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INNER MAPPINGS OF BOL LOOPS

Alexander Kreuzer

K-loops have their origin in the theory of sharply 2-transitive groups. In that paper a proof is given that K-loops and Bruck loops are the same. For the proof it is necessary to show that in a (left) Bruck loop the left inner mappings $L(b)L(a)L(ab)^{-1}$ are automorphisms. This paper generalizes results of Glauberman [6], Kist [12] and Kreuzer [14].

0. INTRODUCTION

In order to describe sharply 2-transitive groups, H. Karzel introduced in [8] the notion of a neardomain $(F,\oplus,\cdot)$(cf. [22]). The crucial difficulty of a neardomain is the additive structure $(F,\oplus)$, which need not be associative and until today no example of a proper neardomain is known (cf. [10,11]). To obtain partial results, W. Kerby and H. Wefelscheid considered separately the additive structure $(F,\oplus)$ and called such loops K-loops (see definition in section 2). Since 1988 the interest on K-loops has been revived because A. A. Ungar has found a famous physical example.

A.A. Ungar investigated the relativistic addition $\oplus$ of the velocities $\mathbb{R}^3_c := \{ v \in \mathbb{R}^3 : |v| < c \}$ He showed that $(\mathbb{R}^3_c, \oplus)$ is a non-associative and non-commutative loop with characteristic automorphisms, which he calls a gyrogroup. Ungar proved that for any two velocities $a,b \in \mathbb{R}^3_c$ there is an automorphism $\delta_{a,b}$ of $(\mathbb{R}^3_c, \oplus)$, the socalled Thomas rotation, satisfying $a \oplus (b \oplus x) = (a \oplus b) \oplus x\delta_{a,b}$ (cf. [19, 20, 21]), i.e. $\delta_{a,b}$ is a left inner mapping of the loop. H. Wefelscheid recognized then that $(\mathbb{R}^3_c, \oplus)$ is a K-loop.

At first it was discovered by G. Kist that there is a connection between K-loops and Bruck loops [12, p. 27]. G. Kist remarks, that already from results of G. Glauberman [6] one can deduce that every finite Bruck loop of odd order is a K-loop. As a generalisation it is proved in [14, Theorem 1] that every Bruck loop with no element of order 2 is a K-loop.

In this note we prove that K-loops and Bruck loops are the same. For that mainly we have to show that the left inner mappings of a (left) Bruck loop are automorphisms of the loop, denoted as axiom (I). (In general the right inner mappings of a left Bruck loop are not automorphisms, hence Bruck loops are clearly not A-loops in then sense of Bruck and Paige [3], but left A-loops by Definition 1.1.4 of Nagy and Strambach [16], and in particular homogenous loops.)

In section 1. and 2. we give the definitions and some easy results, partly known, which we need in section 3. The main results are the Theorems **3.1** and **3.3.** Theorem **3.1** gives also a different proof for a part of Theorem 2.2.iii of Goodaire and Robinson [7] that the left inner mappings of a left conjugacy closed loop are automorphisms (cf. Corollary **3.2**). (An investigation of left conjugacy closed loops and their properties can be found in [16].)

Differently to other papers on K-loops [9, 13, 14, 15], in this paper we use "$\cdot$" instead of "+" for the binary operation, as it is customary for loops.

1. Left inner mappings

Let $(K,\cdot)$ be a loop with the identity element 1, and for $x\in K$ let $x^\lambda, x^\varrho \in K$ be the unique elements with $x^\lambda x = x x^\varrho = 1$. If $x^\lambda = x^\varrho$, then $x^{-1} = x^\lambda = x^\varrho$ is the inverse of x. Let $N_\mu = \{b\in K : a\cdot bc = ab\cdot c \text{ for all } a,c\in K\}$ denote the middle nucleus. For any fixed element $a\in K$, the map

$$L(a): K \to K; \quad x \to xL(a) := a\cdot x \tag{1.1}$$

is called **left translation**. The group $\mathsf{M}_\lambda := \langle L(x) : x\in K\rangle$ of all permutations of K which is generated by all left translations (and their inverses) is called the **left multiplication group** of $(K,\cdot)$.

Let $\mathsf{K} := \{L(x) : x\in K\}$ be the subset of all left translations of M_λ. Clearly, $b\in N_\mu$ iff $ab\cdot c = cL(ab) = a\cdot bc = cL(b)L(a)$, i.e., iff $L(ab) = L(b)L(a)$ for every $a\in K$. Assume $L(b)L(a) = L(x)\in \mathsf{K}$, then $1L(b)L(a) = ab = 1L(x) = x$, i.e., $x = ab$. Hence

$$b\in N_\mu \quad \text{if and only if} \quad L(b)L(a)\in \mathsf{K} \text{ for every } a\in K \tag{1.2}$$

We call the permutations of $\mathsf{A} := \{\alpha\in \mathsf{M}_\lambda : 1\alpha = 1\}$ the **left inner mappings** of $(K,\cdot)$.

1.1 Lemma. $\mathsf{M}_\lambda = \mathsf{A}\mathsf{K}$ and $\mathsf{M}_\lambda = \mathsf{K}\mathsf{A}$ are exact decompositions, i.e. for every $\mu\in\mathsf{M}_\lambda$ there are unique elements $L(a), L(b)\in\mathsf{K}$, $\alpha,\beta\in\mathsf{A}$ with $\mu = \alpha L(a) = L(b)\beta$ and we have $a = b^\varrho\mu^2$.

Proof. For $\mu\in\mathsf{M}_\lambda$ let $a = 1\mu$, $s = 1\mu^{-1}\in K$, i.e., $s\mu = 1$. Set $b = s^\lambda$, then $\mu = \mu L(a)^{-1}L(a) = L(b)L(b)^{-1}\mu$ with $\alpha = \mu L(a)^{-1}$, $\beta = L(b)^{-1}\mu\in\mathsf{A}$, since $1\mu L(a)^{-1} = aL(a)^{-1} = 1$ and $sL(s^\lambda) = 1$, hence $1L(b)^{-1}\mu = 1L(s^\lambda)^{-1}\mu = s\mu = 1$. Clearly $b^\varrho\mu^2 = s\mu\mu = 1\mu^{-1}\mu\mu = 1\mu = a$.

Assume $\mu = \alpha L(a) = \alpha' L(a')$, then $\alpha'^{-1}\alpha = L(a')L(a)^{-1}$ and $1 = 1L(a')L(a)^{-1}$, i.e., $1L(a) = a = a' = 1L(a')$ and $\alpha' = \alpha$. Hence $a\in L$, $\alpha\in\mathsf{A}$ and also $b\in L$, $\beta\in\mathsf{A}$ are uniquely determined.

For fixed elements $a,b \in K$ let

$$\delta_{a,b} := L(b)L(a)L(ab)^{-1}. \tag{1.3}$$

In this paper we prefer to write $\delta_{a,b}$ rather than $L(b,a)$ to match up papers on K-loops. Let $A' := \langle \delta_{x,y} : x,y \in K \rangle$ be the subgroup of M_λ which is generated by all permutations $\delta_{x,y}$. Analogous to [2, IV, Lemma 1.2] and [17, I.5.2] we get:

1.2 Lemma. $A = \langle \delta_{x,y} : x,y \in K \rangle$.

Proof. Clearly $1\delta_{x,y} = 1$, hence $A' \subset A$. Now we show $A'K\,K \subset A'K$ and $A'KK^{-1} \subset A'K$, hence with respect to $M_\lambda = \langle K \rangle$ we get $A'K = M_\lambda$ and with Lemma 1.1 $A' = A$. Let $\alpha \in A'$, $x,y \in L$, then $\alpha L(y)L(x) = \alpha L(y)L(x)L(xy)^{-1}L(xy) = \alpha\delta_{x,y} L(xy) \in A'K$ and for $z \in K$ with $x = yz$ also $\alpha L(x)L(y)^{-1} = \alpha L(yz)L(y)^{-1}L(z)^{-1}L(z) = \alpha\delta_{z,y}^{-1} L(z) \in A'K$.

Clearly definition (1.3) implies for $a,b,x \in K$:

$$a \cdot bx = ab \cdot x\delta_{a,b} \tag{1.4}$$

$$\delta_{a,1} = \delta_{1,a} = id \tag{1.5}$$

1.3 Lemma. In a loop $(K,\cdot)$ the following statements (i),(ii), and (iii) are equivalent:
(i) $\delta_{a^\lambda,a} = id$ **(ii)** $a^\lambda \cdot ax = x$ (left inverse property) **(iii)** $L(a^\lambda) = L(a)^{-1}$

Proof. By (1.4) $a^\lambda \cdot ax = a^\lambda a \cdot x\delta_{a^\lambda,a} = x$ for every $x \in K$ if $\delta_{a^\lambda,a} = id$. Obviously **(ii)** implies **(iii)**. $\delta_{a^\lambda,a} = L(a)L(a^\lambda)L(a^\lambda a) = id$, iff $L(a^\lambda) = L(a)^{-1}$.

We recall that the left inverse property implies $a^\lambda = a^\varrho = a^{-1}$.

2. K-loops, Bruck loops and left conjugacy closed loops

A loop $(K,\cdot)$ is called a **left A-loop** if **(I)**, a left **K-loop** if **(I)**, **(II)** and **(III)**, a left **Bol loop** if **(B)**, and a left **Bruck loop** if **(B)** and **(III)** are satisfied:

(I) For all $x,y \in K$, $\delta_{x,y}$ is an automorphism of $(K,\cdot)$.

(II) $\delta_{x,y} = \delta_{x,yx}$ for all $x,y \in K$.

(III) (Automorphic inverse property) $(ab)^{-1} = a^{-1}b^{-1}$ for all $a,b \in K$.

(B) (left Bol identity) $a(b \cdot ac) = (a \cdot ba)c$ for all $a,b,c \in K$.

We consider also the following axiom

(IB) $a^\lambda(b \cdot ac) = (a^\lambda \cdot ba)c$ for all $a,b,c \in K$.

In the following we omitt the word "left" and refer by the phrase Bol (Bruck, K-) loop always to left Bol (Bruck, K-) loops.

By **(II)** and (1.5), $\delta_{a,a^\lambda} = \delta_{a,a^\lambda a} = \delta_{a,1} = id$, hence by (1.4) $a = a\cdot a^\lambda a = aa^\lambda \cdot a\delta_{a,a^\lambda} = aa^\lambda \cdot a$. We obtain (cf. [15,(2.10)]) $aa^\lambda = 1$, $\delta_{a^\lambda,a} = \delta_{a^\lambda,aa^\lambda} = \delta_{a^\lambda,1} = id$ and $\delta_{a,a} = \delta_{a,aa^\lambda} = \delta_{a,1} = id$, i.e. by Lemma **1.3**, $a^\lambda \cdot ax = x$ and $a\cdot ax = a^2\cdot x$, properties which are well known for Bol loops (cf. [2, 17, 18]). If we set $b=1$ in **(IB)**, we get also $a^\lambda \cdot ac = c$. Hence:

2.1 Lemma. In K-loops, Bol loops and loops with **(IB)** the left inverse property $a^\lambda \cdot ac = c$, in K-loops and Bol loops also the left alternative law $a\cdot ac = a^2c$ is satisfied.

2.2 Example. In a loop with **(IB)** the left alternative law need not be valid, which we see in the following example. For $n \in \mathbb{N}$ let $K := \mathbb{Z}_{8n}$ and

$$\bullet : K\times K \to K; \quad a\bullet b = \begin{cases} a+b+4nb & \text{if } a\in\{2n,6n\} \\ a+b & \text{else} \end{cases} \tag{1.6}$$

we get a loop $(K,\bullet)$ with neutral element 0, with $a\bullet a = 2a$ and $a^{-1} = -a$. It is easy to compute that **(IB)** is satisfied. But we have $n\bullet(n\bullet 1) = 2n+1 \neq (n\bullet n)\bullet 1 = 2n\bullet 1 = 2n+1+4n = 6n+1$, i.e., the left alternative law is not satisfied.

2.3 Lemma. Let $(K,\cdot)$ be a loop. Then:

a) **(B)** $\Longleftrightarrow \delta_{a,ba} = \delta_{b,a}^{-1} \Longleftrightarrow L(a)\mathcal{K}L(a)\subset\mathcal{K}$ for all $a,b\in K$.

b) **(IB)** $\Longleftrightarrow \delta_{a^\lambda,ba} = \delta_{b,a}^{-1} \Longleftrightarrow L(a)\mathcal{K}\,L(a^\lambda)\subset\mathcal{K}$ for all $a,b\in K$.

c) **(IB)** $\Rightarrow L(a)\mathcal{K}\,L(a)^{-1}\subset\mathcal{K}$ for every $a\in K$.

Proof. a). By (1.4) $a(b\cdot ac) = a(ba\cdot c\delta_{b,a}) = (a\cdot ba)c\delta_{b,a}\delta_{a,ba} \overset{!}{=} (a\cdot ba)c$ for every $c\in K$, iff $\delta_{b,a}\delta_{a,ba} = id$. Since $a(b\cdot ac) = cL(a)L(b)(La)$ and $(a\cdot ba)c = cL(a\cdot ba)$, **(B)** is equivalent to $L(a)L(b)L(a)\in\mathcal{K}$ for every $a,b\in K$.

b). $a^\lambda(b\cdot ac) = (a^\lambda\cdot ba)c\delta_{b,a}\delta_{a^\lambda,ba} \overset{!}{=} (a\cdot ba)c$ iff $\delta_{b,a}\delta_{a^\lambda,ba} = id$. By $a^\lambda(b\cdot ac) = cL(a)L(b)(La^\lambda)$ and $(a^\lambda\cdot ba)c = cL(a^\lambda\cdot ba)$, **(IB)** turns out to be equivalent to $L(a)L(b)L(a^\lambda)\in\mathcal{K}$ for all $a,b\in K$.

c). By Lemma 2.1 $a^\lambda = a^{-1}$, hence **b)** implies **c)**.

Loops with $L(a)\mathcal{K}L(a)^{-1}\subset\mathcal{K}$ are called **left conjugacy closed loops** (cf. [7, 16]). Bol loops with **(IB)** are therefore exactly the **Burn loops** of [16], defined as left conjugacy closed loops with **(B)**. With **(IB)** we give another proof for Theorem 1.4.4 of [16]:

2.4 Lemma. In a Bol loop $(K,\cdot)$, **(IB)** is satisfied if and only if $\{x^2 : x\in K\}\subset N_\mu$, where N_μ is the middle Nucleus.

Proof. In a (left) Bol loop $b\cdot a^2c = a^{-1}a\cdot(b(a\cdot ac)) = a^{-1}(a(b(a\cdot ac))) \overset{(B)}{=} a^{-1}((a\cdot ba)\cdot ac)$ and

$ba^2\cdot c = (a^{-1}(a(b\cdot aa)))c \overset{(B)}{=} (a^{-1}((a\cdot ba)a))c$. We set $b' = a\cdot ba$. Hence for every $a\in K$ we have $a^2 \in N_\mu$, iff $b\cdot a^2c = a^{-1}(b'\cdot ac) \overset{!}{=} ba^2\cdot c = (a^{-1}\cdot b'a)c$, i.e., iff **(IB)** is satisfied.

2.5 Remark. There do exist proper Bol loops satisfying **(IB)**. For instance the examples for Bruck loops of order $8n, n\in\mathbb{N}$, given in [15, **(5.2.ii)**, **(5.5)**] and all six examples for Bol loops of order 8 of [4] satisfy **(IB)**. Whereas the examples of Bol loops of [1] do not satisfy **(IB)**.

By [14, (1.2)], [15, (2.12)]:

2.6 Lemma. Every K-loop satisfies the Bol identity and is a Bruck loop.

3. Left inner automorphisms

Now we describe properties of the loop $(K,\cdot)$ in the left multiplication group $M_\lambda = KA$.

3.1 Theorem. An inner mapping $\alpha\in A$ is an automorphism of $(K,\cdot)$ if and only if $\alpha^{-1}K\alpha \subset K$.

Proof. Let $x,y\in K$ and $\alpha\in A$. Then $(xy)\alpha = x\alpha\cdot y\alpha$ is eqivalent to $xy = (x\alpha\cdot y\alpha)\alpha^{-1}$, hence

$$L(x) = \alpha L(x\alpha)\alpha^{-1}\ \text{, i.e.,}\quad \alpha^{-1}L(x)\alpha = L(x\alpha)\in K \tag{3.1}$$

if and only if α is an isomorphism. Assume $\alpha^{-1}L(x)\alpha = L(x')\in K$ for some $x'\in K$, then $1 = 1\alpha^{-1}$ and $1\alpha^{-1}L(x)\alpha = x\alpha = 1L(x') = x'$ and (3.1) is satisfied, i.e. , α is an automorphism.

3.2 Corollary. In every loop $(K,\cdot)$ with **(IB)**, A is a group of automorphisms of $(K,\cdot)$, i.e. the axiom **(I)** is satisfied and $(K,\cdot)$ is a left A-loop.

Proof. Let $a,b\in K$. By Lemma **1.2** it suffices to show that $\delta_{a,b}$ is an automorphism. By Lemma **2.1** and Lemma **1.3**, $\delta_{a,b}^{-1} = L(ab)L(a^\lambda)L(b^\lambda)$. Lemma **2.3.b** implies now $\delta_{a,b}K\delta_{a,b}^{-1} = L(b)L(a)L((ab)^\lambda)\ K\ L(ab)L(a^\lambda)L(b^\lambda) \subset K$ and with Theorem **3.1** the assertion follows.

3.3 Theorem. Let $(K,\cdot)$ be a Bol loop and let $a,b\in K$. Then the inner mapping $\delta_{a,b}$ is an automorphism of $(K,\cdot)$ if and only if

$$ab\cdot(a^{-1}b^{-1}) \in N_\mu = N_\lambda, \tag{3.2}$$

where $N_\mu (N_\lambda)$ denotes the middle (left) nucleus.

Proof. For $L(x)\in K$ let $\gamma := \delta_{a,b}L(x)\ \delta_{a,b}^{-1} = L(b)L(a)L(ab)^{-1}\ L(x)\ L(ab)L(a)^{-1}L(b)^{-1} \in M_\lambda$. By Theorem **3.1**, $\delta_{a,b}$ is an automorphism iff $\gamma\in K$, and by Lemma **2.3.a** $\gamma\in K$ iff $L(ab)^{-1}L(a)L(b)\gamma L(b)L(a)L(ab)^{-1} =$

$$L(ab)^{-1}L(a)L(b)^2L(a)L(ab)^{-1}\ L(x)\in K. \tag{3.3}$$

For $z \in K$, the Bol identity implies $zL(ab)^{-1}L(a)L(b)^2L(a)L(ab)^{-1} = (ab)^{-1}\cdot\big(a\{b^2[a\cdot(ab)^{-1}z]\}\big) \overset{(B)}{=}$ $(ab)^{-1}\cdot[(a\cdot b^2a)\cdot(ab)^{-1}z] \overset{(B)}{=} [(ab)^{-1}\cdot(a\cdot b^2a)(ab)^{-1}]z = zL\big((ab)^{-1}\cdot(a\cdot b^2a)(ab)^{-1}\big)$ and by (1.2) it follows that (3.3) is valid iff

$$s := (ab)^{-1}\cdot(a\cdot b^2a)(ab)^{-1} \in N_\mu. \tag{3.4}$$

With **(B)** and Lemma **1.3**, $(a\cdot b^2a)\cdot a^{-1}b^{-1} = a\cdot b^2(a\cdot a^{-1}b^{-1}) = ab$ it follows $1 = (ab)^{-1}\cdot\{(a\cdot b^2a)\cdot[(ab)^{-1}\cdot(ab)(a^{-1}b^{-1})]\} = [(ab)^{-1}\cdot(a\cdot b^2a)(ab)^{-1}]\cdot(ab)(a^{-1}b^{-1})$, i.e., $s^{-1} = ab\cdot(a^{-1}b^{-1})$. Because N_μ is a subgroup of K, $s\in N_\mu$ iff $s^{-1}\in N_\mu$. We summarize that $\delta_{a,b}$ is an automorphism iff $ab\cdot(a^{-1}b^{-1}) \in N_\mu$.

Since $A = \langle \delta_{a,b} : a, b \in K\rangle$, Theorem 3.3. implies:

3.4 Corollary. In every Bruck loop $(K,\cdot)$, A is a group of automorphisms of $(K,\cdot)$ and the axiom **(I)** is satisfied, i.e., $(K,\cdot)$ is a left A-loop.

3.5 Theorem. Bruck loop and K-loops are the same.

Proof. By Lemma **2.6** a K-loop is a Bruck loop. By [14, (2.12)] in a loop with **(I)**, **(III)** and the (left) inverse property, **(II)** and **(B)** are equivalent, hence in a loop with **(I)**, **(III)** and **(B)**, **(II)** is satisfied, i.e. by Theorem **3.3**, a Bruck loop is a K-loop.

The question whether the axioms **(II)** and **(III)** also imply **(I)** is answered to the negative by the following:

3.6 Example. Let $(R,+,\cdot)$ be an associative and commutative ring with zero element 0, with $x\cdot x = 0 = x+x$ for every $x\in R$ and with four elements p,q,r,s satisfying $pqrs\neq 0$. (For instance for $n\in\mathbb{N}$ with $n\geq 4$ let $R := \mathbb{Z}_2^{2^n-1}$ be the vector space over $\mathbb{Z}_2$ with dimension 2^n-1. We write the vectors of a basis B in the following way:

$$B = \big\{ [k_1,k_2,\dots,k_n] : k_i\in\{0,1\} \text{ for } i\in\{1,\dots,n\} \text{ and } [k_1,\dots,k_n]\neq[0,\dots,0] \big\}.$$

Let 0 be the zero vector. We define by $b\cdot 0 = 0\cdot b$ for every $b\in B$ and

$$[k_1,k_2,\dots,k_n]\cdot[\ell_1,\ell_2,\dots,\ell_n] := \begin{cases} 0 & \text{if } k_i+\ell_i = 2 \text{ for some } i\in\{1,\dots,n\} \\ [k_1+\ell_1,k_2+\ell_2,\dots,k_n+\ell_n] & \text{else} \end{cases}$$

an associative and commutative multiplication on B and extend this multiplication to a distributive multiplication of R. Then obviously $x\cdot x = 0$ and $[1,0,0,0,\dots]\cdot[0,1,0,0,\dots]\cdot[0,0,1,0,\dots]\cdot[0,0,0,1,\dots] = [1,1,1,1,\dots]\neq 0.$)

Now we define on $K := R\times R$ the following operation:

$$\oplus : R\times R\to R,\ (a_1,a_2)\oplus(b_1,b_2) = (a_1+a_2+a_1a_2b_1b_2,\ b_1+b_2) \tag{3.6}$$

Then for $a=(a_1,a_2), b=(b_1,b_2)\in K$, $(x_1,x_2)=(a_1+b_1+a_1b_1a_2b_2,\ a_2+b_2)$ is the unique solution of the equation $(a_1,a_2)\oplus(x_1,x_2)=(b_1,b_2)$ and $(0,0)$ is the zero element, i.e., $(K,\oplus)$ is a commutative loop. Every element of $K\setminus\{(0,0)\}$ has order 2, hence $(K,\oplus)$ satisfies **(III)**.

We compute that

$$(x_1,x_2)\delta_{a,b}=(x_1+a_1a_2(b_1x_2+b_2x_1)+(a_1b_2+a_2b_1)x_1x_2,\ x_2) \tag{3.7}$$

and $\delta_{a,b}=\delta_{a,b\oplus a}$, i.e., **(II)** is satisfied. But for the elements $p,q,r,s\in R$ with $pqrs\neq 0$ we have: $(p,0)\oplus\big((q,r)\oplus\big((p,0)\oplus(0,s)\big)\big)=(q+pqrs,r+s)\neq(q,r+s)=\big((p,0)\oplus\big((q,r)\oplus(p,0)\big)\big)\oplus(0,s)$, i.e., the Bol identity **(B)** is not satisfied and by Lemma **2.6** neither is **(I)**.

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