psi(r,\theta)$ is given by

$$\psi(r,\theta)=\frac{q}{4\pi\varepsilon_0 r}\sum_{s=0}^{\infty}(-1)^s\frac{(2s-1)!!}{(2s)!!}\left(\frac{a}{r}\right)^{2s}P_{2s}(\cos\theta),\qquad r>a. \tag{15.57}$$

The magnetic analog of this problem appears in Example 15.4.2. ■

Exercises

15.2.1 Using a Rodrigues formula, show that the $P_n(x)$ are orthogonal and that

$$\int_{-1}^{1}[P_n(x)]^2dx=\frac{2}{2n+1}.$$

Hint. Integrate by parts.

15.2.2 You have constructed a set of orthogonal functions by the Gram-Schmidt process (Section 5.2), taking $u_n(x) = x^n$, $n = 0, 1, 2, \ldots,$ in increasing order with $w(x) = 1$ and an interval $-1 \leq x \leq 1$. Prove that the nth such function constructed in this way is proportional to $P_n(x)$.

Hint. Use mathematical induction (Section 1.4).

15.2.3 Expand the Dirac delta function $\delta(x)$ in a series of Legendre polynomials using the interval $-1 \leq x \leq 1$.

15.2.4 Verify the Dirac delta function expansions

$$\delta(1-x) = \sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(x),$$

$$\delta(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{2} P_n(x).$$

These expressions appear in a resolution of the Rayleigh plane wave expansion (Exercise 15.2.24) into incoming and outgoing spherical waves.

Note. Assume that the **entire** Dirac delta function is covered when integrating over $[-1, 1]$.

15.2.5 Neutrons (mass 1) are being scattered by a nucleus of mass A $(A > 1)$. In the center-of-mass system the scattering is isotropic. Then, in the laboratory system the average of the cosine of the angle of deflection of the neutron is

$$\langle \cos\psi \rangle = \frac{1}{2} \int_0^{\pi} \frac{A\cos\theta + 1}{(A^2 + 2A\cos\theta + 1)^{1/2}} \sin\theta \, d\theta.$$

Show, by expansion of the denominator, that $\langle \cos\psi \rangle = 2/(3A)$.

15.2.6 A particular function $f(x)$ defined over the interval $[-1, 1]$ is expanded in a Legendre series over this same interval. Show that the expansion is unique.

15.2.7 A function $f(x)$ is expanded in a Legendre series $f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$. Show that

$$\int_{-1}^{1} [f(x)]^2 \, dx = \sum_{n=0}^{\infty} \frac{2a_n^2}{2n+1}.$$

This is a statement that the Legendre polynomials form a complete set.

15.2.8 (a) For

$$f(x) = \begin{cases} +1, & 0 < x < 1, \\ -1, & -1 < x < 0, \end{cases}$$

show that

$$\int_{-1}^{1}\Big[f(x)\Big]^2\,dx = 2\sum_{n=0}^{\infty}(4n+3)\left[\frac{(2n-1)!!}{(2n+2)!!}\right]^2.$$

(b) By testing the series, prove that it is convergent.

(c) The value of the integral in part (a) is 2. Check the rate at which the series converges by summing its first 10 terms.

15.2.9 Prove that

$$\int_{-1}^{1} x(1-x^2)P_n' P_m'\,dx = \frac{2n(n^2-1)}{4n^2-1}\,\delta_{m,n-1} + \frac{2n(n+2)(n+1)}{(2n+1)(2n+3)}\,\delta_{m,n+1}.$$

15.2.10 The coincidence counting rate, $W(\theta)$, in a gamma-gamma angular correlation experiment has the form

$$W(\theta) = \sum_{n=0}^{\infty} a_{2n} P_{2n}(\cos\theta).$$

Show that data in the range $\pi/2 \le \theta \le \pi$ can, in principle, define the function $W(\theta)$ (and permit a determination of the coefficients a_{2n}). This means that although data in the range $0 \le \theta < \pi/2$ may be useful as a check, they are not essential.

15.2.11 A conducting sphere of radius r_0 is placed in an initially uniform electric field, $\mathbf{E}_0$. Show the following:

(a) The induced surface charge density is $\sigma = 3\varepsilon_0 E_0 \cos\theta$,

(b) The induced electric dipole moment is $P = 4\pi r_0^3 \varepsilon_0 E_0$.

Note. The induced electric dipole moment can be calculated either from the surface charge [part (a)] or by noting that the final electric field $\mathbf{E}$ is the result of superimposing a dipole field on the original uniform field.

15.2.12 Obtain as a Legendre expansion the electrostatic potential of the circular ring of Example 15.2.3, for points (r,θ) with $r < a$.

15.2.13 Calculate the **electric field** produced by the charged conducting ring of Example 15.2.3 for

(a) $r > a$, (b) $r < a$.

15.2.14 As an extension of Example 15.2.3, find the potential $\psi(r,\theta)$ produced by a charged conducting disk, Fig. 15.4, for $r > a$, where a is the radius of the disk. The charge density σ (on each side of the disk) is

$$\sigma(\rho) = \frac{q}{4\pi a(a^2-\rho^2)^{1/2}}, \qquad \rho^2 = x^2 + y^2.$$

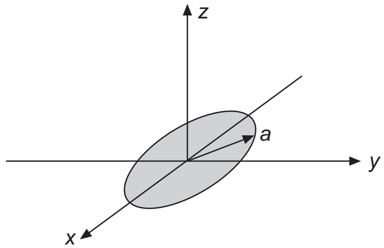

FIGURE 15.4 Charged conducting disk.

Hint. The definite integral you get can be evaluated as a beta function, Section 13.3. For more details see section 5.03 of Smythe in Additional Readings.

$$ANS. \quad \psi(r,\theta) = \frac{q}{4\pi\varepsilon_0 r}\sum_{l=0}^{\infty}(-1)^l \frac{1}{2l+1}\left(\frac{a}{r}\right)^{2l} P_{2l}(\cos\theta).$$

15.2.15 The hemisphere defined by $r = a$, $0 \le \theta < \pi/2$, has an electrostatic potential $+V_0$. The hemisphere $r = a$, $\pi/2 < \theta \le \pi$ has an electrostatic potential $-V_0$. Show that the potential at interior points is

$$V = V_0 \sum_{n=0}^{\infty} \frac{4n+3}{2n+2}\left(\frac{r}{a}\right)^{2n+1} P_{2n}(0) P_{2n+1}(\cos\theta)$$

$$= V_0 \sum_{n=0}^{\infty} (-1)^n \frac{(4n+3)(2n-1)!!}{(2n+2)!!}\left(\frac{r}{a}\right)^{2n+1} P_{2n+1}(\cos\theta).$$

Hint. You need Exercise 15.1.12.

15.2.16 A conducting sphere of radius a is divided into two electrically separate hemispheres by a thin insulating barrier at its equator. The top hemisphere is maintained at a potential V_0, and the bottom hemisphere at $-V_0$.

(a) Show that the electrostatic potential **exterior** to the two hemispheres is

$$V(r,\theta) = V_0 \sum_{s=0}^{\infty} (-1)^s (4s+3) \frac{(2s-1)!!}{(2s+2)!!}\left(\frac{a}{r}\right)^{2s+2} P_{2s+1}(\cos\theta).$$

(b) Calculate the electric charge density σ on the outside surface. Note that your series diverges at $\cos\theta = \pm 1$, as you expect from the infinite capacitance of this system (zero thickness for the insulating barrier).

$$ANS. \quad \text{(b) } \sigma = \varepsilon_0 E_n = -\varepsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=a}$$

$$= \varepsilon_0 V_0 \sum_{s=0}^{\infty} (-1)^s (4s+3) \frac{(2s-1)!!}{(2s)!!} P_{2s+1}(\cos\theta).$$

15.2.17 By writing $\varphi_s(x) = \sqrt{(2s+1)/2}\, P_s(x)$, a Legendre polynomial is renormalized to unity. Explain how $|\varphi_s\rangle\langle\varphi_s|$ acts as a projection operator. In particular, show that if $|f\rangle = \sum_n a'_n|\varphi_n\rangle$, then

$$|\varphi_s\rangle\langle\varphi_s|f\rangle = a'_s|\varphi_s\rangle.$$

15.2.18 Expand x^8 as a Legendre series. Determine the Legendre coefficients from Eq. (15.40),

$$a_m = \frac{2m+1}{2}\int_{-1}^{1} x^8 P_m(x)\,dx.$$

Check your values against AMS-55, Table 22.9. (For the complete reference, see Additional Readings.) This illustrates the expansion of a simple function $f(x)$.

Hint. Gaussian quadrature can be used to evaluate the integral.

15.2.19 Calculate and tabulate the electrostatic potential created by a ring of charge, Example 15.2.3, for $r/a = 1.5(0.5)5.0$ and $\theta = 0°(15°)90°$. Carry terms through $P_{22}(\cos\theta)$.

Note. The convergence of your series will be slow for $r/a = 1.5$. Truncating the series at P_{22} limits you to about four-significant-figure accuracy.

Check value. For $r/a = 2.5$ and $\theta = 60°$, $\psi = 0.40272(q/4\pi\varepsilon_0 r)$.

15.2.20 Calculate and tabulate the electrostatic potential created by a charged disk (Exercise 15.2.14), for $r/a = 1.5(0.5)5.0$ and $\theta = 0°(15°)90°$. Carry terms through $P_{22}(\cos\theta)$.

Check value. For $r/a = 2.0$ and $\theta = 15°$, $\psi = 0.46638(q/4\pi\varepsilon_0 r)$.

15.2.21 Calculate the first five (nonvanishing) coefficients in the Legendre series expansion of $f(x) = 1 - |x|$, evaluating the coefficients in the series by numerical integration. Actually these coefficients can be obtained in closed form. Compare your coefficients with those listed in Exercise 18.4.26.

ANS. $a_0 = 0.5000$, $a_2 = -0.6250$, $a_4 = 0.1875$, $a_6 = -0.1016$, $a_8 = 0.0664$.

15.2.22 Calculate and tabulate the exterior electrostatic potential created by the two charged hemispheres of Exercise 15.2.16, for $r/a = 1.5(0.5)5.0$ and $\theta = 0°(15°)90°$. Carry terms through $P_{23}(\cos\theta)$.

Check value. For $r/a = 2.0$ and $\theta = 45°$, $V = 0.27066V_0$.

15.2.23 (a) Given $f(x) = 2.0$, $|x| < 0.5$ and $f(x) = 0$, $0.5 < |x| < 1.0$, expand $f(x)$ in a Legendre series and calculate the coefficients a_n through a_{80} (analytically).

(b) Evaluate $\sum_{n=0}^{80} a_n P_n(x)$ for $x = 0.400(0.005)0.600$. Plot your results.

Note. This illustrates the Gibbs phenomenon of Section 19.3 and the danger of trying to calculate with a series expansion in the vicinity of a discontinuity.

15.2.24 A plane wave may be expanded in a series of spherical waves by the Rayleigh equation,

$$e^{ikr\cos\gamma} = \sum_{n=0}^{\infty} a_n j_n(kr) P_n(\cos\gamma).$$

Show that $a_n = i^n(2n+1)$.

Hint.

1. Use the orthogonality of the P_n to solve for $a_n j_n(kr)$.
2. Differentiate n times with respect to (kr) and set $r = 0$ to eliminate the r-dependence.
3. Evaluate the remaining integral by Exercise 15.1.15.

Note. This problem may also be treated by noting that both sides of the equation satisfy the Helmholtz equation. The equality can be established by showing that the solutions have the same behavior at the origin and also behave alike at large distances.

15.2.25 Verify the Rayleigh equation of Exercise 15.2.24 by starting with the following steps:

(a) Differentiate with respect to (kr) to establish

$$\sum_n a_n j_n'(kr) P_n(\cos\gamma) = i \sum_n a_n j_n(kr) \cos\gamma \, P_n(\cos\gamma).$$

(b) Use a recurrence relation to replace $\cos\gamma \, P_n(\cos\gamma)$ by a linear combination of P_{n-1} and P_{n+1}.

(c) Use a recurrence relation to replace j_n' by a linear combination of j_{n-1} and j_{n+1}.

15.2.26 From Exercise 15.2.24 show that

$$j_n(kr) = \frac{1}{2i^n} \int_{-1}^{1} e^{ikr\mu} P_n(\mu) d\mu.$$

This means that (apart from a constant factor) the spherical Bessel function $j_n(kr)$ is an integral transform of the Legendre polynomial $P_n(\mu)$.

15.2.27 Rewriting the formula of Exercise 15.2.26 as

$$j_n(z) = \tfrac{1}{2}(-i)^n \int_0^{\pi} e^{iz\cos\theta} P_n(\cos\theta) \sin\theta \, d\theta, \quad n = 0, 1, 2, \ldots,$$

verify it by transforming the right-hand side into

$$\frac{z^n}{2^{n+1} n!} \int_0^{\pi} \cos(z\cos\theta) \sin^{2n+1}\theta \, d\theta$$

and using Exercise 14.7.9.

15.3 PHYSICAL INTERPRETATION OF GENERATING FUNCTION

The generating function for the Legendre polynomials has an interesting and important interpretation. If we introduce spherical polar coordinates (r, θ, φ) and place a charge q at the point a on the positive z axis (see Fig. 15.5), the potential at a point (r, θ) (it is independent of φ) can be calculated, using the law of cosines, as

$$\psi(r,\theta) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r_1} = \frac{q}{4\pi\varepsilon_0}\left(r^2 + a^2 - 2ar\cos\theta\right)^{-1/2}. \tag{15.58}$$

The expression in Eq. (15.58) is essentially that appearing in the generating function; to identify the correspondence we rewrite that equation as

$$\psi(r,\theta) = \frac{q}{4\pi\varepsilon_0 r}\left(1 - 2\frac{a}{r}\cos\theta + \frac{a^2}{r^2}\right)^{-1/2} = \frac{q}{4\pi\varepsilon_0 r}\, g\left(\cos\theta, \frac{a}{r}\right) \tag{15.59}$$

$$= \frac{q}{4\pi\varepsilon_0 r}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n, \tag{15.60}$$

where we reached Eq. (15.60) by inserting the generating-function expansion.

The series in Eq. (15.60) only converges for $r > a$, with a rate of convergence that improves as r/a increases. If, on the other hand, we desire an expression for $\psi(r,\theta)$ when $r < a$, we can perform a different rearrangement of Eq. (15.58), to

$$\psi(r,\theta) = \frac{q}{4\pi\varepsilon_0 a}\left(1 - 2\frac{r}{a}\cos\theta + \frac{r^2}{a^2}\right)^{-1/2}, \tag{15.61}$$

which we again recognize as the generating-function expansion, but this time with the result

$$\psi(r,\theta) = \frac{q}{4\pi\varepsilon_0 a}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r}{a}\right)^n, \tag{15.62}$$

valid when $r < a$.

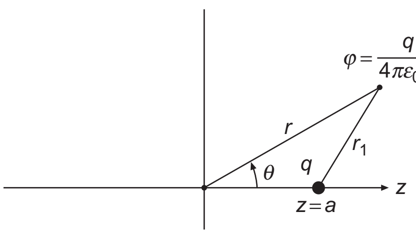

FIGURE 15.5 Electrostatic potential, charge q displaced from origin.

Expansion of $1/|\mathbf{r}_1 - \mathbf{r}_2|$

Equations (15.60) and (15.62) describe the interaction of a charge q at position $\mathbf{a} = a\hat{\mathbf{e}}_z$ with a unit charge at position $\mathbf{r}$. Dropping the factors needed for an electrostatics calculation, these equations yield formulas for $1/|\mathbf{r} - \mathbf{a}|$. The fact that $\mathbf{a}$ is aligned with the z-axis is actually of no importance for the computation of $1/|\mathbf{r} - \mathbf{a}|$; the relevant quantities are r, a, and the angle θ between $\mathbf{r}$ and $\mathbf{a}$. Thus, we may rewrite either Eq. (15.60) or (15.62) in a more neutral notation, to give the value of $1/|\mathbf{r}_1 - \mathbf{r}_2|$ in terms of the magnitudes r_1, r_2 and the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$, which we now call χ. If we define $r_>$ and $r_<$ to be respectively the larger and the smaller of r_1 and r_2, Eqs. (15.60) and (15.62) can be combined into the single equation

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{1}{r_>} \sum_{n=0}^{\infty} \left(\frac{r_<}{r_>}\right)^n P_n(\cos\chi), \tag{15.63}$$

which will converge everywhere except when $r_1 = r_2$.

Electric Multipoles

Returning to Eq. (15.60) and restricting consideration to $r > a$, we may note that its initial term (with $n = 0$) gives the potential we would get if the charge q were at the origin, and that further terms must describe corrections arising from the actual position of the charge. One way to obtain further understanding of the second and later terms in the expansion is to consider what would happen if we added a second charge, $-q$, at $z = -a$, as shown in Fig. 15.6. The potential due to the second charge will be given by an expression similar to that in Eq. (15.58), except that the signs of q and $\cos\theta$ must be reversed (the angle opposite r_2 in the figure is $\pi - \theta$). We now have

$$\psi = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

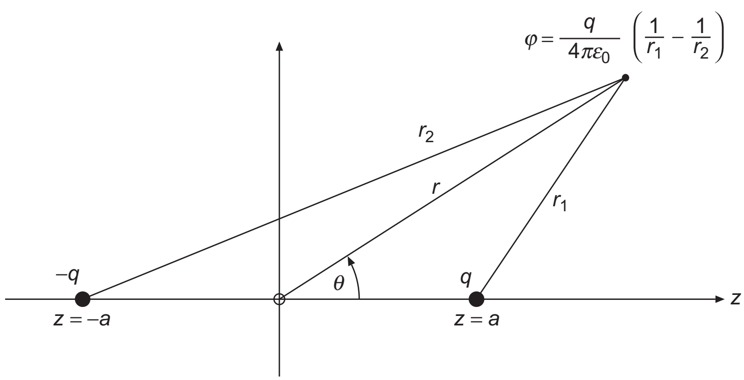

FIGURE 15.6 Electric dipole.

$$= \frac{q}{4\pi\varepsilon_0 r}\left[\left(1 - 2\frac{a}{r}\cos\theta + \frac{a^2}{r^2}\right)^{-1/2} - \left(1 + 2\frac{a}{r}\cos\theta + \frac{a^2}{r^2}\right)^{-1/2}\right]$$

$$= \frac{q}{4\pi\varepsilon_0 r}\left[\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n - \sum_{n=0}^{\infty} P_n(\cos\theta)\left(-\frac{a}{r}\right)^n\right]. \tag{15.64}$$

If we combine the two summations in Eq. (15.64), alternate terms cancel, and we get

$$\psi = \frac{2q}{4\pi\varepsilon_0 r}\left[\frac{a}{r}P_1(\cos\theta) + \frac{a^3}{r^3}P_3(\cos\theta) + \cdots\right]. \tag{15.65}$$

This configuration of charges is called an **electric dipole**, and we note that its leading dependence on r goes as r^{-2}. The strength of the dipole (called the **dipole moment**) can be identified as $2qa$, equal to the magnitude of each charge multiplied by their separation ($2a$). If we let $a \to 0$ while keeping the product $2qa$ constant at a value μ, all but the first term becomes negligible, and we have

$$\psi = \frac{\mu}{4\pi\varepsilon_0}\frac{P_1(\cos\theta)}{r^2}, \tag{15.66}$$

the potential of a **point dipole** of dipole moment μ, located at the origin of the coordinate system (at $r = 0$). Note that because we have limited the discussion to situations of cylindrical symmetry, our dipole is oriented in the polar direction; more general orientations can be considered after we have developed formulas for solutions of the associated Legendre equation (cases where the parameter m in Eq. (15.4) is nonzero).

We can extend the above analysis by combining a pair of dipoles of opposite orientation, for example, in the configuration shown in Fig. 15.7, thereby causing cancellation of their leading terms, leaving a potential whose leading contribution will be proportional to $r^{-3}P_2(\cos\theta)$. A charge configuration of this sort is called an **electric quadrupole**, and the P_2 term of the generating function expansion can be identified as the contribution of a **point quadrupole**, also located at $r = 0$. Further extensions, to 2^n-poles, with contributions proportional to $P_n(\cos\theta)/r^{n+1}$, permit us to identify each term of the generating expansion with the potential of a point multipole. We thus have a **multipole expansion**. Again we observe that because we have limited discussion to situations with cylindrical symmetry our multipoles are presently required to be linear; that restriction will be eliminated when this topic is revisited in Chapter 16.

We look next at more general charge distributions, for simplicity limiting consideration to charges q_i placed at respective positions a_i on the polar axis of our coordinate system.

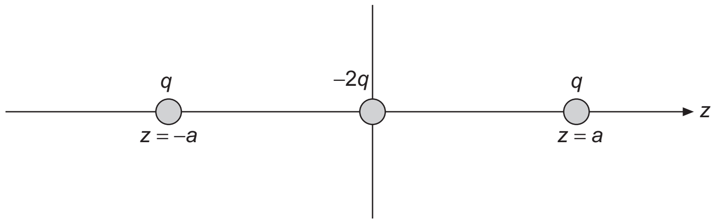

FIGURE 15.7 Linear electric quadrupole.

Adding together the generating-function expansions of the individual charges, our combined expansion takes the form

$$\psi = \frac{1}{4\pi\varepsilon_0 r}\left[\sum_i q_i + \sum_i \frac{q_i a_i}{r} P_1(\cos\theta) + \sum_i \frac{q_i a_i^2}{r^2} P_2(\cos\theta) + \cdots\right]$$
$$= \frac{1}{4\pi\varepsilon_0 r}\left[\mu_0 + \frac{\mu_1}{r} P_1(\cos\theta) + \frac{\mu_2}{r^2} P_2(\cos\theta) + \cdots\right], \tag{15.67}$$

where the μ_i are called the **multipole moments** of the charge distribution; μ_0 is the 2^0-pole, or **monopole** moment, with a value equal to the total net charge of the distribution; μ_1 is the 2^1-pole, or dipole moment, equal to $\sum_i q_i a_i$; μ_2 is the 2^2-pole, or quadrupole moment, given as $\sum_i q_i a_i^2$, etc. Our general (linear) multipole expansion will converge for values of r that are larger than all the a_i values of the individual charges. Put another way, the expansion will converge at points further from the coordinate origin than all parts of the charge distribution.

We next ask: What happens if we move the origin of our coordinate system? Or, equivalently, consider replacing r by $|\mathbf{r}-\mathbf{r_p}|$. For $r > r_p$, the binomial expansion of $1/|\mathbf{r}-\mathbf{r_p}|^n$ will have the generic form

$$\frac{1}{|\mathbf{r}-\mathbf{r_p}|^n} = \frac{1}{r^n} + C\frac{r_p}{r^{n+1}} + \cdots,$$

with the result that only the leading nonzero term of Eq. (15.67) will be unaffected by the change of expansion center. Translated into everyday language, this means that the lowest nonzero moment of the expansion will be independent of the choice of origin, but all higher moments will change when the expansion center is moved. Specifically, the total net charge (monopole moment) will always be independent of the choice of expansion center. The dipole moment will be independent of the expansion point only when the net charge is zero; the quadrupole moment will have such independence only if both the net charge and dipole moments vanish, etc.

We close this section with three observations.

- First, while we have illustrated our discussion with discrete arrays of point charges, we could have reached the same conclusions using continuous charge distributions, with the result that the summations over charges would become integrals over the **charge density**.
- Second, if we remove our restriction to linear arrays, our expansion would involve components of the multipole moments in different directions. In three-dimensional space, the dipole moment would have three components: a generalizes to (a_x, a_y, a_z), while the higher-order multipoles will have larger numbers of components ($a^2 \to a_x a_x,\ a_x a_y, \ldots$). The details of that analysis will be taken up when the necessary background is in place.
- Third, the multipole expansion is not restricted to electrical phenomena, but applies anywhere we have an inverse-square force. For example, planetary configurations are described in terms of mass multipoles. And gravitational radiation depends on the time behavior of mass quadrupoles.

Exercises

15.3.1 Develop the electrostatic potential for the array of charges shown in Fig. 15.7. This is a linear electric quadrupole.

15.3.2 Calculate the electrostatic potential of the array of charges shown in Fig. 15.8. Here is an example of two equal but oppositely directed quadrupoles. The quadrupole contributions cancel. The octopole terms do not cancel.

15.3.3 Show that the electrostatic potential produced by a charge q at $z = a$ for $r < a$ is

$$\varphi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 a}\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n P_n(\cos\theta).$$

15.3.4 Using $\mathbf{E} = -\nabla\varphi$, determine the components of the electric field corresponding to the (pure) electric dipole potential,

$$\varphi(\mathbf{r}) = \frac{2aq\,P_1(\cos\theta)}{4\pi\varepsilon_0 r^2}.$$

Here it is assumed that $r \gg a$.

$$ANS.\quad E_r = +\frac{4aq\cos\theta}{4\pi\varepsilon_0 r^3},\quad E_\theta = +\frac{2aq\sin\theta}{4\pi\varepsilon_0 r^3},\quad E_\varphi = 0.$$

15.3.5 Operating in **spherical polar coordinates**, show that

$$\frac{\partial}{\partial z}\left[\frac{P_l(\cos\theta)}{r^{l+1}}\right] = -(l+1)\frac{P_{l+1}(\cos\theta)}{r^{l+2}}.$$

This is the key step in the mathematical argument that the derivative of one multipole leads to the next higher multipole.

Hint. Compare with Exercise 3.10.28.

15.3.6 A point electric dipole of strength $p^{(1)}$ is placed at $z = a$; a second point electric dipole of equal but opposite strength is at the origin. Keeping the product $p^{(1)}a$ constant, let $a \to 0$. Show that this results in a point electric quadrupole.

Hint. Exercise 15.3.5 (when proved) will be helpful.

15.3.7 A point electric octupole may be constructed by placing a point electric quadrupole (pole strength $p^{(2)}$ in the z-direction) at $z = a$ and an equal but opposite point electric quadrupole at $z = 0$ and then letting $a \to 0$, subject to $p^{(2)}a =$ constant. Find the electrostatic potential corresponding to a point electric octupole. Show from the construction of the point electric octupole that the corresponding potential may be obtained by differentiating the point quadrupole potential.

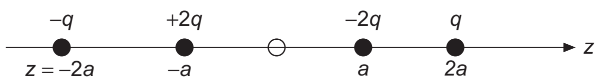

FIGURE 15.8 Linear electric octopole.

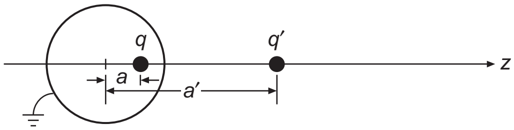

FIGURE 15.9 Image charges for Exercise 15.3.8.

15.3.8 A point charge q is in the interior of a hollow conducting sphere of radius r_0. The charge q is displaced a distance a from the center of the sphere. If the conducting sphere is grounded, show that the potential in the interior produced by q and the distributed induced charge is the same as that produced by q and its image charge q'. The image charge is at a distance $a' = r_0^2/a$ from the center, collinear with q and the origin (Fig. 15.9).

Hint. Calculate the electrostatic potential for $a < r_0 < a'$. Show that the potential vanishes for $r = r_0$ if we take $q' = -qr_0/a$.

15.4 Associated Legendre Equation

We need to extend our analysis to the associated Legendre equation because it is important to be able to remove the restriction to azimuthal symmetry that pervaded the discussion of the previous sections of this chapter. We therefore return to Eq. (15.4), which, before determining what its eigenvalue should be, assumed the form

$$(1-x^2)P''(x) - 2xP'(x) + \left[\lambda - \frac{m^2}{1-x^2}\right]P(x) = 0. \tag{15.68}$$

Trial and error (or great insight) suggests that the troublesome factor $1 - x^2$ in the denominator of this equation can be eliminated by making a substitution of the form $P = (1-x^2)^p\,\mathcal{P}$, and further experimentation shows that a suitable choice for the exponent p is $m/2$. By straightforward differentiation, we find

$$P = (1-x^2)^{m/2}\mathcal{P}, \tag{15.69}$$

$$P' = (1-x^2)^{m/2}\mathcal{P}' - mx(1-x^2)^{m/2-1}\mathcal{P}, \tag{15.70}$$

$$\begin{aligned} P'' &= (1-x^2)^{m/2}\mathcal{P}'' - 2mx(1-x^2)^{m/2-1}\mathcal{P}' \\ &\quad + \left[-m(1-x^2)^{m/2-1} + (m^2-2m)x^2(1-x^2)^{m/2-2}\right]\mathcal{P}. \end{aligned} \tag{15.71}$$

Substitution of Eqs. (15.69)–(15.71) into Eq. (15.68), we obtain an equation that is potentially easier to solve, namely,

$$(1-x^2)\mathcal{P}'' - 2x(m+1)\mathcal{P}' + \left[\lambda - m(m+1)\right]\mathcal{P} = 0. \tag{15.72}$$

We continue by seeking to solve Eq. (15.72) by the method of Frobenius, assuming a solution in the series form $\sum_j a_j x^{k+j}$. The indicial equation for this ODE has solutions

$k=0$ and $k=1$. For $k=0$, substitution into the series solution leads to the recurrence formula

$$a_{j+2}=a_j\left[\frac{j^2+(2m+1)j-\lambda+m(m+1)}{(j+1)(j+2)}\right]. \tag{15.73}$$

Just as for the original Legendre equation, we need solutions $\mathcal{P}(\cos\theta)$ that are nonsingular for the range $-1\le\cos\theta\le+1$, but the recurrence formula leads to a power series that in general is divergent at ±1.[2]

To avoid the divergence, we must cause the numerator of the fraction in Eq. (15.73) to become zero for some nonnegative even integer j, thereby causing $\mathcal{P}$ to be a polynomial. By direct substitution into Eq. (15.73), we can verify that a zero numerator is obtained for $j=l-m$ when λ is assigned the value $l(l+1)$, a condition that can only be met if l is an integer at least as large as m and of the same parity. Further analysis for the other indicial equation solution, $k=1$, extends our present result to values of l that are larger than m and of opposite parity.

Summarizing our results to this point, we have found that the regular solutions to the associated Legendre equation depend on integer indices l and m. Letting P_l^m, called an **associated Legendre function**, denote such a solution (note that the superscript m is **not** an exponent), we define

$$P_l^m(x)=(1-x^2)^{m/2}\mathcal{P}_l^m(x), \tag{15.74}$$

where $\mathcal{P}_l^m$ is a polynomial of degree $l-m$ (consistent with our earlier observation that l must be at least as large as m), and with an explicit form and scale that we will now address.

A convenient explicit formula for $\mathcal{P}_l^m$ can be obtained by repeated differentiation of the regular Legendre equation. Admittedly, this strategy would have been difficult to devise without prior knowledge of the solution, but there are certain advantages to using the experience of those who have gone before. So, without apology, we apply Leibniz's formula for the mth derivative of a product (proved in Exercise 1.4.2),

$$\frac{d^m}{dx^m}\Big[A(x)B(x)\Big]=\sum_{s=0}^{m}\binom{m}{s}\frac{d^{m-s}A(x)}{dx^{m-s}}\,\frac{d^sB(x)}{dx^s}, \tag{15.75}$$

to the Legendre equation,

$$(1-x^2)P_l''-2xP_l'+l(l+1)P_l=0,$$

reaching

$$(1-x^2)u''-2x(m+1)u'+\Big[l(l+1)-m(m+1)\Big]u=0, \tag{15.76}$$

where

$$u\equiv\frac{d^m}{dx^m}\,P_l(x). \tag{15.77}$$

[2] The solution to the associated Legendre equation is $(1-x^2)^{m/2}\mathcal{P}(x)$, suggesting the possibility that the $(1-x^2)^{m/2}$ factor might compensate the divergence in $\mathcal{P}(x)$, yielding a convergent limit. It can be shown that such a compensation does not occur.

Comparing Eq. (15.76) with Eq. (15.72), we see that when $\lambda = l(l+1)$ they are identical, meaning that the polynomial solutions $\mathcal{P}$ of Eq. (15.72) for given l can be identified with the corresponding u. Specifically,

$$\mathcal{P}_l^m = (-1)^m \frac{d^m}{dx^m} P_l(x), \tag{15.78}$$

where the factor $(-1)^m$ is inserted to maintain agreement with AMS-55 (see Additional Readings), which has become the most widely accepted notational standard.[3]

We can now write a complete, explicit form for the associated Legendre functions:

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x). \tag{15.79}$$

Since the P_l^m with $m = 0$ are just the original Legendre functions, it is customary to omit the upper index when it is zero, so, for example, $P_l^0 \equiv P_l$.

Note that the condition on l and m can be stated in two ways:

(1) For each m, there are an infinite number of acceptable solutions to the associated Legendre ODE with l values ranging from m to infinity, or

(2) For each l, there are acceptable solutions with m values ranging from $l = 0$ to $l = m$.

Because m enters the associated Legendre equation only as m^2, we have up to this point tacitly considered only values $m \geq 0$. However, if we insert the Rodrigues formula for P_l into Eq. (15.73), we get the formula

$$P_l^m(x) = \frac{(-1)^m}{2^l\, l!} (1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l, \tag{15.80}$$

which gives results for $-m$ that do not appear similar to those for $+m$. However, it can be shown that if we apply Eq. (15.75) for m values between zero and $-l$, we get

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x). \tag{15.81}$$

Equation (15.81) shows that P_l^m and P_l^{-m} are proportional; its proof is the topic of Exercise 15.4.3. The main reason for discussing both is that recurrence formulas we will develop for P_l^m with contiguous values of m will give results for $m < 0$ that can best be understood if we remember the relative scaling of P_l^m and P_l^{-m}.

Associated Legendre Polynomials

For further development of properties of the P_l^m, it is useful to develop a generating function for the polynomials $\mathcal{P}_l^m(x)$, which we can do by differentiating the Legendre generating function with respect to x. The result is

$$g_m(x,t) \equiv \frac{(-1)^m (2m-1)!!}{(1-2xt+t^2)^{m+1/2}} = \sum_{s=0}^{\infty} \mathcal{P}_{s+m}^m(x) t^s. \tag{15.82}$$

[3] However, we note that the popular text, Jackson's *Electrodynamics* (see Additional Readings), does not include this phase factor. The factor is introduced to cause the definition of spherical harmonics (Section 15.5) to have the usual phase convention.

The factors t that result from differentiating the generating function have been used to change the powers of t that multiply the $\mathcal{P}$ on the right-hand side.

If we now differentiate Eq. (15.82) with respect to t, we obtain initially

$$(1-2tx+t^2)\frac{\partial g_m}{\partial t} = (2m+1)(x-t)g_m(x,t),$$

which we can use together with Eq. (15.82) in a now-familiar way to obtain the recurrence formula,

$$(s+1)\mathcal{P}^m_{s+m+1}(x) - (2m+1+2s)x\mathcal{P}^m_{s+m}(x) + (s+2m)\mathcal{P}^m_{s+m-1} = 0. \tag{15.83}$$

Making the substitution $l = s+m$, we bring Eq. (15.83) to the more useful form,

$$(l-m+1)\mathcal{P}^m_{l+1} - (2l+1)x\mathcal{P}^m_l + (l+m)\mathcal{P}^m_{l-1} = 0. \tag{15.84}$$

For $m = 0$ this relation agrees with Eq. (15.18).

From the form of $g_m(x,t)$, it is also clear that

$$(1-2xt+t^2)g_{m+1}(x,t) = -(2m+1)g_m(x,t). \tag{15.85}$$

From Eqs. (15.85) and (15.82) we may extract the recursion formula

$$\mathcal{P}^{m+1}_{s+m+1}(x) - 2x\mathcal{P}^{m+1}_{s+m}(x) + \mathcal{P}^{m+1}_{s+m-1}(x) = -(2m+1)\mathcal{P}^m_{s+m}(x),$$

which relates the associated Legendre polynomials with upper index $m+1$ to those with upper index m. Again we may simplify by making the substitution $l = s+m$:

$$\mathcal{P}^{m+1}_{l+1}(x) - 2x\mathcal{P}^{m+1}_l(x) + \mathcal{P}^{m+1}_{l-1}(x) = -(2m+1)\mathcal{P}^m_l(x). \tag{15.86}$$

Associated Legendre Functions

The recurrence relations for the associated Legendre polynomials or alternatively, differentiation of formulas for the original Legendre polynomials, enable the construction of recurrence formulas for the associated Legendre functions. The number of such formulas is extensive because these functions have two indices, and there exists a wide variety of formulas with different index combinations. Results of importance include the following:

$$P_l^{m+1}(x) + \frac{2mx}{(1-x^2)^{1/2}}P_l^m(x) + (l+m)(l-m+1)P_l^{m-1}(x) = 0, \tag{15.87}$$

$$(2l+1)xP_l^m(x) = (l+m)P^m_{l-1}(x) + (l-m+1)P^m_{l+1}(x), \tag{15.88}$$

$$(2l+1)(1-x^2)^{1/2}P_l^m(x) = P^{m+1}_{l-1}(x) - P^{m+1}_{l+1}(x) \tag{15.89}$$

$$= (l-m+1)(l-m+2)P^{m-1}_{l+1}(x) - (l+m)(l+m-1)P^{m-1}_{l-1}(x), \tag{15.90}$$

$$(1-x^2)^{1/2}\Big(P_l^m(x)\Big)' = \frac{1}{2}(l+m)(l-m+1)P_l^{m-1}(x) - \frac{1}{2}P_l^{m+1}(x), \tag{15.91}$$

$$= (l+m)(l-m+1)P_l^{m-1}(x) + \frac{mx}{(1-x^2)^{1/2}}P_l^m(x). \tag{15.92}$$

Table 15.3 Associated Legendre Functions

$P_1^1(x) = -(1-x^2)^{1/2} = -\sin\theta$
$P_2^1(x) = -3x(1-x^2)^{1/2} = -3\cos\theta\sin\theta$
$P_2^2(x) = 3(1-x^2) = 3\sin^2\theta$
$P_3^1(x) = -\frac{3}{2}(5x^2-1)(1-x^2)^{1/2} = -\frac{3}{2}(5\cos^2\theta - 1)\sin\theta$
$P_3^2(x) = 15x(1-x^2) = 15\cos\theta\sin^2\theta$
$P_3^3(x) = -15(1-x^2)^{3/2} = -15\sin^3\theta$
$P_4^1(x) = -\frac{5}{2}(7x^3-3x)(1-x^2)^{1/2} = -\frac{5}{2}(7\cos^3\theta - 3\cos\theta)\sin\theta$
$P_4^2(x) = \frac{15}{2}(7x^2-1)(1-x^2) = \frac{15}{2}(7\cos^2\theta - 1)\sin^2\theta$
$P_4^3(x) = -105x(1-x^2)^{3/2} = -105\cos\theta\sin^3\theta$
$P_4^4(x) = 105(1-x^2)^2 = 105\sin^4\theta$

It is obvious that, using Eq. (15.90), all the P_l^m with $m > 0$ can be generated from those with $m = 0$ (the Legendre polynomials), and that these, in turn, can be built recursively from $P_0(x) = 1$ and $P_1(x) = x$. In this fashion (or in other ways as suggested below), we can build a table of associated Legendre functions, the first members of which are listed in Table 15.3. The table shows the $P_l^m(x)$ both as functions of x and as functions of θ, where $x = \cos\theta$.

It is often easier to use recurrence formulas other than Eq. (15.90) to obtain the P_l^m, keeping in mind that when a formula contains P_{m-1}^m for $m > 0$, that quantity can be set to zero. It is also easy to obtain explicit formulas for certain values of l and m which can then be alternate starting points for recursion. See the example that follows.

Example 15.4.1 Recurrence Starting from P_m^m

The associated Legendre function $P_m^m(x)$ is easily evaluated:

$$P_m^m(x) = \frac{(-1)^m}{2^m\, m!}(1-x^2)^{m/2}\frac{d^{2m}}{dx^{2m}}(x^2-1)^m = \frac{(-1)^m}{2^m\, m!}(2m)!\,(1-x^2)^{m/2}$$
$$= (-1)^m(2m-1)!!\,(1-x^2)^{m/2}. \tag{15.93}$$

We can now use Eq. (15.88) with $l = m$ to obtain P_{m+1}^m, dropping the term containing P_{m-1}^m because it is zero. We get

$$P_{m+1}^m(x) = (2m+1)x\,P_m^m(x) = (-1)^m(2m+1)!!\,x(1-x^2)^{m/2}. \tag{15.94}$$

Further increases in l can now be obtained by straightforward application of Eq. (15.88).

Illustrating for a series of P_l^m with $m = 2$: $P_2^2(x) = (-1)^2(3!!)(1-x^2) = 3(1-x^2)$, in agreement with the table value. P_3^2 can be computed from Eq. (15.94) as $P_3^2(x) = (-1)^2(5!!)x(1-x^2)$, which simplifies to the tabulated result. Finally, P_4^2 is obtained from the following case of Eq. (15.88):

$$7x P_3^2(x) = 5P_2^2(x) + 2P_4^2(x),$$

the solution of which for $P_4^2(x)$ is again in agreement with the tabulated value. ■

Parity and Special Values

We have already established that P_l has even parity if l is even and odd parity if l is odd. Since we can form P_l^m by differentiating P_l m times, with each differentiation changing the parity, and thereafter multiplying by $(1-x^2)^{m/2}$, which has even parity, P_l^m must have a parity that depends on $l+m$, namely,

$$P_l^m(-x) = (-1)^{l+m} P_l^m(x). \tag{15.95}$$

We occasionally encounter a need for the value of $P_l^m(x)$ at $x = \pm 1$ or at $x = 0$. At $x = \pm 1$ the result is simple: The factor $(1-x^2)^{m/2}$ causes $P_l^m(\pm 1)$ to vanish unless $m = 0$, in which case we recover the values $P_l(1) = 1$, $P_l(-1) = (-1)^l$. At $x = 0$, the value of P_l^m depends on whether $l+m$ is even or odd. The result, proof of which is left to Exercises 15.4.4 and 15.4.5, is

$$P_l^m(0) = \begin{cases} (-1)^{(l+m)/2} \dfrac{(l+m-1)!!}{(l-m)!!}, & l+m \text{ even}, \\ 0, & l+m \text{ odd}. \end{cases} \tag{15.96}$$

Orthogonality

For each m, the P_l^m of different l can be proved orthogonal by identifying them as eigenfunctions of a Sturm-Liouville system. However, it is instructive to demonstrate the orthogonality explicitly, and to do so by a method that also yields their normalization. We start by writing the orthgonality integral, with the P_l^m given by the Rodrigues formula in Eq. (15.80). For compactness and clarity, we introduce the abbreviated notation $R = x^2 - 1$, thereby getting

$$\int_{-1}^{1} P_p^m(x) P_q^m(x)\, dx = \frac{(-1)^m}{2^{p+q} p!\, q!} \int_{-1}^{1} R^m \left(\frac{d^{p+m} R^p}{dx^{p+m}} \right) \left(\frac{d^{q+m} R^q}{dx^{q+m}} \right) dx. \tag{15.97}$$

We consider first the case $p < q$, for which we plan to prove the integral in Eq. (15.97) vanishes. We proceed by carrying out repeated integrations by parts, in which we differentiate

$$u = R^m \left(\frac{d^{p+m} R^p}{dx^{p+m}} \right) \tag{15.98}$$

$p+m+1$ times while integrating a like number of times the remainder of the integrand,

$$dv = \left(\frac{d^{q+m} R^q}{dx^{q+m}} \right) dx. \tag{15.99}$$

For each of these $p+m+1 \le q+m$ partial integrations the integrated (uv) terms will vanish because there will be at least one factor R that is not differentiated and will therefore vanish at $x=\pm 1$. After the repeated differentiation, we will have

$$\frac{d^{p+m+1}}{dx^{p+m+1}}\,u = \frac{d^{p+m+1}}{dx^{p+m+1}}\left[R^m\left(\frac{d^{p+m}R^p}{dx^{p+m}}\right)\right], \tag{15.100}$$

in which a quantity whose largest power of x is x^{2p+2m} contains also a $(2p+2m+1)$-fold differentiation. There is no way these components can yield a nonzero result. Since both the integrated terms and the transformed integral vanish, we get an overall vanishing result, confirming the orthogonality. Note that the orthogonality is with unit weight, independent of the value of m.

We now examine Eq. (15.97) for $p=q$, repeating the process we just carried out, but this time performing $p+m$ partial integrations. Again all the integrated terms vanish, but now there is a nonvanishing contribution from the repeated differentiation of u, see Eq. (15.98). Since the overall power of x is still x^{2p+2m} and the total number of differentiations is also $2p+2m$, the only contributing terms are those in which the factor R^m is differentiated $2m$ times and the factor R^p is differentiated $2p$ times. Thus, applying Leibniz's formula, Eq. (15.75), to the $p+m$-fold differentiation of u, but keeping only the contributing term, we have

$$\begin{aligned}\frac{d^{p+m}}{dx^{p+m}}\left[R^m\left(\frac{d^{p+m}R^p}{dx^{p+m}}\right)\right] &= \binom{p+m}{2m}\left(\frac{d^{2m}R^m}{dx^{2m}}\right)\left(\frac{d^{2p}R^p}{dx^{2p}}\right)\\ &= \frac{(p+m)!}{(2m)!\,(p-m)!}\,(2m)!\,(2p)! = \frac{(p+m)!}{(p-m)!}\,(2p)!.\end{aligned} \tag{15.101}$$

Inserting this result into the integration by parts, remembering that the transformed integration is accompanied by the sign factor $(-1)^{p+m}$, and recognizing that the repeated integration of dv, Eq. (15.99) with $q=p$, just yields R^p, we have, returning to Eq. (15.97),

$$\int_{-1}^{1}\left[P_p^m(x)\right]^2 dx = \frac{(-1)^{2m+p}}{2^{2p}\,p!\,p!}\,\frac{(p+m)!}{(p-m)!}\,(2p)!\int_{-1}^{1} R^p\,dx. \tag{15.102}$$

To complete the evaluation, we identify the integral of R^p as a beta function, with an evaluation given in Exercise 13.3.3 as

$$\int_{-1}^{1} R^p\,dx = (-1)^p\,\frac{2(2p)!!}{(2p+1)!!} = (-1)^p\,\frac{2^{2p+1}\,p!\,p!}{(2p+1)!}. \tag{15.103}$$

Inserting this result, and combining with the previously established orthogonality relation, we have

$$\int_{-1}^{1} P_p^m(x)P_q^m(x)\,dx = \frac{2}{2p+1}\,\frac{(p+m)!}{(p-m)!}\,\delta_{pq}. \tag{15.104}$$

Making the substitution $x = \cos\theta$, we obtain this formula in spherical polar coordinates:

$$\int_0^\pi P_p^m(\cos\theta) P_q^m(\cos\theta)\sin\theta\, d\theta = \frac{2}{2p+1}\frac{(p+m)!}{(p-m)!}\,\delta_{pq}. \tag{15.105}$$

Another way to look at the orthogonality of the associated Legendre functions is to rewrite Eq. (15.104) in terms of the associated Legendre polynomials $\mathcal{P}_l^m$. Invoking Eq. (15.74), Eq. (15.104) becomes

$$\int_{-1}^1 \mathcal{P}_p^m \mathcal{P}_q^m (1-x^2)^m dx = \frac{2}{2p+1}\frac{(p+m)!}{(p-m)!}\,\delta_{pq}, \tag{15.106}$$

showing that these **polynomials** are, for each m, orthogonal with the weight factor $(1-x^2)^m$. From that viewpoint, we can observe that each value of m corresponds to a set of polynomials that are orthogonal with a different weight. However, since our main interest is in the functions that are in general **not** polynomials but are solutions of the associated Legendre equation, it is usually more relevant to us to note that these functions, which include the factor $(1-x^2)^{m/2}$, are orthogonal **with unit weight**.

It is possible, but not particularly useful, to note that we can also have orthogonality of the P_l^m with respect to the upper index when the lower index is held constant:

$$\int_{-1}^1 P_l^m(x) P_l^n(x)(1-x^2)^{-1} dx = \frac{(l+m)!}{m(l-m)!}\,\delta_{mn}. \tag{15.107}$$

This equation is not very useful because in spherical polar coordinates the boundary condition on the azimuthal coordinate φ causes there already to be orthogonality with respect to m, and we are not usually concerned with orthogonality of the P_l^m with respect to m.

Example 15.4.2 CURRENT LOOP—MAGNETIC DIPOLE

An important problem in which we encounter associated Legendre functions is in the magnetic field of a circular current loop, a situation that may at first seem surprising since this problem has azimuthal symmetry.

Our starting point is the formula relating a current element $I ds$ to the vector potential $\mathbf{A}$ that it produces (this is discussed in the chapter on Green's functions, and also in texts such as Jackson's *Classical Electrodynamics*; see Additional Readings). This formula is

$$d\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{I d\mathbf{s}}{|\mathbf{r}-\mathbf{r_s}|}, \tag{15.108}$$

where $\mathbf{r}$ is the point at which $\mathbf{A}$ is to be evaluated and $\mathbf{r}_s$ is the position of element ds of the current loop. We place our current loop, of radius a, in the equatorial plane of a spherical polar coordinate system, as shown in Fig. 15.10. Our task is to determine $\mathbf{A}$ as a function of position, and therefrom to obtain the components of the magnetic induction field $\mathbf{B}$.

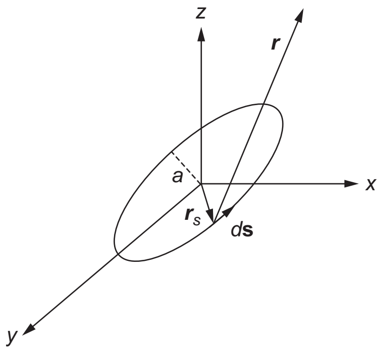

FIGURE 15.10 Circular current loop.

It is in principle possible to figure out the geometry and integrate Eq. (15.108) for the present problem, but a more practical approach will be to determine from general considerations the functional form of an expansion describing the solution, and then to determine the coefficients in the expansion by requiring correct results for points of high symmetry, where the calculation is not too difficult. This is an approach similar to that employed in Example 15.2.3, where we first identified the functional form of an expansion giving the potential generated by a circular ring of charge, after which we found the coefficients in the expansion from the easily computed potential on the axis of the ring.

From the form of Eq. (15.108) and the symmetry of the problem, we see immediately that for all **r**, **A** must lie in a plane of constant z, and in fact it must be in the $\hat{\mathbf{e}}_\varphi$ direction, with A_φ independent of φ, i.e.,

$$\mathbf{A} = A_\varphi(r,\theta)\,\hat{\mathbf{e}}_\varphi. \tag{15.109}$$

If **A** had a component other than A_φ, it would have a nonzero divergence, as then **A** would have a nonzero inward or outward flux, resulting in a singularity on the axis of the loop.

Since everywhere except on the current loop itself there is no current, Maxwell's equation for the curl of **B** reduces to

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = 0,$$

and, since **A** has only a φ component, it further reduces to

$$\nabla \times \left[\nabla \times A_\varphi(r,\theta)\,\hat{\mathbf{e}}_\varphi\right] = 0. \tag{15.110}$$

The left-hand side of Eq. (15.110) was the subject of Example 3.10.4, and its evaluation was presented as Eq. (3.165). Setting that result to zero gives the equation that must be satisfied by $A_\varphi(r,\theta)$:

$$\frac{\partial^2 A_\varphi}{\partial r^2} + \frac{2}{r}\frac{\partial A_\varphi}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial A_\varphi}{\partial\theta}\right) - \frac{1}{r^2\sin^2\theta}A_\varphi = 0. \tag{15.111}$$

Equation (15.111) may now be solved by the method of separation of variables; setting $A_\varphi(r,\theta) = R(r)\Theta(\theta)$, we have

$$r^2 \frac{d^2R}{dr^2} + 2\frac{dR}{dr} - l(l+1)R = 0, \tag{15.112}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + l(l+1)\Theta - \frac{\Theta}{\sin^2\theta} = 0. \tag{15.113}$$

Because the second of these equations can be recognized as the associated Legendre equation, in the form given as Eq. (15.2), we have set the separation constant to the value it must have, namely $l(l+1)$, with l integral. The first equation is also familiar, with solutions for a given l being r^l and r^{-l-1}. The second equation has solutions $P_l^1(\cos\theta)$, i.e., its specific form dictates that the associated Legendre functions which solve it must have upper index $m = 1$. Since our main interest is in the pattern of **B** at r values larger than a, the radius of the current loop, we retain only the radial solution r^{-l-1}, and write

$$A_\varphi(r,\theta) = \sum_{l=1}^{\infty} c_l \left(\frac{a}{r}\right)^{l+1} P_l^1(\cos\theta). \tag{15.114}$$

When we obtain a more detailed solution, we will find that it converges only for $r > a$, so Eq. (15.114) and the value of **B** derived therefrom will only be valid outside a sphere containing the current loop. If we were also interested in solving this problem for $r < a$, we would need to construct a series solution using only the powers r^l.

From Eq. (15.114) we can compute the components of **B**. Clearly, $B_\varphi = 0$. And, using Eq. (3.159), we have

$$B_r(r,\theta) = \nabla \times A_\varphi \hat{\mathbf{e}}_\varphi \Big|_r = \frac{\cot\theta}{r} A_\varphi + \frac{1}{r}\frac{\partial A_\varphi}{\partial\theta}, \tag{15.115}$$

$$B_\theta(r,\theta) = \nabla \times A_\varphi \hat{\mathbf{e}}_\varphi \Big|_\theta = -\frac{1}{r}\frac{\partial(rA_\varphi)}{\partial r}. \tag{15.116}$$

To evaluate the θ derivative in Eq. (15.115), we need

$$\frac{dP_l^1(\cos\theta)}{d\theta} = -\sin\theta \frac{dP_l^1(\cos\theta)}{d\cos\theta} = -l(l+1)P_l(\cos\theta) - \cot\theta\, P_l^1(\cos\theta), \tag{15.117}$$

a special case of Eq. (15.92) with $m = 1$ and $x = \cos\theta$. It is now straightforward to insert the expansion for A_φ into Eqs. (15.115) and (15.116); because of Eq. (15.117) the $\cot\theta$ term of Eq. (15.115) cancels, and we reach

$$B_r(r,\theta) = -\frac{1}{r}\sum_{l=1}^{\infty} l(l+1)\, c_l \left(\frac{a}{r}\right)^{l+1} P_l(\cos\theta), \tag{15.118}$$

$$B_\theta(r,\theta) = \frac{1}{r}\sum_{l=1}^{\infty} l\, c_l \left(\frac{a}{r}\right)^{l+1} P_l^1(\cos\theta). \tag{15.119}$$

To complete our analysis, we must determine the values of the c_l, which we do by using the Biot-Savart law to calculate B_r at points along the polar axis, where B_r is synonymous with B_z. Since $\theta = 0$ on the positive polar axis and $P_l(\cos\theta) = 1$, Eq. (15.118) reduces to

$$B_r(z,0) = -\frac{1}{z}\sum_{l=1}^{\infty} l(l+1)\,c_l\left(\frac{a}{z}\right)^{l+1} = -\frac{a^2}{z^3}\sum_{s=0}^{\infty}(s+1)(s+2)c_{s+1}\left(\frac{a}{z}\right)^{s}. \quad (15.120)$$

The symmetry of the problem permits one more simplification; the value of B_z must be the same at $-z$ as at z, from which we conclude that the coefficients $c_2, c_4, \ldots$ must all vanish, and we can rewrite Eq. (15.120) as

$$B_r(z,0) = -\frac{a^2}{z^3}\sum_{s=0}^{\infty} 2(s+1)(2s+1)c_{2s+1}\left(\frac{a}{z}\right)^{2s}. \quad (15.121)$$

The Biot-Savart law (in SI units) gives the contribution from the current element $I\,ds$ to $\mathbf{B}$ at a point whose displacement from the current element is $\mathbf{r_s}$ as

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\,I\,\frac{d\mathbf{s}\times\hat{\mathbf{r}}_s}{r_s^2}. \quad (15.122)$$

We now compute $\mathbf{B}$ by integration of ds around the current loop. The geometry is shown in Fig. 15.11. Note that dB_z, which will be the same for all current elements $I\,ds$, has the value

$$dB_z = \frac{\mu_0 I}{4\pi r_s^2}\sin\chi\,ds,$$

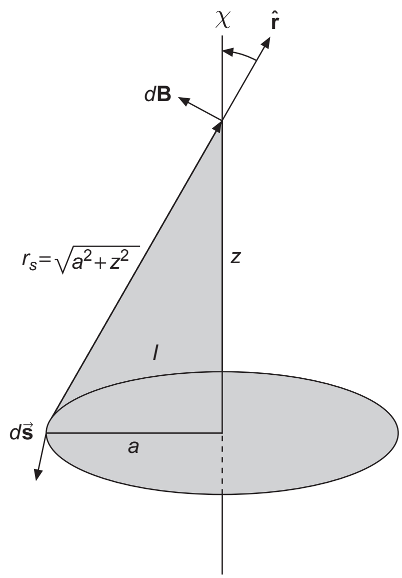

FIGURE 15.11 Biot-Savart law applied to a circular loop.

where χ is the labeled angle in Fig. 15.11 and r_s has the value indicated in the figure. The integration over s simply yields a factor $2\pi a$, and we see that $\sin\chi = a/(a^2+z^2)^{1/2}$, so

$$B_z = \frac{\mu_0 I a^2}{2}(a^2+z^2)^{-3/2} = \frac{\mu_0 I a^2}{2z^3}\left(1+\frac{a^2}{z^2}\right)^{-3/2}$$

$$= \frac{\mu_0 I a^2}{2z^3}\sum_{s=0}^{\infty}(-1)^s \frac{(2s+1)!!}{(2s)!!}\left(\frac{a}{z}\right)^{2s}. \tag{15.123}$$

The binomial expansion in the second line of Eq. (15.123) is convergent for $z > a$.

We are now ready to reconcile Eqs. (15.121) and (15.123), finding that

$$-2(s+1)(2s+1)c_{2s+1} = \frac{\mu_0 I}{2}(-1)^s \frac{(2s+1)!!}{(2s)!!},$$

which reduces to

$$c_{2s+1} = \frac{\mu_0 I}{2}(-1)^{s+1}\frac{(2s-1)!!}{(2s+2)!!}. \tag{15.124}$$

We write final formulas for **A** and **B** in a form that recognizes that $c_{2s} = 0$, applicable for $r > a$:

$$A_\varphi(r,\theta) = \frac{a^2}{r^2}\sum_{s=0}^{\infty} c_{2s+1}\left(\frac{a}{r}\right)^{2s} P_{2s+1}^1(\cos\theta), \tag{15.125}$$

$$B_r(r,\theta) = -\frac{a^2}{r^2}\sum_{s=0}^{\infty}(2s+1)(2s+2)c_{2s+1}\left(\frac{a}{r}\right)^{2s} P_{2s+1}(\cos\theta), \tag{15.126}$$

$$B_\theta(r,\theta) = \frac{a^2}{r^3}\sum_{s=0}^{\infty}(2s+1)c_{2s+1}\left(\frac{a}{r}\right)^{2s} P_{2s+1}^1(\cos\theta). \tag{15.127}$$

These formulas can also be written in terms of complete elliptic integrals. See Smythe (Additional Readings) and Section 18.8 of this book.

A comparison of magnetic current loop and finite electric dipole fields may be of interest. For the magnetic loop dipole, the preceding analysis gives

$$B_r(r,\theta) = \frac{\mu_0 I a^2}{2r^3}\left[P_1 - \frac{3}{2}\left(\frac{a}{r}\right)^2 P_3 + \cdots\right], \tag{15.128}$$

$$B_\theta(r,\theta) = \frac{\mu_0 I a^2}{4r^3}\left[-P_1^1 + \frac{3}{4}\left(\frac{a}{r}\right)^2 P_3^1 + \cdots\right]. \tag{15.129}$$

From the finite electric dipole potential, Eq. (15.65), one can find

$$E_r(r,\theta) = \frac{qa}{\pi\varepsilon_0 r^3}\left[P_1 + 2\left(\frac{a}{r}\right)^2 P_3 + \cdots\right], \tag{15.130}$$

$$E_\theta(r,\theta) = \frac{qa}{2\pi\varepsilon_0 r^3}\left[-P_1^1 - \left(\frac{a}{r}\right)^2 P_3^1 + \cdots\right]. \tag{15.131}$$

The leading terms of both fields agree, and this is the basis for identifying both as dipole fields.

As with electric multipoles, it is sometimes convenient to discuss **point** magnetic multipoles. A point dipole can be formed by taking the limit $a \to 0$, $I \to \infty$, with Ia^2 held constant. The **magnetic moment m** is taken to be $I\pi a^2\mathbf{n}$, where $\mathbf{n}$ is a unit vector perpendicular to the plane of the current loop and in the sense given by the right-hand rule. ■

Exercises

15.4.1 Apply the Frobenius method to Eq. (15.72) to obtain Eq. (15.73) and verify that the numerator of that equation becomes zero if $\lambda = l(l+1)$ and $j = l - m$.

15.4.2 Starting from the entries for P_2^2 and P_2^1 in Table 15.3, apply a recurrence formula to obtain P_2^0 (which is P_2), P_2^{-1}, and P_2^{-2}. Compare your results with the value of P_2 from Table 15.1 and with values of P_2^{-1} and P_2^{-2} obtained by applying Eq. (15.81) to entries from Table 15.3.

15.4.3 Prove that

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x),$$

where $P_l^m(x)$ is defined by

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l.$$

Hint. One approach is to apply Leibniz's formula to $(x+1)^l(x-1)^l$.

15.4.4 Show that

$$P_{2l}^1(0) = 0,$$

$$P_{2l+1}^1(0) = (-1)^{l+1} \frac{(2l+1)!!}{(2l)!!},$$

by each of these three methods:

(a) Use of recurrence relations,
(b) Expansion of the generating function,
(c) Rodrigues formula.

15.4.5 Evaluate $P_l^m(0)$ for $m > 0$.

$$ANS. \quad P_l^m(0) = \begin{cases} (-1)^{(l+m)/2} \dfrac{(l+m-1)!!}{(l-m)!!}, & l+m \text{ even}, \\ 0, & l+m \text{ odd}. \end{cases}$$

15.4.6 Starting from the potential of a finite dipole, Eq. (15.65), verify the formulas for the electric field components given as Eqs. (15.130) and (15.131).

15.4.7 Show that

$$P_l^l(\cos\theta) = (-1)^l (2l-1)!!\, \sin^l\theta, \quad l = 0, 1, 2, \ldots.$$

15.4.8 Derive the associated Legendre recurrence relation,

$$P_l^{m+1}(x) + \frac{2mx}{(1-x^2)^{1/2}} P_l^m(x) + \Big[l(l+1) - m(m-1) \Big] P_l^{m-1}(x) = 0.$$

15.4.9 Develop a recurrence relation that will yield $P_l^1(x)$ as

$$P_l^1(x) = f_1(x,l) P_l(x) + f_2(x,l) P_{l-1}(x).$$

Follow either of the procedures (a) or (b):

(a) Derive a recurrence relation of the preceding form. Give $f_1(x,l)$ and $f_2(x,l)$ explicitly.
(b) Find the appropriate recurrence relation in print.

(1) Give the source.
(2) Verify the recurrence relation.

ANS. (a) $P_l^1(x) = \dfrac{lx}{(1-x^2)^{1/2}} P_l - \dfrac{l}{(1-x^2)^{1/2}} P_{l-1}.$

15.4.10 Show that $\sin\theta \dfrac{d}{d\cos\theta} P_n(\cos\theta) = P_n^1(\cos\theta)$.

15.4.11 Show that

(a) $$\int_0^\pi \left(\frac{dP_l^m}{d\theta} \frac{dP_{l'}^m}{d\theta} + \frac{m^2 P_l^m P_{l'}^m}{\sin^2\theta} \right) \sin\theta \, d\theta = \frac{2l(l+1)}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{l\,l'},$$

(b) $$\int_0^\pi \left(\frac{P_l^1}{\sin\theta} \frac{dP_{l'}^1}{d\theta} + \frac{P_{l'}^1}{\sin\theta} \frac{dP_l^1}{d\theta} \right) \sin\theta \, d\theta = 0.$$

These integrals occur in the theory of scattering of electromagnetic waves by spheres.

15.4.12 As a repeat of Exercise 15.2.9, show, using associated Legendre functions, that

$$\int_{-1}^{1} x(1-x^2) P_n'(x) P_m'(x)\, dx = \frac{n+1}{2n+1} \frac{2}{2n-1} \frac{n!}{(n-2)!} \delta_{m,n-1}$$
$$+ \frac{n}{2n+1} \frac{2}{2n+3} \frac{(n+2)!}{n!} \delta_{m,n+1}.$$

15.4.13 Evaluate $\displaystyle\int_0^\pi \sin^2\theta \; P_n^1(\cos\theta)\, d\theta.$

15.4.14 The associated Legendre function $P_l^m(x)$ satisfies the self-adjoint ODE

$$(1-x^2)\frac{d^2P_l^m(x)}{dx^2} - 2x\frac{dP_l^m(x)}{dx} + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P_l^m(x) = 0.$$

From the differential equations for $P_l^m(x)$ and $P_l^k(x)$ show that for $k \neq m$,

$$\int_{-1}^{1} P_l^m(x)P_l^k(x)\frac{dx}{1-x^2} = 0.$$

15.4.15 Determine the vector potential and the magnetic induction field of a magnetic quadrupole by differentiating the magnetic dipole potential.

ANS. $$\mathbf{A}_{MQ} = -\frac{\mu_0}{2}(Ia^2)(dz)\frac{P_2^1(\cos\theta)}{r^3}\hat{\mathbf{e}}_\varphi + \text{higher-order terms},$$

$$\mathbf{B}_{MQ} = \mu_0(Ia^2)(dz)\left[\frac{3P_2(\cos\theta)}{r^4}\hat{\mathbf{e}}_r - \frac{P_2^1(\cos\theta)}{r^4}\hat{\mathbf{e}}_\theta\right] + \cdots.$$

This corresponds to placing a current loop of radius a at $z \to dz$ and an oppositely directed current loop at $z \to -dz$. The vector potential and magnetic induction field of a point dipole are given by the leading terms in these expansions if we take the limit $dz \to 0$, $a \to 0$, and $I \to \infty$ subject to $Ia^2\,dz =$ constant.

15.4.16 A single circular wire loop of radius a carries a constant current I.

(a) Find the magnetic induction $\mathbf{B}$ for $r < a$, $\theta = \pi/2$.
(b) Calculate the integral of the magnetic flux $(\mathbf{B}\cdot d\boldsymbol{\sigma})$ over the area of the current loop, that is,

$$\int_0^a r\,dr\int_0^{2\pi} d\varphi\, B_z\left(r, \theta = \frac{\pi}{2}\right).$$

ANS. ∞.

The Earth is within such a ring current, in which I approximates millions of amperes arising from the drift of charged particles in the Van Allen belt.

15.4.17 The vector potential $\mathbf{A}$ of a magnetic dipole, dipole moment $\mathbf{m}$, is given by $\mathbf{A}(\mathbf{r}) = (\mu_0/4\pi)(\mathbf{m}\times\mathbf{r}/r^3)$. Show by direct computation that the magnetic induction $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ is given by

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\mathbf{m}\right) - \mathbf{m}}{r^3}.$$

15.4.18 (a) Show that in the point dipole limit the magnetic induction field of the current loop becomes

$$B_r(r,\theta) = \frac{\mu_0}{2\pi}\frac{m}{r^3}P_1(\cos\theta),$$
$$B_\theta(r,\theta) = -\frac{\mu_0}{2\pi}\frac{m}{r^3}P_1^1(\cos\theta),$$

with $m = I\pi a^2$.

(b) Compare these results with the magnetic induction of the point magnetic dipole of Exercise 15.4.17. Take $\mathbf{m} = \hat{\mathbf{z}}m$.

15.4.19 A uniformly charged spherical shell is rotating with constant angular velocity.

(a) Calculate the magnetic induction $\mathbf{B}$ along the axis of rotation outside the sphere.
(b) Using the vector potential series of Example 15.4.2, find $\mathbf{A}$ and then $\mathbf{B}$ for all points outside the sphere.

15.4.20 In the liquid-drop model of the nucleus, a spherical nucleus is subjected to small deformations. Consider a sphere of radius r_0 that is deformed so that its new surface is given by

$$r = r_0\Big[1 + \alpha_2 P_2(\cos\theta)\Big].$$

Find the area of the deformed sphere through terms of order α_2^2.

Hint.

$$dA = \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{1/2} r\sin\theta d\theta d\varphi.$$

ANS. $A = 4\pi r_0^2\left[1 + \frac{4}{5}\alpha_2^2 + \mathcal{O}\left(\alpha_2^3\right)\right]$.

Note. The area element dA follows from noting that the line element ds for fixed φ is given by

$$ds = (r^2\,d\theta^2 + \,dr^2)^{1/2} = \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{1/2} d\theta.$$

15.5 Spherical Harmonics

Our earlier discussion of separated-variable methods for solving the Laplace, Helmholtz, or Schrödinger equations in spherical polar coordinates showed that the possible angular solutions $\Theta(\theta)\Phi(\varphi)$ are always the same in spherically symmetric problems; in particular we found that the solutions for Φ depended on the single integer index m, and can be written in the form

$$\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}}\,e^{im\varphi}, \quad m = \ldots, -2, -1, 0, 1, 2, \ldots, \tag{15.132}$$

or, equivalently,

$$\Phi_m(\varphi) = \begin{cases} \dfrac{1}{\sqrt{2\pi}}, & m = 0, \\ \dfrac{1}{\sqrt{\pi}} \cos m\varphi, & m > 0, \\ \dfrac{1}{\sqrt{\pi}} \sin |m|\varphi, & m < 0. \end{cases} \tag{15.133}$$

The above equations contain the constant factors needed to make Φ_m normalized, and those of different m^2 are automatically orthogonal because they are eigenfunctions of a Sturm-Liouville problem. It is straightforward to verify that in either Eq. (15.132) or Eq. (15.133) our choices of the functions for $+m$ and $-m$ make Φ_m and Φ_{-m} orthogonal. Formally, our definitions are such that

$$\int_0^{2\pi} \Big[\Phi_m(\varphi)\Big]^* \Phi_{m'}(\varphi)d\varphi = \delta_{mm'}. \tag{15.134}$$

In Section 15.4 we found that the solutions $\Theta(\theta)$ could be identified as associated Legendre functions that can be labeled by the two integer indices l and m, with $-l \le m \le l$. From the orthonormality integral for these functions, Eq. (15.105), we can define the normalized solutions

$$\Theta_{lm}(\cos\theta) = \sqrt{\frac{2l+1}{2}\,\frac{(l-m)!}{(l+m)!}}\; P_l^m(\cos\theta), \tag{15.135}$$

satisfying the relation

$$\int_0^{\pi} \Big[\Theta_{lm}(\cos\theta)\Big]^* \Theta_{l'm}(\cos\theta)\sin\theta\, d\theta = \delta_{ll'}. \tag{15.136}$$

We have previously noted that an orthonormality condition of this type only applies if both functions Θ have the same value of the index m. The complex conjugate is not really necessary in Eq. (15.136) because the Θ are real, but we write it anyway to maintain consistent notation. Note also that when the argument of P_l^m is $x = \cos\theta$, then $(1-x^2)^{1/2} = \sin\theta$, so the P_l^m are polynomials of overall degree l in $\cos\theta$ and $\sin\theta$.

The product $\Theta_{lm}\Phi_m$ is called a **spherical harmonic**, with that name usually implying that Φ_m is taken with the definition as a complex exponential; see Eq. (15.132). Therefore we define

$$Y_l^m(\theta,\varphi) \equiv \sqrt{\frac{2l+1}{4\pi}\,\frac{(l-m)!}{(l+m)!}}\; P_l^m(\cos\theta)e^{im\varphi}. \tag{15.137}$$

These functions, being normalized solutions of a Sturm-Liouville problem, are orthonormal over the spherical surface, with

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta \Big[Y_{l_1}^{m_1}(\theta,\varphi)\Big]^* Y_{l_2}^{m_2}(\theta,\varphi) = \delta_{l_1 l_2}\delta_{m_1 m_2}. \tag{15.138}$$

The definition we introduced for the associated Legendre functions leads to specific signs for the Y_l^m that are sometimes identified as the Condon-Shortley phase, after the authors of a classic text on atomic spectroscopy. This sign convention has been found to simplify various calculations, particularly in the quantum theory of angular momentum. One of the effects of this phase factor is to introduce an alternation of sign with m among the positive-m spherical harmonics. The word "harmonic" enters the name of Y_l^m because solutions of Laplace's equation are sometimes called harmonic functions.

The squares of the real parts of the first few spherical harmonics are sketched in Figure 15.12; their functional forms are given in Table 15.4.

Cartesian Representations

For some purposes it is useful to express the spherical harmonics using Cartesian coordinates, which can be done by writing $\exp(\pm i\varphi)$ as $\cos\varphi \pm i\sin\varphi$ and using the formulas for x, y, z in spherical polar coordinates (retaining, however, an overall dependence on r, necessary because the angular quantities must be independent of scale). For example,

$$\cos\theta = z/r, \quad \sin\theta\exp(\pm i\varphi) = \sin\theta\cos\varphi \pm i\sin\theta\sin\varphi = \frac{x}{r} \pm i\,\frac{y}{r}; \tag{15.139}$$

these quantities are all homogeneous (of degree zero) in the coordinates.

Continuing to higher values of l, we obtain fractions in which the numerators are homogeneous products of x, y, z of overall degree l, divided by a common factor r^l. Table 15.4 includes the Cartesian expression for each of its entries.

Overall Solutions

As we have already seen in Section 9.4, the separation of a Laplace, Helmholtz, or even a Schrödinger equation in spherical polar coordinates can be written in terms of equations of the generic form

$$R'' + \frac{2}{r}R' + \Big[f(r) - l(l+1)\Big]R = 0, \tag{15.140}$$

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2}{d\varphi^2} + l(l+1)\right]Y_l^m(\theta,\varphi) = 0. \tag{15.141}$$

The function $f(r)$ in Eq. (15.140) is zero for the Laplace equation, k^2 for the Helmholtz equation, and $E - V(r)$ (V = potential energy, E = total energy, an eigenvalue) for the Schrödinger equation. We have combined the θ and φ equations into Eq. (15.141) and identified one of its solutions as Y_l^m. What is important to note right now is that the combined angular equation (and its boundary conditions and therefore its solutions) will be the same for all spherically symmetric problems, and that the angular solution affects the radial equation only through the separation constant $l(l+1)$. Thus, the radial equation will have solutions that depend on l but are independent of the index m.

In Section 9.4 we solved the radial equation for the Laplace and Helmholtz equations, with the results given in Table 9.2. For the Laplace equation $\nabla^2\psi = 0$, the general solution

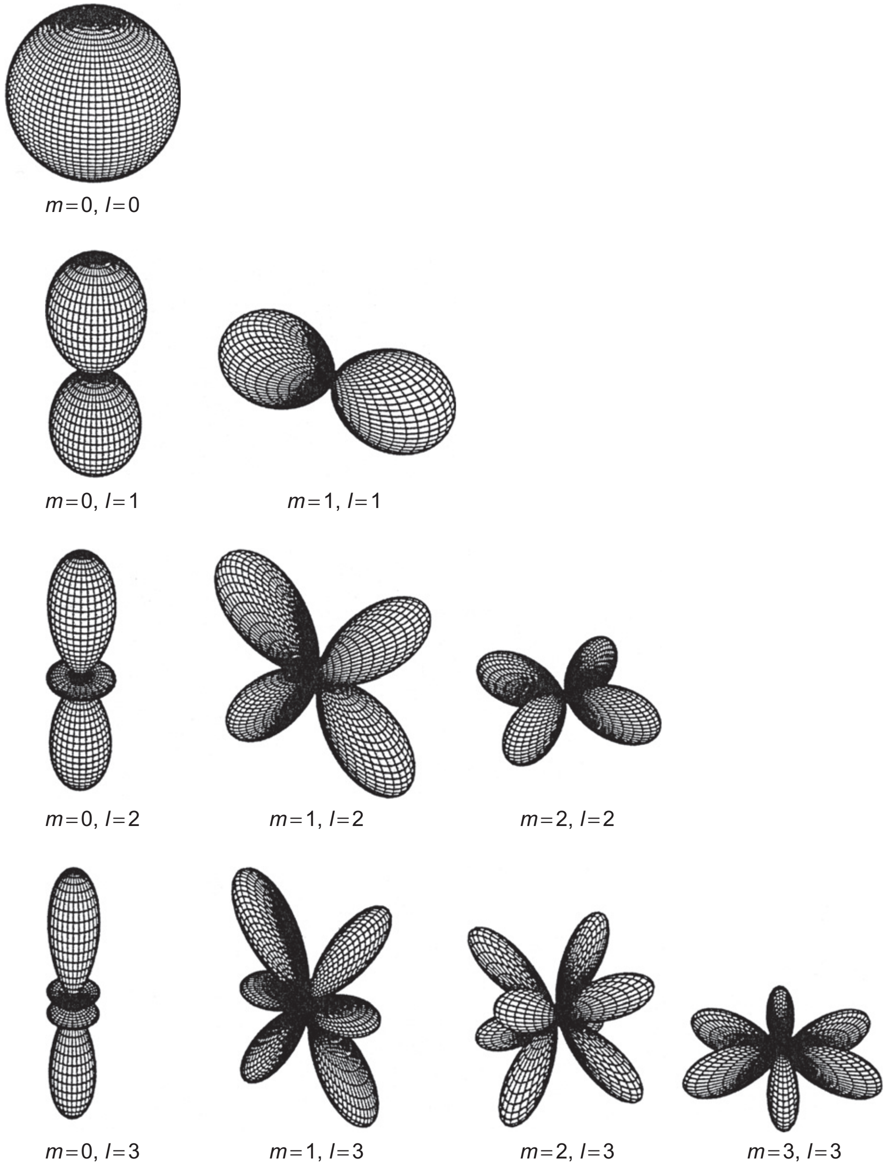

FIGURE 15.12 Shapes of $|\Re e Y_l^m(\theta, \varphi)|^2$ for $0 \le l \le 3$, $m = 0 \ldots l$.

in spherical polar coordinates is a sum, with arbitrary coefficients, of the solutions for the various possible values of l and m:

$$\psi(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(a_{lm} r^l + b_{lm} r^{-l-1} \right) Y_l^m(\theta, \varphi); \tag{15.142}$$

Table 15.4 Spherical Harmonics (Condon-Shortley Phase)

$$Y_0^0(\theta,\varphi)=\frac{1}{\sqrt{4\pi}}$$

$$Y_1^1(\theta,\varphi)=-\sqrt{\frac{3}{8\pi}}\sin\theta\,e^{i\varphi}=-\sqrt{\frac{3}{8\pi}}\,(x+iy)/r$$

$$Y_1^0(\theta,\varphi)=\sqrt{\frac{3}{4\pi}}\cos\theta=\sqrt{\frac{3}{4\pi}}\,z/r$$

$$Y_1^{-1}(\theta,\varphi)=+\sqrt{\frac{3}{8\pi}}\sin\theta\,e^{-i\varphi}=\sqrt{\frac{3}{8\pi}}\,(x-iy)/r$$

$$Y_2^2(\theta,\varphi)=\sqrt{\frac{5}{96\pi}}\,3\sin^2\theta\,e^{2i\varphi}=3\sqrt{\frac{5}{96\pi}}\,(x^2-y^2+2ixy)/r^2$$

$$Y_2^1(\theta,\varphi)=-\sqrt{\frac{5}{24\pi}}\,3\sin\theta\cos\theta\,e^{i\varphi}=-\sqrt{\frac{5}{24\pi}}\,3z(x+iy)/r^2$$

$$Y_2^0(\theta,\varphi)=\sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)=\sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}z^2-\frac{1}{2}r^2\right)/r^2$$

$$Y_2^{-1}(\theta,\varphi)=\sqrt{\frac{5}{24\pi}}\,3\sin\theta\cos\theta\,e^{-i\varphi}=+\sqrt{\frac{5}{24\pi}}\,3z(x-iy)/r^2$$

$$Y_2^{-2}(\theta,\varphi)=\sqrt{\frac{5}{96\pi}}\,3\sin^2\theta\,e^{-2i\varphi}=3\sqrt{\frac{5}{96\pi}}\,(x^2-y^2-2ixy)/r^2$$

$$Y_3^3(\theta,\varphi)=-\sqrt{\frac{7}{2880\pi}}\,15\sin^3\theta\,e^{3i\varphi}=-\sqrt{\frac{7}{2880\pi}}\,15[x^3-3xy^2+i(3x^2y-y^3)]/r^3$$

$$Y_3^2(\theta,\varphi)=\sqrt{\frac{7}{480\pi}}\,15\cos\theta\sin^2\theta\,e^{2i\varphi}=\sqrt{\frac{7}{480\pi}}\,15z(x^2-y^2+2ixy)/r^3$$

$$Y_3^1(\theta,\varphi)=-\sqrt{\frac{7}{48\pi}}\left(\frac{15}{2}cos^2\theta-\frac{3}{2}\right)\sin\theta\,e^{i\varphi}=-\sqrt{\frac{7}{48\pi}}\left(\frac{15}{2}z^2-\frac{3}{2}r^2\right)(x+iy)/r^3$$

$$Y_3^0(\theta,\varphi)=\sqrt{\frac{7}{4\pi}}\left(\frac{5}{2}\cos^3\theta-\frac{3}{2}\cos\theta\right)=\sqrt{\frac{7}{4\pi}}\,z\left(\frac{5}{2}z^2-\frac{3}{2}r^2\right)/r^3$$

$$Y_3^{-1}(\theta,\varphi)=+\sqrt{\frac{7}{48\pi}}\left(\frac{15}{2}cos^2\theta-\frac{3}{2}\right)\sin\theta\,e^{-i\varphi}=\sqrt{\frac{7}{48\pi}}\left(\frac{15}{2}z^2-\frac{3}{2}r^2\right)(x-iy)/r^3$$

$$Y_3^{-2}(\theta,\varphi)=\sqrt{\frac{7}{480\pi}}\,15\cos\theta\sin^2\theta\,e^{-2i\varphi}=\sqrt{\frac{7}{480\pi}}\,15z(x^2-y^2-2ixy)/r^3$$

$$Y_3^{-3}(\theta,\varphi)=+\sqrt{\frac{7}{2880\pi}}\,15\sin^3\theta\,e^{-3i\varphi}=\sqrt{\frac{7}{2880\pi}}\,15[x^3-3xy^2-i(3x^2y-y^3)]/r^3$$

for the Helmholtz equation $(\nabla^2+k^2)\psi=0$, the radial equation has the form given in Eq. (14.148), so the general solution assumes the form

$$\psi(r,\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\Big(a_{lm}\,j_l(kr)+b_{lm}\,y_l(kr)\Big)\,Y_l^m(\theta,\varphi). \tag{15.143}$$

Laplace Expansion

Part of the importance of spherical harmonics lies in the completeness property, a consequence of the Sturm-Liouville form of Laplace's equation. Here this property means that any function $f(\theta,\varphi)$ (with sufficient continuity properties) evaluated over the surface of a sphere can be expanded in a uniformly convergent double series of spherical harmonics.[4]

[4]For a proof of this fundamental theorem, see E. W. Hobson (Additional Readings), chapter VII.

This expansion, known as a **Laplace series**, takes the form

$$f(\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}c_{lm}Y_l^m(\theta,\varphi), \tag{15.144}$$

with

$$c_{lm}=\left\langle Y_l^m\middle| f(\theta,\varphi)\right\rangle=\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta\,d\theta\,Y_l^m(\theta,\varphi)^*f(\theta,\varphi). \tag{15.145}$$

A frequent use of the Laplace expansion is in specializing the general solution of the Laplace equation to satisfy boundary conditions on a spherical surface. This situation is illustrated in the following example.

Example 15.5.1 Spherical Harmonic Expansion

Consider the problem of determining the electrostatic potential within a charge-free spherical region of radius r_0, with the potential on the spherical bounding surface specified as an arbitrary function $V(r_0,\theta,\varphi)$ of the angular coordinates θ and φ. The potential $V(r,\theta,\varphi)$ is the solution of the Laplace equation satisfying the boundary condition at $r=r_0$ and regular for all $r\le r_0$. This means it must be of the form of Eq. (15.142), with the coefficients b_{lm} set to zero to ensure a solution that is nonsingular at $r=0$.

We proceed by obtaining the spherical harmonic expansion of $V(r_0,\theta,\varphi)$, namely Eq. (15.144), with coefficients

$$c_{lm}=\left\langle Y_l^m(\theta,\varphi)\middle|V(r_0,\theta,\varphi)\right\rangle.$$

Then, comparing Eq. (15.142), evaluated for $r=r_0$,

$$V(r_0,\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}a_{lm}\,r_0^l\,Y_l^m(\theta,\varphi),$$

with the expression from Eq. (15.144),

$$V(r_0,\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}c_{lm}Y_l^m(\theta,\varphi),$$

we see that $a_{lm}=c_{lm}/r_0^l$, so

$$V(r,\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}c_{lm}\left(\frac{r}{r_0}\right)^l Y_l^m(\theta,\varphi).$$

■

Example 15.5.2 LAPLACE SERIES—GRAVITY FIELDS

This example illustrates the notion that sometimes it is appropriate to replace the spherical harmonics by their real counterparts (in terms of sine and cosine functions). The gravity fields of the Earth, the Moon, and Mars have been described by a Laplace series of the form

$$U(r,\theta,\varphi)=\frac{GM}{R}\left[\frac{R}{r}-\sum_{l=2}^{\infty}\sum_{m=0}^{l}\left(\frac{R}{r}\right)^{l+1}\left[C_{lm}Y_{ml}^{e}(\theta,\varphi)+S_{lm}Y_{ml}^{o}(\theta,\varphi)\right]\right]. \tag{15.146}$$

Here M is the mass of the body, R is its equatorial radius, and G is the gravitational constant. The real functions Y_{ml}^{e} and Y_{ml}^{o} are defined by Morse and Feshbach (see Additional Readings) as the unnormalized forms

$$Y_{ml}^{e}(\theta,\varphi)=P_l^m(\cos\theta)\cos m\varphi, \quad Y_{ml}^{o}(\theta,\varphi)=P_l^m(\cos\theta)\sin m\varphi.$$

Note that Morse and Feshbach place the m index before l. The normalization integrals for Y^e and Y^o are the topic of Exercise 15.5.6.

Satellite measurements have led to the numerical values for C_{20}, C_{22}, and S_{22} shown in Table 15.5.

Table 15.5 Gravity Field Coefficients, Eq. (15.145).

Coefficient[a]	Earth	Moon	Mars
C_{20}	1.083×10^{-3}	$(0.200\pm0.002)\times10^{-3}$	$(1.96\pm0.01)\times10^{-3}$
C_{22}	0.16×10^{-5}	$(2.4\pm0.5)\times10^{-5}$	$(-5\pm1)\times10^{-5}$
S_{22}	-0.09×10^{-5}	$(0.5\pm0.6)\times10^{-5}$	$(3\pm1)\times10^{-5}$

[a] C_{20} represents an equatorial bulge, whereas C_{22} and S_{22} represent an azimuthal dependence of the gravitational field.

■

Symmetry of Solutions

The angular solutions of given l but different m are closely related in that they lead to the same solution for the radial equation. Except when $l=0$, the individual solutions Y_l^m are not spherically symmetric, and we must recognize that a spherically symmetric problem can have solutions with less than the full spherical symmetry. A classical example of this phenomenon is provided by the Earth-Sun system, which has a spherically symmetric gravitational potential. However, the actual orbit of the Earth is planar. This apparent dilemma is resolved by noting that a solution exists for any orientation of the Earth's orbital plane; that actually occurring was determined by "initial conditions."

Returning now to the Laplace equation, we see that a radial solution for given l, i.e., r^l or r^{-l-1}, is associated with $2l+1$ different angular solutions Y_l^m $(-l\le m\le l)$, no one of which (for $l\neq0$) has spherical symmetry. The most general solution for this l must be a linear combination of these $2l+1$ mutually orthogonal functions. Put another way, the solution space of the angular solution of the Laplace equation for given l is a Hilbert space containing the $2l+1$ members $Y_l^{-l}(\theta,\varphi),\ldots,Y_l^l(\theta,\varphi)$. Now, if we write the Laplace equation in a coordinate system (θ',φ') oriented differently than the original coordinates, we must still have the same angular solution set, meaning that $Y_l^m(\theta',\varphi')$ must be a linear combination of the original Y_l^m. Thus, we may write

$$Y_l^m(\theta',\varphi') = \sum_{m'=-l}^{l} D^l_{m'm} Y_l^{m'}(\theta,\varphi), \tag{15.147}$$

where the coefficients D depend on the coordinate rotation involved. Note that a coordinate rotation cannot change the r dependence of our solution to the Laplace equation, so Eq. (15.147) does not need to include a sum over all values of l. As a specific example, we see (Fig. 15.12) that for $l=1$ we have three solutions that appear similar, but with different orientations. Alternatively, from Table 15.4 we see that the angular solutions Y_1^m have forms proportional to z/r, $(x+iy)/r$, and $(x-iy)/r$, meaning that they can be combined to form arbitrary combinations of x/r, y/r, and z/r. Since a rotation of the coordinate axes converts x, y, and z into linear combinations of each other, we can understand why the set of three functions Y_1^m $(m=0,1,-1)$ is closed under coordinate rotations.

For $l=2$, there are five possible m values, so the angular functions of this l value form a closed space containing five independent members. A fuller discussion of these spaces spanned by angular functions is part of what will be considered in Chapter 16.

Applying the preceding analysis to solutions of the Schrödinger equation, the eigenvalues of which are determined by solving its radial ODE for various values of the separation constant $l(l+1)$, we see that all solutions for the same l but different m will have the same eigenvalues E and radial functions, but will differ in the orientation of their angular parts. States of the same energy are called **degenerate**, and the independence of E with respect to m will cause a $(2l+1)$-fold degeneracy of the eigenstates of given l.

Example 15.5.3 SOLUTIONS FOR $l=1$ AT ARBITRARY ORIENTATION

Let's do this problem in Cartesian coordinates. The angular solution Y_1^0 to Laplace's equation is shown in Table 15.4 to be proportional to z/r, which for our present purposes we write $(\mathbf{r}\cdot\hat{\mathbf{e}}_z)/r$, where $\hat{\mathbf{e}}_z$ is a unit vector in the z direction. We seek a similar solution, with $\hat{\mathbf{e}}_z$ replaced by an arbitrary unit vector $\hat{\mathbf{e}}_u = \cos\alpha\,\hat{\mathbf{e}}_x + \cos\beta\,\hat{\mathbf{e}}_y + \cos\gamma\,\hat{\mathbf{e}}_z$, where $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the direction cosines of $\hat{\mathbf{e}}_u$. We get immediately

$$\frac{(\mathbf{r}\cdot\hat{\mathbf{e}}_u)}{r} = \frac{x}{r}\cos\alpha + \frac{y}{r}\cos\beta + \frac{z}{r}\cos\gamma.$$

Consulting the Cartesian-coordinate expressions for the spherical harmonics in Table 15.4, we see that the above expression can be written

$$\frac{(\mathbf{r}\cdot\hat{\mathbf{u}})}{r} = \sqrt{\frac{8\pi}{3}}\left(\frac{Y_1^{-1}-Y_1^1}{2}\right)\cos\alpha + \sqrt{\frac{8\pi}{3}}\left(\frac{-Y_1^{-1}-Y_1^1}{2i}\right)\cos\beta + \sqrt{\frac{4\pi}{3}}Y_1^0\cos\gamma.$$

This shows that all three Y_1^m are needed to reproduce Y_1^0 at an arbitrary orientation. Similar manipulations can be carried out for other l and m values. ■

Further Properties

The main properties of the spherical harmonics follow directly from those of the functions Θ_{lm} and Φ_m. We summarize briefly:

Special values. At $\theta = 0$, the polar direction in the spherical coordinates, the value of φ becomes immaterial, and all Y_l^m that have φ dependence must vanish. Using also the fact that $P_l(1) = 1$, we find in general

$$Y_l^m(0, \varphi) = \sqrt{\frac{2l+1}{4\pi}}\,\delta_{m0}. \tag{15.148}$$

A similar argument for $\theta = \pi$ leads to

$$Y_l^m(\pi, \varphi) = (-1)^l\sqrt{\frac{2l+1}{4\pi}}\,\delta_{m0}. \tag{15.149}$$

Recurrence formulas. Using the recurrence formulas developed for the associated Legendre functions, we get for the spherical harmonics with arguments (θ, φ):

$$\cos\theta\, Y_l^m = \left[\frac{(l-m+1)(l+m+1)}{(2l+1)(2l+3)}\right]^{1/2} Y_{l+1}^m + \left[\frac{(l-m)(l+m)}{(2l-1)(2l+1)}\right]^{1/2} Y_{l-1}^m, \tag{15.150}$$

$$e^{\pm i\varphi}\sin\theta\, Y_l^m = \mp\left[\frac{(l\pm m+1)(l\pm m+2)}{(2l+1)(2l+3)}\right]^{1/2} Y_{l+1}^{m\pm 1} \pm \left[\frac{(l\mp m)(l\mp m-1)}{(2l-1)(2l+1)}\right]^{1/2} Y_{l-1}^{m\pm 1}. \tag{15.151}$$

Some integrals. These recurrence relations permit the ready evaluation of some integrals of practical importance. Our starting point is the orthonormalization condition, Eq. (15.138). For example, the matrix elements describing the dominant (electric dipole) mode of interaction of an electromagnetic field with a charged system in a spherical harmonic state are proportional to

$$\int \left[Y_{l'}^{m'}\right]^* \cos\theta\, Y_l^m\, d\Omega.$$

Using Eq. (15.150) and invoking the orthonormality of the Y_l^m, we find

$$\int \left[Y_{l'}^{m'}\right]^* \cos\theta\, Y_l^m\, d\Omega = \left[\frac{(l-m+1)(l+m+1)}{(2l+1)(2l+3)}\right]^{1/2} \delta_{m'm}\,\delta_{l',l+1} + \left[\frac{(l-m)(l+m)}{(2l-1)(2l+1)}\right]^{1/2} \delta_{m'm}\,\delta_{l',l-1}. \tag{15.152}$$

Equation (15.152) provides a basis for the well-known selection rule for dipole radiation.

Additional formulas involving products of three spherical harmonics and the detailed behavior of these quantities under coordinate rotations are more appropriately discussed in connection with a study of angular momentum and are therefore deferred to Chapter 16.

Exercises

15.5.1 Show that the parity of $Y_l^m(\theta, \varphi)$ is $(-1)^l$. Note the disappearance of any m dependence.

Hint. For the parity operation in spherical polar coordinates, see Exercise 3.10.25.

15.5.2 Prove that $Y_l^m(0, \varphi) = \left(\frac{2l+1}{4\pi}\right)^{1/2} \delta_{m0}$.

15.5.3 In the theory of Coulomb excitation of nuclei we encounter $Y_l^m(\pi/2, 0)$. Show that

$$Y_l^m\left(\frac{\pi}{2}, 0\right) = \left(\frac{2l+1}{4\pi}\right)^{1/2} \frac{[(l-m)!\,(l+m)!]^{1/2}}{(l-m)!!\,(l+m)!!}(-1)^{(l-m)/2}, \quad l+m \text{ even},$$
$$= 0, \quad l+m \text{ odd}.$$

15.5.4 The orthogonal azimuthal functions yield a useful representation of the Dirac delta function. Show that

$$\delta(\varphi_1 - \varphi_2) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\varphi_1 - \varphi_2)}.$$

Note. This formula assumes that φ_1 and φ_2 are restricted to $0 \le \varphi < 2\pi$. Without this restriction there will be additional delta-function contributions at intervals of 2π in $\varphi_1 - \varphi_2$.

15.5.5 Derive the spherical harmonic closure relation

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \left[Y_l^m(\theta_1, \varphi_1)\right]^* Y_l^m(\theta_2, \varphi_2) = \frac{1}{\sin\theta_1} \delta(\theta_1 - \theta_2)\,\delta(\varphi_1 - \varphi_2)$$
$$= \delta(\cos\theta_1 - \cos\theta_2)\,\delta(\varphi_1 - \varphi_2).$$

15.5.6 In some circumstances it is desirable to replace the imaginary exponential of our spherical harmonic by sine or cosine. Morse and Feshbach (see Additional Readings) define

$$Y_{ml}^e = P_l^m(\cos\theta)\cos m\varphi, \quad m \ge 0,$$
$$Y_{ml}^o = P_l^m(\cos\theta)\sin m\varphi, \quad m > 0,$$

and their normalization integrals are

$$\int_0^{2\pi}\int_0^{\pi} [Y_{mn}^{e \text{ or } o}(\theta, \varphi)]^2 \sin\theta\, d\theta\, d\varphi = \frac{4\pi}{2(2n+1)} \frac{(n+m)!}{(n-m)!}, \; n = 1, 2, \ldots$$
$$= 4\pi, \quad n = 0.$$

These spherical harmonics are often named according to the patterns of their positive and negative regions on the surface of a sphere: zonal harmonics for $m = 0$, sectoral harmonics for $m = n$, and tesseral harmonics for $0 < m < n$. For Y^e_{mn}, $n = 4$, $m = 0, 2, 4$, indicate on a diagram of a hemisphere (one diagram for each spherical harmonic) the regions in which the spherical harmonic is positive.

15.5.7 A function $f(r,\theta,\varphi)$ may be expressed as a Laplace series

$$f(r,\theta,\varphi) = \sum_{l,m} a_{lm} r^l Y_l^m(\theta,\varphi).$$

Letting $\langle\cdots\rangle_{\text{sphere}}$ denote the average over a sphere centered on the origin, show that

$$\Big\langle f(r,\theta,\varphi)\Big\rangle_{\text{sphere}} = f(0,0,0).$$

15.6 Legendre Functions of the Second Kind

The Legendre equation, a linear second-order ODE, has two independent solutions. Writing this equation in the form

$$y'' - \frac{2x}{1-x^2}\,y' - \frac{l(l+1)}{1-x^2}\,y = 0, \tag{15.153}$$

and restricting consideration to integer $l \ge 0$, our objective is to find a second solution that is linearly independent from the Legendre polynomials $P_l(x)$. Using the procedure of Section 7.6, and denoting the second solution $Q_l(x)$, we have

$$\begin{aligned} Q_l(x) &= P_l(x)\int^x \frac{\exp\left[\displaystyle\int^x 2x/(1-x^2)\,dx\right]}{[P_l(x)]^2}\,dx \\ &= P_l(x)\int^x \frac{dx}{(1-x^2)[P_l(x)]^2}\,dx. \end{aligned} \tag{15.154}$$

Since any linear combination of P_l and the right-hand side of Eq. (15.154) is equally valid as a second solution of the Legendre ODE, we note that Eq. (15.154) defines both the scale and the specific functional form of Q_l.

Using Eq. (15.154), we can obtain explicit formulas for the Q_l. We find (remembering that $P_0 = 1$ and expanding the denominator in partial fractions):

$$Q_0(x) = \int^x \frac{1}{1-x^2}\,dx = \frac{1}{2}\int\left[\frac{1}{1+x} + \frac{1}{1-x}\right]dx = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right). \tag{15.155}$$

Continuing to Q_1, the partial fraction expansion is a bit more involved, but leads to a simple result. Noting that $P_1(x) = x$, we have

$$Q_1(z) = x \int^{x} \frac{dx}{(1-x^2)x^2}\,dx = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1. \tag{15.156}$$

With significantly more work, we can obtain Q_2:

$$Q_2(x) = \frac{1}{2}P_2(x)\ln\left(\frac{1+x}{1-x}\right) - \frac{3x}{2}. \tag{15.157}$$

This process can in principle be repeated for larger l, but it is easier and more instructive to verify that the forms of Q_0, Q_1, and Q_2 are consistent with the Legendre-function recurrence relations,[5] and then to obtain Q_l of larger l by recurrence. The recurrence formulas, originally written for P_l in Eq. (15.18), are

$$(l+1)Q_{l+1}(x) - (2l+1)xQ_l(x) + lQ_{l-1}(x) = 0, \tag{15.158}$$

$$(2l+1)Q_l(x) = Q'_{l+1}(x) - Q'_{n-1}(x). \tag{15.159}$$

Verification that Q_0, Q_1, and Q_2 satisfy these recurrence formulas is straightforward and is left as a exercise. Extension to higher l leads to the formula

$$Q_l(x) = \frac{1}{2}P_l(x)\ln\left(\frac{1+x}{1-x}\right) - \frac{2l-1}{1\cdot l}P_{l-1}(x) - \frac{2l-5}{3(l-1)}P_{l-3}(x) - \cdots. \tag{15.160}$$

Many applications using the functions $Q_l(x)$ involve values of x outside the range $-1 < x < 1$. If Eq. (15.160) is extended, say, beyond $+1$, then $1-x$ will become negative and make a contribution $\pm i\pi$ to the logarithm, thereby making a contribution $\pm i\pi P_l$ to Q_l. Our solution will still remain a solution if this contribution is removed, and it is therefore convenient to define the second solution for x outside the range $(-1,+1)$ with

$$\ln\left(\frac{1+x}{1-x}\right) \quad \text{replaced by} \quad \ln\left(\frac{x+1}{x-1}\right).$$

From a complex-variable perspective, the logarithmic term in the solutions Q_l is related to the singularity in the ODE at $z = \pm 1$, reflecting the fact that to make the solutions single-valued it will be necessary to make a branch cut, traditionally taken on the real axis from -1 to $+1$. Then the Q_l with the $(1+x)/(1-x)$ logarithm are recovered on $-1 < x < 1$ if we average the results from the $(z+1)/(z-1)$ form on the two sides of the branch cut.

The behavior of the Q_l is illustrated by plots for $x < 1$ in Fig. 15.13 and for $x > 1$ in Fig. 15.14. Note that there is no singularity at $x = 0$ but all the Q_l exhibit a logarithmic singularity at $x = 1$.

[5] In Section 15.1 we showed that any set of functions that satisfies the recurrence relations reproduced here also satisfies the Legendre ODE.

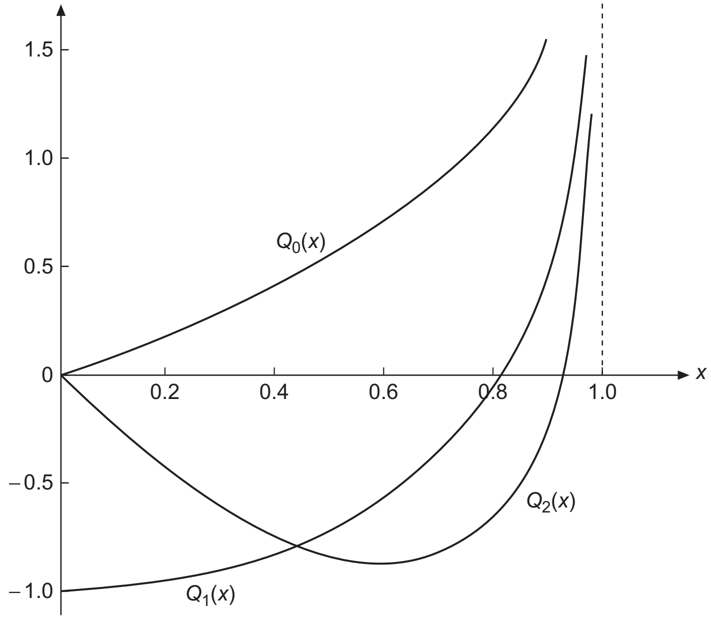

FIGURE 15.13 Legendre functions $Q_l(x)$, $0 \leq x < 1$.

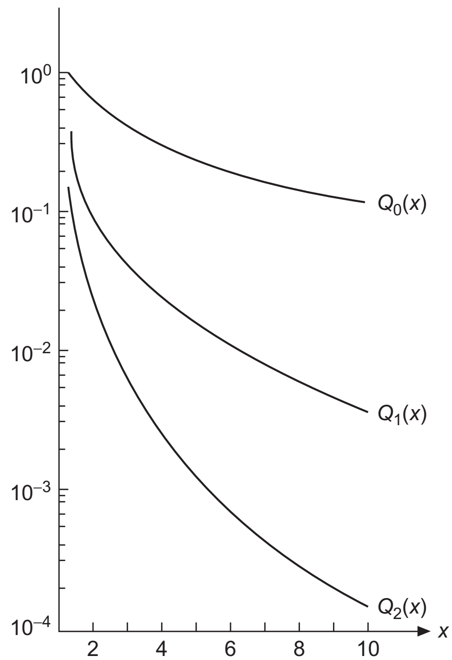

FIGURE 15.14 Legendre functions $Q_l(x)$, $x > 1$.

Properties

1. An examination of the formulas for $Q_l(x)$ reveals that if l is even, then $Q_l(x)$ is an odd function of x, while $Q_l(x)$ of odd l are even functions of x. More succinctly, $Q_l(-x) = (-1)^{l+1} Q_l(x)$.
2. The presence of the logarithmic term causes $Q_l(1) = \infty$ for all l.
3. Because $x = 0$ is a regular point of the Legendre ODE, $Q_l(0)$ must for all l be finite. The symmetry of Q_l causes $Q_l(0)$ to vanish for even l; it is shown in the next subsection that for odd l,

$$Q_{2s+1}(0) = (-1)^{s+1} \frac{(2s)!!}{(2s+1)!!}. \tag{15.161}$$

4. From the result of Exercise 15.6.3, it can be shown that $Q_l(\infty) = 0$.

Alternate Formulations

Because the singular points of the Legendre ODE nearest to the origin are at the points ± 1, it should be possible to describe $Q_l(x)$ as a power series about the origin, with convergence for $|x| < 1$. Moreover, since the only other singular point of the Legendre equation is a regular singular point at infinity, it should also be possible to express one of its solutions as a power series in $1/x$, i.e., a series about the point at infinity, which must converge for $|x| > 1$.

To obtain a power series about $x = 0$, we return to the discussion of the Legendre ODE presented in Section 8.3, where we saw that an expansion of the form

$$y(x) = \sum_{j=0}^{\infty} a_j x^{s+j} \tag{15.162}$$

led to an indicial equation with solutions $s = 0$ and $s = 1$, and with the a_j satisfying the recurrence formula, for eigenvalue $l(l+1)$,

$$a_{j+2} = a_j \frac{(s+j)(s+j+1) - l(l+1)}{(s+j+2)(s+j+1)}, \quad j = 0, 2, \ldots. \tag{15.163}$$

When l is even, we found that $P_l(x)$ was obtained as the solution $y(x)$ from the indicial-equation solution $s = 0$, and we did not make use (for even l) of the solution from $s = 1$ because that solution was not a polynomial and did not converge at $x = 1$. However, we are now seeking a second solution and are no longer restricting attention to those that converge at $x = \pm 1$. Thus, a second solution linearly independent of P_l must be that produced (again, for even l) as the series obtained when $s = 1$. This second solution will have odd parity, and therefore must be proportional to $Q_l(x)$.

Continuing, for even l, with $s = 1$, Eq. (15.163) becomes

$$a_{j+2} = a_j \frac{(l+j+2)(l-j-1)}{(j+2)(j+3)},$$

corresponding to

$$Q_l(x) = b_l \left[x - \frac{(l-1)(l+2)}{3!} x^3 + \frac{(l-3)(l-1)(l+2)(l+4)}{5!} x^5 - \cdots \right]. \qquad (15.164)$$

Here b_l is the value of the coefficient of the expansion needed to give the formula for Q_l the proper scaling. For odd l, the corresponding formula, with $s = 0$, is an even function of x, and must therefore be proportional to Q_l:

$$Q_l(x) = b_l \left[1 - \frac{l(l+1)}{2!} x^2 + \frac{(l-2)l(l+1)(l+3)}{4!} x^4 \cdots \right]. \qquad (15.165)$$

To find the values of the scale factors b_l, we turn now to the explicit forms for Q_0 and Q_1, Eqs. (15.155) and (15.156). Expanding the logarithm, we find (again keeping only the lowest-order terms)

$$Q_0(x) = x + \cdots, \qquad Q_1(x) = -1 + \cdots .$$

From the recurrence formula, Eq. (15.158), keeping only the lowest-order contributions, we find

$$\begin{aligned}
2Q_2 &= 3xQ_1 - Q_0 \longrightarrow Q_2 = -2x + \cdots \\
3Q_3 &= 5xQ_2 - 2Q_1 \longrightarrow Q_3 = 2/3 + \cdots \\
4Q_4 &= 7xQ_3 - 3Q_2 \longrightarrow Q_4 = 8x/3 + \cdots \\
\cdots &= \cdots .
\end{aligned}$$

These results generalize to

$$b_l = \begin{cases} (-1)^p \dfrac{(2p)!!}{(2p-1)!!} & l \text{ even}, \quad l = 2p, \\[2ex] (-1)^{p+1} \dfrac{(2p)!!}{(2p+1)!!} & l \text{ odd}, \quad l = 2p+1. \end{cases} \qquad (15.166)$$

One may now combine the values of the coefficients b_l with the expansions in Eqs. (15.164) and (15.165) to obtain entirely explicit series expansions of $Q_l(x)$ about $x = 0$. This is the topic of Exercise 15.6.2.

As mentioned earlier, the point $x = \infty$ is a regular singular point, and expansion about this point yields an expansion of $Q_l(x)$ in inverse powers of x. That expansion is considered in Exercise 15.6.3.

Exercises

15.6.1 Show that if l is even, $Q_l(-x) = -Q_l(x)$, and that if l is odd, $Q_l(-x) = Q_l(x)$.

15.6.2 Show that

(a)
$$Q_{2p}(x) = (-1)^p\, 2^{2p} \sum_{s=0}^{p} (-1)^s \frac{(p+s)!\,(p-s)!}{(2s+1)!\,(2p-2s)!}\, x^{2s+1}$$
$$+ 2^{2p} \sum_{s=p+1}^{\infty} \frac{(p+s)!\,(2s-2p)!}{(2s+1)!\,(s-p)!}\, x^{2s+1}, \quad |x| < 1,$$

(b)
$$Q_{2p+1}(x) = (-1)^{p+1} 2^{2p} \sum_{s=0}^{p} (-1)^s \frac{(p+s)!\,(p-s)!}{(2s)!\,(2p-2s+1)!}\, x^{2s}$$
$$+ 2^{2p+1} \sum_{s=p+1}^{\infty} \frac{(p+s)!\,(2s-2p-2)!}{(2s)!\,(s-p-1)!}\, x^{2s}, \quad |x| < 1.$$

15.6.3 (a) Starting with the assumed form

$$Q_l(x) = \sum_{j=0}^{\infty} b_{lj} x^{k-j},$$

show that

$$Q_l(x) = b_{l0} x^{-l-1} \sum_{s=0}^{\infty} \frac{(l+s)!\,(l+2s)!\,(2l+1)!}{s!\,(l!)^2\,(2l+2s+1)!}\, x^{-2s}.$$

(b) The standard choice of b_{l0} is

$$b_{l0} = \frac{2^l\,(l!)^2}{(2l+1)!},$$

leading to the final result

$$Q_l(x) = x^{-l-1} \sum_{s=0}^{\infty} \frac{(l+2s)!}{(2s)!!\,(2l+2s+1)!!}\, x^{-2s}.$$

Show that this choice of b_{l0} brings this negative power-series form of $Q_n(x)$ into agreement with the closed-form solutions.

15.6.4 (a) Using the recurrence relations, prove (independent of the Wronskian relation) that

$$n\Big[P_n(x) Q_{n-1}(x) - P_{n-1}(x) Q_n(x) \Big] = P_1(x) Q_0(x) - P_0(x) Q_1(x).$$

(b) By direct substitution show that the right-hand side of this equation equals 1.

Additional Readings

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Margenau, H., and G. M. Murphy, *The Mathematics of Physics and Chemistry*, 2nd ed. Princeton, NJ: Van Nostrand (1956).

Morse, P. M., and H. Feshbach, *Methods of Theoretical Physics*, 2 vols. New York: McGraw-Hill (1953). This work is detailed but at a rather advanced level.

Smythe, W. R., *Static and Dynamic Electricity*, 3rd ed. New York: McGraw-Hill (1968), reprinted, Taylor & Francis (1989), paperback. Advanced, detailed, and difficult. Includes use of elliptic integrals to obtain closed formulas.

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