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Hyperbolic distances in Hilbert spaces

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Dedicated to János Aczél on the occasion of his 75^{th} birthday, in friendship

Summary. We present a functional equations approach to the non-negative functions $h\,(x, y)$ and $E\,(x, y)$ satisfying

$$\cosh h\,(x, y) = \sqrt{1 + x^2}\,\sqrt{1 + y^2} - xy,$$
$$E\,(x, y) = ||x - y||.$$

The underlying structure is a pre-Hilbert space X of dimension at least 2. An important tool is the group of translations

$$T_t(x) = x + \left((xe)(\cosh t - 1) + \sqrt{1 + x^2}\sinh t\right) e,$$

$t \in \mathbb{R}$, where $T_t : X \to X$ satisfies the translation equation with a fixed $e \in X$ such that $e^2 = 1$. One of the results is that a function

$$d : X \times X \to \mathbb{R}_{\geq 0} := \{r \in \mathbb{R} \mid r \geq 0\}$$

which is invariant under orthogonal mappings and the described translations for a fixed e, must be of the form

$$d\,(x, y) = g\left((h\,(x, y)\right)$$

with an arbitrary function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. If, moreover, d is additive on the line $\{\xi e \mid \xi \in \mathbb{R}\}$, then d is essentially equal to h.

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1. Suppose that X is a *pre-Hilbert space*, i.e. a real vector space equiped with an inner product

$$\sigma : X \times X \to \mathbb{R},\ \sigma\,(x, y) =: xy$$

satisfying $x^2 = xx > 0$ for all $x \neq 0$ in X. In addition we assume that the dimension of X is at least 2. Hence there exist elements e_1, e_2 of X with

$$e_1^2 = 1 = e_2^2 \text{ and } e_1 e_2 = 0. \tag{1}$$

We define the *hyperbolic distance* $h(x,y) \in \mathbb{R}$ of $x, y \in X$ by means of $h(x,y) \geq 0$ and

$$\cosh h(x,y) = \sqrt{1+x^2}\,\sqrt{1+y^2} - xy, \tag{2}$$

where cosh denotes the hyperbolic cosine. The right-hand side of (2) must be greater or equal to 1: the inequality of Cauchy–Schwarz,

$$(xy)^2 \leq x^2 y^2,$$

namely implies $(xy)^2 \leq x^2y^2 + (x-y)^2$, i.e.

$$xy + 1 \leq |xy+1| \leq \sqrt{1+x^2}\,\sqrt{1+y^2}.$$

Among the results of this note are a characterization of the function $h(x,y)$, more precisely a functional equations approach to $h(x,y)$, and, moreover, a similar approach to the euclidean distance function

$$E(x,y) := \sqrt{(x-y)^2} = \| x - y \| . \tag{3}$$

We are thus able to carry over results in [2] from $\mathbb{R}^n$ to arbitrary pre-Hilbert spaces of dimension greater than 1 (Theorems 2, 3, 4). This, however, is accomplished by developing additional methods in comparison with [2]. Especially, translation groups $\mathfrak{T}(e)$ are crucial. Moreover, the hyperbolic group $H(X)$ of X will be determined (Theorem 1) and the fundamental objects of the hyperbolic geometry of X, like hyperbolic lines, hyperbolic subspaces, spherical-hyperbolic subspaces, will be described (Theorem 5 and Propositions 2, 3, 4).

2. Let e be an element of X such that $e^2 = 1$ holds true. For $t \in \mathbb{R}$ we call the mapping

$$T_t(x) = x + \Big((xe)(\cosh t - 1) + \sqrt{1+x^2}\,\sinh t \Big) e \tag{4}$$

from X into itself a *hyperbolic translation* of X with axis e. For arbitrary y in X we denote by y_1 the real number ye. A simple calculation yields

$$1 + \big[T_t(x)\big]^2 = \Big(x_1 \sinh t + \sqrt{1+x^2}\,\cosh t \Big)^2 . \tag{5}$$

Since $x_1^2 = (xe)^2 \leq x^2 \cdot e^2 = x^2$, we have

$$0 \leq x_1^2 + \Big[1 + x^2 - x_1^2 \Big] \cosh^2 t$$

and hence $-x_1 \sinh t \leq |x_1 \sinh t| \leq \sqrt{1+x^2} \cosh t$, i.e.

$$0 \leq x_1 \sinh t + \sqrt{1+x^2} \cosh t.$$

This leads to

$$\sqrt{1+[T_t(x)]^2} = x_1 \sinh t + \sqrt{1+x^2} \cosh t, \tag{6}$$

on account of (5). A simple calculation now implies

$$\cosh h \Big(T_t(x), T_t(y)\Big) = \cosh h\,(x,y)$$

for all $x, y \in X$, and hence that hyperbolic translations with axis e preserve hyperbolic distances.

Since, by applying (6),

$$T_{t+s}(x) = T_t\Big(T_s(x)\Big)$$

holds true for all $t, s \in \mathbb{R}$ and all $x \in X$, the set of all hyperbolic translations with axis e must be a group of bijective mappings of X with respect to the permutation product. Notice that T_0 is the identity mapping, and that $T_{-t}(y)$ is the uniquely determined solution x of $T_t(x) = y$ for given $y \in X$. We denote the group of all hyperbolic translations with axis e by $\mathfrak{T}\,(e)$.

If $x, y \in X$ satisfy $y - x \in \mathbb{R}e$, then there exists exactly one $t \in \mathbb{R}$ such that

$$T_t(x) = y$$

holds true. On account of (4) and in view of

$$y - x =: \lambda e,$$

$\lambda + xe = (xe) \cosh t + \sqrt{1+x^2} \sinh t$ must be solved with respect to t. Since $(xe)^2 \leq x^2$, we define $\alpha \in \mathbb{R}$ by means of

$$xe =: a \sinh \alpha \text{ with } a \geq 1 \text{ and } a^2 := 1 + x^2 - (xe)^2.$$

Hence $\lambda + xe = a \sinh\,(t + \alpha)$ and t is thus uniquely determined.

3. We would like to define an orthogonal mapping ω of X as a surjective mapping $\omega : X \to X$ with $\omega\,(0) = 0$ and such that

$$E\Big(\omega\,(x),\, \omega\,(y)\Big) = E\,(x,y)$$

holds true for all $x, y \in X$ of euclidean distance 1 or 3. A theorem of H. Berens and the author (see, e.g., [3], 48 ff) then implies that orthogonal mappings of X are bijective and linear and that they preserve euclidean distances. (In this connection also compare E. Schröder [5]). Denote by $O(X)$ the group of all orthogonal mappings of X. If ω is in $O(X)$ then

$$E(x,0) = E\Big(\omega(x),\, 0\Big)$$

implies $x^2 = \big[\omega(x)\big]^2$ for all $x \in X$. This together with

$$E(x,y) = E\Big(\omega(x),\, \omega(y)\Big)$$

then yields $xy = \omega(x)\,\omega(y)$ for all $x, y \in X$. We hence have

$$\cosh h(x,y) = \cosh h\Big(\omega(x),\, \omega(y)\Big)$$

and thus $h(x,y) = h\Big(\omega(x),\, \omega(y)\Big)$ for all $x, y \in X$ and all $\omega \in O(X)$. This implies that all orthogonal mappings of X preserve hyperbolic distances.

A *hyperbolic isometry* of X is a mapping of X into itself such that hyperbolic distances are preserved. A hyperbolic isometry need not to be bijective. Take for instance the pre-Hilbert space X of all sequences

$$(x_1, x_2, x_3, \dots)$$

of real numbers such that almost all x_i of the sequence are 0, with the usual operations, and with the usual inner product

$$(x_1, \dots)(y_1, \dots) = \sum_{i=1}^{\infty} x_i\, y_i.$$

The mapping γ of X into itself with

$$\gamma(x_1, x_2, x_3, \dots) := (x_1, 0, x_2, 0, x_3, 0, \dots)$$

is not bijective, but it preserves hyperbolic distances.

A *hyperbolic transformation* of X is a surjective hyperbolic isometry. The group of all these transformations will be denoted by $H(X)$.

Theorem 1. *Let $e \in X$ be given with $e^2 = 1$. Then*

$$H(X) = O(X) \cdot \mathfrak{T}(e) \cdot O(X).$$

Proof. 1. If p is in X, then there exists γ in $O(X)$ with $\gamma(p) = \| p \| e$. — This is trivial in the case $p = -\| p \| e$ by just applying $\gamma(x) := -x$. Otherwise put

$$b := p + \| p \| e \text{ and } \| b \| \cdot a := b$$

and, moreover, $\gamma(x) := -x + 2(xa)\,a$. Now observe that γ is an involution and that it preserves euclidean distances.

2. Suppose that δ is in $H(X)$ and that $\delta(0) =: p$. Then there exists $\gamma \in O(X)$ with

$$\gamma\delta(0) = \| p \| \, e.$$

According to Section 2 there exists $T_t \in \mathfrak{T}(e)$ with

$$T_t \gamma\delta(0) = 0.$$

The mapping $\varphi := T_t\gamma\delta$ is bijective and it preserves hyperbolic distances. Hence

$$\cosh h(x,y) = \cosh h\Big(\varphi(x),\ \varphi(y)\Big),$$

i.e. $\sqrt{1+x^2}\,\sqrt{1+y^2} - xy = \sqrt{1+\xi^2}\,\sqrt{1+\eta^2} - \xi\eta$ with $\xi := \varphi(x)$ and $\eta := \varphi(y)$. Because of

$$h(0,z) = h\Big(0, \varphi(z)\Big)$$

we get $z^2 = \big[\varphi(z)\big]^2$ for all $z \in X$. This implies $xy = \xi\eta$ for all x, y in X. The mapping φ hence preserves euclidean distances and is thus in $O(X)$. □

4. Denote by $\mathbb{R}_{\geq 0}$ the set of all real numbers $r \geq 0$. A function $d : X \times X \to \mathbb{R}_{\geq 0}$ is called a *distance function* of X. We will say that such a distance function is of type D_1 if, and only if, the functional equation

(D_1) $d(x,y) = d\Big(\varphi(x),\ \varphi(y)\Big)$ *for all* $\varphi \in O(X)$ *and all* $x, y \in X$

holds true (see [2]). Obviously, h and E are of type D_1.

Theorem 2. *Define*

$$K := \big\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_1, \xi_2 \in \mathbb{R}_{\geq 0} \text{ and } \xi_3^2 \leq \xi_1\xi_2\big\}.$$

Suppose that $f : K \to \mathbb{R}_{\geq 0}$ *is chosen arbitrarily. Then*

$$d(x,y) = f(x^2, y^2, xy) \tag{7}$$

is a distance function of X *of type* D_1. *If, vice versa,* d *is a distance function of* X *of type* D_1, *there exists* $f : K \to \mathbb{R}_{\geq 0}$ *such that* (7) *holds true for all* $x, y \in X$.

Proof. Obviously, (7) is of type D_1. So assume that d is a distance function of X of type D_1. Suppose that (ξ_1, ξ_2, ξ_3) is in K and that $e_1, e_2 \in X$ satisfy (1). Put

$$x_0 := 0 \text{ and } y_0 := e_1\sqrt{\xi_2}$$

in the case $\xi_1 = 0$. Observe here $\xi_3 = 0$, in view of $\xi_3^2 \leq \xi_1 \xi_2$. Then define

$$f(\xi_1, \xi_2, \xi_3) := d(x_0, y_0). \tag{8}$$

In the remaining case $\xi_1 > 0$ put $x_0 := e_1 \sqrt{\xi_1}$,

$$y_0 \sqrt{\xi_1} := e_1 \xi_3 + e_2 \sqrt{\xi_1 \xi_2 - \xi_3^2}$$

and, again, (8). The function $f : K \to \mathbb{R}_{\geq 0}$ is hence defined for all elements of K. We now have to prove that (7) holds true. Let x, y be elements of X and put

$$\xi_1 := x^2,\ \xi_2 := y^2,\ \xi_3 := xy.$$

Because of the Cauchy–Schwarz inequality, (ξ_1, ξ_2, ξ_3) must be in K. If we are able to prove that there exists $\varphi \in O(X)$ with

$$\varphi(x_0) = x \text{ and } \varphi(y_0) = y, \tag{9}$$

where x_0, y_0 are the already defined elements with respect to (ξ_1, ξ_2, ξ_3), then

$$d(x, y) = d(x_0, y_0) = f(\xi_1, \xi_2, \xi_3) = f(x^2, y^2, xy)$$

holds true and (7) is established. — In order to find $\varphi \in O(X)$ with (9), we observe

$$x^2 = x_0^2,\ y^2 = y_0^2,\ xy = x_0\, y_0. \tag{10}$$

According to step 1 of the proof of Theorem 1 we may assume

$$x = x_0 \neq 0 \text{ and } y \neq y_0 \neq 0, \tag{11}$$

without loss of generality. Put $z := y - y_0$ and define

$$M := \{m \in X \mid m \perp z\}.$$

Then M is a maximal subspace of X because

$$p \in X \backslash M$$

implies $pz^2 - (pz)\, z \in M$ and hence $p \in \mathbb{R}z \oplus M$. Furthermore observe $x \in M$, in view of (10) and (11). For

$$v = \alpha z + m,\ m \in M,$$

define $\varphi(v) = -\alpha z + m$. Then $\varphi \in O(X)$ satisfies $\varphi(x) = x$, since $x \in M$, and $\varphi(y_0) = y$, in view of

$$y_0 = -\frac{1}{2} z + \frac{1}{2}(y + y_0),\ y + y_0 \perp z.$$

□

Proposition 1. *X is a metric space with respect to the distance function $h(x, y)$.*

The proof of this proposition is, mutatis mutandis, the same as that given in [2] in the case of a more specialized situation, namely $X = \mathbb{R}^n$.

Remark. Observe that X is also a metric space under the rather strange distance function

$$d(x, y) := 3 \cdot h(x, y) + 5 \cdot E(x, y)$$

(for all $x, y \in X$) which is of type D_1 as well.

5. If e is an element of X with $e^2 = 1$, then we already defined the hyperbolic translation group $\mathfrak{T}(e)$. The euclidean translation group $\mathfrak{S}(e)$ is the set of all mappings

$$S_t(x) = x + te, \; t \in \mathbb{R},$$

of X into itself.

For a distance function d define

$$\Big(\mathbf{D}_2(e, \mathrm{hyp})\Big) \;\; d(x, y) = d\Big(\tau(x), \, \tau(y)\Big) \;\text{ for all } x, y \in X \text{ and all } \tau \in \mathfrak{T}(e),$$

$$\Big(\mathbf{D}_2(e, \; \mathrm{eucl})\Big) \;\; d(x, y) = d\Big(\tau(x), \, \tau(y)\Big) \;\text{ for all } x, y \in X \text{ and all } \tau \in \mathfrak{S}(e).$$

Theorem 3. *Let $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be given. Then*

$$d(x, y) = g\Big(E(x, y)\Big) \tag{12}$$

satisfies D_1 *and* $D_2(e, \mathrm{eucl})$ *for every $e \in X$ with $e^2 = 1$. Similarly,*

$$d(x, y) = g\Big(h(x, y)\Big) \tag{13}$$

has properties D_1 *and* $D_2(e, \mathrm{hyp})$ *for all e in question. There are no other distance functions satisfying* D_1 *and* $D_2(e, \mathrm{eucl})$*,* $D_2(e, \mathrm{hyp})$, respectively, for a fixed given *e.*

Proof. a) Suppose that d satisfies D_1 and $D_2(e, \mathrm{eucl})$ for a fixed given e. If $y \in X$ is not 0, then

$$\mathfrak{S}\left(\frac{y}{\| y \|}\right) = \omega \, \mathfrak{S}(e) \, \omega^{-1}$$

for a suitable $\omega \in O(X)$. Hence

$$d(x, y) = d\Big(x + (-y), \, y + (-y)\Big) = d(x - y, 0),$$

a formula which also holds true in the case $y = 0$. Thus $d(x, y) = f\Big((x-y)^2, 0, 0\Big)$ because of Theorem 2. Define

$$g(\xi) := f(\xi^2, 0, 0)$$

for all real $\xi \geq 0$. Hence

$$d(x, y) = g\left(\sqrt{(x-y)^2}\right) = g\Big(E(x, y)\Big).$$

b) Suppose that d is a distance function satisfying D_1 and $\mathrm{D}_2\,(e,\ \text{hyp})$ for a fixed given e. We define a function

$$g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$$

as follows: for $\xi \geq 0$ put

$$g(\xi) := d(0, e \cdot \sinh \xi).$$

If $x, y \in X$, then

$$h(x, y) = h(0, e \cdot \sinh \xi)$$

in the case $\xi := h(x, y)$. Take a $\varphi_1 \in O(X)$ that transforms x in $e\sqrt{x^2}$, then a $\tau \in \mathfrak{T}(e)$ which maps this latter element into 0. With another $\varphi_2 \in O(X)$ we get

$$\varphi_2\tau\varphi_1(x) = 0 \text{ and } \varphi_2\tau\varphi_1(y) =: e\eta$$

with $\eta \geq 0$. Since

$$\xi = h(x, y) = h(0, e\eta)$$

it follows $\cosh \xi = \cosh h(0, e\eta) = \sqrt{1 + \eta^2}$, i.e. $\eta = \sinh \xi$. Hence with $\gamma := \varphi_2\tau\varphi_1$

$$d(x, y) = d\Big(\gamma(x),\ \gamma(y)\Big) = d(0, e \sinh \xi) = g(\xi) = g\Big(h(x, y)\Big).$$

□

6. A distance function d of X will be called *additive* on the *half-line*

$$l_+ := \{\lambda e \mid \lambda \geq 0\}$$

if, and only if, the following property holds true.

($\mathbf{D}_3$ (e)) *Suppose that α, β, γ are real numbers with $0 = \alpha \leq \beta \leq \gamma$. Then*

$$d(\alpha e, \gamma e) = d(\alpha e, \beta e) + d(\beta e, \gamma e). \tag{14}$$

Theorem 4. *Let $e \in X$ be an element with $e^2 = 1$ and suppose that d is a distance function of X satisfying* D_1, D_3 *(e) and* $D_2(e,\ \text{eucl})$, $D_2(e,\ \text{hyp})$, *respectively. Then*

$$d\,(x, y) = k \cdot E\,(x, y)$$

or

$$d\,(x, y) = k \cdot h\,(x, y)$$

holds true with a fixed real number $k \geq 0$.

Proof. We would like to prove that

$$g\,(\xi + \eta) = g\,(\xi) + g\,(\eta), \tag{15}$$

holds true for all non-negative real numbers ξ and η. In the euclidean case there exist $0 = \alpha \leq \beta \leq \gamma$ with

$$\xi = E\,(0, \beta e) \text{ and } \eta = E\,(\beta e, \gamma e).$$

In view of $\xi + \eta = E\,(0, \gamma e)$ this implies (15), on account of (12) and (14). Mutatis mutandis, the same argument may be applied to the hyperbolic case. Since all solutions

$$g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$$

of (15) are given by $g\,(\xi) = k\xi$, where k is a constant ≥ 0 (J. Aczél [1]), Theorem 4 is proved. □

7. The set

$$S\,(m, \varrho) := \{x \in X \mid h\,(m, x) = \varrho\} \tag{16}$$

is called the *hyperbolic hypersphere* with *center* $m \in X$ and *radius* $\varrho > 0$.

Proposition 2. *$S\,(m, \varrho)$ is the hyperellipsoid*

$$S\,(m, \varrho) = \{x \in X \mid E\,(f, x) + E\,(g, x) = 2\alpha\} \tag{17}$$

with $f := me^{-\varrho}$, $g := me^{\varrho}$ and $\alpha := \sinh \varrho \cdot \sqrt{1 + m^2}$, where e^t denotes the exponential function $\exp\,(t)$.

Proof. a) Put $S := \sinh \varrho$ and $C := \cosh \varrho$. If then

$$E\,(f, x) + E\,(g, x) = 2\alpha \tag{18}$$

holds true, a simple calculation (apply $e^{\varrho} = C + S$, $e^{-\varrho} = C - S$ and $p := x - mC$) yields

$$|mx + C| = \sqrt{1 + m^2}\,\sqrt{1 + x^2}. \tag{19}$$

If here $-mx - C$ were equal to $\sqrt{1+m^2}\sqrt{1+x^2}$, then the contradiction

$$1 \leq \cosh h\,(m, -x) = \sqrt{1+m^2}\sqrt{1+x^2} + mx = -C$$

would be the consequence. Hence (19) yields

$$\cosh h\,(m, x) = C,$$

i.e. $x \in S\,(m, \varrho)$.

b) Assume vice versa $C = \sqrt{1+m^2}\sqrt{1+x^2} - mx$. Then the simple calculation from a), but now in the other direction, leads to

$$\sqrt{(p+mS)^2}\sqrt{(p-mS)^2} = |S^2 \cdot (2+m^2) - p^2|. \tag{20}$$

In the case

$$S^2 \cdot (2+m^2) - p^2 \geq 0, \tag{21}$$

(18) is a consequence of (20). In order to prove (21) we observe

$$(mx)^2 \leq m^2x^2 + S^2$$

and $(1+x^2)(1+m^2) = (mx+C)^2$, i.e.

$$x^2 - 2\,(mx)\,C + m^2 = (mx)^2 + S^2 - m^2x^2 \leq 2S^2,$$

i.e. (21). □

Obviously, $S\,(0, \varrho)$ is a euclidean hypersphere with euclidean center 0 and euclidean radius $\sinh \varrho$. In the case $m \neq 0$ the pairwise distinct elements

$$0, f = me^{-\varrho},\ m, g = me^{\varrho}$$

are all on the euclidean half-line

$$M := \{m\sigma \mid \sigma \geq 0\}.$$

If we define

$$m\sigma_1 \textit{ before } m\sigma_2$$

if, and only if, $\sigma_1 < \sigma_2$, then

$$0 \textit{ before } f \textit{ before } m \textit{ before } g$$

holds true. Suppose that $a \neq 0$ is in X and that λ, μ are real numbers with $0 < \lambda < \mu$. Then there exists exactly one $\alpha > 0$ such that

$$\{x \in X \mid E\,(a\lambda, x) + E\,(a\mu, x) = 2\alpha\}$$

is a hyperbolic hypersphere $S(m, \varrho)$. Here, obviously, the equations

$$2\alpha = (\mu - \lambda)\sqrt{(\lambda\mu)^{-1} + a^2},$$

$m = a\sqrt{\lambda\mu}$ and $\varrho = \frac{1}{2}(\ln \mu - \ln \lambda)$ hold true.

8. We also would like to work with

$$S(m, 0) := \{x \in X \mid h(m, x) = 0\} = \{m\}.$$

If p, q are distinct elements of X, then

$$g(p, q) := \Big\{x \in X \mid S\Big(p,\, h(p, x)\Big) \cap S\Big(q,\, h(q, x)\Big) = \{x\}\Big\}$$

(see [4], 20) will be called a *hyperbolic line* of X.

Theorem 5. *All hyperbolic lines of X are given by*

$$l(a) = \{a\xi \mid \xi \in \mathbb{R}\} \text{ with } a \neq 0 \text{ in } X$$

and by

$$l(a, b) = \{a \cosh \xi + b \sinh \xi \mid \xi \in \mathbb{R}\}$$

with $a, b \in X$ and $a \neq 0$, $b^2 = 1$, $ab = 0$.

Proof. a) Suppose that $a \neq 0$ is in X and that $x \in g(0, a)$. Because of

$$\frac{2xa}{a^2}a - x \in S\Big(0, h(0, x)\Big) \cap S\Big(a, h(a, x)\Big) = \{x\}$$

we get $x \in l(a)$. Assume $\xi a \notin g(0, a)$. Hence there is an $y \neq \xi a$ with

$$(\xi a)\, a = ya \text{ and } (\xi a)^2 = y^2.$$

But

$$(ya)^2 = \xi^2 a^2 a^2 = y^2 a^2$$

implies, according to Cauchy–Schwarz, that a and y are linearly dependent, i.e. that $y = \xi a$.

b) If $g(p, q)$ is a hyperbolic line and δ a hyperbolic transformation, then, obviously,

$$\delta\Big(g(p, q)\Big) = g\Big(\delta(p),\, \delta(q)\Big)$$

and $p, q \in g(p, q)$. Take $\delta \in H(X)$ with $\delta(p) = 0$. Then

$$\delta\Big(g(p, q)\Big) = l\Big(\delta(q)\Big).$$

All hyperbolic lines of X are hence images of lines $l\,(a)$ under hyperbolic transformations. In view of Theorem 1 we hence have to determine all

$$\gamma_1 T_t \gamma_2 \Big(l\,(a) \Big)$$

with $\gamma_1, \gamma_2 \in O\,(X)$ and $T_t \in \mathfrak{T}\,(e)$. Obviously for $\gamma \in O\,(X)$,

$$\gamma\,\Big(l\,(a)\Big) = l\,\Big(\gamma\,(a)\Big),$$
$$\gamma\,\Big(l\,(a,b)\Big) = l\,\Big(\gamma\,(a),\,\gamma\,(b)\Big).$$

So it remains to determine $T_t\Big(l\,(a)\Big)$. The cases $t = 0$ or $e \in l\,(a)$ are trivial and we hence will exclude them. Let j be an element in the subspace generated by e and a such that $j^2 = 1$ and $ej = 0$. Without loss of generality assume $a =: \alpha e + j$. Then

$$T_t(\xi a) = (\xi \alpha C + S\sqrt{1 + \xi^2 a^2})\, e + \xi j$$

with $S := \sinh t \neq 0$ and $C := \cosh t > 1$. We observe that

$$\{T_t(\xi a) =: x_1(\xi)\, e + x_2(\xi)\, j \mid \xi \in \mathbb{R}\}$$

is the branch $x_1 > x_2 \alpha C$ (for $t > 0$) or the branch $x_1 < x_2 \alpha C$ (for $t < 0$) of the hyperbola with equation

$$x_1^2 - 2\alpha C x_1 x_2 + (\alpha^2 - S^2)\, x_2^2 = S^2,$$

which can be written in the form

$$\frac{y_1^2}{k} - y_2^2 = 1 \tag{22}$$

with $ka^2 := S^2$ and

$$\sqrt{\alpha^2 + C^2} \cdot (y_1\, y_2) = (x_1\, x_2) \begin{pmatrix} C & \alpha \\ -\alpha & C \end{pmatrix}.$$

But the branches of (22) are exactly hyperbolic lines $l\,(v,w)$.

c) Suppose that a, b are elements of X with $a \neq 0$, $b^2 = 1$, $ab = 0$. Define $t \in \mathbb{R}$ by $\sinh t = 1$. For T_t in $\mathfrak{T}\,(b)$ we then have

$$T_t(\xi b) = \xi b + \sqrt{1 + \xi^2} a,$$

i.e. $l\,(a,b) = T_t\Big(l\,(b)\Big)$. □

A hyperbolic line $l\,(a,b)$ never contains 0, since a,b are linearly independent. On the basis of this information it is easy to prove that through two distinct elements p,q of X there is exactly one hyperbolic line: without loss of generality we may assume that $p=0$. But then there is only the line $l\,(q)$ through p and q.

The nearest element of $l\,(a,b)$ to 0, from the euclidean point of view (and also from the hyperbolic point of view), is the element a, and it is a vertex of the underlying hyperbola of $l\,(a,b)$ as well. The other vertex is $-a$, and the foci of the hyperbola in question are

$$\pm\frac{a}{\parallel a \parallel}\sqrt{(a+b)^2}=\pm a\sqrt{1+\frac{1}{a^2}}.$$

The asymptotes are $l\,(a+b)$ and $l\,(a-b)$. It is then easy to prove that $l\,(a,b)=l\,(c,d)$ holds true if, and only if, $a=c$ and $b=\pm d$.

A hyperbolic line $l\,(a)$ can be written in the form

$$l\,(a)=\{0\cdot\cosh\xi+a\cdot\sinh\xi \mid \xi\in\mathbb{R}\}.$$

We thus have formally $l\,(a)=l\,(0,a)$. This is the reason that all hyperbolic lines are of the form

$$l\,(a,b)=\{a\cosh\xi+b\sinh\xi \mid \xi\in\mathbb{R}\}$$

with elements $a,b\in X$ such that $b^2=1$ and $ab=0$ hold true. b is a tangent vector in $\xi=0$, i.e. in a and a will be called the *vertex* of $l\,(a,b)$, even in the case $a=0$. If we determine the hyperbolic distance of $x\,(\alpha)$ and $x\,(\beta)$, where

$$x\,(\xi)=a\cosh\xi+b\sinh\xi, \tag{23}$$

we get

$$h\Big(x\,(\alpha),\;x\,(\beta)\Big)=|\beta-\alpha|. \tag{24}$$

In order to find the hyperbolic line $l\,(a,b)$ through the elements $p\neq q$ of X we proceed as follows: if p,q are linearly dependent, then $l\,(0,b)$ is this line with $0\neq c\in\{p,q\}$, $\parallel c\parallel\cdot b:=c$. In the case that p,q are linearly independent, we have, in view of (24),

$$p=a\cosh\xi+b\sinh\xi, \tag{25}$$

$$q=a\cosh(\xi+\varrho)+b\sinh\,(\xi+\varrho) \tag{26}$$

with $\varrho=h\,(p,q)$. (We could also work with $\varrho=-h\,(p,q)$.) This implies

$$a\sinh\varrho=p\sinh\,(\xi+\varrho)-q\sinh\xi, \tag{27}$$

$$b\sinh\varrho=-p\cosh\,(\xi+\varrho)+q\cosh\xi. \tag{28}$$

Now $ab = 0$ yields

$$0 = p^2 \sinh(2\xi + 2\varrho) - 2pq \sinh(2\xi + \varrho) + q^2 \sinh 2\xi,$$

i.e.

$$4\xi = \ln(pe^{-\varrho} - q)^2 - \ln(pe^{\varrho} - q)^2.$$

Knowing in this way ϱ and ξ, (27), (28) lead to a, b, since $\varrho \neq 0$.

A *hyperbolic spear* is an oriented hyperbolic line. If we agree that b has in a the orientation of the curve (23), then $l(a,b)$ may serve as representation of this spear. The other spear then would be $l(a,-b)$.

If $p \neq q$ are elements of X, then the *hyperbolic segment* $[p,q]$ is defined by means of

$$[p,q] := \{x(\eta) \mid \xi \leq \eta \leq \xi + \varrho\},$$

where we observe (23), (25), (26) and $\varrho > 0$.

If we have $p \in l(a,b)$ with (23), (25), then

$$\{x(\eta) \mid \eta \geq \xi\} \text{ and } \{x(\eta) \mid \eta \leq \xi\}$$

are called the *hyperbolic half-lines* of $l(a,b)$ with *starting point* p.

The theory of *hyperbolic angles* for X may now be developed as we did it in our book [4], Section 3.3.

A *hyperbolic subspace* of X is a set $M \subseteq X$ such that for all $p \neq q$ in M the line $g(p,q)$ is a subset of M. Of course, $\emptyset$ and M are subspaces, also every single element of X, but hyperbolic lines as well. Since every hyperbolic line is contained in a one- or two-dimensional linear subspace of the vector space X, the following Proposition must hold true.

Proposition 3. *All hyperbolic subspaces of X are given by the linear subspaces of X and their images under hyperbolic transformations of X.*

A *spherical-hyperbolic subspace* is a set

$$M \cap S(m, \varrho),$$

where M is a hyperbolic subspace containing m. Without loss of generality we may assume $m = 0$. Hence the following Proposition holds true.

Proposition 4. *All spherical-hyperbolic subspaces of X are given by the spherical-euclidean subspaces of X with center 0 and their images under hyperbolic transformations of X.*

Remark. Similar expressions, as those for hyperbolic lines, may be derived for other hyperbolic subspaces. Again, the images of such subspaces (through 0) under mappings T_t are crucial for this purpose.

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