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K-loops derived from Frobenius groups ☆

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Dedicated to Mario Marchi on the occasion of his 60th birthday

Abstract

We consider a generalization of the representation of the so-called co-Minkowski plane (due to H. and R. Struve) to an abelian group $(V,+)$ and a commutative subgroup G of $\mathrm{Aut}(V,+)$. If $P = G \times V$ satisfies suitable conditions then an invariant reflection structure (in the sense of Karzel (Discrete Math. 208/209 (1999) 387–409)) can be introduced in P which carries the algebraic structure of K-loop on P (cf. Theorem 1). We investigate the properties of the K-loop $(P,+)$ and its connection with the semi-direct product of V and G. If G is a fixed point free automorphism group then it is possible to introduce in $(P,+)$ an incidence bundle in such a way that the K-loop $(P,+)$ becomes an incidence fibered loop (in the sense of Zizioli (J. Geom. 30 (1987) 144–151)) (cf. Theorem 3).

Keywords: K-loops; Invariant reflection structures; Co-Minkowski planes

0. Introduction

In [3] there was introduced the concept of an *invariant reflection structure* $(P, {}^0; 0)$, that is a set P with a fixed element 0 and a map ${}^0 : P \to \mathrm{Sym}\, P;\ x \to x^0$ such that $x^0(0) = x$, $x^0 \circ x^0 = id$ and $x^0 \circ y^0 \circ x^0 = (x^0 y^0(x))^0$ for all $x, y \in P$, and it was proved that $(P,+)$ for $a + b := a^0 \circ 0^0(b)$ becomes a K-loop.

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If one takes a so-called co-Minkowski plane (cf. [8,9]) $(M, \mathscr{L}, \equiv)$ then in the motion group Γ of $(M, \mathscr{L}, \equiv)$ to each point $x \in M$ there exists exactly one reflection $\tilde{x}$ in x and the point set M splits into two subsets P and P^- with the properties:

1. $M = P \dot{\cup} P^-$
2. $\forall \sigma \in \Gamma$, $\sigma(P) = P$ and $\sigma(P^-) = P^-$
3. any two points $a, b \in P$ (resp. P^-) have exactly one midpoint m in P (resp. in P^-), i.e. $\tilde{m}(a) = b$.

Therefore, after fixing a point $0 \in P$, denoting for any $x \in P$ the midpoint of 0 and x in P by x' and setting $x^0 := \tilde{x}'$ then $(P, {}^0; 0)$ is an invariant reflection structure. Since in the classical co-Minkowski plane the subset P has the analytical representation $P = \mathbf{R}_+ \times \mathbf{R}$ ($\mathbf{R}_+ := \{x \in \mathbf{R} \mid x > 0\}$) and the reflection in the point $(\alpha, a) \in P$ has the form

$$(*) \qquad \widetilde{(\alpha, a)} : \begin{cases} P \to P \\ (\xi, x) \to (\alpha^2 \xi^{-1}, -x + (\alpha \xi^{-1} + \xi \alpha^{-1})a) \end{cases}$$

this procedure can be generalized. We replace $(\mathbf{R}, +)$ by an arbitrary abelian group $(V, +)$ and $(\mathbf{R}_+, \cdot)$ by a commutative subgroup $(G, \cdot)$ of $\mathrm{Aut}(V, +)$. Then in the product set $P := G \times V$ we can associate by $(*)$ to each element $(\alpha, a) \in P$ an involutory permutation $\widetilde{(\alpha, a)}$.

Here we discuss the following problems:

1. Under which conditions we derive from $G \times V$ an invariant reflection structure and so turn $P = G \times V$ in a K-loop $(P, +)$ (cf. Theorem 1).
2. In the case that $(P, +)$ is a K-loop what can be said of its structure (cf. Section 2).
3. In the co-Minkowski plane the intersections of P with lines, passing through the fixed point 0, form subgroups of the loop $(P, +)$. In the general case, is there also a fibration of $(P, +)$ in subgroups or in subloops?
4. The set P can be turned via the semi-direct product $G \ltimes V$ in a group $(P, \cdot)$ (which can be considered as an affine permutation group of $(V, +)$) by setting: $(\alpha, a)^{\cdot} : V \to V$; $x \to a + \alpha x$. What are the relations between $(P, \cdot)$ and $(P, +)$ in particular when $(P, \cdot)$ is a subset of a kinematic stripe space (cf. [4,5])?

1. Basic definitions and preliminary results

Let $(L, +)$ be a loop; for any $a \in L$ we denote by $-a \in L$ the element of L such that $a + (-a) = 0$; moreover let $a^+ : L \to L$; $x \to a + x$ and $L^+ := \{a^+ \mid a \in L\}$.

Since $(L, +)$ is a loop, $L^+ \subseteq \mathrm{Sym}\, L$, hence $\delta_{a,b} := ((a+b)^+)^{-1} \circ a^+ \circ b^+ \in \mathrm{Sym}\, L$ and the *structure group* $\Delta := \langle \{\delta_{a,b} \mid a, b \in L\} \rangle$ is a subgroup of $\mathrm{Sym}\, L$. For any $a \in L$ let $Z(a) := \{x \in L \mid a + x = x + a\}$.

According to Kerby and Wefelscheid, we say that a loop $(L, +)$ is a *K-loop* if the following conditions hold:

$$\text{for all } a, b \in L: \ -(a+b) = -a + (-b); \quad \delta_{a,b} = \delta_{a,b+a} \in \mathrm{Aut}(L, +).$$

By [3] one can derive a K-loop from a so-called *invariant reflection structure* $(P, {}^0; 0)$ that is a set $P \neq \emptyset$, a fixed element $0 \in P$ and a map ${}^0: P \to J := \{\sigma \in \mathrm{Sym}\, P \mid \sigma^2 = id\}$; $x \to x^0$ such that the following conditions hold:

(B1) $\forall x \in P,\ x^0(0) = x$;
(B2) $\forall a \in P,\ a^0 \circ P^0 \circ a^0 = P^0$ (where $P^0 := \{a^0 \mid a \in P\}$).

Then we have (cf. [3], Section 6):

(1.1) For all $a, b \in P$ let $a^+ := a^0 \circ 0^0$, $a + b := a^+(b)$, $-a := 0^0(a)$ then

(i) $(P, +)$ is a K-loop;
(ii) $\forall a \in P: \ -a + a = a + (-a) = 0,\ \delta_{a,a} = id,\ \delta_{a,-a} = id$;
(iii) $\forall a, b \in P: \ a^+ \circ b^+ \circ a^+ = (a + (b + a))^+$ (Bol identity).

Given a loop $(L, +)$, a set $\mathscr{F} \subseteq 2^L$ is called a *bundle with respect to* 0 or simply 0-*bundle* if:

(F1) $\forall X \in \mathscr{F}$: $|X| \geqslant 2$;
(F2) $\bigcup \mathscr{F} = L$;
(F3) $\forall A, B \in \mathscr{F},\ A \neq B$: $A \cap B = \{0\}$.

If furthermore the following conditions (cf. [10]):

(F4) $\forall a \in L,\ \forall X \in \mathscr{F}$: $0 \in a + X \Rightarrow a + X \in \mathscr{F}$;
(F5) $\forall X \in \mathscr{F},\ \forall \delta \in \Delta$: $\delta(X) \in \mathscr{F}$;

are satisfied then $\mathscr{F}$ is called an *incidence* 0-*bundle and* $(L, +, \mathscr{F})$ a *fibered loop* if moreover all $X \in \mathscr{F}$ are subloops of $(L, +)$.

Remark 1. We observe that if $(L, +, \mathscr{F})$ is a fibered loop then condition (F4) is trivially verified.

A triple $(P, \mathscr{L}, +)$ is an *incidence loop* (*group*) if $(P, \mathscr{L})$ is an incidence space, $(P, +)$ is a loop (group) and for any $a \in P$, a^+ is a collineation of $(P, \mathscr{L})$, i.e. $a^+ \in \mathrm{Aut}(P, \mathscr{L})$.

Incidence loops and loops with an incidence 0-bundle are the same by the following (see [10]):

(1.2) Let $(L, +)$ be a loop then:

(i) if $(L, \mathscr{L}, +)$ is an incidence loop then $\mathscr{L}(0) := \{X \in \mathscr{L} \mid 0 \in X\}$ is an incidence 0-bundle;
(ii) if $\mathscr{F} \subseteq 2^L$ is an incidence 0-bundle then $(L, \mathscr{L}, +)$ with $\mathscr{L} := \{a + X \mid a \in L,\ X \in \mathscr{F}\}$ is an incidence loop.

An incidence group $(P, \mathscr{L}, +)$ is said to be a *kinematic space* (cf. [2]) if for any $X \in \mathscr{L}(0) := \{A \in \mathscr{L} \mid 0 \in A\}$:

(i) X is a subgroup of $(P, +)$,
(ii) for any $a \in P$, $a + X - a \in \mathscr{L}(0)$.

2. Derivation from a pair of groups

Let $(V, +)$ be an abelian group and let $(G, \cdot) \leqslant \mathrm{Aut}(V, +)$ verifying the following conditions: $(G, \cdot)$ is abelian and uniquely divisible by 2 (i.e. $\forall \gamma \in G\ \exists_1 \xi \in G$ such that $\xi^2 = \gamma$; we shall write $\sqrt{\gamma} := \xi$).

We explicitly note that since $(V, +)$ is commutative, $(\mathrm{End}\, V, +, \cdot)$ is a ring and since $(G, \cdot)$ is abelian the subring $\langle G \rangle_+$ of $\mathrm{End}(V, +)$ generated by G is commutative.

Let us now consider the cartesian product

$$P := G \times V := \{(\alpha, a) \mid \alpha \in G, a \in V\}.$$

Our aim is to introduce a reflection structure on P, thus for any $(\alpha, a) \in P$ we define the map $\widetilde{(\alpha, a)} : P \to P$; $(\xi, x) \to \widetilde{(\alpha, a)}(\xi, x) := (\alpha^2 \xi^{-1}, (1 + \alpha\xi^{-1})(a) - \alpha\xi^{-1}(x))$ where $(1 + \alpha\xi^{-1}) \in \mathrm{End}\, V$ (here 1 denotes, as usual, the identity of $(G, \cdot)$).

In the following, for any $\gamma \in \mathrm{End}\, V$ and for any $x \in V$, we shall write γx instead of $\gamma(x)$ in order to simplify notations.

(2.1) For $(\alpha, a), (\beta, b) \in P$:

(i) $\widetilde{(\alpha, a)} \in J^* := \{\sigma \in \mathrm{Sym}\, P \mid \sigma^2 = id\} \setminus \{id\}$;
(ii) $\mathrm{Fix}\, \widetilde{(\alpha, a)} = \{(\alpha, x) \in P \mid x + x = a + a\}$;
(iii) $\widetilde{(\alpha, a)} \circ \widetilde{(\beta, b)} \circ \widetilde{(\alpha, a)} = \widetilde{(\alpha^2\beta^{-1}, (1 + \alpha\beta^{-1})a - \alpha\beta^{-1}b)}$.

Proof. (ii) We have $(\xi, x) \in \mathrm{Fix}\widetilde{(\alpha, a)}$ if and only if $\alpha^2\xi^{-1} = \xi$ and $(1 + \alpha\xi^{-1})a - (\alpha\xi^{-1})x = x$. These equalities imply $\xi = \alpha$ and $x + x = a + a$. □

(2.2) For any $(\beta, b) \in P$ there exists exactly one $(\xi, x) \in P$ such that $\widetilde{(\xi, x)}(1, 0) = (\beta, b)$ if and only if $1 + \sqrt{\beta} \in \mathrm{Aut}(V, +)$.

Proof. From $\widetilde{(\xi, x)}(1, 0) = (\xi^2, (1 + \xi)x) = (\beta, b)$ we have $\xi^2 = \beta$ and $(1 + \xi)x = b$; thus, $\xi = \sqrt{\beta}$ and $(1 + \sqrt{\beta})x = b$. Hence our assumption is valid if and only if for any $\beta \in G$, $1 + \sqrt{\beta} \in \mathrm{Aut}(V, +)$. □

By (2.1) and (2.2) we are now able to define an invariant reflection structure on P and therefore, by (1.1), an addition $+$ such that $(P, +)$ becomes a K-loop.

Theorem 1. *If the pair (G,V) satisfies the following conditions*:

1. $(G,\cdot)$ *is uniquely divisible by* 2;
2. $1+G\subseteq \mathrm{Aut}(V,+)$

and if we set ${}^0: P\to J; (\alpha,a)\to(\alpha,a)^0 := (\sqrt{\alpha},\widetilde{(1+\sqrt{\alpha})^{-1}a})$ *then* $(P,{}^0;(1,0))$ *is an invariant reflection structure and if we define*:

$$(\alpha,a)+(\beta,b) := (\alpha,a)^0\circ(1,0)^0(\beta,b) = (\alpha\beta,[(1+\sqrt{\alpha}\beta)/(1+\sqrt{\alpha})]a+\sqrt{\alpha}b)$$

then $(P,+)$ *is a* K-*loop with the properties*:

(i) $-(\alpha,a)=(\alpha^{-1},-\alpha^{-1}a)$;
(ii) $(1,V)$ *and* $(G,0)$, *respectively, are abelian subgroups of the loop* $(P,+)$ *isomorphic to* $(V,+)$ *and* $(G,\cdot)$, *respectively*;
(iii) *for any* $(\alpha,a),(\beta,b),(\gamma,c)\in P$,

$$\delta_{(\beta,b),(\gamma,c)}(\alpha,a)=(\alpha,(1-\alpha)d+a)$$

where

$$d := \frac{1}{1+\sqrt{\beta\gamma}}\left(\frac{1-\sqrt{\gamma}}{1+\sqrt{\beta}}b-\frac{1-\sqrt{\beta}}{1+\sqrt{\gamma}}c\right).$$

Proof. By assumptions 1, 2 of theorem 1 and the proof of (2.2) it follows that for any (α,a) the map $(\alpha,a)^0 := (\sqrt{\alpha},\widetilde{(1+\sqrt{\alpha})^{-1}a})$ is the uniquely determined involution of $\tilde{P}$ mapping $(1,0)$ onto (α,a). Consequently $(P,{}^0;(1,0))$ satisfies (B1) and by (2.1(iii)) also (B2), and so by (1.1), $(P,+)$ is a K-loop.

(iii) The formula can be obtained by direct calculation. □

From now on we assume always that (G,V) satisfies conditions 1 and 2 of Theorem 1 and $|G|>1$.

Now we study the action of the structure group Δ on P.

For each $d\in V$ let $\vartheta_d: P\to P; (\xi,x)\to(\xi,(1-\xi)d+x)$.

Then ϑ_d is an automorphism of $(P,+)$ and for $d_1,d_2\in V$ we have

$$\vartheta_{d_1+d_2}=\vartheta_{d_1}\circ\vartheta_{d_2}.$$

If $|G|>1$, ϑ_d is the identity if and only if $d=0$, and then

$$\vartheta:\begin{cases} V\ \to\ \mathrm{Aut}(P,+),\\ d\ \to\ \vartheta_d,\end{cases}$$

is a monomorphism of $(V,+)$ in $\mathrm{Aut}(P,+)$ consequently $\bar{\Delta} := \vartheta(V)$ is a commutative subgroup of $\mathrm{Aut}(P,+)$ and $\vartheta': V\to\bar{\Delta}$ with $\vartheta'(d) := \vartheta_d$ an isomorphism. By Theorem 1 (iii) the structure group Δ is a subgroup of $\bar{\Delta}$ and so $V' := \vartheta'^{-1}(\Delta)$ a subgroup of $(V,+)$. Moreover by Theorem 1(iii), $(1+\sqrt{\beta})(1+\sqrt{\gamma})(1+\sqrt{\beta\gamma})d=(1-\gamma)b-(1-\beta)c$, hence for any $\xi\in G$, $v\in V$ we set $\gamma=\xi$, $c=0$, and any $\beta\in G$, $b=(1+\sqrt{\beta})(1+\sqrt{\gamma})$

$(1+\sqrt{\beta\gamma})v$ and get $d=(1-\gamma)v$. This shows $\vartheta((1-G)V)\subseteq\Delta$. Since $(1+G)\subseteq \mathrm{Aut}(P,+)$, $d=(1+\sqrt{\beta})^{-1}(1+\sqrt{\gamma})^{-1}(1+\sqrt{\beta\gamma})^{-1}((1-\gamma)b-(1-\beta)c)\in\langle(1-G)V\rangle$ for any $(\beta,b),(\gamma,c)\in P$ hence $\vartheta^{-1}(\Delta)=V'=\langle(1-G)V\rangle$. Thus, we can state the following theorem.

Theorem 2. *Let $|G|>1$, $V':=\langle(1-G)V\rangle$ and $\alpha\in G^*:=G\backslash\{id\}$. Then Δ has the following properties*:

(i) $(V',+)\cong\Delta\leqslant\bar{\Delta}\cong(V,+)$;
(ii) $\Delta(\alpha,V)=(\alpha,V)=\bar{\Delta}(\alpha,V)=(\alpha,V)$; $\Delta(\alpha,V')=(\alpha,V')$;
(iii) $\Delta_{|(\alpha,V)}\cong((1-\alpha)V',+)\cong V'/\ker(1-\alpha)$;
(iv) $\bar{\Delta}\cong\bar{\Delta}_{|(\alpha,V)}\Leftrightarrow \mathrm{Fix}\,\alpha=\{0\}\Rightarrow \mathrm{Fix}\,\alpha_{|V'}=\{0\}\Leftrightarrow\Delta\cong\Delta_{|(\alpha,V)}$;
$\mathrm{Fix}\,\alpha=\{0\}\Rightarrow V\cong(1-\alpha)V\leqslant V'\leqslant V$;
(v) $\Delta_{|(\alpha,V)}$ *acts transitively on* $(\alpha,V)\Leftrightarrow(1-\alpha)V=V$ $(\Rightarrow V'=V)$;
(vi) $\Delta_{|(\alpha,V)}$ *acts regularly on* $(\alpha,V)\Leftrightarrow(1-\alpha)\in\mathrm{Aut}(V,+)\Rightarrow V'=V$ *and* $\Delta=\bar{\Delta}$.

Proof. (iii) By (ii) $\phi:\Delta\to\Delta_{|(\alpha,V)}$ is a homomorphism and if $\delta\in\Delta$, $d:=\vartheta'^{-1}(\delta)\in V'$ then for any $x\in V$: $\delta(\alpha,x)=(\alpha,(1-\alpha)d+x)$ showing $\Delta_{|(\alpha,V)}\cong((1-\alpha)V',+)$ and $\delta_{|(\alpha,V)}=\mathrm{id}_{|(\alpha,V)}\Leftrightarrow(1-\alpha)d=0\Leftrightarrow d\in\ker(1-\alpha)$.

(iv) If $\mathrm{Fix}\,\alpha=\{0\}$ then $(1-\alpha)$ is a monomorphism of V hence $V\cong(1-\alpha)V\leqslant\langle(1-G)V\rangle=V'\leqslant V$. $\square$

(2.3) Let $(\alpha,a)\in P\backslash(1,V)$ be given and let

$$[(\alpha,a)]:=\{(\xi,x)\in P\mid(1-\xi)a=(1-\alpha)x\}.$$

Then:

(i) $[(\alpha,a)]=[-(\alpha,a)]$; $[(\alpha,0)]=(G,\mathrm{Fix}\,\alpha)$;
(ii) $[(\alpha,a)]$ is a subloop of $(P,+)$ such that for any $\delta\in\bar{\Delta}$ and $d:=\vartheta^{-1}(\delta)$:

$$\delta[(\alpha,a)]=[\delta(\alpha,a)]=[(\alpha,(1-\alpha)d+a)];$$

(iii) $(\alpha,a)+(\beta,b)=(\beta,b)+(\alpha,a)\Leftrightarrow(1-\sqrt{\alpha\beta})((1-\beta)a-(1-\alpha)b)=0$;
(iv) $Z(\alpha,a)\supseteq[(\alpha,a)]\cup(\alpha^{-1},V)$, $Z(\alpha,a)\cap Z(-(\alpha,a))\supseteq[(\alpha,a)]$;
(v) $[(\alpha,a)]\cap(\beta,V)\neq\emptyset\Leftrightarrow(1-\beta)a\in(1-\alpha)V$;
(vi) $(\beta,b)\in[(\alpha,a)]\cap(\beta,V)\Rightarrow[(\alpha,a)]\cap(\beta,V)=(\beta,b+\mathrm{Fix}\,\alpha)$;
(vii) $\forall a\in(1-\alpha)V$: $[(\alpha,a)]\cap(\beta,V)\neq\emptyset$.

Proof. (i) $(\xi,x)\in[-(\alpha,a)]=[(\alpha^{-1},-\alpha^{-1}(a))]$ (by definition) $\Leftrightarrow(1-\xi)(-\alpha^{-1}(a))=(1-\alpha^{-1})x\Leftrightarrow(1-\xi)a=(-\alpha+1)x=(1-\alpha)x\Leftrightarrow(\xi,x)\in[(\alpha,a)]$. Hence $[-(\alpha,a)]=[(\alpha,a)]$.

(ii) Let $(\xi,x),(\eta,y)\in[(\alpha,a)]$ i.e. $(1-\xi)a=(1-\alpha)x$ and $(1-\eta)a=(1-\alpha)y$, then $(\xi,x)+(\eta,y)=(\xi\eta,(1+\sqrt{\xi\eta})/(1+\sqrt{\xi})x+\sqrt{\xi}y)$ and $(1-\alpha)((1+\sqrt{\xi\eta})/(1+\sqrt{\xi})x+\sqrt{\xi}y)=(1+\sqrt{\xi\eta})/(1+\sqrt{\xi})(1-\alpha)x+\sqrt{\xi}(1-\alpha)y=(1+\sqrt{\xi\eta})/(1+\sqrt{\xi})(1-\xi)a+\sqrt{\xi}(1-\eta)a=(1-\xi\eta)a$ so $(\xi,x)+(\eta,y)\in[(\alpha,a)]$.

Moreover $(\xi,x)\in[(\alpha,a)]$ implies $-(\xi,x)\in[(\alpha,a)]$.

Let us now consider the equations

$(\xi,x)+(\alpha_1,a_1)=(\alpha_2,a_2)$, $(\alpha_1,a_1)+(\eta,y)=(\alpha_2,a_2)$ with $(\alpha_i,a_i)\in[(\alpha,a)]$ and $i=1,2$. Since $(P,+)$ is a K-loop we know (cf. [6]) that the solutions are given by $(\xi,x)=-(\alpha_1,a_1)+(((\alpha_1,a_1)+(\alpha_2,a_2))-(\alpha_1,a_1))$, $(\eta,y)=-(\alpha_1,a_1)+(\alpha_2,a_2)$; thus, by our previous considerations, $(\xi,x),(\eta,y)\in[(\alpha,a)]$ and $([(\alpha,a)],+)$ is a subloop of $(P,+)$. $\delta(\xi,x)=(\xi,(1-\xi)d+x)\in[(\alpha,(1-\alpha)d+a)]\Leftrightarrow(1-\xi)((1-\alpha)d+a)=(1-\alpha)((1-\xi)d+x)\Leftrightarrow(1-\xi)a=(1-\alpha)x\Leftrightarrow(\xi,x)\in[(\alpha,a)]$.

(iii) $(\alpha,a)+(\beta,b)=(\beta,b)+(\alpha,a)\Leftrightarrow(1+\sqrt{\alpha}\beta)/(1+\sqrt{\alpha})a+\sqrt{\alpha}b=(1+\sqrt{\beta}\alpha)/(1+\sqrt{\beta})b+\sqrt{\beta}a\Leftrightarrow(1-\sqrt{\alpha\beta})(1-\sqrt{\beta})/(1+\sqrt{\alpha})a=(1-\sqrt{\alpha\beta})(1-\sqrt{\alpha})/(1+\sqrt{\beta})b\Leftrightarrow(1-\sqrt{\alpha\beta})(1+\sqrt{\alpha})(1+\sqrt{\beta})((1-\beta)a-(1-\alpha)b)=0$; since $(1+G)\subseteq\mathrm{Aut}(V,+)$, the last equation is equivalent to $(1-\sqrt{\alpha\beta})((1-\beta)a-(1-\alpha)b)=0$.

(iv) By (iii) $Z(\alpha,a)=\{(\xi,x)\in P\mid(1-\sqrt{\alpha\xi})((1-\xi)a-(1-\alpha)x=0\}$. Hence $[(\alpha,a)]\subseteq Z(\alpha,a)$ and also $\{(\alpha^{-1},x)\mid x\in V\}\subseteq Z(\alpha,a)$.

Moreover, $Z(-(\alpha,a))=Z(\alpha^{-1},\alpha^{-1}(-a))\supseteq[-(\alpha,a)]\cup(\alpha,V)$ and by (i) we have: $[(\alpha,a)]\subseteq Z(\alpha,a)\cap Z(-(\alpha,a))$.

(v)–(vi) Let $(\beta,b),(\beta,x)\in[(\alpha,a)]\cap(\beta,V)$, then $(1-\alpha)b=(1-\beta)a$ and $(1-\alpha)x=(1-\beta)a$, i.e. $(1-\beta)a\in(1-\alpha)V$ and $(1-\alpha)x=(1-\alpha)b$ that is $(1-\alpha)(x-b)=0$.

(vii) By assumption there is $b\in V$ such that $a=(1-\alpha)b$ hence $(\alpha,a)=(\alpha,(1-\alpha)b)$ and $(1-\beta)a=(1-\beta)(1-\alpha)b=(1-\alpha)(1-\beta)b\in(1-\alpha)V$; so by (v) we have $[(\alpha,a)]\cap(\beta,V)\neq\emptyset$. □

It follows from (2.3(vi)(vii)):

(2.4) Let $\alpha\in G^*$; then the following statements are equivalent:

(i) $\mathrm{Fix}\,\alpha=\{0\}$;
(ii) $\forall\beta\in G,\ \forall a\in V\ |[(\alpha,a)]\cap(\beta,V)\ |\leqslant 1$;
(iii) $\forall\beta\in G,\ \forall a\in(1-\alpha)V\ |[(\alpha,a)]\cap(\beta,V)|=1$.

We introduce now the following:

Definition. An element $(\alpha,a)\in P\backslash\{(1,0)\}$ is called *transversal* if $[(\alpha,a)]\cap(\beta,V)\neq\emptyset$ for any $\beta\in G$, or equivalently, by (2.3.v), $(1-G)a\subseteq(1-\alpha)V$. Then we say that $[(\alpha,a)]$ is transversal too.

From this definition it follows that any transversal $(\alpha,a)\in P$ must have $\alpha\neq 1$ and $(\alpha,0)$ is transversal for any $\alpha\in G^*$.

(2.5) Let $\alpha\in G^*$ and $a\in V$ then

(i) if $a\in(1-\alpha)V$ then (α,a) is transversal;
(ii) if $(1-\alpha)$ is surjective then (α,a) is transversal.

(2.6) For any $\delta\in\bar{\varDelta}$ and for any transversal $(\alpha,a)\in P$, $\delta(\alpha,a)$ is transversal.

Proof. By (2.4(ii)) and Theorem 2(ii), for any $\beta\in G$ $[\delta(\alpha,a)]\cap(\beta,V)=\delta[(\alpha,a)]\cap\delta(\beta,V)=\delta([(\alpha,a)]\cap(\beta,V))\neq\emptyset$. □

3. The K-loop (P, +) and the group $G \ltimes V$

By the assumption of Section 2 we can turn $P = G \times V$ also in a group $(P, \cdot)$ via the semidirect product:

$$(\alpha, a) \cdot (\beta, b) := (\alpha\beta, a + \alpha b).$$

Then the reflection $\widetilde{(\alpha, a)}$ defined in Section 2 is exactly the map:

$$\widetilde{(\alpha, a)} : \begin{cases} P \to P \\ (\xi, x) \to (\alpha, a) \cdot (\xi, x)^{-1} \cdot (\alpha, a) \end{cases}$$

and, if $\alpha \neq 1$, the centralizer of (α, a) in the group $(P, \cdot)$ is exactly the set $[(\alpha, a)]$ (cf. (2.3)). Assumptions 1 and 2 of Theorem 1 are equivalent to requiring the group $(P, \cdot) = G \ltimes V$ to be uniquely divisible by 2.

Remark 2. It is well known that to any group G one can associate a discrete symmetric space (see e.g. [7]), namely the so-called *special reflection groupoid* in the sense of [1], by setting, for any $a \in G$, $\tilde{a} : G \to G$; $x \to ax^{-1}a$. If (and only if) G is uniquely divisible by 2, then we can define, for any $a \in G$, $a^0 : G \to G$; $x \to \widetilde{\sqrt{a}}(x)$, so that $(G, {}^0; 1)$ becomes an invariant reflection structure in the sense of Section 1. So we note that from any group G one can derive, in the sense of Section 2, a K-loop if G is uniquely divisible by 2.

The semidirect product $(P = G \ltimes V, \cdot)$ has a representation as an affine permutation group of V by:

$$(\alpha, a)^{\cdot} : \begin{cases} V \to V, \\ x \to \alpha x + a. \end{cases}$$

Then, for each $a \in V$, the stabilizer $P_a := \{(\xi, x) \in P \mid (\xi, x)^{\cdot}(a) = a\}$ is a commutative subgroup of $(P, \cdot)$ which intersects the normal subgroup $(1, V)$ in the neutral element $(1, 0)$ of $(P, \cdot)$ and $(P, +)$. But we can say more:

(3.1) For any $a \in V$ we have $P_a = \{(\xi, (1 - \xi)a) \mid \xi \in G\}$ and:

(i) $\forall \alpha \in G^*$, $P_a \subseteq [(\alpha, (1 - \alpha)a)]$ and the equality holds if $\mathrm{Fix}\, \alpha = \{0\}$.
(ii) The operation "$\cdot$" and the loop operation "+" coincide in P_a, and $(P_a, +)$ is a commutative subgroup of $(P, +)$ (and of any transversal subloop $[(\alpha, (1 - \alpha)a)]$ with $\alpha \in G^*$).

Proof. (i) Let $(\beta, b) \in [(\alpha, (1 - \alpha)a)]$, i.e. $(1 - \alpha)b = (1 - \beta)(1 - \alpha)a$, then $(1 - \alpha)(b - (1 - \beta)a) = 0$ and this gives $b = (1 - \beta)a$ if $\mathrm{Fix}\, \alpha = \{0\}$.

(ii) For $\xi, \xi' \in G$ we have
$(\xi, (1 - \xi)a) \cdot (\xi', (1 - \xi')a) = (\xi\xi', (1 - \xi\xi')a)$
$(\xi, (1 - \xi)a) + (\xi', (1 - \xi')a) = (\xi\xi', (1 + \sqrt{\xi}\xi')(1 - \sqrt{\xi})a + \sqrt{x}(1 - \xi')a) = (\xi\xi', (1 - \xi\xi')a)$. $\square$

(3.2) Let $(\alpha, a) \in P \setminus (1, V)$, then

(i) $([(\alpha, a)], \cdot)$ is a subgroup of $(P, \cdot)$;
(ii) the operations "$\cdot$" and "$+$" coincide on $[(\alpha, a)]$ if and only if $([(\alpha, a)], \cdot)$ is abelian;
(iii) if $\mathrm{Fix}\, \alpha = \{0\}$ then $([(\alpha, a)], \cdot)$ is abelian.

Proof. Let $(\xi, x), (\eta, y) \in [(\alpha, a)]$, i.e.

$(*)$ $(1 - \xi)a = (1 - \alpha)x$ and $(1 - \eta)a = (1 - \alpha)y$.

(ii) $x + \xi y = y + \eta x \Leftrightarrow (1 - \eta)x = (1 - \xi)y$.
Moreover $(1 + \sqrt{\xi}\eta)/(1 + \sqrt{\xi})x + \sqrt{\xi}y - (x + \xi y) = (\sqrt{\xi}(\eta - 1))/(1 + \sqrt{\xi})x + \sqrt{\xi}(1 - \sqrt{\xi})y = 0 \Leftrightarrow \sqrt{\xi}(\eta - 1)x + \sqrt{\xi}(1 - \xi)y = 0 \Leftrightarrow (1 - \xi)y = (1 - \eta)x$.
(iii) $(*)$ implies $(1 - \alpha)(1 - \eta)x = (1 - \eta)(1 - \xi)a = (1 - \alpha)(1 - \xi)y$ and so, by $\mathrm{Fix}\, \alpha = \{0\}$, $(1 - \eta)x = (1 - \xi)y$. □

4. A bundle of $(P, +)$

In this section, we assume that, in addition to conditions 1 and 2 of Theorem 1, the following condition is satisfied.

3. $\forall \alpha \in G^*$ $\mathrm{Fix}\, \alpha = \{0\}$ (i.e. $(1 - \alpha)$ is a monomorphism of $(V, +)$).

Let

$$\mathscr{F} := \{[(\alpha, a)] \mid (\alpha, a) \in P \setminus (1, V)\} \cup \{(1, V)\}$$

then we have

(4.1) $\mathscr{F}$ is a $(1, 0)$-bundle of $(P, +)$ consisting of abelian subgroups.

Proof. By Theorem 1(iii) and (3.2(ii)), the elements of $\mathscr{F}$ are all abelian subgroups. Since conditions (F1,2) of Section 1 are trivially verified, we have only to check (F3). By (2.4(ii)), for any $(\alpha, a) \in P \setminus (1, V)$, $[(\alpha, a)] \cap (1, V) = \{(1, 0)\}$.

Let $(\beta, b) \in [(\alpha, a)]$ with $\beta \neq 1$ and let $(\xi, x) \in [(\beta, b)]$, i.e. $(1 - \beta)a = (1 - \alpha)b$ and $(1 - \xi)b = (1 - \beta)x$. Then $(\alpha, a) \in [(\beta, b)]$ and $(1 - \alpha)(1 - \beta)x = (1 - \xi)(1 - \beta)a$, and so, by $\mathrm{Fix}\, \beta = \{0\}$, $(1 - \alpha)x = (1 - \xi)a$, i.e. $(\xi, x) \in [(\alpha, a)]$, i.e. $[(\beta, b)] \subseteq [(\alpha, a)]$. By $(\alpha, a) \in [(\beta, b)]$ we have $[(\beta, b)] = [(\alpha, a)]$. □

By Theorem 1(iii), we know that for any $\delta \in \bar{\Delta}$, $\delta(1, V) = (1, V)$ and by (2.3(ii)) for any $[(\alpha, a)] \in \mathscr{F} \setminus \{(1, V)\}$ $\delta([(\alpha, a)]) = [(\alpha, (1 - \alpha)d + a)] \in \mathscr{F} \setminus \{(1, V)\}$.
Thus condition (F5) is satisfied for the elements of $\mathscr{F}$ and by (4.1), (1.2(ii)) and Theorem 1. we can state:

Theorem 3. *The set* $\mathscr{F}$ *is an incidence* $(1,0)$*-bundle of the K-loop* $(P,+)$ *consisting of abelian subgroups and* $(P,\mathscr{L},+)$, *where* $\mathscr{L}:=\{(\alpha,a)+X \mid (\alpha,a)\in P,\ X\in\mathscr{F}\}$, *is an incidence loop with* $\Delta\leqslant \mathrm{Aut}(P,\mathscr{L},+)$.

We observe that the elements of $\mathscr{F}$, that are the centralizers in the group $(P,\cdot)$, can be also characterized with respect to the loop operation in the following way:

(4.2) (i) For any $\alpha\in G^*$:

$$[(\alpha,a)]=Z(\alpha,a)\cap Z(-(\alpha,a)).$$

(ii) For any $a\neq 0$ $(1,V)=Z(1,a)$.

Proof. (i) By (2.3(iii)), we have that $(\xi,x)\in Z(\alpha,a)$ if and only if $(1-\sqrt{\alpha\xi})((1-\xi)a-(1-\alpha)x)=0$. So two cases can occur:

(a) $\xi\neq\alpha^{-1}$ then $\sqrt{\alpha\xi}\neq 1$ and so, since Fix $\sqrt{\alpha\xi}=\{0\}$, we have $(1-\xi)a-(1-\alpha)x=0$ i.e. $(\xi,x)\in[(\alpha,a)]$.

(b) $\xi=\alpha^{-1}$ then $(\alpha^{-1},V)\subseteq Z(\alpha,a)$.

Thus, with (2.3(iv)) $Z(\alpha,a)=[(\alpha,a)]\cup(\alpha^{-1},V)$ and so $[(\alpha,a)]=Z(\alpha,a)\cap Z(-(\alpha,a))$.

(ii) $(\xi,x)\in Z(1,a)\Leftrightarrow(1-\sqrt{\xi})(1-\xi)=0\Leftrightarrow\xi=1$ by condition 3. □

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