Our fourth and final task remains to determine A_n and B_n. At $t = 0$,

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \tag{7.3.20}$$

and

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin\left(\frac{n\pi x}{L}\right) = g(x). \tag{7.3.21}$$

Both of these series are Fourier half-range sine expansions over the interval $(0, L)$. Applying the results from Section 2.3,

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \tag{7.3.22}$$

and

$$\frac{n\pi c}{L}B_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \tag{7.3.23}$$

or

$$B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx. \tag{7.3.24}$$

As an example, let us take the initial conditions:

$$f(x) = \begin{cases} 0, & 0 < x \le L/4 \\ 4h\left(\frac{x}{L} - \frac{1}{4}\right), & L/4 \le x \le L/2 \\ 4h\left(\frac{3}{4} - \frac{x}{L}\right), & L/2 \le x \le 3L/4 \\ 0, & 3L/4 \le x < L \end{cases} \tag{7.3.25}$$

and

$$g(x) = 0, \qquad 0 < x < L. \tag{7.3.26}$$

In this particular example, $B_n = 0$ for all n because $g(x) = 0$. On the other hand,

$$A_n = \frac{8h}{L}\int_{L/4}^{L/2}\left(\frac{x}{L} - \frac{1}{4}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$
$$+ \frac{8h}{L}\int_{L/2}^{3L/4}\left(\frac{3}{4} - \frac{x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx \tag{7.3.27}$$
$$= \frac{8h}{n^2\pi^2}\left[2\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{3n\pi}{4}\right) - \sin\left(\frac{n\pi}{4}\right)\right] \tag{7.3.28}$$
$$= \frac{8h}{n^2\pi^2}\left[2\sin\left(\frac{n\pi}{2}\right) - 2\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{4}\right)\right] \tag{7.3.29}$$
$$= \frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\left[1 - \cos\left(\frac{n\pi}{4}\right)\right] \tag{7.3.30}$$
$$= \frac{32h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin^2\left(\frac{n\pi}{8}\right), \tag{7.3.31}$$

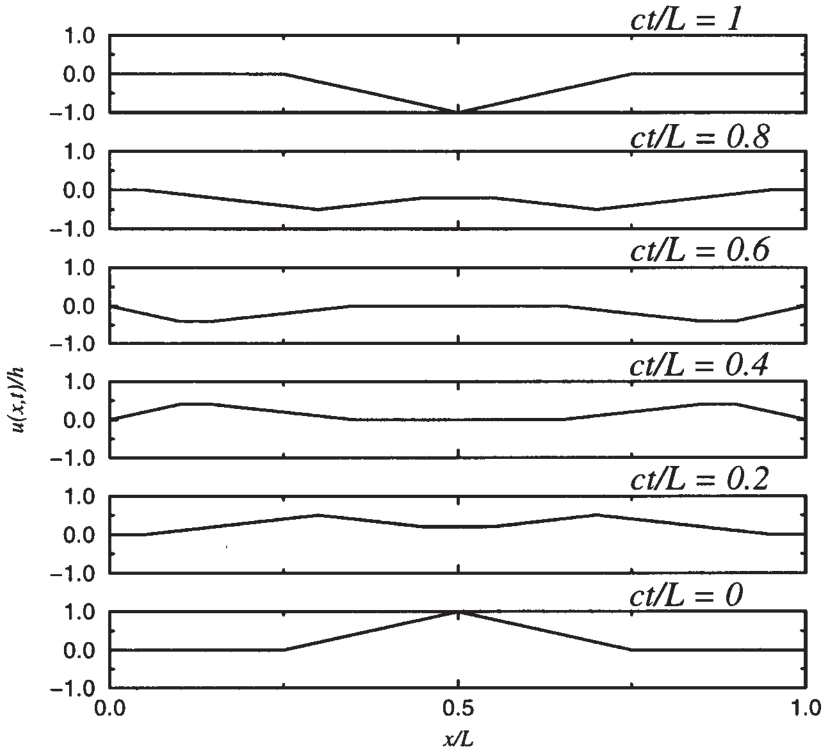

Figure 7.3.1: The vibration of a string $u(x,t)/h$ at various positions x/L at the times $ct/L = 0, 0.2, 0.4, 0.6, 0.8$, and 1. For times $1 < ct/L < 2$ the pictures appear in reverse order.

because $\sin(A)+\sin(B) = 2\sin[\frac{1}{2}(A+B)]\cos[\frac{1}{2}(A-B)]$ and $1-\cos(2A) = 2\sin^2(A)$. Therefore,

$$u(x,t) = \frac{32h}{\pi^2}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right)\sin^2\left(\frac{n\pi}{8}\right)\frac{1}{n^2}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right). \tag{7.3.32}$$

Because $\sin(n\pi/2)$ vanishes for n even, so does A_n. If (7.3.32) were evaluated on a computer, considerable time and effort would be wasted. Consequently it is preferable to rewrite (7.3.32) so that we eliminate these vanishing terms. The most convenient method introduces the general expression $n = 2m-1$ for any odd integer, where $m = 1, 2, 3, \ldots$, and notes that $\sin[(2m-1)\pi/2] = (-1)^{m+1}$. Therefore, (7.3.32) becomes

$$u(x,t) = \frac{32h}{\pi^2}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^2}\sin^2\left[\frac{(2m-1)\pi}{8}\right]$$

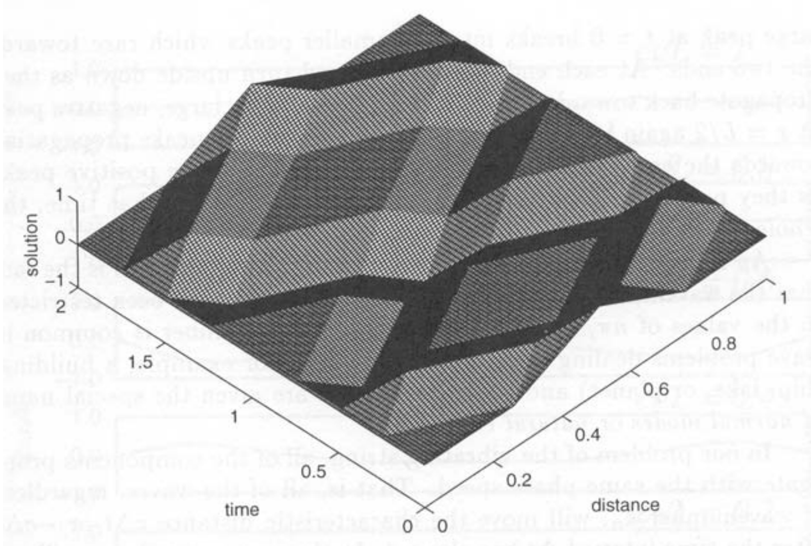

Figure 7.3.2: Two-dimensional plot of the vibration of a string $u(x,t)/h$ at various times ct/L and positions x/L.

$$\times \sin\left[\frac{(2m-1)\pi x}{L}\right] \cos\left[\frac{(2m-1)\pi ct}{L}\right]. \quad \textbf{(7.3.33)}$$

Although we have completely solved the problem, it is useful to rewrite (7.3.33) as

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} A_n \left\{ \sin\left[\frac{n\pi}{L}(x-ct)\right] + \sin\left[\frac{n\pi}{L}(x+ct)\right] \right\} \quad \textbf{(7.3.34)}$$

through the application of the trigonometric identity $\sin(A)\cos(B) = \frac{1}{2}\sin(A-B) + \frac{1}{2}\sin(A+B)$. From general physics we find expressions like $\sin[k_n(x-ct)]$ or $\sin(kx-\omega t)$ arising in studies of simple wave motions. The quantity $\sin(kx-\omega t)$ is the mathematical description of a propagating wave in the sense that we must move to the right at the speed c if we wish to keep in the same position relative to the nearest crest and trough. The quantities k, ω, and c are the wavenumber, frequency, and phase speed or wave-velocity, respectively. The relationship $\omega = kc$ holds between the frequency and phase speed.

It may seem paradoxical that we are talking about traveling waves in a problem dealing with waves confined on a string of length L. Actually we are dealing with standing waves because at the same time that a wave is propagating to the right its mirror image is running to the left so that there is no resultant progressive wave motion. Figures 7.3.1 and 7.3.2 illustrate our solution; Figure 7.3.1 gives various cross sections of the continuous solution plotted in Figure 7.3.2. The single large peak at $t = 0$ breaks into two smaller peaks which race towards the two ends. At each end, they reflect and turn upside down as they propagate back towards $x = L/2$ at $ct/L = 1$. This large, negative peak at $x = L/2$ again breaks apart, with the two smaller peaks propagating towards the endpoints. They reflect and again become positive peaks as they propagate back to $x = L/2$ at $ct/L = 2$. After that time, the whole process repeats itself.

An important dimension to the vibrating string problem is the fact that the wavenumber k_n is not a free parameter but has been restricted to the values of $n\pi/L$. This restriction on wavenumber is common in wave problems dealing with limited domains (for example, a building, ship, lake, or planet) and these oscillations are given the special name of *normal modes* or *natural vibrations.*

In our problem of the vibrating string, all of the components propagate with the same phase speed. That is, all of the waves, regardless of wavenumber k_n, will move the characteristic distance $c\Delta t$ or $-c\Delta t$ after the time interval Δt has elapsed. In the next example we will see that this is not always true.

• Example 7.3.2: Dispersion

In the preceding example, the solution to the vibrating string problem consisted of two simple waves, each propagating with a phase speed c to the right and left. In problems where the equations of motion are a little more complicated than (7.3.1), all of the harmonics no longer propagate with the same phase speed but at a speed that depends upon the wavenumber. In such systems the phase relation varies between the harmonics and these systems are referred to as *dispersive.*

A modification of the vibrating string problem provides a simple illustration. We now subject each element of the string to an additional applied force which is proportional to its displacement:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - hu, \quad 0 < x < L, 0 < t, \tag{7.3.35}$$

where $h > 0$ is constant. For example, if we embed the string in a thin sheet of rubber, then in addition to the restoring force due to tension, there will be a restoring force due to the rubber on each portion of the string. From its use in the quantum mechanics of "scalar" mesons, (7.3.35) is often referred to as the *Klein-Gordon* equation.

We shall again look for particular solutions of the form $u(x,t) = X(x)T(t)$. This time, however,

$$XT'' - c^2 X'' T + hXT = 0 \tag{7.3.36}$$

or

$$\frac{T''}{c^2 T} + \frac{h}{c^2} = \frac{X''}{X} = -\lambda, \tag{7.3.37}$$

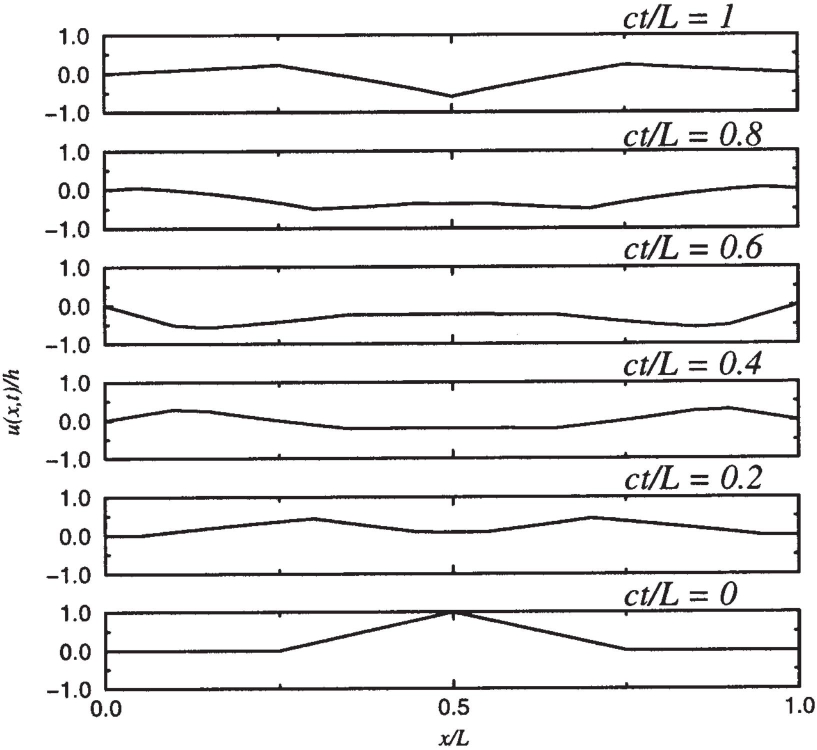

Figure 7.3.3: The vibration of a string $u(x,t)/h$ embedded in a thin sheet of rubber at various positions x/L at the times $ct/L = 0$, 0.2, 0.4, 0.6, 0.8, and 1 for $hL^2/c^2 = 10$. The same parameters were used as in Figure 7.3.1.

which leads to two ordinary differential equations

$$X'' + \lambda X = 0 \tag{7.3.38}$$

and

$$T'' + (\lambda c^2 + h)T = 0. \tag{7.3.39}$$

If we attach the string at $x = 0$ and $x = L$, the $X(x)$ solution is

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right) \tag{7.3.40}$$

with $k_n = n\pi/L$ and $\lambda_n = n^2\pi^2/L^2$. On the other hand, the $T(t)$ solution becomes

$$T_n(t) = A_n \cos\left(\sqrt{k_n^2 c^2 + h}\, t\right) + B_n \sin\left(\sqrt{k_n^2 c^2 + h}\, t\right) \tag{7.3.41}$$

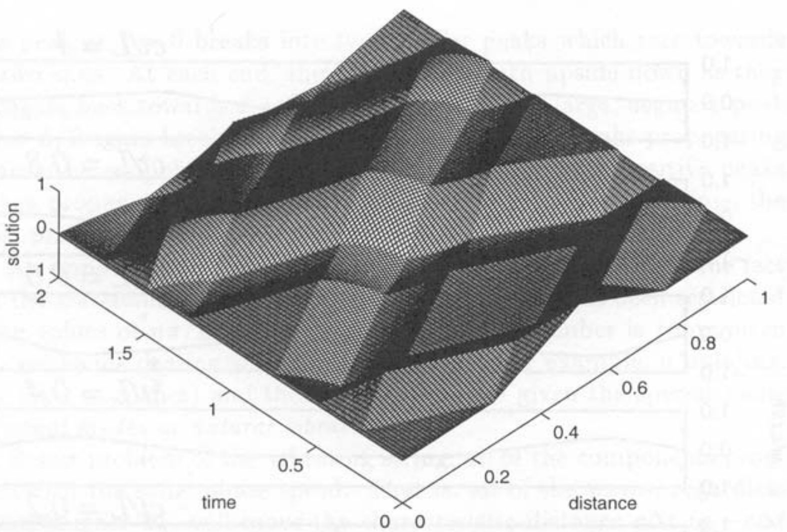

Figure 7.3.4: The two-dimensional plot of the vibration of a string $u(x,t)/h$ embedded in a thin sheet of rubber at various times ct/L and positions x/L for $hL^2/c^2 = 10$.

so that the product solution is

$$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\sqrt{k_n^2c^2 + h}\,t\right) + B_n \sin\left(\sqrt{k_n^2c^2 + h}\,t\right)\right]. \tag{7.3.42}$$

Finally, the general solution becomes

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\sqrt{k_n^2c^2 + h}\,t\right) + B_n \sin\left(\sqrt{k_n^2c^2 + h}\,t\right)\right] \tag{7.3.43}$$

from the principle of linear superposition. Let us consider the case when $B_n = 0$. Then we can write (7.3.43)

$$u(x,t) = \sum_{n=1}^{\infty} \frac{A_n}{2}\left[\sin\left(k_n x + \sqrt{k_n^2c^2 + h}\,t\right) + \sin\left(k_n x - \sqrt{k_n^2c^2 + h}\,t\right)\right]. \tag{7.3.44}$$

Comparing our results with (7.3.34), the distance that a particular mode k_n moves during the time interval Δt depends not only upon external parameters such as h, the tension and density of the string, but also upon its wavenumber (or equivalently, wavelength). Furthermore, the frequency of a particular harmonic is larger than that when $h = 0$.

This result is not surprising, because the added stiffness of the medium should increase the natural frequencies.

The importance of dispersion lies in the fact that if the solution $u(x,t)$ is a superposition of progressive waves in the same direction, then the phase relationship between the different harmonics will change with time. Because most signals consist of an infinite series of these progressive waves, dispersion causes the signal to become garbled. We show this by comparing the solution (7.3.43) given in Figures 7.3.3 and 7.3.4 for the initial conditions (7.3.25) and (7.3.26) with $hL^2/c^2 = 10$ to the results given in Figures 7.3.1 and 7.3.2. Note how garbled the picture becomes at $ct/L = 2$ in Figure 7.3.4 compared to the nondispersive solution at the same time in Figure 7.3.2.

• Example 7.3.3: Damped wave equation

In the previous example a slight modification of the wave equation resulted in a wave solution where each Fourier harmonic propagates with its own particular phase speed. In this example we introduce a modification of the wave equation that will result not only in dispersive waves but also in the exponential decay of the amplitude as the wave propagates.

So far we have neglected the reaction of the surrounding medium (air or water, for example) on the motion of the string. For small-amplitude motions this reaction opposes the motion of each element of the string and is proportional to the element's velocity. The equation of motion, when we account for the tension and friction in the medium but not its stiffness or internal friction, is

$$\frac{\partial^2 u}{\partial t^2} + 2h\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, 0 < t. \tag{7.3.45}$$

Because (7.3.45) first arose in the mathematical description of the telegraph,[4] it is generally known as the *equation of telegraphy*. The effect of friction is, of course, to damp out the free vibration.

Let us assume a solution of the form $u(x,t) = X(x)T(t)$ and separate the variables to obtain the two ordinary differential equations:

$$X'' + \lambda X = 0 \tag{7.3.46}$$

and

$$T'' + 2hT' + \lambda c^2 T = 0 \tag{7.3.47}$$

[4] The first published solution was by Kirchhoff, G., 1857: Über die Bewegung der Electrität in Drähten. *Ann. Phys. Chem.*, **100**, 193–217. English translation: Kirchhoff, G., 1857: On the motion of electricity in wires. *Philos. Mag., Ser. 4*, **13**, 393–412.

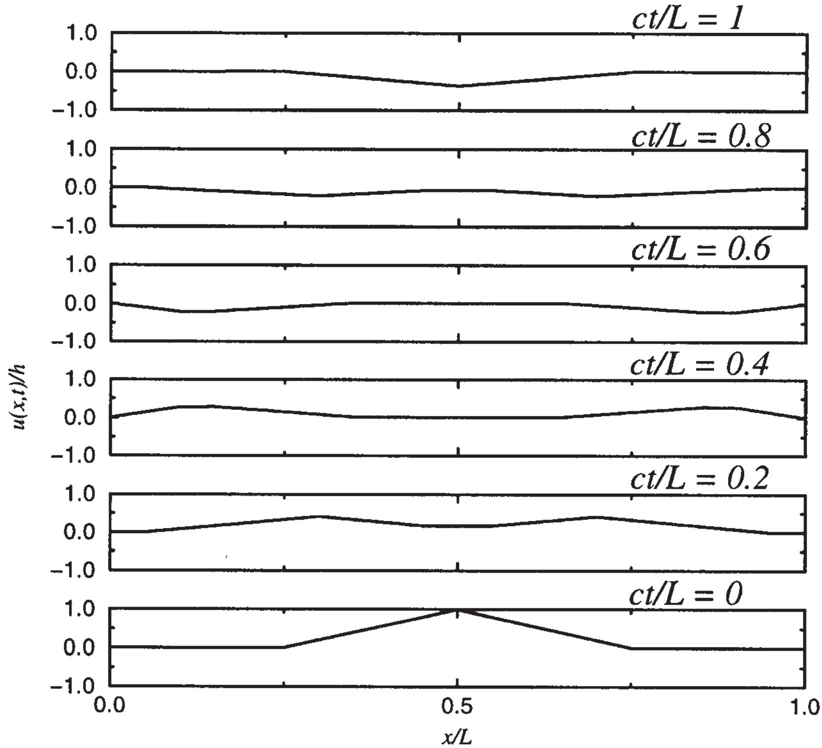

Figure 7.3.5: The vibration of a string $u(x,t)/h$ with frictional dissipation at various positions x/L at the times $ct/L = 0$, 0.2, 0.4, 0.6, 0.8, and 1 for $hL/c = 1$. The same parameters were used as in Figure 7.3.1.

with $X(0) = X(L) = 0$. Friction does not affect the shape of the normal modes; they are still

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right) \tag{7.3.48}$$

with $k_n = n\pi/L$ and $\lambda_n = n^2\pi^2/L^2$.

The solution for the $T(t)$ equation is

$$T_n(t) = e^{-ht}\left[A_n \cos\left(\sqrt{k_n^2c^2 - h^2}\,t\right) + B_n \sin\left(\sqrt{k_n^2c^2 - h^2}\,t\right)\right] \tag{7.3.49}$$

with the condition that $k_n c > h$. If we violate this condition, the solutions are two exponentially decaying functions in time. Because most physical problems usually fulfill this condition, we will concentrate on this solution.

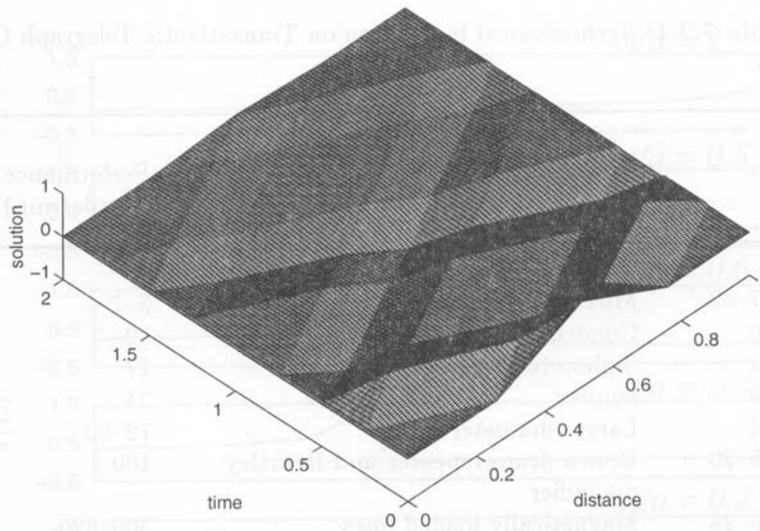

Figure 7.3.6: The vibration of a string $u(x,t)/h$ with frictional dissipation at various times ct/L and positions x/L for $hL/c = 1$.

From the principle of linear superposition, the general solution is

$$u(x,t) = e^{-ht} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[A_n \cos\left(\sqrt{k_n^2 c^2 - h^2}\, t\right) + B_n \sin\left(\sqrt{k_n^2 c^2 - h^2}\, t\right)\right], \quad (7.3.50)$$

where $\pi c > hL$. From (7.3.50) we see two important effects. First, the presence of friction slows all of the harmonics. Furthermore, friction dampens all of the harmonics. Figures 7.3.5 and 7.3.6 illustrate the solution using the initial conditions given by (7.3.25) and (7.3.26) with $hL/c = 1$. This is a rather large coefficient of friction and these figures show the rapid damping that results with a small amount of dispersion.

This damping and dispersion of waves also occurs in solutions of the equation of telegraphy where the solutions are progressive waves. Because early telegraph lines were short, time delay effects were negligible. However, when engineers laid the first transoceanic cables in the 1850s, the time delay became seconds and differences in the velocity of propagation of different frequencies, as predicted by (7.3.50), became noticeable to the operators. Table 7.3.1 gives the transmission rate for various transatlantic submarine telegraph lines. As it shows, increases in the transmission rates during the nineteenth century were due primarily to improvements in terminal technology.

When they instituted long-distance telephony just before the turn of the twentieth century, this difference in velocity between frequencies

Table 7.3.1: Technological Innovation on Transatlantic Telegraph Cables.

Year	Technological Innovation	Performance (words/min)
1857–58	Mirror galvanometer	3–7
1870	Condensers	12
1872	Siphon recorder	17
1879	Duplex	24
1894	Larger diameter cable	72–90
1915–20	Brown drum repeater and Heurtley magnifier	100
1923–28	Magnetically loaded lines	300–320
1928–32	Electronic signal shaping amplifiers and time division multiplexing	480
1950	Repeaters on the continental shelf	100–300
1956	Repeatered telephone cables	21600

From Coates, V. T. and Finn, B., 1979: *A Retrospective Technology Assessment: Submarine Telegraphy. The Transatlantic Cable of 1866,* San Francisco Press, Inc.

should have limited the circuits to a few tens of miles.[5] However, in 1899, Prof. Michael Pupin, at Columbia University, showed that by adding inductors ("loading coils") to the line at regular intervals the velocities at the different frequencies could be equalized.[6] Heaviside[7] and the French engineer Vaschy[8] made similar suggestions in the nineteenth century. Thus, adding resistance and inductance, which would seem to make things worse, actually made possible long-distance telephony. Today

[5] Rayleigh, J. W., 1884: On telephoning through a cable. *Br. Assoc. Rep.*, 632–633; Jordan, D. W., 1982: The adoption of self-induction by telephony, 1886–1889. *Ann. Sci.*, **39**, 433–461.

[6] There is considerable controversy on this subject. See Brittain, J. E., 1970: The introduction of the loading coil: George A. Campbell and Michael I. Pupin. *Tech. Culture*, **11**, 36–57.

[7] First published 3 June 1887. Reprinted in Heaviside, O., 1970: *Electrical Papers, Vol. II*, Chelsea Publishing, Bronx, NY, pp. 119–124.

[8] See Devaux-Charbonnel, X. G. F., 1917: La contribution des ingénieurs français à la téléphonie à grande distance par câbles souterrains: Vaschy et Barbarat. *Rev. Gén. Électr.*, **2**, 288–295.

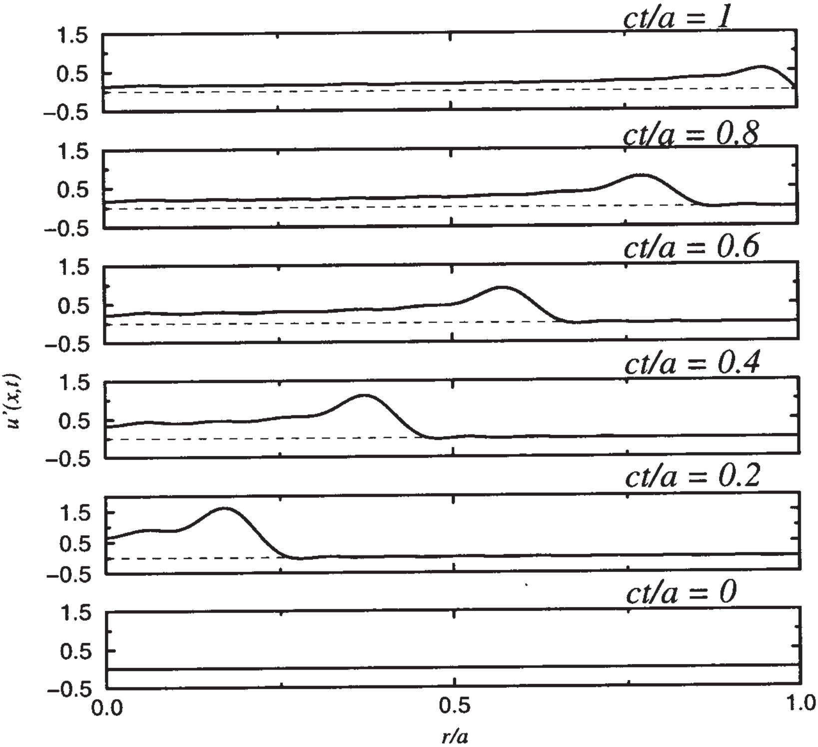

Figure 7.3.7: The axisymmetric vibrations $u'(r,t) = ca\rho u(r,t)/P$ of a circular membrane when struck by a hammer at various positions r/a at the times ct/a = 0, 0.2, 0.4, 0.6, 0.8, and 1 for $\epsilon = a/4$.

you can see these loading coils as you drive along the street; they are the black cylinders, approximately one between each pair of telephone poles, spliced into the telephone cable. The loading of long submarine telegraph cables had to wait for the development of permalloy and mu-metal materials of high magnetic induction.

• **Example 7.3.4: Axisymmetric vibrations of a circular membrane**

The wave equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \quad 0 \le r < a, 0 < t \tag{7.3.51}$$

governs axisymmetric vibrations of a circular membrane, where $u(r,t)$ is the vertical displacement of the membrane, r is the radial distance, t is time, c is the square root of the ratio of the tension of the membrane to its density, and a is the radius of the membrane. We shall solve (7.3.51)

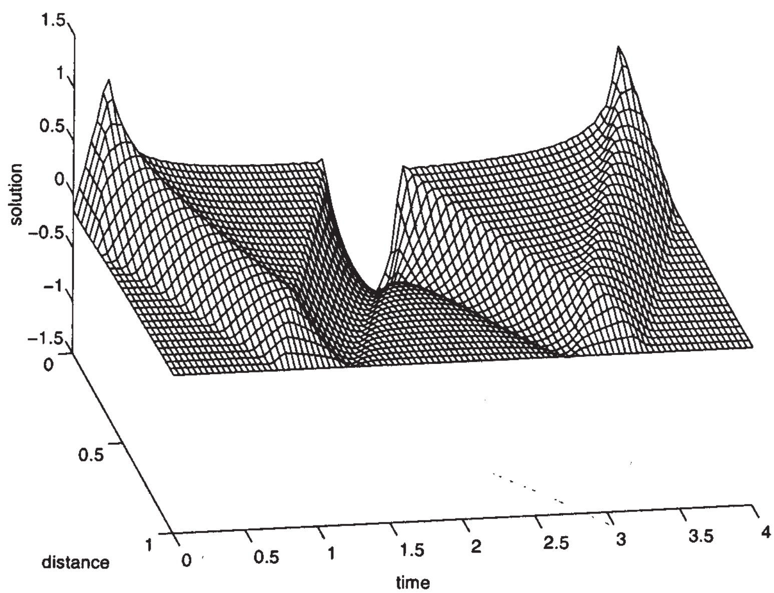

Figure 7.3.8: The axisymmetric vibrations $ca\rho u(r,t)/P$ of a circular membrane when struck by a hammer at various times ct/a and positions r/a for $\epsilon = a/4$.

when the membrane is initially at rest, $u(r,0) = 0$, and struck so that its initial velocity is

$$\frac{\partial u(r,0)}{\partial t} = \begin{cases} P/(\pi\epsilon^2\rho), & 0 \le r < \epsilon \\ 0, & \epsilon < r < a. \end{cases} \tag{7.3.52}$$

If this problem can be solved by separation of variables, then $u(r,t) = R(r)T(t)$. Following the substitution of this $u(r,t)$ into (7.3.51), separation of variables leads to

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \frac{1}{c^2T}\frac{d^2T}{dt^2} = -k^2 \tag{7.3.53}$$

or

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + k^2R = 0 \tag{7.3.54}$$

and

$$\frac{d^2T}{dt^2} + k^2c^2T = 0. \tag{7.3.55}$$

The separation constant $-k^2$ must be negative so that we obtain solutions that remain bounded in the region $0 \le r < a$ and can satisfy the boundary condition. This boundary condition is $u(a,t) = R(a)T(t) = 0$ or $R(a) = 0$.

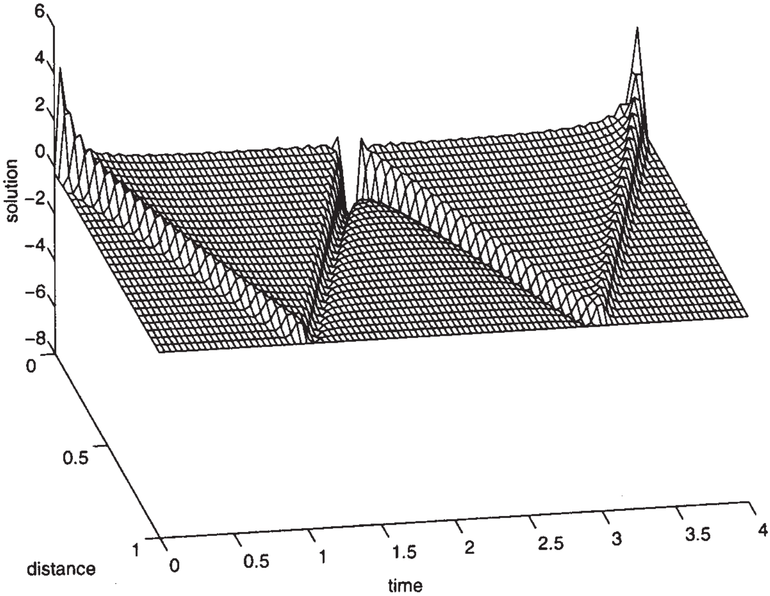

Figure 7.3.9: Same as Figure 7.3.8 except $\epsilon = a/20$.

The solutions of (7.3.54)–(7.3.55), subject to the boundary condition, are

$$R_n(r) = J_0\left(\frac{\lambda_n r}{a}\right) \tag{7.3.56}$$

and

$$T_n(t) = A_n \sin\left(\frac{\lambda_n ct}{a}\right) + B_n \cos\left(\frac{\lambda_n ct}{a}\right), \tag{7.3.57}$$

where λ_n satisfies the equation $J_0(\lambda) = 0$. Because $u(r,0) = 0$ and $T_n(0) = 0$, $B_n = 0$. Consequently, the product solution is

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\lambda_n r}{a}\right) \sin\left(\frac{\lambda_n ct}{a}\right). \tag{7.3.58}$$

To determine A_n, we use the condition

$$\frac{\partial u(r,0)}{\partial t} = \sum_{n=1}^{\infty} \frac{\lambda_n c}{a} A_n J_0\left(\frac{\lambda_n r}{a}\right) = \begin{cases} P/(\pi\epsilon^2\rho), & 0 \le r < \epsilon \\ 0, & \epsilon < r < a. \end{cases} \tag{7.3.59}$$

Equation (7.3.59) is a Fourier-Bessel expansion in the orthogonal function $J_0(\lambda_n r/a)$, where A_n equals

$$\frac{\lambda_n c}{a} A_n = \frac{2}{a^2 J_1^2(\lambda_n)} \int_0^{\epsilon} \frac{P}{\pi\epsilon^2\rho} J_0\left(\frac{\lambda_n r}{a}\right) dr \tag{7.3.60}$$

from (6.5.35) and (6.5.43) in Section 6.5. Carrying out the integration,

$$A_n = \frac{2PJ_1(\lambda_n\epsilon/a)}{c\pi\epsilon\rho\lambda_n^2 J_1^2(\lambda_n)} \tag{7.3.61}$$

or

$$u(r,t) = \frac{2P}{c\pi\epsilon\rho}\sum_{n=1}^{\infty}\frac{J_1(\lambda_n\epsilon/a)}{\lambda_n^2 J_1^2(\lambda_n)}J_0\left(\frac{\lambda_n r}{a}\right)\sin\left(\frac{\lambda_n ct}{a}\right). \tag{7.3.62}$$

Figures 7.3.7, 7.3.8, and 7.3.9 illustrate the solution (7.3.62) for various times and positions when $\epsilon = a/4$ and $\epsilon = a/20$. Figures 7.3.8 and 7.3.9 show that striking the membrane with a hammer generates a pulse that propagates out to the rim, reflects, inverts, and propagates back to the center. This process then repeats itself forever.

Problems

Solve the wave equation $u_{tt} = c^2 u_{xx}$, $0 < x < L$, $0 < t$ subject to the boundary conditions that $u(0,t) = u(L,t) = 0, t < 0$ and the following initial conditions for $0 < x < L$:

1. $u(x,0) = 0, \quad u_t(x,0) = 1$

2. $u(x,0) = 1, \quad u_t(x,0) = 0$

3. $u(x,0) = \begin{cases} 3hx/2L, & 0 < x < 2L/3 \\ 3h(L-x)/L, & 2L/3 < x < L, \end{cases} \quad u_t(x,0) = 0$

4. $u(x,0) = [3\sin(\pi x/L) - \sin(3\pi x/L)]/4, \quad u_t(x,0) = 0,$

5. $u(x,0) = \sin(\pi x/L), \quad u_t(x,0) = \begin{cases} 0, & 0 < x < L/4 \\ a, & L/4 < x < 3L/4 \\ 0, & 3L/4 < x < L \end{cases}$

6. $u(x,0) = 0, \quad u_t(x,0) = \begin{cases} ax/L, & 0 < x < L/2 \\ a(L-x)/L, & L/2 < x < L \end{cases}$

7. $u(x,0) = \begin{cases} x, & 0 < x < L/2 \\ L-x, & L/2 < x < L, \end{cases} \quad u_t(x,0) = 0$

8. Solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \pi, 0 < t$$

subject to the boundary conditions

$$\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(\pi,t)}{\partial x} = 0, \qquad 0 < t$$

and the initial conditions

$$u(x,0) = 0 \quad \text{and} \quad \frac{\partial u(x,0)}{\partial t} = 1 + \cos^3(x), \qquad 0 < x < \pi.$$

[Hint: You must include the separation constant of zero.]

9. The differential equation for the longitudinal vibrations of a rod within a viscous fluid is

$$\frac{\partial^2 u}{\partial t^2} + 2h\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, 0 < t,$$

where c is the velocity of sound in the rod and h is the damping coefficient. If the rod is fixed at $x = 0$ so that $u(0,t) = 0$ and allowed to freely oscillate at the other end $x = L$ so that $u_x(L,t) = 0$, find the vibrations for any location x and subsequent time t if the rod has the initial displacement of $u(x,0) = x$ and the initial velocity $u_t(x,0) = 0$ for $0 < x < L$. Assume that $h < c\pi/(2L)$. Why?

10. A closed pipe of length L contains air whose density is slightly greater than that of the outside air in the ratio of $1+s_0$ to 1. Everything being at rest, we suddenly draw aside the disk closing one end of the pipe. We want to determine what happens *inside* the pipe after we remove the disk.

As the air rushes outside, it generates sound waves within the pipe. The wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

governs these waves, where c is the speed of sound and $u(x,t)$ is the velocity potential. Without going into the fluid mechanics of the problem, the boundary conditions are

a. No flow through the closed end: $u_x(0,t) = 0$.

b. No infinite acceleration at the open end: $u_{xx}(L,t) = 0$.

c. Air is initially at rest: $u_x(x,0) = 0$.

d. Air initially has a density greater than the surrounding air by the amount s_0: $u_t(x,0) = -c^2 s_0$.

Find the velocity potential at all positions within the pipe and all subsequent times.

11. One of the classic applications of the wave equation has been the explanation of the acoustic properties of string instruments. Usually we excite a string in one of three ways: by plucking (as in the harp, zither, etc.), by striking with a hammer (piano), or by bowing (violin, violoncello, etc.). In all these case, the governing partial differential equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with the boundary conditions $u(0,t) = u(L,t) = 0$, $0 < t$. For each of following methods of exciting a string instrument, find the complete solution to the problem:

(a) Plucked string

For the initial conditions:

$$u(x,0) = \begin{cases} \beta x/a, & 0 < x < a \\ \beta(L-x)/(L-a), & a < x < L \end{cases}$$

and

$$u_t(x,0) = 0, \qquad 0 < x < L,$$

show that

$$u(x,t) = \frac{2\beta L^2}{\pi^2 a(L-a)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi a}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right)$$

We note that the harmonics are absent where $\sin(n\pi a/L) = 0$. Thus, if we pluck the string at the center, all of the harmonics of even order will be absent. Furthermore, the intensity of the successive harmonics will vary as n^{-2}. The higher harmonics (overtones) are therefore relatively feeble compared to the $n = 1$ term (the fundamental).

(b) String excited by impact

The effect of the impact of a hammer depends upon the manner and duration of the contact, and is more difficult to estimate. However, as a first estimate, let

$$u(x,0) = 0, \qquad 0 < x < L$$

and

$$u_t(x,0) = \begin{cases} \mu, & a - \epsilon < x < a + \epsilon \\ 0, & \text{otherwise}, \end{cases}$$

where $\epsilon \ll 1$. Show that the solution in this case is

$$u(x,t) = \frac{4\mu L}{\pi^2 c}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi\epsilon}{L}\right)\sin\left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right).$$

As in part (a), the nth mode is absent if the origin is at a node. The intensity of the overtones are now of the same order of magnitude; higher harmonics (overtones) are relatively more in evidence than in part (a).

(c) Bowed violin string

The theory of the vibration of a string when excited by bowing is poorly understood. The bow drags the string for a time until the string springs back. After awhile the process repeats. It can be shown[9] that the proper initial conditions are

$$u(x,0) = 0, \qquad 0 < x < L$$

and

$$u_t(x,0) = 4\beta c(L-x)/L^2, \qquad 0 < x < L,$$

where β is the maximum displacement. Show that the solution is now

$$u(x,t) = \frac{8\beta}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right).$$

[9] See Lamb, H., 1960: *The Dynamical Theory of Sound.* Dover Publishers, Mineola, NY, Section 27.

7.4 D'ALEMBERT'S FORMULA

In the previous section we sought solutions to the homogeneous wave equation in the form of a product $X(x)T(t)$. For the one-dimensional wave equation there is a more general method for constructing the solution. D'Alembert[10] derived it in 1747.

Let us determine a solution to the homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, 0 < t \tag{7.4.1}$$

which satisfies the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u(x,0)}{\partial t} = g(x), \quad -\infty < x < \infty. \tag{7.4.2}$$

We begin by introducing two new variables ξ, η defined by $\xi = x+ct$ and $\eta = x-ct$ and set $u(x,t) = w(\xi,\eta)$. The variables ξ and η are called the *characteristics* of the wave equation. Using the chain rule,

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \tag{7.4.3}$$

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta} \tag{7.4.4}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) \tag{7.4.5}$$

$$= \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2} \tag{7.4.6}$$

and similarly

$$c^2 \frac{\partial^2}{\partial t^2} = c^2\left(\frac{\partial^2}{\partial \xi^2} - 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}\right), \tag{7.4.7}$$

so that the wave equation becomes

$$\frac{\partial^2 w}{\partial \xi \partial \eta} = 0. \tag{7.4.8}$$

The general solution of (7.4.8) is

$$w(\xi,\eta) = F(\xi) + G(\eta). \tag{7.4.9}$$

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[10] D'Alembert, J., 1747: Recherches sur la courbe que forme une corde tenduë mise en vibration. *Hist. Acad. R. Sci. Belles Lett., Berlin*, 214–219.

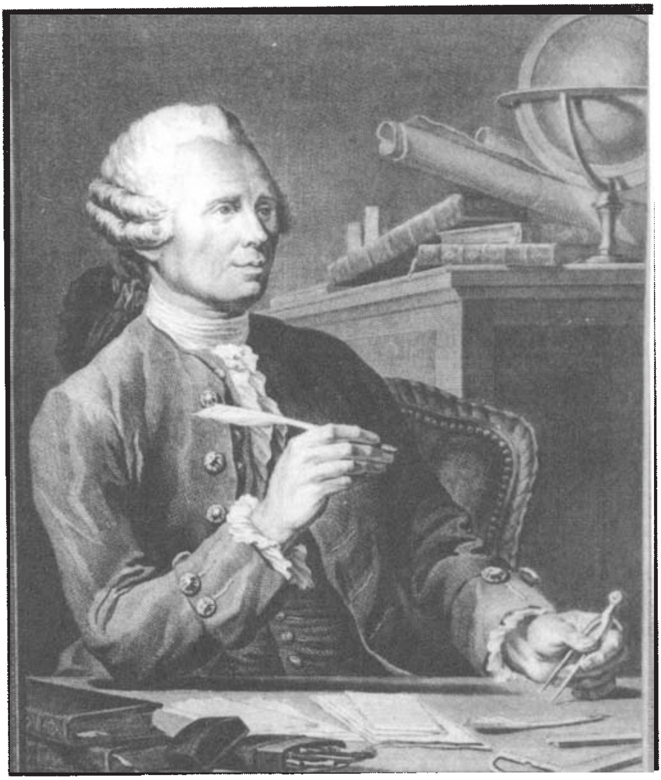


Figure 7.4.1: Although largely self-educated in mathematics, Jean Le Rond d'Alembert (1717–1783) gained equal fame as a mathematician and *philosophe* of the continental Enlightenment. By the middle of the eighteenth century, he stood with such leading European mathematicians and mathematical physicists as Clairaut, D. Bernoulli, and Euler. Today we best remember him for his work in fluid dynamics and applying partial differential equations to problems in physics. (Portrait courtesy of the Archives de l'Académie des sciences, Paris.)

Thus, the general solution of (7.4.1) is of the form

$$u(x,t) = F(x+ct) + G(x-ct), \qquad \textbf{(7.4.10)}$$

where F and G are arbitrary functions of one variable and are assumed to be twice differentiable. Setting $t=0$ in (7.4.10) and using the initial condition that $u(x,0) = f(x)$,

$$F(x) + G(x) = f(x). \qquad \textbf{(7.4.11)}$$

The partial derivative of (7.4.10) with respect to t yields

$$\frac{\partial u(x,t)}{\partial t} = cF'(x+ct) - cG'(x-ct). \tag{7.4.12}$$

Here primes denote differentiation with respect to the argument of the function. If we set $t=0$ in (7.4.12) and apply the initial condition that $u_t(x,0) = g(x)$,

$$cF'(x) - cG'(x) = g(x). \tag{7.4.13}$$

Integrating (7.4.13) from 0 to any point x gives

$$F(x) - G(x) = \frac{1}{c}\int_0^x g(\tau)\,d\tau + C, \tag{7.4.14}$$

where C is the constant of integration. Combining this result with (7.4.11),

$$F(x) = \frac{f(x)}{2} + \frac{1}{2c}\int_0^x g(\tau)\,d\tau + \frac{C}{2} \tag{7.4.15}$$

and

$$G(x) = \frac{g(x)}{2} - \frac{1}{2c}\int_0^x g(\tau)\,d\tau - \frac{C}{2}. \tag{7.4.16}$$

If we replace the variable x in the expression for F and G by $x+ct$ and $x-ct$, respectively, and substitute the results into (7.4.10), we finally arrive at the formula

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau. \tag{7.4.17}$$

This is known as *d'Alembert's formula* for the solution of the wave equation (7.4.1) subject to the initial conditions (7.4.2). It gives a *representation* of the solution in terms of *known* initial conditions.

- **Example 7.4.1**

To illustrate d'Alembert's formula, let us find the solution to the wave equation (7.4.1) satisfying the initial conditions $u(x,0) = \sin(x)$ and $u_t(x,0) = 0$, $-\infty < x < \infty$. By d'Alembert's formula (7.4.17),

$$u(x,t) = \frac{\sin(x-ct)+\sin(x+ct)}{2} = \sin(x)\cos(ct). \tag{7.4.18}$$

• **Example 7.4.2**

Let us find the solution to the wave equation (7.4.1) when $u(x,0) = 0$ and $u_t(x,0) = \sin(2x)$, $-\infty < x < \infty$. By d'Alembert's formula, the solution is

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(2\tau)\,d\tau = \frac{\sin(2x)\sin(2ct)}{2}. \tag{7.4.19}$$

In addition to providing a method of solving the wave equation, d'Alembert's solution may also be used to gain physical insight into the vibration of a string. Consider the case when we release a string with zero velocity after giving it an initial displacement of $f(x)$. According to (7.4.17), the displacement at a point x at any time t is

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2}. \tag{7.4.20}$$

Because the function $f(x-ct)$ is the same as the function of $f(x)$ translated to the right by a distance equal to ct, $f(x-ct)$ represents a wave of form $f(x)$ traveling to the right with the velocity c, a forward wave. Similarly, we can interpret the function $f(x+ct)$ as representing a wave with the shape $f(x)$ traveling to the left with the velocity c, a backward wave. Thus, the solution (7.4.17) is a superposition of forward and backward waves traveling with the same velocity c and having the shape of the initial profile $f(x)$ with half of the amplitude. Clearly the characteristics $x+ct$ and $x-ct$ give the propagation paths along which the waveform $f(x)$ propagates.

• **Example 7.4.3**

To illustrate our physical interpretation of d'Alembert's solution, suppose that the string has an initial displacement defined by

$$f(x) = \begin{cases} a - |x|, & -a \le x \le a \\ 0, & \text{otherwise.} \end{cases} \tag{7.4.21}$$

In Figure 7.4.2(A) the forward and backward waves, indicated by the dashed line, coincide at $t = 0$. As time advances, both waves move in opposite directions. In particular, at $t = a/(2c)$, they have moved through a distance $a/2$, resulting in the displacement of the string shown in Figure 7.4.2(B). Eventually, at $t = a/c$, the forward and backward waves completely separate. Finally, Figures 4.7.2(D) and 4.7.2(E) show how the waves radiate off to infinity at the speed of c. Note that at each point the string returns to its original position of rest after the passage of each wave.

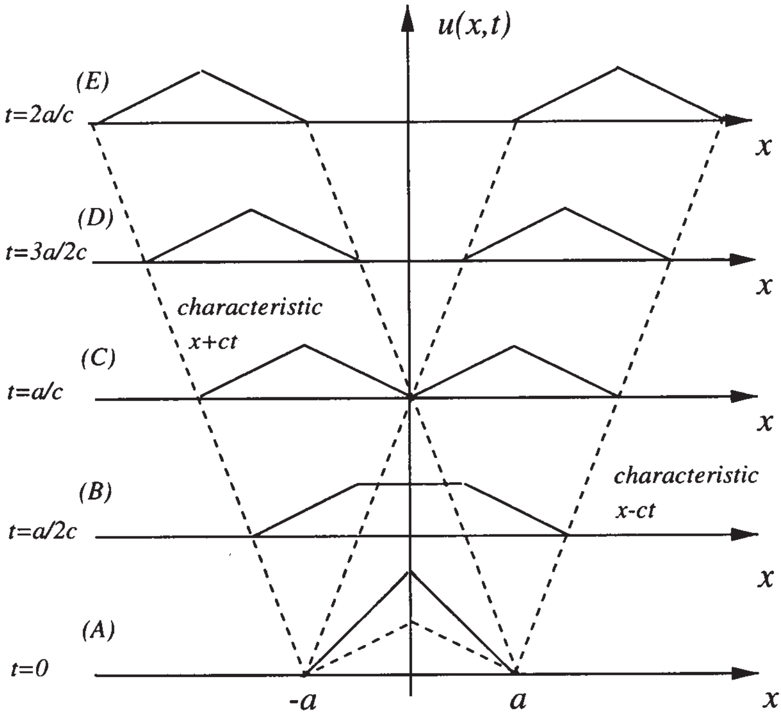


Figure 7.4.2: The propagation of waves due to an initial displacement according to d'Alembert's formula.

Consider now the opposite situation when $u(x,0) = 0$ and $u_t(x,0) = g(x)$. The displacement is

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau. \tag{7.4.22}$$

If we introduce the function

$$\varphi(x) = \frac{1}{2c}\int_0^x g(\tau)\,d\tau, \tag{7.4.23}$$

then we can write (7.4.22) as

$$u(x,t) = \varphi(x+ct) - \varphi(x-ct), \tag{7.4.24}$$

which again shows that the solution is a superposition of a forward wave $-\varphi(x-ct)$ and a backward wave $\varphi(x+ct)$ traveling with the same velocity c. The function φ, which we compute from (7.4.23) and the initial velocity $g(x)$, determines the exact form of these waves.

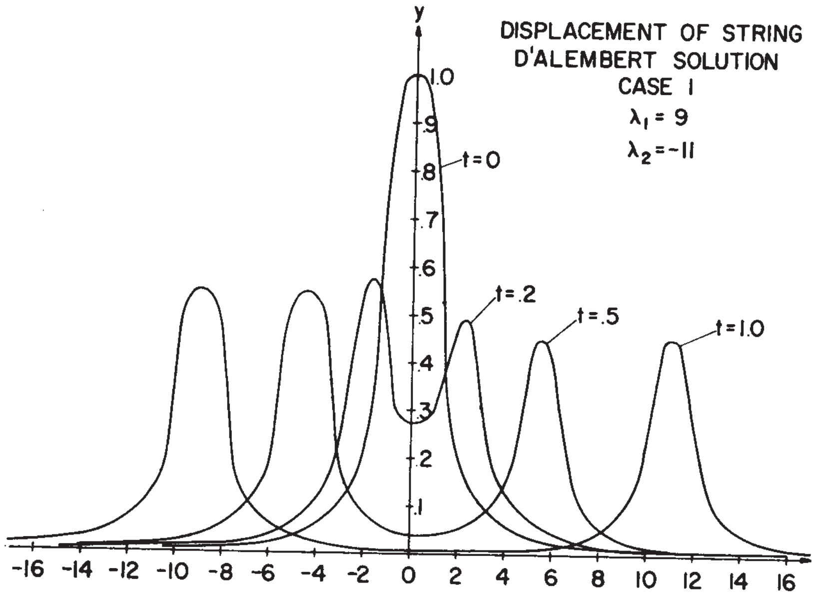


Figure 7.4.3: Displacement of an infinite, moving threadline when $c = 10$ and $V = 1$.

• **Example 7.4.4: Vibration of a moving threadline**

The characterization and analysis of the oscillations of a string or yarn have an important application in the textile industry because they describe the way that yarn winds on a bobbin[11]. As we showed in Section 7.4.1, the governing equation, the "threadline equation," is

$$\frac{\partial^2 y}{\partial t^2} + \alpha \frac{\partial^2 y}{\partial x \partial t} + \beta \frac{\partial^2 y}{\partial x^2} = 0, \tag{7.4.25}$$

where $\alpha = 2V$, $\beta = V^2 - gT/\rho$, V is the windup velocity, g is the gravitational attraction, T is the tension in the yarn, and ρ is the density of the yarn. We now introduce the characteristics $\xi = x + \lambda_1 t$ and $\eta = x + \lambda_2 t$, where λ_1 and λ_2 are yet undetermined. Upon substituting ξ and η into (7.4.25),

$$\begin{aligned}(\lambda_1^2 + 2V\lambda_1 + V^2 - gT/\rho)u_{\xi\xi} + (\lambda_2^2 + 2V\lambda_2 + V^2 - gT/\rho)u_{\eta\eta} \\ + [2V^2 - 2gT/\rho + 2V(\lambda_1 + \lambda_2) + 2\lambda_1\lambda_2]u_{\xi\eta} = 0.\end{aligned} \tag{7.4.26}$$

[11] Reprinted from *J. Franklin Inst.*, **275**, Swope, R. D., and W. F. Ames, Vibrations of a moving threadline, 36–55, ©1963, with kind permission from Elsevier Science Ltd, The Boulevard, Langford Lane, Kidlington OX5 1GB, UK.

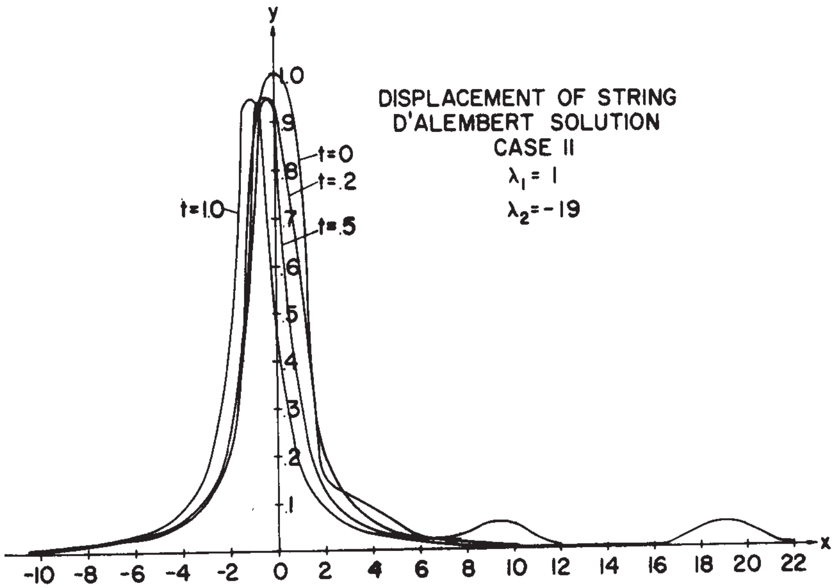


Figure 7.4.4: Displacement of an infinite, moving threadline when $c = 11$ and $V = 10$.

If we choose λ_1 and λ_2 to be roots of the equation:

$$\lambda^2 + 2V\lambda + V^2 - gT/\rho = 0, \tag{7.4.27}$$

(7.4.26) reduces to the simple form

$$u_{\xi\eta} = 0, \tag{7.4.28}$$

which has the general solution

$$u(x,t) = F(\xi) + G(\eta) = F(x + \lambda_1 t) + G(x + \lambda_2 t). \tag{7.4.29}$$

Solving (7.4.27) yields

$$\lambda_1 = c - V \qquad \text{and} \qquad \lambda_2 = -c - V, \tag{7.4.30}$$

where $c = \sqrt{gT/\rho}$. If the initial conditions are

$$u(x,0) = f(x) \qquad \text{and} \qquad u_t(x,0) = g(x), \tag{7.4.31}$$

then

$$u(x,t) = \frac{1}{2c}\left[\lambda_1 f(x + \lambda_2 t) - \lambda_2 f(x + \lambda_1 t) + \int_{x+\lambda_2 t}^{x+\lambda_1 t} g(\tau)\, d\tau\right]. \tag{7.4.32}$$

Because λ_1 does not generally equal to λ_2, the two waves that constitute the motion of the string move with different speeds and have different shapes and forms. For example, if

$$f(x) = \frac{1}{x^2+1} \quad \text{and} \quad g(x) = 0, \tag{7.4.33}$$

$$u(x,t) = \frac{1}{2c}\left\{\frac{c-V}{1+[x-(c+V)t]^2} + \frac{c+V}{1+[x-(c-V)t]^2}\right\}. \tag{7.4.34}$$

Figures 7.4.3 and 7.4.4 illustrate this solution for several different parameters.

Problems

Use d'Alembert's formula to solve the wave equation (7.4.1) for the following initial conditions defined for $|x| < \infty$.

1.	$u(x,0) = 2\sin(x)\cos(x)$	$u_t(x,0) = \cos(x)$
2.	$u(x,0) = x\sin(x)$	$u_t(x,0) = \cos(2x)$
3.	$u(x,0) = 1/(x^2+1)$	$u_t(x,0) = e^x$
4.	$u(x,0) = e^{-x}$	$u_t(x,0) = 1/(x^2+1)$
5.	$u(x,0) = \cos(\pi x/2)$	$u_t(x,0) = \sinh(ax)$
6.	$u(x,0) = \sin(3x)$	$u_t(x,0) = \sin(2x) - \sin(x)$

7.5 THE LAPLACE TRANSFORM METHOD

The solution of linear partial differential equations by Laplace transforms is the most commonly employed analytic technique after the method of separation of variables. Because the transform consists solely of an integration with respect to time, we obtain a transform which varies both in x and s, namely

$$U(x,s) = \int_0^\infty u(x,t)e^{-st}\,dt. \tag{7.5.1}$$

Partial derivatives involving time have transforms similar to those that we encountered in the case of functions of a single variable. They include

$$\mathcal{L}[u_t(x,t)] = sU(x,s) - u(x,0) \tag{7.5.2}$$

and

$$\mathcal{L}[u_{tt}(x,t)] = s^2U(x,s) - su(x,0) - u_t(x,0). \tag{7.5.3}$$

These transforms introduce the initial conditions via $u(x,0)$ and $u_t(x,0)$. On the other hand, derivatives involving x become

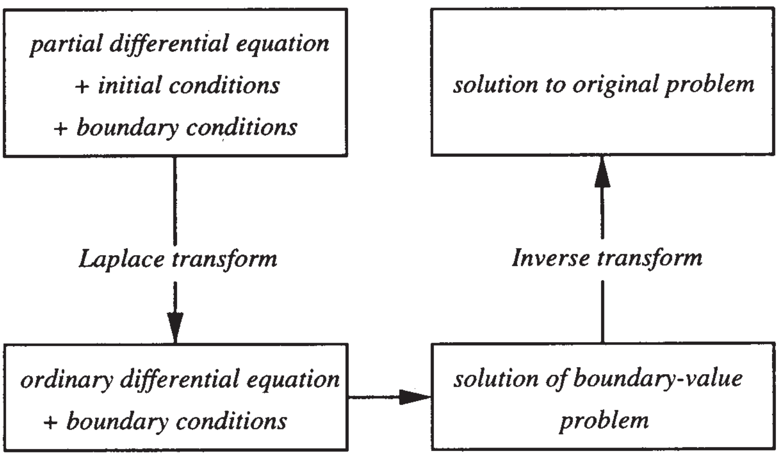
$$\mathcal{L}[u_x(x,t)] = \frac{d}{dx}\{\mathcal{L}[u(x,t)]\} = \frac{dU(x,s)}{dx} \tag{7.5.4}$$

and

$$\mathcal{L}[u_{xx}(x,t)] = \frac{d^2}{dx^2}\{\mathcal{L}[u(x,t)]\} = \frac{d^2U(x,s)}{dx^2}. \tag{7.5.5}$$

Because the transformation has eliminated the time variable, only $U(x,s)$ and its derivatives remain in the equation. Consequently, we have transformed the partial differential equation into a boundary-value problem for an ordinary differential equation. Because this equation is often easier to solve than a partial differential equation, the use of Laplace transforms has considerably simplified the original problem. Of course, the Laplace transforms must exist for this technique to work.

To summarize this method, we have constructed the following schematic:



In the following examples, we will illustrate transform methods by solving the classic equation of telegraphy as it applies to a uniform transmission line. The line has a resistance R, an inductance L, a capacitance C, and a leakage conductance G per unit length. We denote the current in the direction of positive x by I; V is the voltage drop across the transmission line at the point x. The dependent variables I and V are functions of both distance x along the line and time t.

To derive the differential equations that govern the current and voltage in the line, consider the points A at x and B at $x + \Delta x$ in Figure 7.5.1. The current and voltage at A are $I(x,t)$ and $V(x,t)$; at B, $I + \frac{\partial I}{\partial x}\Delta x$ and $V + \frac{\partial V}{\partial x}\Delta x$. Therefore, the voltage drop from A to B is $-\frac{\partial V}{\partial x}\Delta x$ and the current in the line is $I + \frac{\partial I}{\partial x}\Delta x$. Neglecting terms that are proportional to $(\Delta x)^2$,

$$\left(L\frac{\partial I}{\partial t} + RI\right)\Delta x = -\frac{\partial V}{\partial x}\Delta x. \tag{7.5.6}$$

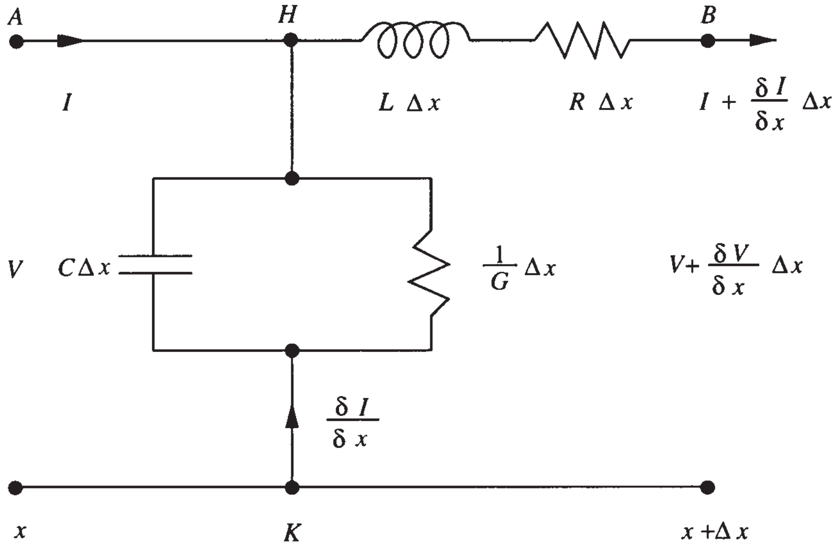


Figure 7.5.1: Schematic of an uniform transmission line.

The voltage drop over the parallel portion HK of the line is V while the current in this portion of the line is $-\frac{\partial I}{\partial x}\Delta x$. Thus,

$$\left(C\frac{\partial V}{\partial t} + GV\right)\Delta x = -\frac{\partial I}{\partial x}\Delta x. \tag{7.5.7}$$

Therefore, the differential equations for I and V are

$$L\frac{\partial I}{\partial t} + RI = -\frac{\partial V}{\partial x} \tag{7.5.8}$$

and

$$C\frac{\partial V}{\partial t} + GV = -\frac{\partial I}{\partial x}. \tag{7.5.9}$$

Turning to the initial conditions, we solve these simultaneous partial differential equations with the initial conditions:

$$I(x,0) = I_0(x) \tag{7.5.10}$$

and

$$V(x,0) = V_0(x) \tag{7.5.11}$$

for $0 < t$. There are also boundary conditions at the ends of the line; we will introduce them for each specific problem. For example, if the line is short-circuited at $x = a$, $V = 0$ at $x = a$; if there is an open circuit at $x = a$, $I = 0$ at $x = a$.

To solve (7.5.8)–(7.5.9) by Laplace transforms, we take the Laplace transform of both sides of these equations, which yields

$$(Ls+R)\overline{I}(x,s) = -\frac{d\overline{V}(x,s)}{dx} + LI_0(x) \tag{7.5.12}$$

and

$$(Cs+G)\overline{V}(x,s) = -\frac{d\overline{I}(x,s)}{dx} + CV_0(x). \tag{7.5.13}$$

Eliminating $\overline{I}$ gives an ordinary differential equation in $\overline{V}$:

$$\frac{d^2\overline{V}}{dx^2} - q^2\overline{V} = L\frac{dI_0(x)}{dx} - C(Ls+R)V_0(x), \tag{7.5.14}$$

where $q^2 = (Ls+R)(Cs+G)$. After finding $\overline{V}$, we may compute $\overline{I}$ from

$$\overline{I} = -\frac{1}{Ls+R}\frac{d\overline{V}}{dx} + \frac{LI_0(x)}{Ls+R}. \tag{7.5.15}$$

At this point we treat several classic cases.

• Example 7.5.1: The semi-infinite transmission line

We consider the problem of a semi-infinite line $x > 0$ with no initial current and charge. The end $x = 0$ has a constant voltage E for $0 < t$.

In this case,

$$\frac{d^2\overline{V}}{dx^2} - q^2\overline{V} = 0, \qquad x > 0. \tag{7.5.16}$$

The boundary conditions at the ends of the line are

$$V(0,t) = E, \qquad 0 < t \tag{7.5.17}$$

and $V(x,t)$ is finite as $x \to \infty$. The transform of these boundary conditions is

$$\overline{V}(0,s) = E/s \quad \text{and} \quad \lim_{x\to\infty} \overline{V}(x,s) < \infty. \tag{7.5.18}$$

The general solution of (7.5.16) is

$$\overline{V}(x,s) = Ae^{-qx} + Be^{qx}. \tag{7.5.19}$$

The requirement that $\overline{V}$ remains finite as $x \to \infty$ forces $B = 0$. The boundary condition at $x = 0$ gives $A = E/s$. Thus,

$$\overline{V}(x,s) = \frac{E}{s}\exp[-\sqrt{(Ls+R)(Cs+G)}\,x]. \tag{7.5.20}$$

We will discuss the general case later. However, for the so-called "lossless" line, where $R = G = 0$,

$$\overline{V}(x,s) = \frac{E}{s}\exp(-sx/c), \tag{7.5.21}$$

where $c = 1/\sqrt{LC}$. Consequently,

$$V(x,t) = EH\left(t - \frac{x}{c}\right), \tag{7.5.22}$$

where $H(t)$ is Heaviside's step function. The physical interpretation of this solution is as follows: $V(x,t)$ is zero up to the time x/c at which time a wave traveling with speed c from $x = 0$ would arrive at the point x. $V(x,t)$ has the constant value E afterwards.

For the so-called "distortionless" line, $R/L = G/C = \rho$,

$$V(x,t) = Ee^{-\rho x/c}H\left(t - \frac{x}{c}\right). \tag{7.5.23}$$

In this case, the disturbance not only propagates with velocity c but also attenuates as we move along the line.

Suppose now, that instead of applying a constant voltage E at $x = 0$, we apply a time-dependent voltage, $f(t)$. The only modification is that in place of (7.5.20),

$$\overline{V}(x,s) = F(s)e^{-qx}. \tag{7.5.24}$$

In the case of the distortionless line, $q = (s+\rho)/c$, this becomes

$$\overline{V}(x,s) = F(s)e^{-(s+\rho)x/c} \tag{7.5.25}$$

and

$$V(x,t) = e^{-\rho x/c}f\left(t - \frac{x}{c}\right)H\left(t - \frac{x}{c}\right). \tag{7.5.26}$$

Thus, our solution shows that the voltage at x is zero up to the time x/c. Afterwards $V(x,t)$ follows the voltage at $x = 0$ with a time lag of x/c and decreases in magnitude by $e^{-\rho x/c}$.

• **Example 7.5.2: The finite transmission line**

We now discuss the problem of a finite transmission line $0 < x < l$ with zero initial current and charge. We ground the end $x = 0$ and maintain the end $x = l$ at constant voltage E for $0 < t$.

The transformed partial differential equation becomes

$$\frac{d^2\overline{V}}{dx^2} - q^2\overline{V} = 0, \qquad 0 < x < l. \tag{7.5.27}$$

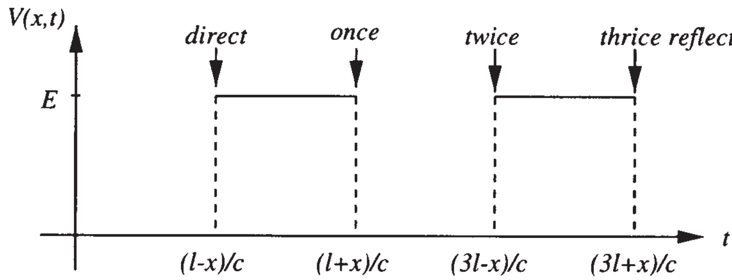


Figure 7.5.2: The voltage within a lossless, finite transmission line of length l as a function of time t.

The boundary conditions are

$$V(0,t) = 0 \quad \text{and} \quad V(l,t) = E, \qquad 0 < t. \tag{7.5.28}$$

The Laplace transform of these boundary conditions is

$$\overline{V}(0,s) = 0 \quad \text{and} \quad \overline{V}(l,s) = E/s. \tag{7.5.29}$$

The solution of (7.5.27) which satisfies the boundary conditions is

$$\overline{V}(x,s) = \frac{E \sinh(qx)}{s \sinh(ql)}. \tag{7.5.30}$$

Let us rewrite (7.5.30) in a form involving negative exponentials and expand the denominator by the binomial theorem,

$$\overline{V}(x,s) = \frac{E}{s} e^{-q(l-x)} \frac{1 - \exp(-2qx)}{1 - \exp(-2ql)} \tag{7.5.31}$$

$$= \frac{E}{s} e^{-q(l-x)} \left(1 - e^{-2qx}\right) \left(1 + e^{-2ql} + e^{-4ql} + \cdots\right) \tag{7.5.32}$$

$$= \frac{E}{s} \left[e^{-q(l-x)} - e^{-q(l+x)} + e^{-q(3l-x)} - e^{-q(3l+x)} + \cdots\right]. \tag{7.5.33}$$

In the special case of the lossless line where $q = s/c$,

$$\overline{V}(x,s) = \frac{E}{s} \left[e^{-s(l-x)/c} - e^{-s(l+x)/c} + e^{-s(3l-x)/c} - e^{-s(3l+x)/c} + \cdots\right] \tag{7.5.34}$$

or

$$V(x,t) = E\left[H\left(t - \frac{l-x}{c}\right) - H\left(t - \frac{l+x}{c}\right) + H\left(t - \frac{3l-x}{c}\right) - H\left(t - \frac{3l+x}{c}\right) + \cdots\right]. \tag{7.5.35}$$

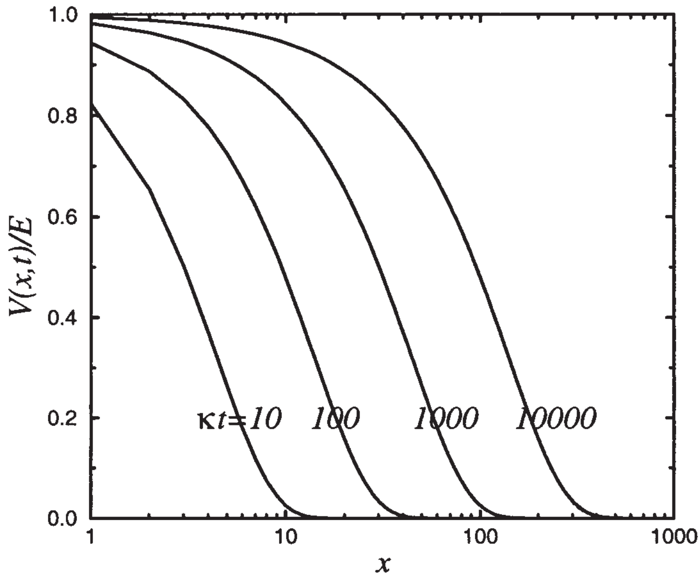


Figure 7.5.3: The voltage within a submarine cable as a function of distance for various κt's.

We illustrate (7.5.35) in Figure 7.5.2. The voltage at x is zero up to the time $(l-x)/c$, at which time a wave traveling directly from the end $x = l$ would reach the point x. The voltage then has the constant value E up to the time $(l+x)/c$, at which time a wave traveling from the end $x = l$ and reflected back from the end $x = 0$ would arrive. From this time up to the time of arrival of a twice-reflected wave, it has the value zero, and so on.

• **Example 7.5.3: The semi-infinite transmission line reconsidered**

In the first example, we showed that the transform of the solution for the semi-infinite line is

$$\overline{V}(x,s) = \frac{E}{s}e^{-qx}, \tag{7.5.36}$$

where $q^2 = (Ls+R)(Cs+G)$. In the case of a lossless line ($R = G = 0$), we found traveling wave solutions.

In this example, we shall examine the case of a submarine cable[12] where $L = G = 0$. In this special case,

$$\overline{V}(x,s) = \frac{E}{s}e^{-x\sqrt{s/\kappa}}, \tag{7.5.37}$$

[12] First solved by Thomson, W., 1855: On the theory of the electric telegraph. *Proc. R. Soc. London*, **A7**, 382–399.

where $\kappa = 1/(RC)$. From a table of Laplace transforms,[13] we can immediately invert (7.5.37) and find that

$$V(x,t) = E \text{ erfc}\left(\frac{x}{2\sqrt{\kappa t}}\right), \tag{7.5.38}$$

where erfc is the complementary error function. Unlike the traveling wave solution, the voltage diffuses into the cable as time increases. We illustrate (7.5.38) in Figure 7.5.3.

• Example 7.5.4: A short-circuited, finite transmission line

Let us find the voltage of a lossless transmission line of length l that initially has the constant voltage E. At $t = 0$, we ground the line at $x = 0$ while we leave the end $x = l$ insulated.

The transformed partial differential equation now becomes

$$\frac{d^2\overline{V}}{dx^2} - \frac{s^2}{c^2}\overline{V} = -\frac{sE}{c^2}, \tag{7.5.39}$$

where $c = 1/\sqrt{LC}$. The boundary conditions are

$$\overline{V}(0,s) = 0 \tag{7.5.40}$$

and

$$\overline{I}(l,s) = -\frac{1}{Ls}\frac{d\overline{V}(l,s)}{dx} = 0 \tag{7.5.41}$$

from (7.5.15).

The solution to this boundary-value problem is

$$\overline{V}(x,s) = \frac{E}{s} - \frac{E\cosh[s(l-x)/c]}{s\cosh(sl/c)}. \tag{7.5.42}$$

The first term on the right side of (7.5.42) is easy to invert and equals E. The second term is much more difficult to handle. We will use Bromwich's integral.

In Section 4.10 we showed that

$$\mathcal{L}^{-1}\left\{\frac{\cosh[s(l-x)/c]}{s\cosh(sl/c)}\right\} = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{\cosh[z(l-x)/c]e^{tz}}{z\cosh(zl/c)}\,dz. \tag{7.5.43}$$

13 See Churchill, R. V., 1972: *Operational Mathematics*, McGraw-Hill Book, New York, Section 27.

To evaluate this integral we must first locate and then classify the singularities. Using the product formula for the hyperbolic cosine,

$$\frac{\cosh[z(l-x)/c]}{z\cosh(zl/c)} = \frac{[1+\frac{4z^2(l-x)^2}{c^2\pi^2}][1+\frac{4z^2(l-x)^2}{9c^2\pi^2}]\cdots}{z[1+\frac{4z^2l^2}{c^2\pi^2}][1+\frac{4z^2l^2}{9c^2\pi^2}]\cdots}. \tag{7.5.44}$$

This shows that we have an infinite number of simple poles located at $z=0$ and $z_n = \pm(2n-1)\pi ci/(2l)$, where $n = 1,2,3,\ldots$. Therefore, Bromwich's contour can lie along, and just to the right of, the imaginary axis. By Jordan's lemma we close the contour with a semicircle of infinite radius in the left half of the complex plane. Computing the residues,

$$\operatorname{Res}\left\{\frac{\cosh[z(l-x)/c]e^{tz}}{z\cosh(zl/c)};0\right\} = \lim_{z\to 0}\frac{\cosh[z(l-x)/c]e^{tz}}{\cosh(zl/c)} = 1 \tag{7.5.45}$$

and

$$\operatorname{Res}\left\{\frac{\cosh[z(l-x)/c]e^{tz}}{z\cosh(zl/c)};z_n\right\}$$

$$= \lim_{z\to z_n}\frac{(z-z_n)\cosh[z(l-x)/c]e^{tz}}{z\cosh(zl/c)} \tag{7.5.46}$$

$$= \frac{\cosh[(2n-1)\pi(l-x)i/(2l)]\exp[\pm(2n-1)\pi cti/(2l)]}{[(2n-1)\pi i/2]\sinh[(2n-1)\pi i/2]} \tag{7.5.47}$$

$$= \frac{(-1)^{n+1}}{(2n-1)\pi/2}\cos\left[\frac{(2n-1)\pi(l-x)}{2l}\right]\exp\left[\pm\frac{(2n-1)\pi cti}{2l}\right]. \tag{7.5.48}$$

Summing the residues and using the relationship that $\cos(t) = (e^{ti}+e^{-ti})/2$,

$$V(x,t) = E - E\left\{1-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\cos\left[\frac{(2n-1)\pi(l-x)}{2l}\right]\right.$$

$$\left.\times\cos\left[\frac{(2n-1)\pi ct}{2l}\right]\right\} \tag{7.5.49}$$

$$= \frac{4E}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\cos\left[\frac{(2n-1)\pi(l-x)}{2l}\right]\cos\left[\frac{(2n-1)\pi ct}{2l}\right]. \tag{7.5.50}$$

An alternative to contour integration is to rewrite (7.5.42) as

$$\overline{V}(x,s) = \frac{E}{s}\left(1-\frac{\exp(-sx/c)\{1+\exp[-2s(l-x)/c]\}}{1+\exp(-2sl/c)}\right) \tag{7.5.51}$$

$$= \frac{E}{s}\left[1-e^{-sx/c}-e^{-s(2l-x)/c}+e^{-s(2l+x)/c}+\cdots\right] \tag{7.5.52}$$

so that

$$V(x,t) = E\left[1 - H\left(t - \frac{x}{c}\right) - H\left(t - \frac{2l - x}{c}\right) + H\left(t - \frac{2l + x}{c}\right) + \cdots\right]. \tag{7.5.53}$$

• **Example 7.5.5: The general solution of the equation of telegraphy**

In this example we solve the equation of telegraphy without any restrictions on R, C, G or L. We begin by eliminating the dependent variable $I(x,t)$ from the set of Equations (7.5.8)–(7.5.9). This yields

$$CL\frac{\partial^2 V}{\partial t^2} + (GL + RC)\frac{\partial V}{\partial t} + RG\,V = \frac{\partial^2 V}{\partial x^2}. \tag{7.5.54}$$

We next take the Laplace transform of (7.5.54) assuming that $V(x,0) = f(x)$ and $V_t(x,0) = g(x)$. The transformed version of (7.5.54) is

$$\frac{d^2\overline{V}}{dx^2} - [CLs^2 + (GL+RC)s + RG]\overline{V} = -CLg(x) - (CLs + GL + RC)f(x) \tag{7.5.55}$$

or

$$\frac{d^2\overline{V}}{dx^2} - \frac{(s+\rho)^2 - \sigma^2}{c^2}\overline{V} = -\frac{g(x)}{c^2} - \left(\frac{s}{c^2} + \frac{2\rho}{c^2}\right) f(x), \tag{7.5.56}$$

where $c^2 = 1/LC$, $\rho = c^2(RC + GL)/2$ and $\sigma = c^2(RC - GL)/2$.

We solve (7.5.56) by Fourier transforms (see Section 3.6) with the requirement that the solution dies away as $|x| \to \infty$. The most convenient way of expressing this solution is the convolution product (see Section 3.5)

$$\overline{V}(x,s) = \left[\frac{g(x)}{c} + \left(\frac{s}{c} + \frac{2\rho}{c}\right) f(x)\right] * \frac{\exp[-|x|\sqrt{(s+\rho)^2 - \sigma^2}/c]}{2\sqrt{(s+\rho)^2 - \sigma^2}}. \tag{7.5.57}$$

From a table of Laplace transforms,

$$\mathcal{L}^{-1}\left[\frac{\exp\left(-b\sqrt{s^2 - a^2}\right)}{\sqrt{s^2 - a^2}}\right] = I_0\left(a\sqrt{t^2 - b^2}\right) H(t - b), \tag{7.5.58}$$

where $b > 0$ and $I_0(\ )$ is the zeroth order modified Bessel function of the first kind. Therefore, by the first shifting theorem,

$$\mathcal{L}^{-1}\left\{\frac{\exp\left[-|x|\sqrt{(s+\rho)^2 - \sigma^2}/c\right]}{\sqrt{(s+\rho)^2 - \sigma^2}}\right\} = e^{-\rho t} I_0\left[\sigma\sqrt{t^2 - (x/c)^2}\right] H\left(t - \frac{|x|}{c}\right). \tag{7.5.59}$$

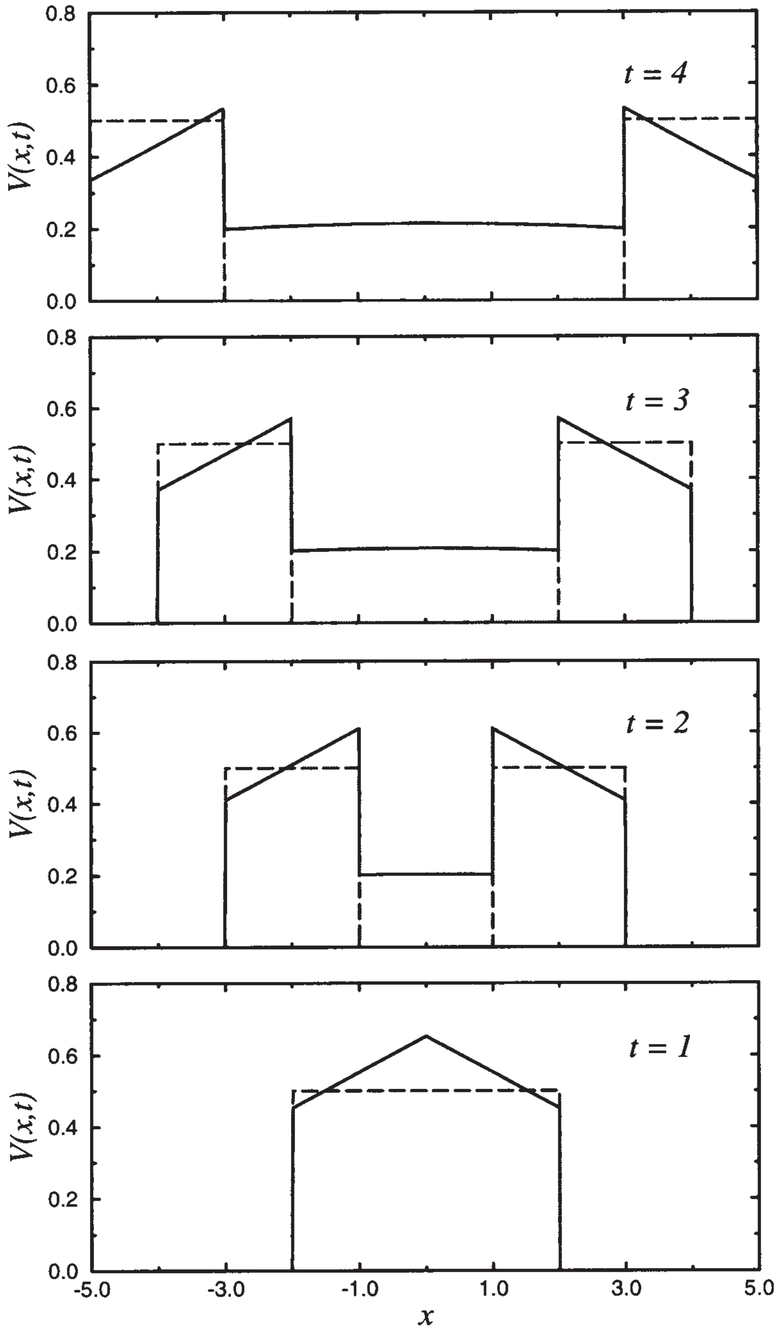


Figure 7.5.4: The evolution of the voltage with time given by the general equation of telegraphy for initial conditions and parameters stated in the text.

Using (7.5.59) to invert (7.5.57), we have that

$$\begin{aligned} V(x,t) &= \tfrac{1}{2c}e^{-\rho t}g(x) * I_0\left[\sigma\sqrt{t^2-(x/c)^2}\right]H(t-|x|/c) \\ &+ \tfrac{1}{2c}e^{-\rho t}f(x) * \frac{\partial}{\partial t}\left\{I_0[\sigma\sqrt{t^2-(x/c)^2}]\right\}H(t-|x|/c) \\ &+ \tfrac{\rho}{c}e^{-\rho t}f(x) * I_0\left[\sigma\sqrt{t^2-(x/c)^2}\right]H(t-|x|/c) \\ &+ \tfrac{1}{2}e^{-\rho t}[f(x+ct)+f(x-ct)]. \end{aligned} \tag{7.5.60}$$

The last term in (7.5.60) arises from noting that $sF(s) = \mathcal{L}[f(t)] + f(0)$. If we explicitly write out the convolution, the final form of the solution is

$$\begin{aligned} V(x,t) &= \tfrac{1}{2}e^{-\rho t}[f(x+ct)+f(x-ct)] \\ &+ \tfrac{1}{2c}e^{-\rho t}\int_{x-ct}^{x+ct}[g(\eta)+2\rho f(\eta)]I_0\left[\sigma\sqrt{c^2t^2-(x-\eta)^2}\Big/c\right]d\eta \\ &+ \tfrac{1}{2c}e^{-\rho t}\int_{x-ct}^{x+ct}f(\eta)\frac{\partial}{\partial t}\left\{I_0\left[\sigma\sqrt{c^2t^2-(x-\eta)^2}\Big/c\right]\right\}d\eta. \end{aligned} \tag{7.5.61}$$

The physical interpretation of the first line of (7.5.61) is straightforward. It represents damped progressive waves; one is propagating to the right and the other to the left. In addition to these progressive waves, there is a contribution from the integrals, even after the waves have passed. These integrals include all of the points where $f(x)$ and $g(x)$ are nonzero within a distance ct from the point in question. This effect persists through all time, although dying away, and constitutes a residue or tail. Figure 7.5.4 illustrates this for $\rho = 0.1$, $\sigma = 0.2$, and $c = 1$. We evaluated the integrals by Simpson's rule for the initial conditions $f(x) = H(x+1) - H(x-1)$ and $g(x) = 0$. We have also included the solution for the lossless case for comparison. If there was no loss, then two pulses would propagate to the left and right as shown by the dashed line. However, with resistance and leakage the waves leave a residue after their leading edge has passed.

Problems

1. Use transform methods to solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, 0 < t$$

for the boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad 0 < t$$

and the initial conditions

$$u(x,0) = 0, \quad \frac{\partial u(0,t)}{\partial t} = 1, \quad 0 < x < 1.$$

2. Use transform methods to solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, 0 < t$$

for the boundary conditions

$$u(0,t) = u_x(1,t) = 0, \qquad 0 < t$$

and the initial conditions

$$u(x,0) = 0, \quad \frac{\partial u(0,t)}{\partial t} = x, \quad 0 < x < 1.$$

3. Use transform methods to solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, 0 < t$$

for the boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad 0 < t$$

and the initial conditions

$$u(x,0) = \sin(\pi x), \quad \frac{\partial u(x,0)}{\partial t} = -\sin(\pi x), \quad 0 < x < 1.$$

4. Use transform methods to solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < a, 0 < t$$

for the boundary conditions

$$u(0,t) = \sin(\omega t), \quad u(a,t) = 0, \qquad 0 < t$$

and the initial conditions

$$u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = 0, \quad 0 < x < a.$$

Assume that $\omega a/c$ is *not* an integer multiple of π. Why?

5. Use transform methods to solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = te^{-x}, \quad 0 < x < \infty, 0 < t$$

for the boundary conditions

$$u(0,t) = 1 - e^{-t}, \quad \lim_{x\to\infty} |u(x,t)| \sim x^n, \ n \text{ finite}, \quad 0 < t$$

and the initial conditions

$$u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = x, \quad 0 < x < \infty.$$

6. Use transform methods to solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = xe^{-t}, \quad 0 < x < \infty, 0 < t$$

for the boundary conditions

$$u(0,t) = \cos(t), \quad \lim_{x\to\infty} |u(x,t)| \sim x^n, \ n \text{ finite}, \quad 0 < t$$

and the initial conditions

$$u(x,0) = 1, \quad \frac{\partial u(x,0)}{\partial t} = 0, \quad 0 < x < \infty.$$

7. Use transform methods to solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, 0 < t$$

for the boundary conditions

$$u(0,t) = 0, \quad \frac{\partial^2 u(L,t)}{\partial t^2} + \frac{k}{m}\frac{\partial u(L,t)}{\partial x} = g, \quad 0 < t$$

and the initial conditions

$$u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = 0, \quad 0 < x < L,$$

where c, k, m, and g are constants.

8. Use transform methods[14] to solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right), \quad 0 < x < 1, 0 < t$$

for the boundary conditions

$$\lim_{x\to\infty} |u(x,t)| < \infty, \quad u(1,t) = A\sin(\omega t), \quad 0 < t$$

and the initial conditions

$$u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = 0, \quad 0 < x < 1.$$

Assume that $2\omega \neq c\beta_n$, where $J_0(\beta_n) = 0$. [Hint: The ordinary differential equation

$$\frac{d}{dx}\left(x\frac{dU}{dx}\right) - \frac{s^2}{c^2}U = 0$$

has the solution

$$U(x,s) = c_1 I_0\left(\frac{s}{c}\sqrt{x}\right) + c_2 K_0\left(\frac{s}{c}\sqrt{x}\right),$$

where $I_0(x)$ and $K_0(x)$ are modified Bessel functions of the first and second kind, respectively. Note that $J_n(iz) = i^n I_n(z)$ and $I_n(iz) = i^n J_n(z)$ for complex z.]

9. A lossless transmission line of length ℓ has a constant voltage E applied to the end $x = 0$ while we insulate the other end $[u_x(\ell,t) = 0]$. Find the voltage at any point on the line if the initial current and charge are zero.

10. Solve the equation of telegraphy without leakage

$$\frac{\partial^2 u}{\partial x^2} = CR\frac{\partial u}{\partial t} + CL\frac{\partial^2 u}{\partial t^2}, \quad 0 < x < \ell, 0 < t$$

subject to the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(\ell,t) = E, \quad 0 < t$$

and the initial conditions

$$u(x,0) = u_t(x,0) = 0, \quad 0 < x < \ell.$$

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[14] Suggested by a problem solved by Brown, J., 1975: Stresses in towed cables during re-entry. *J. Spacecr. Rockets*, **12**, 524–527.

Assume that $4\pi^2 L/CR^2\ell^2 > 1$. Why?

11. The pressure and velocity oscillations from water hammer in a pipe without friction[15] are given by the equations

$$\frac{\partial p}{\partial t} = -\rho c^2 \frac{\partial u}{\partial x}$$

and

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

where $p(x,t)$ denotes the pressure perturbation, $u(x,t)$ is the velocity perturbation, c is the speed of sound in water, and ρ is the density of water. These two first-order partial differential equations may be combined to yield

$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2}.$$

Find the solution to this partial differential equation if $p(0,t) = p_0$ and $u(L,t) = 0$ and the initial conditions are $p(x,0) = p_0$, $p_t(x,0) = 0$ and $u(x,t) = u_0$.

12. Use Laplace transforms to solve the wave equation[16]

$$\frac{\partial^2 (ru)}{\partial t^2} = c^2 \frac{\partial^2 (ru)}{\partial r^2}, \qquad a < r < \infty,\ 0 < t$$

subject to the boundary conditions that

$$-\rho c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{3r}\frac{\partial u}{\partial r} \right)\bigg|_{r=a} = p_0 e^{-\alpha t} H(t) \text{ and } \lim_{r\to\infty} |u(r,t)| < \infty, \quad 0 < t,$$

where $\alpha > 0$, and the initial conditions that

$$u(r,0) = u_t(r,0) = 0, \qquad a < r < \infty.$$

13. Consider a vertical rod or column of length L that is supported at both ends. The elastic waves that arise when the support at the bottom is suddenly removed are governed by the wave equation[17]

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + g, \qquad 0 < x < L,\ 0 < t,$$

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[15] See Rich, G. R., 1945: Water-hammer analysis by the Laplace-Mellin transformation. *Trans. ASME*, **67**, 361–376.

[16] Originally solved using Fourier transforms by Sharpe, J. A., 1942: The production of elastic waves by explosion pressures. I. Theory and empirical field observations. *Geophysics*, **7**, 144–154.

[17] Abstracted with permission from Hall, L. H., 1953: Longitudinal vibrations of a vertical column by the method of Laplace transform. *Am. J. Phys.*, **21**, 287–292. ©1953 American Association of Physics Teachers.

where g denotes the gravitational acceleration, $c^2 = E/\rho$, E is Young's modulus and ρ is the mass density. Find the wave solution if the boundary conditions are

$$\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(L,t)}{\partial x} = 0, \qquad 0 < t$$

and the initial conditions are

$$u(x,0) = -\frac{gx^2}{2c^2}, \qquad \frac{\partial u(x,0)}{\partial t} = 0, \qquad 0 < x < L.$$

14. Solve the telegraph-like equation[18]

$$\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \alpha\frac{\partial u}{\partial x}\right), \qquad 0 < x < \infty,\ 0 < t$$

subject to the boundary conditions

$$\frac{\partial u(0,t)}{\partial x} = -u_0\delta(t), \quad \lim_{x\to\infty} |u(x,t)| < \infty, \qquad 0 < x < \infty$$

and the initial conditions

$$u(x,0) = u_0, \quad u_t(x,0) = 0, \quad 0 < t$$

with $\alpha c > k$.

Step 1: Take the Laplace transform of the partial differential equation and boundary conditions and show that

$$\frac{d^2U(x,s)}{dx^2} + \alpha\frac{dU(x,s)}{dx} - \left(\frac{s^2+ks}{c^2}\right)U(x,s) = -\left(\frac{s+k}{c^2}\right)u_0$$

with $U'(0,s) = -u_0$ and $\lim_{x\to\infty} |U(x,s)| < \infty$.

Step 2: Show that the solution to the previous step is

$$U(x,s) = \frac{u_0}{s} + u_0 e^{-\alpha x/2}\,\frac{\exp\left[-x\sqrt{\left(s+\frac{k}{2}\right)^2 + a^2/c}\right]}{\frac{\alpha}{2} + \sqrt{(s+\frac{k}{2})^2 + a^2/c}},$$

[18] From Abbott, M. R., 1959: The downstream effect of closing a barrier across an estuary with particular reference to the Thames. *Proc. R. Soc. London*, **A251**, 426–439 with permission.

where $4a^2 = \alpha^2 c^2 - k^2 > 0$.

Step 3: Using the first and second shifting theorems and the property that

$$F\left(\sqrt{s^2+a^2}\right) = \mathcal{L}\left[f(t) - a\int_0^t \frac{J_1\left(a\sqrt{t^2-\tau^2}\right)}{\sqrt{t^2-\tau^2}}\tau f(\tau)\,d\tau\right],$$

show that

$$u(x,t) = u_0 + u_0 c e^{-kt/2} H(t - x/c)$$
$$\times\left[e^{-\alpha ct/2} - a\int_{x/c}^t \frac{J_1\left(a\sqrt{t^2-\tau^2}\right)}{\sqrt{t^2-\tau^2}}\tau e^{-\alpha c\tau/2}d\tau\right].$$

15. As an electric locomotive travels down a track at the speed V, the pantograph (the metallic framework that connects the overhead power lines to the locomotive) pushes up the line with a force P. Let us find the behavior[19] of the overhead wire as a pantograph passes between two supports of the electrical cable that are located a distance L apart. We model this system as a vibrating string with a point load:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + \frac{P}{\rho V}\delta\left(t - \frac{x}{V}\right), \qquad 0 < x < L,\ 0 < t.$$

Let us assume that the wire is initially at rest $[u(x,0) = u_t(x,0) = 0$ for $0 < x < L]$ and fixed at both ends $[u(0,t) = u(L,t) = 0$ for $0 < t]$.

Step 1: Take the Laplace transform of the partial differential equation and show that

$$s^2 U(x,s) = c^2\frac{d^2U(x,s)}{dx^2} + \frac{P}{\rho V}e^{-xs/V}.$$

Step 2: Solve the ordinary differential equation in Step 1 as a Fourier half-range sine series:

$$U(x,s) = \sum_{n=1}^{\infty} B_n(s)\sin\left(\frac{n\pi x}{L}\right),$$

[19] From Oda, O. and Ooura, Y., 1976: Vibrations of catenary overhead wire. *Q. Rep., (Tokyo) Railway Tech. Res. Inst.*, **17**, 134–135 with permission.

where

$$B_n(s) = \frac{2P\beta_n}{\rho L(\beta_n^2 - \alpha_n^2)} \left[\frac{1}{s^2+\alpha_n^2} - \frac{1}{s^2+\beta_n^2}\right] \left[1 - (-1)^n e^{-Ls/V}\right],$$

$\alpha_n = n\pi c/L$ and $\beta_n = n\pi V/L$. This solution satisfies the boundary conditions.

Step 3: By inverting the solution in Step 2, show that

$$\begin{aligned}
u(x,t) &= \frac{2P}{\rho L}\sum_{n=1}^{\infty}\left[\frac{\sin(\beta_n t)}{\alpha_n^2-\beta_n^2} - \frac{V}{c}\,\frac{\sin(\alpha_n t)}{\alpha_n^2-\beta_n^2}\right]\sin\left(\frac{n\pi x}{L}\right) \\
&\quad - \frac{2P}{\rho L}H\left(t-\frac{L}{V}\right)\sum_{n=1}^{\infty}(-1)^n\sin\left(\frac{n\pi x}{L}\right) \\
&\qquad \times\left\{\frac{\sin[\beta_n(t-L/V)]}{\alpha_n^2-\beta_n^2} - \frac{V}{c}\,\frac{\sin[\alpha_n(t-L/V)]}{\alpha_n^2-\beta_n^2}\right\} \\
&= \frac{2P}{\rho L}\sum_{n=1}^{\infty}\left[\frac{\sin(\beta_n t)}{\alpha_n^2-\beta_n^2} - \frac{V}{c}\,\frac{\sin(\alpha_n t)}{\alpha_n^2-\beta_n^2}\right]\sin\left(\frac{n\pi x}{L}\right) \\
&\quad - \frac{2P}{\rho L}H\left(t-\frac{L}{V}\right)\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right) \\
&\qquad \times\left\{\frac{\sin(\beta_n t)}{\alpha_n^2-\beta_n^2} - \frac{V}{c}(-1)^n\frac{\sin[\alpha_n(t-L/V)]}{\alpha_n^2-\beta_n^2}\right\}.
\end{aligned}$$

The first term in both summations represents the static uplift on the line; this term disappears after the pantograph has passed. The second term in both summations represents the vibrations excited by the traveling force. Even after the pantograph passes, they will continue to exist.

16. Solve the wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \frac{u}{r^2} = \frac{\delta(r-\alpha)}{\alpha^2}, \qquad 0 \le r < a,\ 0 < t,$$

where $0 < \alpha < a$, subject to the boundary conditions

$$\lim_{r\to 0}|u(r,t)| < \infty \quad \text{and} \quad \frac{\partial u(a,t)}{\partial r} + \frac{h}{a}u(a,t) = 0, \quad 0 < t$$

and the initial conditions

$$u(r,0) = u_t(r,0) = 0, \qquad 0 \le r < a.$$

Step 1: Take the Laplace transform of the partial differential equation and show that

$$\frac{d^2U(r,s)}{dr^2}+\frac{1}{r}\frac{dU(r,s)}{dr}-\left(\frac{s^2}{c^2}+\frac{1}{r^2}\right)U(r,s)=-\frac{\delta(r-\alpha)}{s\alpha^2}, \qquad 0\le r<a$$

with

$$\lim_{r\to 0}|U(r,s)|<\infty \quad \text{and} \quad \frac{dU(a,s)}{dr}+\frac{h}{a}U(a,s)=0.$$

Step 2: Show that the Dirac delta function can be reexpressed as the Fourier-Bessel series:

$$\delta(r-\alpha)=\frac{2\alpha}{a^2}\sum_{n=1}^{\infty}\frac{\beta_n^2\,J_1(\beta_n\alpha/a)}{(\beta_n^2+h^2-1)J_1^2(\beta_n)}J_1(\beta_n r/a), \qquad 0\le r<a,$$

where β_n is the nth root of $\beta J_1'(\beta)+h\,J_1(\beta)=\beta J_0(\beta)+(h-1)J_1(\beta)=0$ and $J_0(\,)$, $J_1(\,)$ are the zeroth and first-order Bessel functions of the first kind, respectively.

Step 3: Show that solution to the ordinary differential equation in Step 1 is

$$U(r,s)=\frac{2}{\alpha}\sum_{n=1}^{\infty}\frac{J_1(\beta_n\alpha/a)J_1(\beta_n r/a)}{(\beta_n^2+h^2-1)\,J_1^2(\beta_n)}\left[\frac{1}{s}-\frac{s}{s^2+c^2\beta_n^2/a^2}\right].$$

Note that this solution satisfies the boundary conditions.

Step 4: Taking the inverse of the Laplace transform in Step 3, show that the solution to the partial differential equation is

$$u(r,t)=\frac{2}{\alpha}\sum_{n=1}^{\infty}\frac{J_1(\beta_n\alpha/a)J_1(\beta_n r/a)}{(\beta_n^2+h^2-1)\,J_1^2(\beta_n)}\left[1-\cos\left(\frac{c\beta_n t}{a}\right)\right].$$

17. A powerful method for solving certain partial differential equations is the joint application of Laplace and Fourier transforms. To illustrate this *joint transform method*, let us find the Green's function for the Klein-Gordon equation

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}-\beta^2 u=-\delta(x)\delta(t), \quad -\infty<x<\infty, 0<t$$

subject to the boundary condition

$$\lim_{x\to\pm\infty}|u(x,t)|<\infty, \quad 0<t$$

and the initial conditions

$$u(x,0) = u_t(x,0) = 0, \qquad -\infty < x < \infty.$$

Step 1: Take the Laplace transform of the partial differential equation and show that

$$\frac{d^2U(x,s)}{dx^2} - \left(\frac{s^2}{c^2} + \beta^2\right) U(x,s) = -\delta(x), \qquad -\infty < x < \infty$$

with the boundary condition

$$\lim_{x\to\pm\infty} |U(x,s)| < \infty.$$

Step 2: Using Fourier transforms, show that the solution to the ordinary differential equation in Step 1 is

$$U(x,s) = \frac{\exp\left(-|x|\sqrt{s^2/c^2+\beta^2}\right)}{2\sqrt{s^2/c^2+\beta^2}}.$$

You may need to review Section 3.6.

Step 3: Using tables, show that the Green's function is

$$u(x,t) = \frac{c}{2} J_0\left(\beta\sqrt{c^2t^2 - x^2}\right) H(ct - x),$$

where $J_0()$ is the zeroth order Bessel function of the first kind.

7.6 NUMERICAL SOLUTION OF THE WAVE EQUATION

Despite the powerful techniques shown in the previous sections for solving the wave equation, often these analytic techniques fail and we must resort to numerical techniques. In counterpoint to the continuous solutions, finite difference methods, a type of numerical solution technique, give discrete numerical values at a specific location (x_m, t_n), called a *grid point*. These numerical values represent a numerical approximation of the continuous solution over the region $(x_m - \Delta x/2, x_m + \Delta x/2)$ and $(t_n - \Delta t/2, t_n + \Delta t/2)$, where Δx and Δt are the distance and time intervals between grid points, respectively. Clearly, in the limit of $\Delta x, \Delta t \to 0$, we recover the continuous solution. However, practical considerations such as computer memory or execution time often require that Δx and Δt, although small, are not negligibly small.

The first task in the numerical solution of a partial differential equation is the replacement of its continuous derivatives with finite differences. The most popular approach employs Taylor expansions. If we focus on the x-derivative, then the value of the solution at $u[(m+1)\Delta x, n\Delta t]$ in terms of the solution at $(m\Delta x, n\Delta t)$ is

$$u[(m+1)\Delta x, n\Delta t] = u(x_m, t_n) + \frac{\Delta x}{1!}\frac{\partial u(x_m, t_n)}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u(x_m, t_n)}{\partial x^2} + \frac{(\Delta x)^3}{3!}\frac{\partial^3 u(x_m, t_n)}{\partial x^3} + \frac{(\Delta x)^4}{4!}\frac{\partial^4 u(x_m, t_n)}{\partial x^4} + \cdots \tag{7.6.1}$$

$$= u(x_m, t_n) + \Delta x \frac{\partial u(x_m, t_n)}{\partial x} + O[(\Delta x)^2], \tag{7.6.2}$$

where $O[(\Delta x)^2]$ gives a measure of the magnitude of neglected terms.[20]

From (7.6.2), one possible approximation for u_x is

$$\frac{\partial u(x_m, t_n)}{\partial x} = \frac{u_{m+1}^n - u_m^n}{\Delta x} + O(\Delta x), \tag{7.6.3}$$

where we have used the standard notation that $u_m^n = u(x_m, t_n)$. This is an example of a *one-sided finite difference* approximation of the partial derivative u_x. The error in using this approximation behaves as Δx.

Another possible approximation for the derivative arises from using $u(m\Delta x, n\Delta t)$ and $u[(m-1)\Delta x, n\Delta t]$. From the Taylor expansion:

$$u[(m-1)\Delta x, n\Delta t] = u(x_m, t_n) - \frac{\Delta x}{1!}\frac{\partial u(x_m, t_n)}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u(x_m, t_n)}{\partial x^2}$$

[20] The symbol O is a mathematical notation indicating relative magnitude of terms, namely that $f(\epsilon) = O(\epsilon^n)$ provided $\lim_{\epsilon \to 0} |f(\epsilon)/\epsilon^n| < \infty$. For example, as $\epsilon \to 0$, $\sin(\epsilon) = O(\epsilon)$, $\sin(\epsilon^2) = O(\epsilon^2)$, and $\cos(\epsilon) = O(1)$.

$$-\frac{(\Delta x)^3}{3!}\frac{\partial^3 u(x_m,t_n)}{\partial x^3}+\frac{(\Delta x)^4}{4!}\frac{\partial^4 u(x_m,t_n)}{\partial x^4}-\cdots, \tag{7.6.4}$$

we can also obtain the one-sided difference formula

$$\frac{u(x_m,t_n)}{\partial x}=\frac{u_m^n-u_{m-1}^n}{\Delta x}+O(\Delta x). \tag{7.6.5}$$

A third possibility arises from subtracting (7.6.4) from (7.6.1):

$$u_{m+1}^n-u_{m-1}^n=2\Delta x\frac{\partial u(x_m,t_n)}{\partial x}+O[(\Delta x)^3] \tag{7.6.6}$$

or

$$\frac{\partial u(x_m,t_n)}{\partial x}=\frac{u_{m+1}^n-u_{m-1}^n}{2\Delta x}+O[(\Delta x)^2]. \tag{7.6.7}$$

Thus, the choice of the finite differencing scheme can produce profound differences in the accuracy of the results. In the present case, *centered finite differences* can yield results that are markedly better than using one-sided differences.

To solve the wave equation, we need to approximate u_{xx}. If we add (7.6.1) and (7.6.4),

$$u_{m+1}^n+u_{m-1}^n=2u_m^n+\frac{\partial^2 u(x_m,t_n)}{\partial x^2}(\Delta x)^2+O[(\Delta x)^4] \tag{7.6.8}$$

or

$$\frac{\partial^2 u(x_m,t_n)}{\partial x^2}=\frac{u_{m+1}^n-2u_m^n+u_{m-1}^n}{(\Delta x)^2}+O[(\Delta x)^2]. \tag{7.6.9}$$

Similar considerations hold for the time derivative. Thus, by neglecting errors of $O[(\Delta x)^2]$ and $O[(\Delta t)^2]$, we may approximate the wave equation by

$$\frac{u_m^{n+1}-2u_m^n+u_m^{n-1}}{(\Delta t)^2}=c^2\frac{u_{m+1}^n-2u_m^n+u_{m-1}^n}{(\Delta x)^2}. \tag{7.6.10}$$

Because the wave equation represents evolutionary change of some quantity, (7.6.10) is generally used as a predictive equation where we forecast u_m^{n+1} by

$$u_m^{n+1}=2u_m^n-u_m^{n-1}+\left(\frac{c\Delta t}{\Delta x}\right)^2\left(u_{m+1}^n-2u_m^n+u_{m-1}^n\right). \tag{7.6.11}$$

Figure 7.6.1 illustrates this numerical scheme.

The greatest challenge in using (7.6.11) occurs with the very first prediction. When $n=0$, clearly u_{m+1}^0, u_m^0 and u_{m-1}^0 are specified from the initial condition $u(m\Delta x,0)=f(x_m)$. But what about u_m^{-1}? Recall

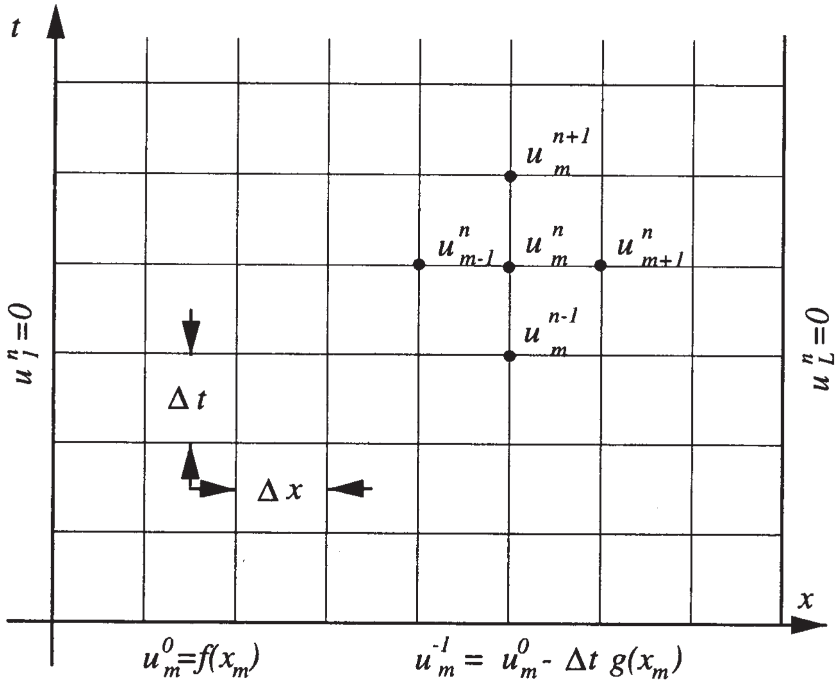

Figure 7.6.1: Schematic of the numerical solution of the wave equation with fixed end points.

that we still have $u_t(x,0) = g(x)$. If we use the backward difference formula (7.6.5),

$$\frac{u_m^0 - u_m^{-1}}{\Delta t} = g(x_m). \tag{7.6.12}$$

Solving for u_m^{-1},

$$u_m^{-1} = u_m^0 - \Delta t g(x_m). \tag{7.6.13}$$

One of the disadvantages of using the backward finite-difference formula is the larger error associated with this term compared to those associated with the finite-differenced form of the wave equation. In the case of the barotropic vorticity equation, a partial differential equation with wave-like solutions, this inconsistency eventually leads to a separation of solution between adjacent time levels.[21] This difficulty is avoided by stopping after a certain number of time steps, averaging the solution, and starting again.

[21] Gates, W. L., 1959: On the truncation error, stability, and convergence of difference solutions of the barotropic vorticity equation. *J. Meteorol.*, **16**, 556–568. See Section 4.

A better solution for computing that first time step employs the centered difference form

$$\frac{u_m^1 - u_m^{-1}}{2\Delta t} = g(x_m) \tag{7.6.14}$$

along with the wave equation

$$\frac{u_m^1 - 2u_m^0 + u_m^{-1}}{(\Delta t)^2} = c^2 \frac{u_{m+1}^0 - 2u_m^0 + u_{m-1}^0}{(\Delta x)^2} \tag{7.6.15}$$

so that

$$u_m^1 = \left(\frac{c\Delta t}{\Delta x}\right)^2 \frac{f(x_{m+1}) + f(x_{m-1})}{2} + \left[1 - \left(\frac{c\Delta t}{\Delta x}\right)^2\right] f(x_m) + \Delta t g(x_m). \tag{7.6.16}$$

Although it appears that we are ready to start calculating, we need to check whether our numerical scheme possesses three properties: convergence, stability, and consistency. By *consistency* we mean that the difference equations approach the differential equation as $\Delta x, \Delta t \to 0$. To prove consistency, we first write u_{m+1}^n, u_{m-1}^n, u_m^{n-1}, and u_m^{n+1} in terms of $u(x,t)$ and its derivatives evaluated at (x_m, t_n). From Taylor expansions,

$$u_{m+1}^n = u_m^n + \Delta x \left.\frac{\partial u}{\partial x}\right|_n^m + \tfrac{1}{2}(\Delta x)^2 \left.\frac{\partial^2 u}{\partial x^2}\right|_n^m + \tfrac{1}{6}(\Delta x)^3 \left.\frac{\partial^3 u}{\partial x^3}\right|_n^m + \cdots, \tag{7.6.17}$$

$$u_{m-1}^n = u_m^n - \Delta x \left.\frac{\partial u}{\partial x}\right|_n^m + \tfrac{1}{2}(\Delta x)^2 \left.\frac{\partial^2 u}{\partial x^2}\right|_n^m - \tfrac{1}{6}(\Delta x)^3 \left.\frac{\partial^3 u}{\partial x^3}\right|_n^m + \cdots, \tag{7.6.18}$$

$$u_m^{n+1} = u_m^n + \Delta t \left.\frac{\partial u}{\partial t}\right|_n^m + \tfrac{1}{2}(\Delta t)^2 \left.\frac{\partial^2 u}{\partial t^2}\right|_n^m + \tfrac{1}{6}(\Delta t)^3 \left.\frac{\partial^3 u}{\partial t^3}\right|_n^m + \cdots \tag{7.6.19}$$

and

$$u_m^{n-1} = u_m^n - \Delta t \left.\frac{\partial u}{\partial t}\right|_n^m + \tfrac{1}{2}(\Delta t)^2 \left.\frac{\partial^2 u}{\partial t^2}\right|_n^m - \tfrac{1}{6}(\Delta t)^3 \left.\frac{\partial^3 u}{\partial t^3}\right|_n^m + \cdots \tag{7.6.20}$$

Substituting (7.6.17)–(7.6.20) into (7.6.10), we obtain

$$\begin{aligned} \frac{u_m^{n+1} - 2u_m^n + u_m^{n-1}}{(\Delta t)^2} - c^2 \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{(\Delta x)^2} &= \left.\left(\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2}\right)\right|_n^m \\ &\quad + \tfrac{1}{12}(\Delta t)^2 \left.\frac{\partial^4 u}{\partial t^4}\right|_n^m - \tfrac{1}{12}(c\Delta x)^2 \left.\frac{\partial^4 u}{\partial x^4}\right|_n^m + \cdots \end{aligned} \tag{7.6.21}$$

The first term on the right side of (7.6.21) vanishes because $u(x,t)$ satisfies the wave equation. As $\Delta x \to 0$, $\Delta t \to 0$, the remaining terms on the right side of (7.6.21) tend to zero and (7.6.10) is a consistent finite difference approximation of the wave equation.

Stability is another question. Under certain conditions the small errors inherent in fixed precision arithmetic (round off) can grow for certain choices of Δx and Δt. During the 1920s the mathematicians Courant, Friedrichs, and Lewy[22] found that if $c\Delta t/\Delta x > 1$, then our scheme is unstable. This CFL criteria has its origin in the fact that if $c\Delta t > \Delta x$, then we are asking signals in the numerical scheme to travel faster than their real-world counterparts and this unrealistic expectation leads to instability!

One method of determining *stability*, commonly called the von Neumann method,[23] involves examining solutions to (7.6.11) that have the form

$$u_m^n = e^{im\theta} e^{in\lambda}, \tag{7.6.22}$$

where θ is an arbitrary real number and λ is a complex number that has yet to be determined. Our choice of (7.6.22) is motivated by the fact that the initial condition u_m^0 can be represented by a Fourier series where a typical term behaves as $e^{im\theta}$.

If we substitute (7.6.22) into (7.6.10) and divide out the common factor $e^{im\theta}e^{in\lambda}$, we have that

$$\frac{e^{i\lambda} - 2 + e^{-i\lambda}}{(\Delta t)^2} = c^2 \frac{e^{i\theta} - 2 + e^{-i\theta}}{(\Delta x)^2} \tag{7.6.23}$$

or

$$\sin^2\left(\frac{\lambda}{2}\right) = \left(\frac{c\Delta t}{\Delta x}\right)^2 \sin^2\left(\frac{\theta}{2}\right). \tag{7.6.24}$$

The behavior of u_m^n is determined by the values of λ given by (7.6.24). If $c\Delta t/\Delta x \le 1$, then λ is real and u_m^n is bounded for all θ as $n \to \infty$. If $c\Delta t/\Delta x > 1$, then it is possible to find a value of θ such that the right side of (7.6.24) exceeds unity and the corresponding λ's occur as complex conjugate pairs. The λ with the negative imaginary part produces a solution with exponential growth because $n = t_n/\Delta t \to \infty$ as $\Delta t \to 0$ for a fixed t_n and $c\Delta t/\Delta x$. Thus, the value of u_m^n becomes infinitely large, even though the initial data may be arbitrarily small.

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[22] Courant, R., Friedrichs, K. O., and Lewy, H., 1928: Über die partiellen Differenzengleichungen der mathematischen Physik. *Math. Annalen*, **100**, 32–74. Translated into English in *IBM J. Res. Dev.*, **11**, 215–234.

[23] After its inventor, J. von Neumann. See O'Brien, G. G., Hyman, M. A., and Kaplan, S., 1950: A study of the numerical solution of partial differential equations. *J. Math. Phys.*, **29**, 223–251.

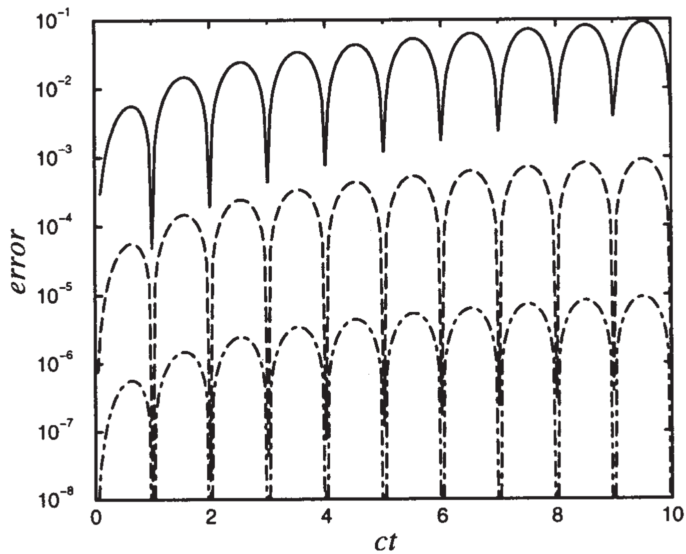

Figure 7.6.2: The growth of error $||e_n||$ as a function of ct for various resolutions. For the top line, $\Delta x = 0.1$; for the middle line, $\Delta x = 0.01$; and for the bottom line, $\Delta x = 0.001$.

Finally, we must check for convergence. A numerical scheme is *convergent* if the numerical solution approaches the continuous solution as $\Delta x, \Delta t \to 0$. The general procedure for proving convergence involves the evolution of the error term e_m^n which gives the difference between the true solution $u(x_m, t_n)$ and the finite difference solution u_m^n. From (7.6.21),

$$e_m^{n+1} = \left(\frac{c\Delta t}{\Delta x}\right)^2 (e_{m+1}^n + e_{m-1}^n) + 2\left[1 - \left(\frac{c\Delta t}{\Delta x}\right)^2\right] e_m^n - e_m^{n-1} + O[(\Delta t)^4] + O[(\Delta x)^2(\Delta t)^2]. \quad \textbf{(7.6.25)}$$

Let us apply (7.6.25) to work backwards from the point (x_m, t_n) by changing n to $n-1$. The nonvanishing terms in e_m^n reduce to a sum of $n+1$ values on the line $n = 1$ plus $\frac{1}{2}(n+1)n$ terms of the form $A(\Delta x)^4$. If we define the max norm $||e_n|| = \max_m |e_m^n|$, then

$$||e_n|| \le nB(\Delta x)^3 + \tfrac{1}{2}(n+1)nA(\Delta x)^4. \quad \textbf{(7.6.26)}$$

Because $n\Delta x \le ct_n$, (7.6.26) simplifies to

$$||e_n|| \le ct_n B(\Delta x)^2 + \tfrac{1}{2}c^2 t_n^2 A(\Delta x)^2. \quad \textbf{(7.6.27)}$$

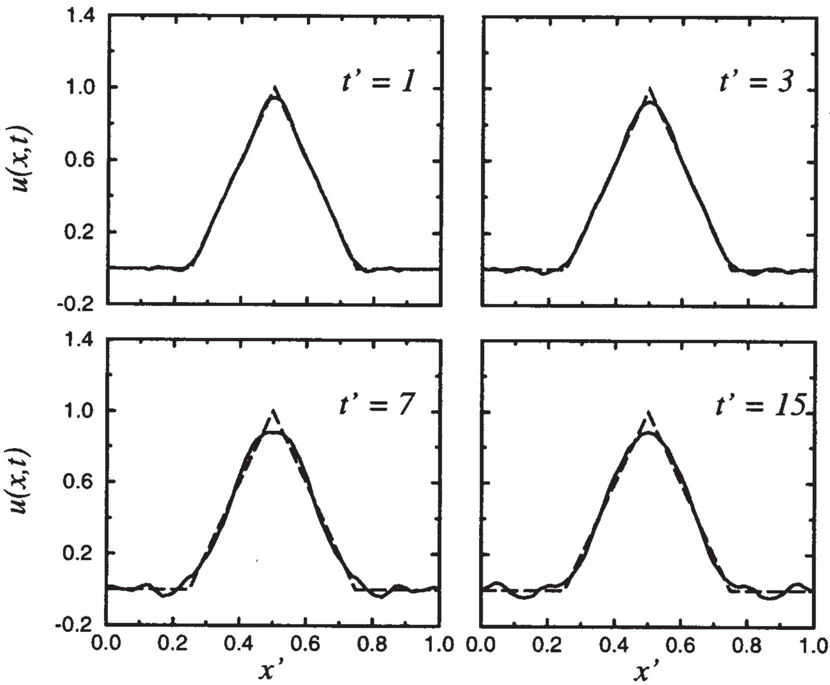

Figure 7.6.3: The numerical solution $u(x,t)/h$ of the wave equation with $c\Delta t/\Delta x = \frac{1}{2}$ using (7.6.11) at various positions $x' = x/L$ and times $t' = ct/L$. We have plotted the exact solution as a dashed line.

Thus, the error tends to zero as $\Delta x \to 0$, verifying convergence. We have illustrated (7.6.27) by using the finite difference equation (7.6.11) to compute $||e_n||$ during a numerical experiment that used $c\Delta t/\Delta x = 0.5$, $f(x) = \sin(\pi x)$ and $g(x) = 0$. Note how each increase of resolution by 10 results in a drop in the error by 100.

In the following examples we apply our scheme to solve a few simple initial and boundary conditions:

• **Example 7.6.1**

For our first example, we resolve (7.3.1) – (7.3.3) and (7.3.25) – (7.3.26) numerically using (7.6.11) with $c\Delta t/\Delta x = 1/2$ and $\Delta x = 0.01$. Figure 7.6.3 shows the resulting numerical solution at the nondimensional times $ct/L = 1, 3, 7$, and 15. We also included the exact solution as a dashed line.

Overall, the numerical solution approximates the exact or analytic solution well. However, we note small-scale noise in the numerical solution. Why does this occur? Recall that the exact solution could be written as an infinite sum of sines in the x dimension. Each successive

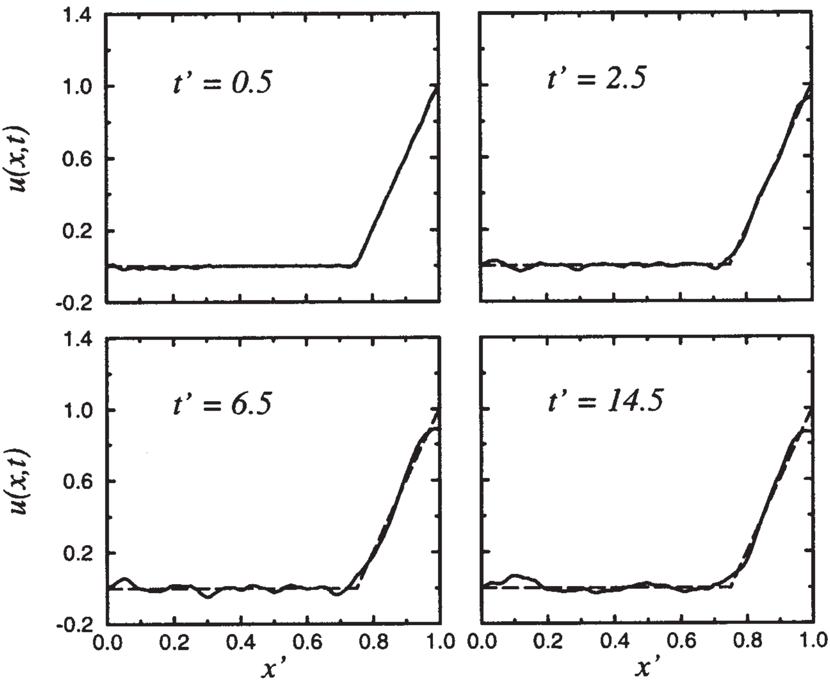

Figure 7.6.4: The numerical solution $u(x,t)/h$ of the wave equation when the right end moves freely with $c\Delta t/\Delta x = \frac{1}{2}$ using (7.6.11) and (7.6.30) at various positions $x' = x/L$ and times $t' = ct/L$. We have plotted the exact solution as a dashed line.

harmonic adds a contribution from waves of shorter and shorter wavelength. In the case of the numerical solution, the longer-wavelength harmonics are well represented by the numerical scheme because there are many grid points available to resolve a given wavelength. As the wavelengths become shorter, the higher harmonics are poorly resolved by the numerical scheme, move at incorrect phase speeds, and their misplacement (dispersion) creates the small-scale noise that you observe rather than giving the sharp angular features of the exact solution. The only method for avoiding this problem is to devise schemes that resolve the smaller-scale waves better.

• **Example 7.6.2**

Let us redo Example 7.6.1 except that we will introduce the boundary condition that $u_x(L,t) = 0$. This corresponds to a string where we fix the left end and allow the right end to freely move up and down. This requires a new difference condition along the right boundary. If we employ centered differencing,

$$\frac{u_{L+1}^n - u_{L-1}^n}{2\Delta x} = 0 \tag{7.6.28}$$

and

$$u_L^{n+1} = 2u_L^n - u_L^{n-1} + \left(\frac{c\Delta t}{\Delta x}\right)^2 \left(u_{L+1}^n - 2u_L^n + u_{L-1}^n\right). \tag{7.6.29}$$

Eliminating u_{L+1}^n between (7.6.28)–(7.6.29),

$$u_L^{n+1} = 2u_L^n - u_L^{n-1} + \left(\frac{c\Delta t}{\Delta x}\right)^2 \left(2u_{L-1}^n - 2u_L^n\right). \tag{7.6.30}$$

Figure 7.6.4 is the same as Figure 7.6.3 except for the new boundary condition. In this case the exact solution is

$$\begin{aligned} u(x,t) = \frac{32h}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \\ \times \left\{2\sin\left[\frac{(2n-1)\pi}{4}\right] - \sin\left[\frac{3(2n-1)\pi}{8}\right] - \sin\left[\frac{(2n-1)\pi}{8}\right]\right\} \\ \times \sin\left[\frac{(2n-1)\pi x}{2L}\right] \cos\left[\frac{(2n-1)\pi ct}{2L}\right]. \end{aligned} \tag{7.6.31}$$

We have highlighted those times when the solution has its maximum amplitude at the free right end. The results are consistent with those presented in Example 7.6.1, especially the small-scale noise due the dispersion. Overall, however, the numerical solution does approximate the exact solution well.

Project: Numerical Solution of First-Order Hyperbolic Equations

The equation $u_t + u_x = 0$ is the simplest possible hyperbolic partial differential equation. Indeed the classic wave equation can be written as a system of these equations: $u_t + cv_x = 0$ and $v_t + cu_x = 0$. In this project you will examine several numerical schemes for solving such a partial differential equation.

Step 1: One of the simplest numerical schemes is the forward-in-time, centered-in-space of

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} + \frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x} = 0.$$

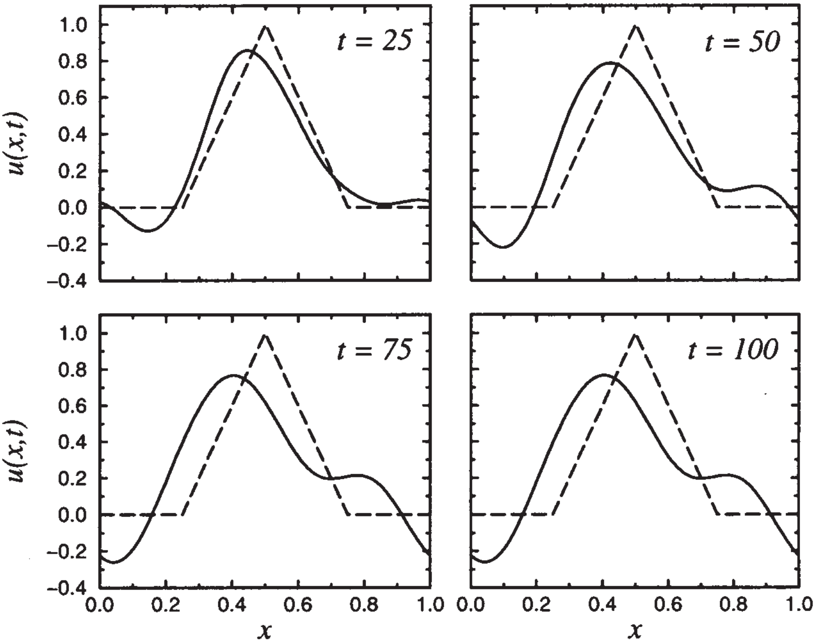

Figure 7.6.5: The numerical solution $u(x,t)$ of the first-order hyperbolic partial differential equation $u_t + u_x = 0$ using the Lax-Wendroff formula. The initial conditions are given by (7.3.25) with $h = 1$ and $\Delta t/\Delta x = \frac{1}{2}$. We have plotted the exact solution as a dashed line.

Use von Neumann's stability analysis to show that this scheme is *always* unstable.

Step 2: The most widely used method for numerically integrating first-order hyperbolic equations is the *Lax-Wendroff* method:

$$u_m^{n+1} = u_m^n - \frac{\Delta t}{2\Delta x}\left(u_{m+1}^n - u_{m-1}^n\right) + \frac{(\Delta t)^2}{2(\Delta x)^2}\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right).$$

This methods introduces errors of $O[(\Delta t)^2]$ and $O[(\Delta x)^2]$. Show that this scheme is stable if it satisfies the CFL criteria of $\Delta t/\Delta x \le 1$.

Using the initial condition given by (7.3.25), write code that uses this scheme to numerically integrate $u_t + u_x = 0$. Plot the results over the interval $0 < x < 1$ given the *periodic* boundary conditions of $u(0,t) = u(1,t)$ for the temporal interval $0 < t \le 100$. Discuss the strengths and weaknesses of the scheme with respect to dissipation or damping of the numerical solution and preserving the phase of the solution. Most numerical methods books will discuss this.[24]

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[24] For example, Lapidus, L. and Pinder, G. F., 1982: *Numerical Solu-*

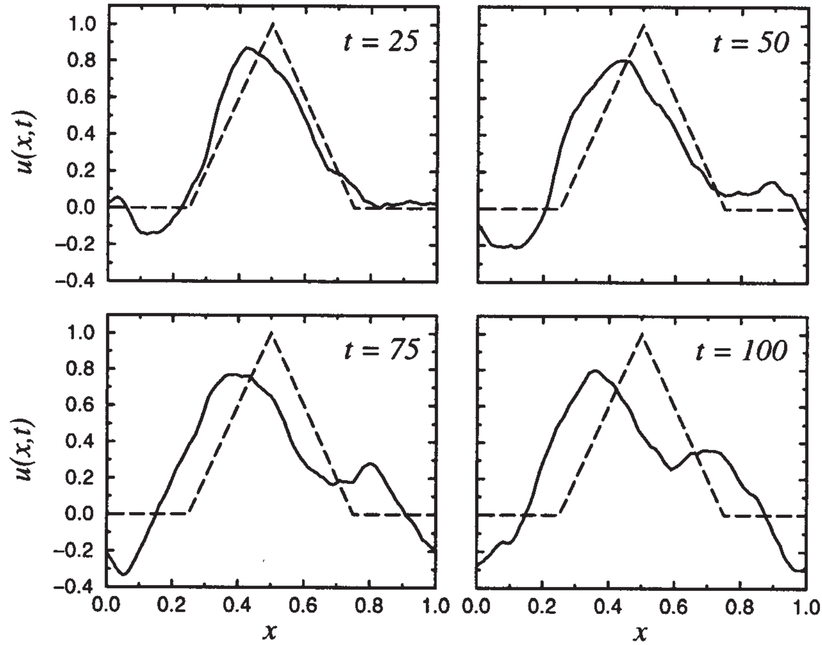

Figure 7.6.6: Same as Figure 7.6.5 except that the centered-in-time, centered-in-space scheme was used.

Step 3: Another simple scheme is the centered-in-time, centered-in-space of

$$\frac{u_m^{n+1} - u_m^{n-1}}{2\Delta t} + \frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x} = 0.$$

This methods introduces errors of $O[(\Delta t)^2]$ and $O[(\Delta x)^2]$. Show that this scheme is stable if it satisfies the CFL criteria of $\Delta t/\Delta x \leq 1$.

Using the initial condition given by (7.3.25), write code that uses this scheme to numerically integrate $u_t + u_x = 0$ over the interval $0 < x < 1$ given the *periodic* boundary conditions of $u(0,t) = u(1,t)$. Plot the results over the spatial interval for the temporal interval $0 < t \leq 100$. One of the difficulties is taking the first time step. Use the scheme in Step 1 to take this first time step. Discuss the strengths and weaknesses of the scheme with respect to dissipation or damping of the numerical solution and preserving the phase of the solution.

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tion of Partial Differential Equations in Science and Engineering, John Wiley & Sons, New York.

Chapter 8

The Heat Equation

In this chapter we deal with the linear parabolic differential equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{8.0.1}$$

in the two independent variables x and t. This equation, known as the one-dimensional heat equation, serves as the prototype for a wider class of *parabolic equations*:

$$a(x,t)\frac{\partial^2 u}{\partial x^2} + b(x,t)\frac{\partial^2 u}{\partial x \partial t} + c(x,t)\frac{\partial^2 u}{\partial t^2} = f\left(x,t,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}\right), \tag{8.0.2}$$

where $b^2 = 4ac$. It arises in the study of heat conduction in solids as well as in a variety of diffusive phenomena. The heat equation is similar to the wave equation in that it is also an equation of evolution. However, the heat equation is not "conservative" because if we reverse the sign of t, we obtain a different solution. This reflects the presence of entropy which must always increase during heat conduction.

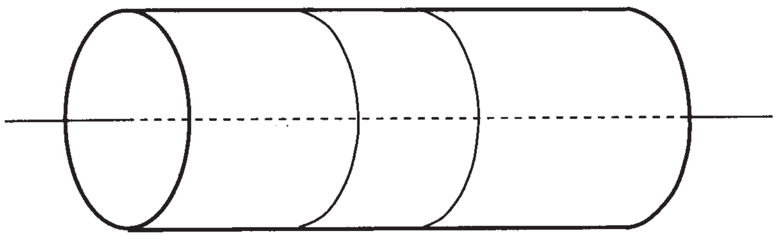

Figure 8.1.1: Heat conduction in a thin bar.

8.1 DERIVATION OF THE HEAT EQUATION

To derive the heat equation, consider a heat-conducting homogeneous rod, extending from $x = 0$ to $x = L$ along the x-axis (see Figure 8.1.1). The rod has uniform cross section A and constant density ρ, is insulated laterally so that heat flows only in the x-direction and is sufficiently thin so that the temperature at all points on a cross section is constant. Let $u(x,t)$ denote the temperature of the cross section at the point x at any instant of time t, and let c denote the specific heat of the rod (the amount of heat required to raise the temperature of a unit mass of the rod by a degree). In the segment of the rod between the cross section at x and the cross section at $x + \Delta x$, the amount of heat is

$$Q(t) = \int_x^{x+\Delta x} c\rho A u(s,t)\, ds. \tag{8.1.1}$$

On the other hand, the rate at which heat flows into the segment across the cross section at x is proportional to the cross section and the gradient of the temperature at the cross section (Fourier's law of heat conduction):

$$-\kappa A \frac{\partial u(x,t)}{\partial x}, \tag{8.1.2}$$

where κ denotes the thermal conductivity of the rod. The sign in (8.1.2) indicates that heat flows in the direction of decreasing temperature. Similarly, the rate at which heat flows out of the segment through the cross section at $x + \Delta x$ equals

$$-\kappa A \frac{\partial u(x+\Delta x,t)}{\partial x}. \tag{8.1.3}$$

The difference between the amount of heat that flows in through the cross section at x and the amount of heat that flows out through the cross section at $x + \Delta x$ must equal the change in the heat content of the segment $x \leq s \leq x + \Delta x$. Hence, by subtracting (8.1.3) from (8.1.2) and equating the result to the time derivative of (8.1.1),

$$\frac{\partial Q}{\partial t} = \int_x^{x+\Delta x} c\rho A \frac{\partial u(s,t)}{\partial t}\, ds = \kappa A \left[\frac{\partial u(x+\Delta x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right]. \tag{8.1.4}$$

Assuming that the integrand in (8.1.4) is a continuous function of s, then by the mean value theorem for integrals,

$$\int_x^{x+\Delta x} \frac{\partial u(s,t)}{\partial t}\, ds = \frac{\partial u(\xi,t)}{\partial t}\Delta x, \qquad x < \xi < x + \Delta x, \tag{8.1.5}$$

so that (8.1.4) becomes

$$c\rho \Delta x \frac{\partial u(\xi,t)}{\partial t} = \kappa \left[\frac{\partial u(x+\Delta x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right]. \tag{8.1.6}$$

Dividing both sides of (8.1.6) by $c\rho\Delta x$ and taking the limit as $\Delta x \to 0$,

$$\frac{\partial u(x,t)}{\partial t} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} \tag{8.1.7}$$

with $a^2 = \kappa/(c\rho)$. Equation (8.1.7) is called the one-dimensional *heat equation*. The constant a^2 is called the *diffusivity* within the solid.

If an external source supplies heat to the rod at a rate $f(x,t)$ per unit volume per unit time, we must add the term $\int_x^{x+\Delta x} f(s,t)\, ds$ to the time derivative term of (8.1.4). Thus, in the limit $\Delta x \to 0$,

$$\frac{\partial u(x,t)}{\partial t} - a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = F(x,t), \tag{8.1.8}$$

where $F(x,t) = f(x,t)/(c\rho)$ is the source density. This equation is called the *nonhomogeneous heat equation.*

8.2 INITIAL AND BOUNDARY CONDITIONS

In the case of heat conduction in a thin rod, the temperature function $u(x,t)$ must satisfy not only the heat equation (8.1.7) but also how the two ends of the rod exchange heat energy with the surrounding medium. If (1) there is no heat source, (2) the function $f(x)$, $0 < x < L$ describes the temperature in the rod at $t = 0$, and (3) we maintain both ends at zero temperature for all time, then the partial differential equation

$$\frac{\partial u(x,t)}{\partial t} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad 0 < x < L,\ 0 < t \tag{8.2.1}$$

describes the temperature distribution $u(x,t)$ in the rod at any later time $0 < t$ subject to the condition

$$u(x,0) = f(x), \qquad 0 < x < L \tag{8.2.2}$$

and

$$u(0,t) = u(L,t) = 0, \qquad 0 < t. \tag{8.2.3}$$

Equations (8.2.1)–(8.2.3) describe the *initial-boundary value problem* for this particular heat conduction problem; (8.2.3) is the boundary condition while (8.2.2) gives the initial condition. Note that in the case of the heat equation, the problem only demands the initial value of $u(x,t)$ and not $u_t(x,0)$, as with the wave equation.

Historically most linear boundary conditions have been classified in one of three ways. The condition (8.2.3) is an example of a *Dirichlet problem*[1] or *condition of the first kind.* This type of boundary condition gives the value of the solution (which is not necessarily equal to zero) along a boundary.

The next simplest condition involves derivatives. If we insulate both ends of the rod so that no heat flows from the ends, then according to (8.1.2) the boundary condition assumes the form

$$\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(L,t)}{\partial x} = 0, \qquad 0 < t. \tag{8.2.4}$$

This is an example of a *Neumann problem*[2] or *condition of the second kind.* This type of boundary condition specifies the value of the normal derivative (which may not be equal to zero) of the solution along the boundary.

Finally, if there is radiation of heat from the ends of the rod into the surrounding medium, we shall show that the boundary condition is of the form

$$\frac{\partial u(0,t)}{\partial x} - hu(0,t) = \text{a constant} \tag{8.2.5}$$

and

$$\frac{\partial u(L,t)}{\partial x} + hu(L,t) = \text{another constant} \tag{8.2.6}$$

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[1] Dirichlet, P. G. L., 1850: Über einen neuen Ausdruck zur Bestimmung der Dichtigkeit einer unendlich dünnen Kugelschale, wenn der Werth des Potentials derselben in jedem Punkte ihrer Oberfläche gegeben ist. *Abh. Königlich. Preuss. Akad. Wiss.*, 99–116.

[2] Neumann, C. G., 1877: *Untersuchungen über das Logarithmische und Newton'sche Potential.* Leibzig.

for $0 < t$, where h is a positive constant. This is an example of a *condition of the third kind* or *Robin problem*[3] and is a linear combination of Dirichlet and Neumann conditions.

8.3 SEPARATION OF VARIABLES

As with the wave equation, the most popular and widely used technique for solving the heat equation is separation of variables. Its success depends on our ability to express the solution $u(x,t)$ as the product $X(x)T(t)$. If we cannot achieve this separation, then the technique must be abandon for others. In the following examples we show how to apply this technique even if it takes a little work to get it right.

• Example 8.3.1

Let us find the solution to the homogeneous heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, 0 < t \tag{8.3.1}$$

which satisfies the initial condition

$$u(x,0) = f(x), \quad 0 < x < L \tag{8.3.2}$$

and the boundary conditions

$$u(0,t) = u(L,t) = 0, \quad 0 < t. \tag{8.3.3}$$

This system of equations models heat conduction in a thin metallic bar where both ends are held at the constant temperature of zero and the bar initially has the temperature $f(x)$.

We shall solve this problem by the method of separation of variables. Accordingly, we seek particular solutions of (8.3.1) of the form

$$u(x,t) = X(x)T(t), \tag{8.3.4}$$

which satisfy the boundary conditions (8.3.3). Because

$$\frac{\partial u}{\partial t} = X(x)T'(t) \tag{8.3.5}$$

and

$$\frac{\partial^2 u}{\partial x^2} = X''(x)T(t), \tag{8.3.6}$$

[3] Robin, G., 1886: Sur la distribution de l'électricité à la surface des conducteurs fermés et des conducteurs ouverts. *Ann. Sci. l'Ecole Norm. Sup., Ser. 3*, **3**, S1–S58.

(8.3.1) becomes

$$T'(t)X(x) = a^2X''(x)T(t). \tag{8.3.7}$$

Dividing both sides of (8.3.7) by $a^2X(x)T(t)$ gives

$$\frac{T'}{a^2T} = \frac{X''}{X} = -\lambda, \tag{8.3.8}$$

where $-\lambda$ is the separation constant. Equation (8.3.8) immediately yields two ordinary differential equations:

$$X'' + \lambda X = 0 \tag{8.3.9}$$

and

$$T' + a^2\lambda T = 0 \tag{8.3.10}$$

for the functions $X(x)$ and $T(t)$, respectively.

We now rewrite the boundary conditions in terms of $X(x)$ by noting that the boundary conditions are $u(0,t) = X(0)T(t) = 0$ and $u(L,t) = X(L)T(t) = 0$ for $0 < t$. If we were to choose $T(t) = 0$, then we would have a trivial solution for $u(x,t)$. Consequently, $X(0) = X(L) = 0$.

There are three possible cases: $\lambda = -m^2$, $\lambda = 0$, and $\lambda = k^2$. If $\lambda = -m^2 < 0$, then we must solve the boundary-value problem:

$$X'' - m^2X = 0, \qquad X(0) = X(L) = 0. \tag{8.3.11}$$

The general solution to (8.3.11) is

$$X(x) = A\cosh(mx) + B\sinh(mx). \tag{8.3.12}$$

Because $X(0) = 0$, it follows that $A = 0$. The condition $X(L) = 0$ yields $B\sinh(mL) = 0$. Since $\sinh(mL) \neq 0$, $B = 0$ and we have a trivial solution for $\lambda < 0$.

If $\lambda = 0$, the corresponding boundary-value problem is

$$X''(x) = 0, \qquad X(0) = X(L) = 0. \tag{8.3.13}$$

The general solution is

$$X(x) = C + Dx. \tag{8.3.14}$$

From $X(0) = 0$, we have that $C = 0$. From $X(L) = 0$, $DL = 0$ or $D = 0$. Again, we obtain a trivial solution.

Finally, we assume that $\lambda = k^2 > 0$. The corresponding boundary-value problem is

$$X'' + k^2X = 0, \qquad X(0) = X(L) = 0. \tag{8.3.15}$$

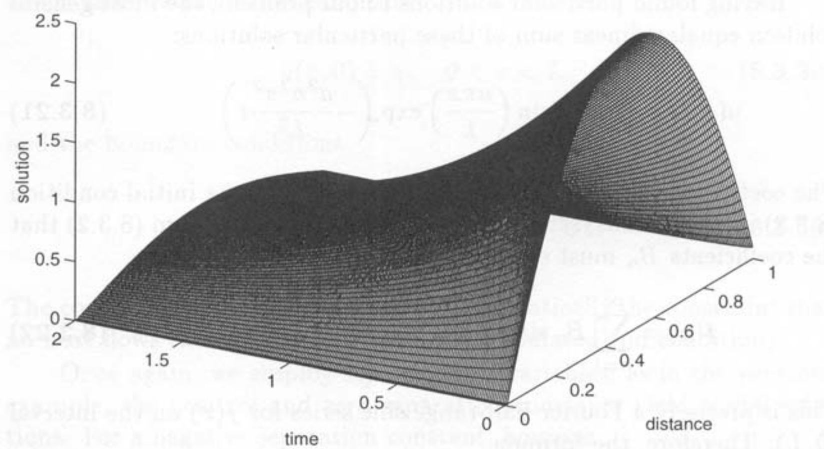

Figure 8.3.1: The temperature $u(x,t)$ within a thin bar as a function of position x/π and time a^2t when we maintain both ends at zero and the initial temperature equals $x(\pi - x)$.

The general solution to (8.3.15) is

$$X(x) = E\cos(kx) + F\sin(kx). \tag{8.3.16}$$

Because $X(0) = 0$, it follows that $E = 0$; from $X(L) = 0$, we obtain $F\sin(kL) = 0$. To have a nontrivial solution, $F \neq 0$ and $\sin(kL) = 0$. This implies that $k_nL = n\pi$, where $n = 1, 2, 3, \ldots$. In summary, the x-dependence of the solution is

$$X_n(x) = F_n \sin\left(\frac{n\pi x}{L}\right), \tag{8.3.17}$$

where $\lambda_n = n^2\pi^2/L^2$.

Turning to the time dependence, we use $\lambda_n = n^2\pi^2/L^2$ in (8.3.10):

$$T_n' + \frac{a^2n^2\pi^2}{L^2}T_n = 0. \tag{8.3.18}$$

The corresponding general solution is

$$T_n(t) = G_n \exp\left(-\frac{a^2n^2\pi^2}{L^2}t\right). \tag{8.3.19}$$

Thus, the functions

$$u_n(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{a^2n^2\pi^2}{L^2}t\right), n = 1, 2, 3, \ldots, \tag{8.3.20}$$

where $B_n = F_nG_n$, are particular solutions of (8.3.1) and satisfy the homogeneous boundary conditions (8.3.3).

As we noted in the case of wave equation, we can solve the x-dependence equation as a regular Sturm-Liouville problem. After finding the eigenvalue λ_n and eigenfunction, we solve for $T_n(t)$. The product solution $u_n(x,t)$ equals the product of the eigenfunction and $T_n(t)$.

Having found particular solutions to our problem, the most general solution equals a linear sum of these particular solutions:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{a^2n^2\pi^2}{L^2}t\right). \qquad (8.3.21)$$

The coefficient B_n is chosen so that (8.3.21) yields the initial condition (8.3.2) if $t = 0$. Thus, setting $t = 0$ in (8.3.21), we see from (8.3.2) that the coefficients B_n must satisfy the relationship

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right), \qquad 0 < x < L. \qquad (8.3.22)$$

This is precisely a Fourier half-range sine series for $f(x)$ on the interval $(0, L)$. Therefore, the formula

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \qquad n = 1, 2, 3, \ldots \qquad (8.3.23)$$

gives the coefficients B_n. For example, if $L = \pi$ and $u(x,0) = x(\pi - x)$, then

$$B_n = \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin(nx)\,dx \qquad (8.3.24)$$

$$= 2\int_0^\pi x\sin(nx)\,dx - \frac{2}{\pi}\int_0^\pi x^2\sin(nx)\,dx \qquad (8.3.25)$$

$$= 4\frac{1-(-1)^n}{n^3\pi}. \qquad (8.3.26)$$

Hence,

$$u(x,t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)x]}{(2n-1)^3}e^{-(2n-1)^2a^2t}. \qquad (8.3.27)$$

Figure 8.3.1 illustrates (8.3.27) for various times. Note that both ends of the bar satisfy the boundary conditions, namely that the temperature equals zero. As time increases, heat flows out from the center of the bar to both ends where it is removed. This process is reflected in the collapse of the original parabolic shape of the temperature profile towards zero as time increases.

- **Example 8.3.2**

As a second example, let us solve the heat equation

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, 0 < t \qquad (8.3.28)$$

which satisfies the initial condition

$$u(x,0) = x, \quad 0 < x < L \tag{8.3.29}$$

and the boundary conditions

$$\frac{\partial u(0,t)}{\partial x} = u(L,t) = 0, \quad 0 < t. \tag{8.3.30}$$

The condition $u_x(0,t) = 0$ expresses mathematically the constraint that no heat flows through the left boundary (insulated end condition).

Once again, we employ separation of variables; as in the previous example, the positive and zero separation constants yield trivial solutions. For a negative separation constant, however,

$$X'' + k^2 X = 0 \tag{8.3.31}$$

with

$$X'(0) = X(L) = 0, \tag{8.3.32}$$

because $u_x(0,t) = X'(0)T(t) = 0$ and $u(L,t) = X(L)T(t) = 0$. This regular Sturm-Liouville problem has the solution

$$X_n(x) = \cos\left[\frac{(2n-1)\pi x}{2L}\right], \qquad n = 1, 2, 3, \ldots \tag{8.3.33}$$

The temporal solution then becomes

$$T_n(t) = B_n \exp\left[-\frac{a^2(2n-1)^2\pi^2 t}{4L^2}\right]. \tag{8.3.34}$$

Consequently, a linear superposition of the particular solutions gives the total solution which equals

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos\left[\frac{(2n-1)\pi x}{2L}\right] \exp\left[-\frac{a^2(2n-1)^2\pi^2}{4L^2}t\right]. \tag{8.3.35}$$

Our final task remains to find the B_n's. Evaluating (8.3.35) at $t = 0$,

$$u(x,0) = x = \sum_{n=1}^{\infty} B_n \cos\left[\frac{(2n-1)\pi x}{2L}\right], \qquad 0 < x < L. \tag{8.3.36}$$

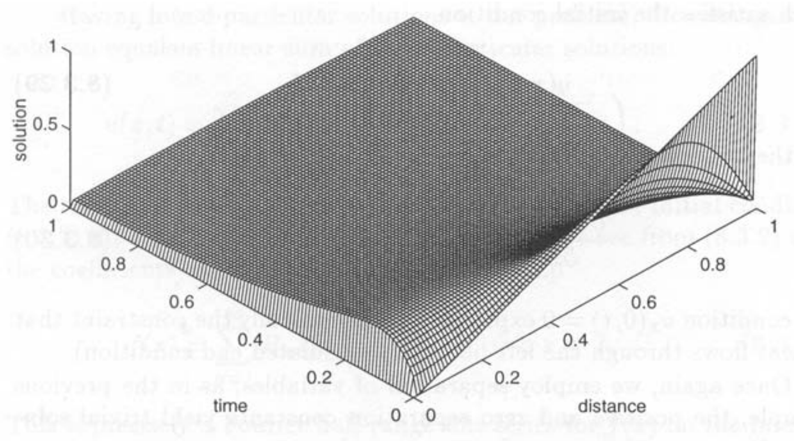

Figure 8.3.2: The temperature $u(x,t)/L$ within a thin bar as a function of position x/L and time a^2t/L^2 when we insulate the left end and hold the right end at the temperature of zero. The initial temperature equals x.

Equation (8.3.36) is not a half-range cosine expansion; it is an expansion in the orthogonal functions $\cos[(2n-1)\pi x/(2L)]$ corresponding to the regular Sturm-Liouville problem (8.3.31)–(8.3.32). Consequently, B_n is given by (6.3.4) with $r(x)=1$ as

$$B_n = \frac{\int_0^L x\cos[(2n-1)\pi x/(2L)]\,dx}{\int_0^L \cos^2[(2n-1)\pi x/(2L)]\,dx} \tag{8.3.37}$$

$$= \frac{\frac{4L^2}{(2n-1)^2\pi^2}\cos\left[\frac{(2n-1)\pi x}{2L}\right]\Big|_0^L + \frac{2Lx}{(2n-1)\pi}\sin\left[\frac{(2n-1)\pi x}{2L}\right]\Big|_0^L}{\frac{x}{2}\Big|_0^L + \frac{L}{2(2n-1)\pi}\sin\left[\frac{(2n-1)\pi x}{L}\right]\Big|_0^L} \tag{8.3.38}$$

$$= \frac{8L}{(2n-1)^2\pi^2}\left\{\cos\left[\frac{(2n-1)\pi}{2}\right]-1\right\} + \frac{4L}{(2n-1)\pi}\sin\left[\frac{(2n-1)\pi}{2}\right] \tag{8.3.39}$$

$$= -\frac{8L}{(2n-1)^2\pi^2} - \frac{4L(-1)^n}{(2n-1)\pi}, \tag{8.3.40}$$

as $\cos[(2n-1)\pi/2]=0$ and $\sin[(2n-1)\pi/2]=(-1)^{n+1}$. Consequently, the final solution is

$$u(x,t) = -\frac{4L}{\pi}\sum_{n=1}^{\infty}\left[\frac{2}{(2n-1)^2\pi}+\frac{(-1)^n}{2n-1}\right]\cos\left[\frac{(2n-1)\pi x}{2L}\right] \times \exp\left[-\frac{(2n-1)^2\pi^2a^2t}{4L^2}\right]. \tag{8.3.41}$$

Figure 8.3.2 illustrates the evolution of the temperature field with time. Initially, heat near the center of the bar flows towards the cooler, insulated end, resulting in an increase of temperature there. On the right side, heat flows out of the bar because the temperature is maintained at zero at $x = L$. Eventually the heat that has accumulated at the left end flows rightward because of the continual heat loss on the right end. In the limit of $t \to \infty$, all of the heat has left the bar.

• **Example 8.3.3**

A slight variation on Example 8.3.1 is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, 0 < t, \tag{8.3.42}$$

where

$$u(x,0) = u(0,t) = 0 \quad \text{and} \quad u(L,t) = \theta. \tag{8.3.43}$$

We begin by blindly employing the technique of separation of variables. Once again, we obtain the ordinary differential equation (8.3.9) and (8.3.10). The initial and boundary conditions become, however,

$$X(0) = T(0) = 0 \tag{8.3.44}$$

and

$$X(L)T(t) = \theta. \tag{8.3.45}$$

Although (8.3.44) is acceptable, (8.3.45) gives us an impossible condition because $T(t)$ cannot be constant. If it were, it would have to equal to zero by (8.3.44).

To find a way around this difficulty, suppose we wanted the solution to our problem at a time long after $t = 0$. From experience we know that heat conduction with time-independent boundary conditions eventually results in an evolution from the initial condition to some time-independent (steady-state) equilibrium. If we denote this steady-state solution by $w(x)$, it must satisfy the heat equation

$$a^2 w''(x) = 0 \tag{8.3.46}$$

and the boundary conditions

$$w(0) = 0 \quad \text{and} \quad w(L) = \theta. \tag{8.3.47}$$

We can integrate (8.3.46) immediately to give

$$w(x) = A + Bx \tag{8.3.48}$$

and the boundary condition (8.3.47) results in

$$w(x) = \frac{\theta x}{L}. \tag{8.3.49}$$

Clearly (8.3.49) cannot hope to satisfy the initial conditions; that was never expected of it. However, if we add a time-varying (transient) solution $v(x,t)$ to $w(x)$ so that

$$u(x,t) = w(x) + v(x,t), \tag{8.3.50}$$

we could satisfy the initial condition if

$$v(x,0) = u(x,0) - w(x) \tag{8.3.51}$$

and $v(x,t)$ tends to zero as $t \to \infty$. Furthermore, because $w''(x) = w(0) = 0$ and $w(L) = \theta$,

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}, \qquad 0 < x < L, 0 < t \tag{8.3.52}$$

with the boundary conditions

$$v(0,t) = 0 \quad \text{and} \quad v(L,t) = 0, \quad 0 < t. \tag{8.3.53}$$

We can solve (8.3.51), (8.3.52), and (8.3.53) by separation of variables; we did it in Example 8.3.1. However, in place of $f(x)$ we now have $u(x,0) - w(x)$ or $-w(x)$ because $u(x,0) = 0$. Therefore, the solution $v(x,t)$ is

$$v(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{a^2 n^2 \pi^2}{L^2} t\right) \tag{8.3.54}$$

with

$$B_n = \frac{2}{L} \int_0^L -w(x) \sin\left(\frac{n\pi x}{L}\right) dx \tag{8.3.55}$$

$$= \frac{2}{L} \int_0^L -\frac{\theta x}{L} \sin\left(\frac{n\pi x}{L}\right) dx \tag{8.3.56}$$

$$= -\frac{2\theta}{L^2}\left[\frac{L^2}{n^2\pi^2} \sin\left(\frac{n\pi x}{L}\right) - \frac{xL}{n\pi} \cos\left(\frac{n\pi x}{L}\right)\right]_0^L \tag{8.3.57}$$

$$= (-1)^n \frac{2\theta}{n\pi}. \tag{8.3.58}$$

Thus, the entire solution is

$$u(x,t) = \frac{\theta x}{L} + \frac{2\theta}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{a^2 n^2 \pi^2}{L^2} t\right). \tag{8.3.59}$$

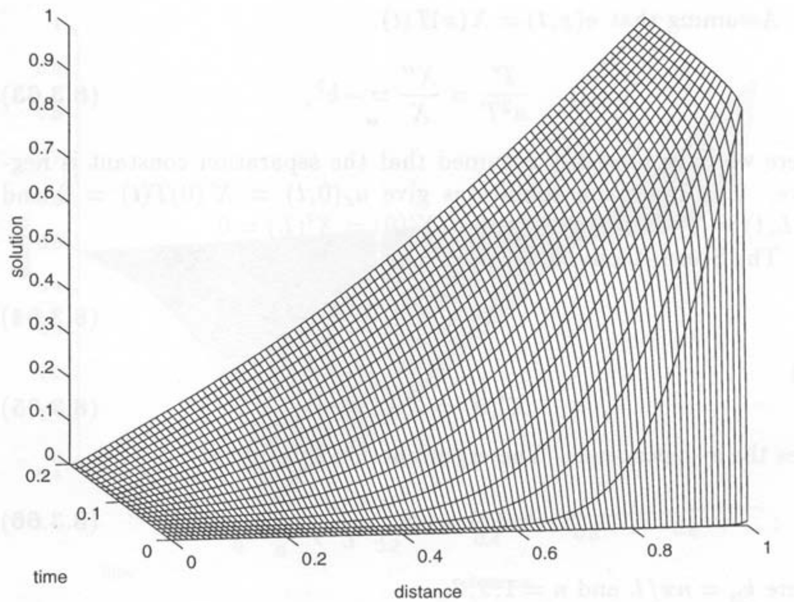

Figure 8.3.3: The temperature $u(x,t)/\theta$ within a thin bar as a function of position x/L and time a^2t/L^2 with the left end held at a temperature of zero and right end held at a temperature θ while the initial temperature of the bar is zero.

The quantity a^2t/L^2 is the *Fourier number*.

Figure 8.3.3 illustrates our solution. Clearly it satisfies the boundary conditions. Initially, heat flows rapidly from right to left. As time increases, the rate of heat transfer decreases until the final equilibrium (steady-state) is established and no more heat flows.

• **Example 8.3.4**

Let us find the solution to the heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, 0 < t \tag{8.3.60}$$

subject to the Neumann boundary conditions

$$\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(L,t)}{\partial x} = 0, \qquad 0 < t \tag{8.3.61}$$

and the initial condition that

$$u(x,0) = x, \qquad 0 < x < L. \tag{8.3.62}$$

We have now insulated *both* ends of the bar.

Assuming that $u(x,t) = X(x)T(t)$,

$$\frac{T'}{a^2T} = \frac{X''}{X} = -k^2, \tag{8.3.63}$$

where we have presently assumed that the separation constant is negative. The Neumann conditions give $u_x(0,t) = X'(0)T(t) = 0$ and $u_x(L,t) = X'(L)T(t) = 0$ so that $X'(0) = X'(L) = 0$.

The Sturm-Liouville problem

$$X'' + k^2X = 0 \tag{8.3.64}$$

and

$$X'(0) = X'(L) = 0 \tag{8.3.65}$$

gives the x-dependence. The eigenfunction solution is

$$X_n(x) = \cos\left(\frac{n\pi x}{L}\right), \tag{8.3.66}$$

where $k_n = n\pi/L$ and $n = 1, 2, 3, \ldots$

The corresponding temporal part equals the solution of

$$T_n' + a^2k_n^2T_n = T_n' + \frac{a^2n^2\pi^2}{L^2}T_n = 0, \tag{8.3.67}$$

which is

$$T_n(t) = A_n \exp\left(-\frac{a^2n^2\pi^2}{L^2}t\right). \tag{8.3.68}$$

Thus, the product solution given by a negative separation constant is

$$u_n(x,t) = X_n(x)T_n(t) = A_n \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{a^2n^2\pi^2}{L^2}t\right). \tag{8.3.69}$$

Unlike our previous problems, there is a nontrivial solution for a separation constant that equals zero. In this instance, the x-dependence equals

$$X(x) = Ax + B. \tag{8.3.70}$$

The boundary conditions $X'(0) = X'(L) = 0$ force A to be zero but B is completely free. Consequently, the eigenfunction in this particular case is

$$X_0(x) = 1. \tag{8.3.71}$$

Because $T_0'(t) = 0$ in this case, the temporal part equals a constant which we shall take to be $A_0/2$. Therefore, the product solution corresponding to the zero separation constant is

$$u_0(x,t) = X_0(x)T_0(t) = A_0/2. \tag{8.3.72}$$

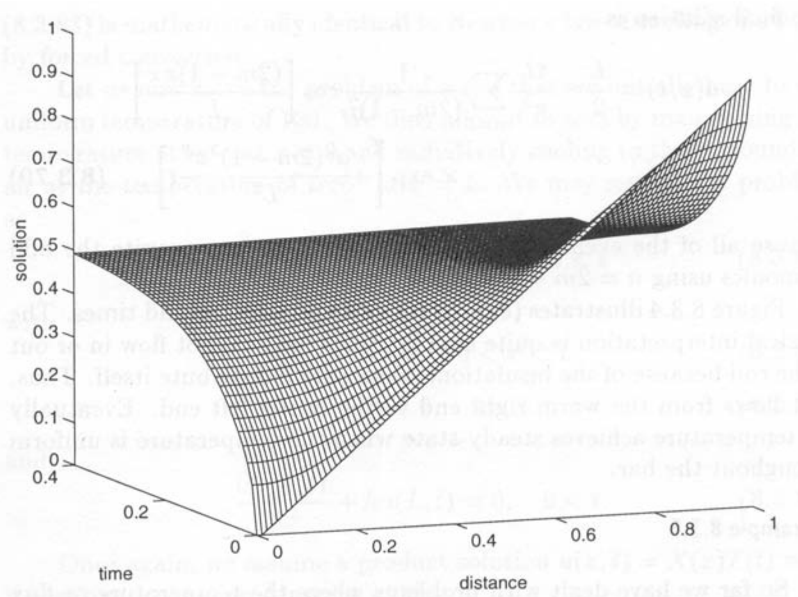

Figure 8.3.4: The temperature $u(x,t)/L$ within a thin bar as a function of position x/L and time a^2t/L^2 when we insulate both ends. The initial temperature of the bar is x.

The most general solution to our problem equals the sum of all of the possible solutions:

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{a^2n^2\pi^2}{L^2}t\right). \tag{8.3.73}$$

Upon substituting $t = 0$ into (8.3.73), we can determine A_n because

$$u(x,0) = x = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \tag{8.3.74}$$

is merely a half-range Fourier cosine expansion of the function x over the interval $(0, L)$. From (2.1.23)–(2.1.24),

$$A_0 = \frac{2}{L}\int_0^L x\,dx = L \tag{8.3.75}$$

and

$$A_n = \frac{2}{L}\int_0^L x \cos\left(\frac{n\pi x}{L}\right) dx \tag{8.3.76}$$

$$= \frac{2}{L}\left[\frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right) + \frac{xL}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_0^L \tag{8.3.77}$$

$$= \frac{2L}{n^2\pi^2}\left[(-1)^n - 1\right]. \tag{8.3.78}$$

The final solution is

$$u(x,t) = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos\left[\frac{(2m-1)\pi x}{L}\right] \times \exp\left[-\frac{a^2(2m-1)^2\pi^2}{L^2}t\right], \qquad \mathbf{(8.3.79)}$$

because all of the even harmonics vanish and we may rewrite the odd harmonics using $n = 2m - 1$, where $m = 1, 2, 3, 4, \ldots$.

Figure 8.3.4 illustrates (8.3.79) for various positions and times. The physical interpretation is quite simple. Since heat cannot flow in or out of the rod because of the insulation, it can only redistribute itself. Thus, heat flows from the warm right end to the cooler left end. Eventually the temperature achieves steady-state when the temperature is uniform throughout the bar.

• Example 8.3.5

So far we have dealt with problems where the temperature or flux of heat has been specified at the ends of the rod. In many physical applications, one or both of the ends may radiate to free space at temperature u_0. According to Stefan's law, the amount of heat radiated from a given area dA in a given time interval dt is

$$\sigma(u^4 - u_0^4)\,dA\,dt, \qquad \mathbf{(8.3.80)}$$

where σ is called the Stefan-Boltzmann constant. On the other hand, the amount of heat that reaches the surface from the interior of the body, assuming that we are at the right end of the bar, equals

$$-\kappa\frac{\partial u}{\partial x}\,dA\,dt, \qquad \mathbf{(8.3.81)}$$

where κ is the thermal conductivity. Because these quantities must be equal,

$$-\kappa\frac{\partial u}{\partial x} = \sigma(u^4 - u_0^4) = \sigma(u - u_0)(u^3 + u^2u_0 + uu_0^2 + u_0^3). \qquad \mathbf{(8.3.82)}$$

If u and u_0 are nearly equal, we may approximate the second bracketed term on the right side of (8.3.82) as $4u_0^3$. We write this approximate form of (8.3.82) as

$$-\frac{\partial u}{\partial x} = h(u - u_0), \qquad \mathbf{(8.3.83)}$$

where h, the *surface conductance* or the *coefficient of surface heat transfer*, equals $4\sigma u_0^3/\kappa$. Equation (8.3.83) is a "radiation" boundary condition. Sometimes someone will refer to it as "Newton's law" because

(8.3.83) is mathematically identical to Newton's law of cooling of a body by forced convection.

Let us now solve the problem of a rod that we initially heat to the uniform temperature of 100. We then allow it to cool by maintaining the temperature at zero at $x = 0$ and radiatively cooling to the surrounding air at the temperature of zero[4] at $x = L$. We may restate the problem as

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, 0 < t \tag{8.3.84}$$

with

$$u(x,0) = 100, \quad 0 < x < L \tag{8.3.85}$$

$$u(0,t) = 0, \quad 0 < t \tag{8.3.86}$$

and

$$\frac{\partial u(L,t)}{\partial x} + hu(L,t) = 0, \quad 0 < t. \tag{8.3.87}$$

Once again, we assume a product solution $u(x,t) = X(x)T(t)$ with a negative separation constant so that

$$\frac{X''}{X} = \frac{T'}{a^2 T} = -k^2. \tag{8.3.88}$$

We obtain for the x-dependence that

$$X'' + k^2 X = 0 \tag{8.3.89}$$

but the boundary conditions are now

$$X(0) = 0 \quad \text{and} \quad X'(L) + hX(L) = 0. \tag{8.3.90}$$

The most general solution of (8.3.89) is

$$X(x) = A\cos(kx) + B\sin(kx). \tag{8.3.91}$$

However, $A = 0$ because $X(0) = 0$. On the other hand,

$$k\cos(kL) + h\sin(kL) = kL\cos(kL) + hL\sin(kL) = 0, \tag{8.3.92}$$

if $B \neq 0$. The nondimensional number hL is the *Biot number* and is completely dependent upon the physical characteristics of the rod.

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[4] Although this would appear to make $h = 0$, we have merely chosen a temperature scale so that the air temperature is zero and the absolute temperature used in Stefan's law is nonzero.

Table 8.3.1: The First Ten Roots of (8.3.93) and C_n for $hL = 1$.

n	α_n	Approximate α_n	C_n
1	2.0288	2.2074	118.9193
2	4.9132	4.9246	31.3402
3	7.9787	7.9813	27.7554
4	11.0856	11.0865	16.2878
5	14.2075	14.2079	14.9923
6	17.3364	17.3366	10.8359
7	23.6044	23.6043	8.0989
8	26.7410	26.7409	7.7483
9	29.8786	29.8776	6.4625
10	33.0170	33.0170	6.2351

In Chapter 6 we saw how to find the roots of the transcendental equation

$$\alpha + hL\tan(\alpha) = 0, \qquad (8.3.93)$$

where $\alpha = kL$. Consequently, if α_n is the nth root of (8.3.93), then the eigenfunction is

$$X_n(x) = \sin(\alpha_n x/L). \qquad (8.3.94)$$

In Table 8.3.1, we list the first ten roots of (8.3.93) for $hL = 1$.

In general, we must solve (8.3.93) either numerically or graphically. If α is large, however, we can find approximate values by noting that

$$\cot(\alpha) = -hL/\alpha \approx 0 \qquad (8.3.95)$$

or

$$\alpha_n = (2n-1)\pi/2, \qquad (8.3.96)$$

where $n = 1, 2, 3, \ldots$ We may obtain a better approximation by setting

$$\alpha_n = (2n-1)\pi/2 - \epsilon_n, \qquad (8.3.97)$$

where $\epsilon_n \ll 1$. Substituting into (8.3.95),

$$[(2n-1)\pi/2 - \epsilon_n]\cot[(2n-1)\pi/2 - \epsilon_n] + hL = 0. \qquad (8.3.98)$$

We can simplify (8.3.98) to

$$\epsilon_n^2 + (2n-1)\pi\epsilon_n/2 + hL = 0 \qquad (8.3.99)$$

because $\cot[(2n-1)\pi/2 - \theta] = \tan(\theta)$ and $\tan(\theta) \approx \theta$ for $\theta \ll 1$. Solving for ϵ_n,

$$\epsilon_n \approx -\frac{2hL}{(2n-1)\pi} \qquad (8.3.100)$$

and

$$\alpha_n \approx \frac{(2n-1)\pi}{2} + \frac{2hL}{(2n-1)\pi}. \qquad (8.3.101)$$

In Table 8.3.1 we compare the approximate roots given by (8.3.101) with the actual roots.

The temporal part equals

$$T_n(t) = C_n \exp\left(-k_n^2 a^2 t\right) = C_n \exp\left(-\frac{\alpha_n^2 a^2 t}{L^2}\right). \qquad (8.3.102)$$

Consequently, the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{\alpha_n x}{L}\right) \exp\left(-\frac{\alpha_n^2 a^2 t}{L^2}\right), \qquad (8.3.103)$$

where α_n is the nth root of (8.3.93).

To determine C_n, we use the initial condition (8.3.85) and find that

$$100 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{\alpha_n x}{L}\right). \qquad (8.3.104)$$

Equation (8.3.104) is an eigenfunction expansion of 100 employing the eigenfunctions from the Sturm-Liouville problem

$$X'' + k^2 X = 0 \qquad (8.3.105)$$

and

$$X(0) = X'(L) + hX(L) = 0. \qquad (8.3.106)$$

Thus, the coefficient C_n is given by (6.3.4) or

$$C_n = \frac{\int_0^L 100 \sin(\alpha_n x/L)\, dx}{\int_0^L \sin^2(\alpha_n x/L)\, dx}, \qquad (8.3.107)$$

as $r(x) = 1$. Performing the integrations,

$$C_n = \frac{100L[1-\cos(\alpha_n)]/\alpha_n}{\frac{1}{2}[L - L\sin(2\alpha_n)/(2\alpha_n)]} = \frac{200[1-\cos(\alpha_n)]}{\alpha_n[1+\cos^2(\alpha_n)/(hL)]}, \qquad (8.3.108)$$

because $\sin(2\alpha_n) = 2\cos(\alpha_n)\sin(\alpha_n)$ and $\alpha_n = -hL\tan(\alpha_n)$. The final solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200[1-\cos(\alpha_n)]}{\alpha_n[1+\cos^2(\alpha_n)/(hL)]} \sin\left(\frac{\alpha_n x}{L}\right) \exp\left(-\frac{\alpha_n^2 a^2 t}{L^2}\right). \qquad (8.3.109)$$

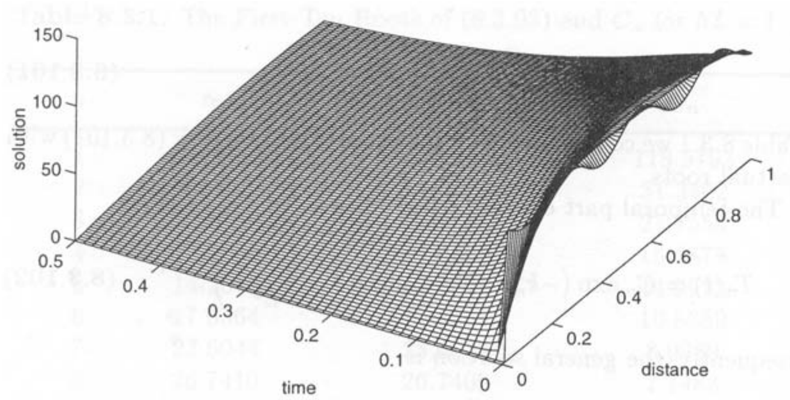

Figure 8.3.5: Tbe temperature $u(x, t)$ within a thin bar as a function of position x/L and time a^2t/L^2 when we allow the bar to radiatively cool at $x = L$ while the temperature is zero at $x = 0$. Initially the temperature was 100.

Figure 8.3.5 illustrates this solution for $hL = 1$ at various times and positions. It is similar to Example 8.3.1 in that the heat lost to the environment occurs either because the temperature at an end is zero or because it radiates heat to space which has the temperature of zero. The oscillations in the initial temperature distribution arise from Gibbs phenomena. We are using eigenfunctions that satisfy the boundary conditions (8.3.90) to fit a curve that equals 100 for all x.

• **Example 8.3.6: Refrigeration of apples**

Some decades ago, shiploads of apples, going from Australia to England, deteriorated from a disease called "brown heart," which occurred under insufficient cooling conditions. Apples, when placed on shipboard, are usually warm and must be cooled to be carried in cold storage. They also generate heat by their respiration. It was suspected that this heat generation effectively counteracted the refrigeration of the apples, resulting in the "brown heart."

This was the problem which induced Awberry[5] to study the heat distribution within a sphere in which heat is being generated. Awberry first assumed that the apples are initially at a uniform temperature. We can take this temperature to be zero by the appropriate choice of temperature scale. At time $t = 0$, the skins of the apples assume the temperature θ immediately when we introduce them into the hold.

[5] Awberry, J. H., 1927: The flow of heat in a body generating heat. *Philos. Mag., Ser. 7*, **4**, 629–638.

Because of the spherical geometry, the nonhomogeneous heat equation becomes

$$\frac{1}{a^2}\frac{\partial u}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{G}{\kappa}, \quad 0 \leq r < b, 0 < t, \tag{8.3.110}$$

where a^2 is the thermal diffusivity, b is the radius of the apple, κ is the thermal conductivity, and G is the heating rate (per unit time per unit volume).

If we try to use separation of variables on (8.3.110), we find that it does not work because of the G/κ term. To circumvent this difficulty, we ask the simpler question of what happens after a very long time. We anticipate that a balance will eventually be established where conduction transports the heat produced within the apple to the surface of the apple where the surroundings absorb it. Consequently, just as we introduced a steady-state solution in Example 8.3.3, we again anticipate a steady-state solution $w(r)$ where the heat conduction removes the heat generated within the apples. The ordinary differential equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dw}{dr}\right) = -\frac{G}{\kappa} \tag{8.3.111}$$

gives the steady-state. Furthermore, just as we introduced a transient solution which allowed our solution to satisfy the initial condition, we must also have one here and the governing equation is

$$\frac{\partial v}{\partial t} = \frac{a^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right). \tag{8.3.112}$$

Solving (8.3.111) first,

$$w(r) = C + \frac{D}{r} - \frac{Gr^2}{6\kappa}. \tag{8.3.113}$$

The constant D equals zero because the solution must be finite at $r = 0$. Because the steady-state solution must satisfy the boundary condition $w(b) = \theta$,

$$C = \theta + \frac{Gb^2}{6\kappa}. \tag{8.3.114}$$

Turning to the transient problem, we introduce a new dependent variable $y(r,t) = rv(r,t)$. This new dependent variable allows us to replace (8.3.112) with

$$\frac{\partial y}{\partial t} = a^2\frac{\partial^2 y}{\partial r^2}, \tag{8.3.115}$$

which we can solve. If we assume that $y(r,t) = R(r)T(t)$ and we only have a negative separation constant, the $R(r)$ equation becomes

$$\frac{d^2R}{dr^2} + k^2R = 0, \tag{8.3.116}$$

which has the solution

$$R(r) = A\cos(kr) + B\sin(kr). \tag{8.3.117}$$

The constant A equals zero because the solution (8.3.117) must vanish at $r = 0$ in order that $v(0,t)$ remains finite. However, because $\theta = w(b) + v(b,t)$ for all time and $v(b,t) = R(b)T(t)/b = 0$, then $R(b) = 0$. Consequently, $k_n = n\pi/b$ and

$$v_n(r,t) = \frac{B_n}{r}\sin\left(\frac{n\pi r}{b}\right)\exp\left(-\frac{n^2\pi^2a^2t}{b^2}\right). \tag{8.3.118}$$

Superposition gives the total solution which equals

$$u(r,t) = \theta + \frac{G}{6\kappa}(b^2 - r^2) + \sum_{n=1}^{\infty}\frac{B_n}{r}\sin\left(\frac{n\pi r}{b}\right)\exp\left(-\frac{n^2\pi^2a^2t}{b^2}\right). \tag{8.3.119}$$

Finally, we determine the B_n's by the initial condition that $u(r,0) = 0$. Therefore,

$$B_n = -\frac{2}{b}\int_0^b r\left[\theta + \frac{G}{6\kappa}(b^2 - r^2)\right]\sin\left(\frac{n\pi r}{b}\right)\,dr \tag{8.3.120}$$

$$= \frac{2\theta b}{n\pi}(-1)^n + \frac{2G}{\kappa}\left(\frac{b}{n\pi}\right)^3(-1)^n. \tag{8.3.121}$$

The final solution is

$$u(r,t) = \theta + \frac{2\theta b}{r\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi r}{b}\right)\exp\left(-\frac{n^2\pi^2a^2t}{b^2}\right)$$
$$+ \frac{G}{6\kappa}(b^2 - r^2) + \frac{2Gb^3}{r\kappa\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\left(\frac{n\pi r}{b}\right)\exp\left(-\frac{n^2\pi^2a^2t}{b^2}\right). \tag{8.3.122}$$

The first line of (8.3.122) gives the temperature distribution due to the imposition of the temperature θ on the surface of the apple while the second line gives the rise in the temperature due to the interior heating.

Returning to our original problem of whether the interior heating is strong enough to counteract the cooling by refrigeration, we merely use the second line of (8.3.122) to find how much the temperature deviates from what we normally expect. Because the highest temperature exists at the center of each apple, its value there is the only one of interest in this problem. Assuming $b = 4$ cm as the radius of the apple, $a^2G/\kappa = 1.33 \times 10^{-5}$ °C/s and $a^2 = 1.55 \times 10^{-3}$ cm^2/s, the temperature effect of the heat generation is very small, only 0.0232 °C when, after about 2 hours, the temperatures within the apples reach equilibrium. Thus, we must conclude that heat generation within the apples is not the cause of brown heart.

We now know that brown heart results from an excessive concentration of carbon dioxide and a deficient amount of oxygen in the storage hold.[6] Presumably this atmosphere affects the metabolic activities that are occurring in the apple[7] and leads to low-temperature breakdown.

• Example 8.3.7

In this example we illustrate how separation of variables may be employed in solving the axisymmetric heat equation in an infinitely long cylinder. In circular coordinates the heat equation is

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \quad 0 \le r < b, 0 < t, \tag{8.3.123}$$

where r denotes the radial distance and a^2 denotes the thermal diffusivity. Let us assume that we have heated this cylinder of radius b to the uniform temperature T_0 and then allowed it to cool by having its surface held at the temperature of zero starting from the time $t = 0$.

We begin by assuming that the solution is of the form $u(r,t) = R(r)T(t)$ so that

$$\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) = \frac{1}{a^2T}\frac{dT}{dt} = -\frac{k^2}{b^2}. \tag{8.3.124}$$

The only values of the separation constant that yield nontrivial solutions are negative. The nontrivial solutions are $R(r) = J_0(kr/b)$, where J_0 is the Bessel function of the first kind and zeroth order. A separation constant of zero gives $R(r) = \ln(r)$ which becomes infinite at the

[6] Thornton, N. C., 1931: The effect of carbon dioxide on fruits and vegetables in storage. *Contrib. Boyce Thompson Inst.*, **3**, 219–244.

[7] Fidler, J. C. and North, C. J., 1968: The effect of conditions of storage on the respiration of apples. IV. Changes in concentration of possible substrates of respiration, as related to production of carbon dioxide and uptake of oxygen by apples at low temperatures. *J. Hortic. Sci.*, **43**, 429–439.

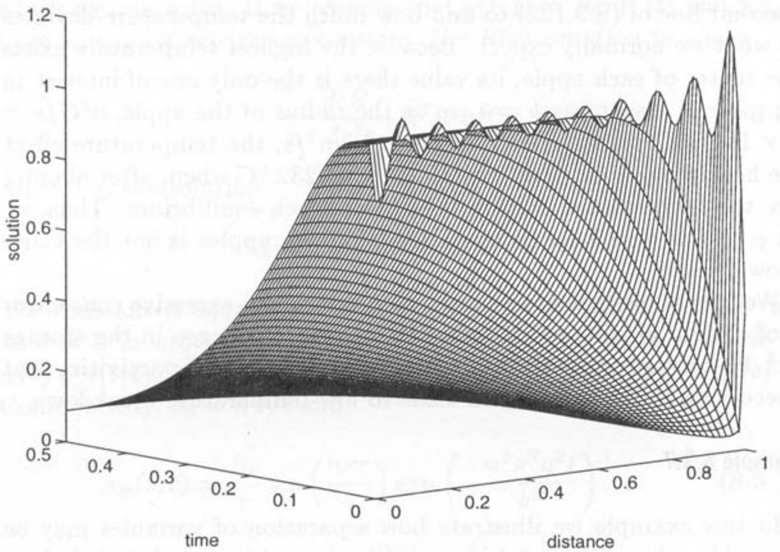

Figure 8.3.6: The temperature $u(r,t)/T_0$ within an infinitely long cylinder at various positions r/b and times a^2t/b^2 that we initially heated to the uniform temperature T_0 and then allowed to cool by forcing its surface to equal zero.

origin. Positive separation constants yield the modified Bessel function $I_0(kr/b)$. Although this function is finite at the origin, it cannot satisfy the boundary condition that $u(b,t) = R(b)T(t) = 0$ or $R(b) = 0$.

The boundary condition that $R(b) = 0$ requires that $J_0(k) = 0$. This transcendental equation yields an infinite number of k_n's. For each of these k_n's, the temporal part of the solution satisfies the differential equation

$$\frac{dT_n}{dt} + \frac{k_n^2 a^2}{b^2} T_n = 0, \tag{8.3.125}$$

which has the solution

$$T_n(t) = A_n \exp\left(-\frac{k_n^2 a^2}{b^2} t\right). \tag{8.3.126}$$

Consequently, the product solutions are

$$u_n(r,t) = A_n J_0\left(k_n \frac{r}{b}\right) \exp\left(-\frac{k_n^2 a^2}{b^2} t\right). \tag{8.3.127}$$

The total solution is a linear superposition of all of the particular solutions or

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0\left(k_n \frac{r}{b}\right) \exp\left(-\frac{k_n^2 a^2}{b^2} t\right). \tag{8.3.128}$$

Our final task remains to determine A_n. From the initial condition that $u(r,0) = T_0$,

$$u(r,0) = T_0 = \sum_{n=1}^{\infty} A_n J_0\left(k_n \frac{r}{b}\right). \tag{8.3.129}$$

From (6.5.35) and (6.5.43),

$$A_n = \frac{2T_0}{J_1^2(k_n)b^2} \int_0^b r J_0\left(k_n \frac{r}{b}\right) dr \tag{8.3.130}$$

$$= \frac{2T_0}{k_n^2 J_1^2(k_n)} \left(\frac{k_n r}{b}\right) J_1\left(k_n \frac{r}{b}\right)\Big|_0^b = \frac{2T_0}{k_n J_1(k_n)} \tag{8.3.131}$$

from (6.5.25). Thus, the final solution is

$$u(r,t) = 2T_0 \sum_{n=1}^{\infty} \frac{1}{k_n J_1(k_n)} J_0\left(k_n \frac{r}{b}\right) \exp\left(-\frac{k_n^2 a^2}{b^2} t\right). \tag{8.3.132}$$

Figure 8.3.6 illustrates the solution (8.3.132) for various Fourier numbers a^2t/b^2. It is similar to Example 8.3.1 except that we are in cylindrical coordinates. Heat flows from the interior and is removed at the cylinder's surface where the temperature equals zero. The initial oscillations of the solution result from Gibbs phenomena because we have a jump in the temperature field at $r = b$.

• **Example 8.3.8**

In this example we find the evolution of the temperature field within a cylinder of radius b as it radiatively cools from an initial uniform temperature T_0. The heat equation is

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \quad 0 \le r < b, 0 < t, \tag{8.3.133}$$

which we shall solve by separation of variables $u(r,t) = R(r)T(t)$. Therefore,

$$\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) = \frac{1}{a^2T}\frac{dT}{dt} = -\frac{k^2}{b^2}, \tag{8.3.134}$$

because only a negative separation constant yields a $R(r)$ which is finite at the origin and satisfies the boundary condition. This solution is $R(r) = J_0(kr/b)$, where J_0 is the Bessel function of the first kind and zeroth order.

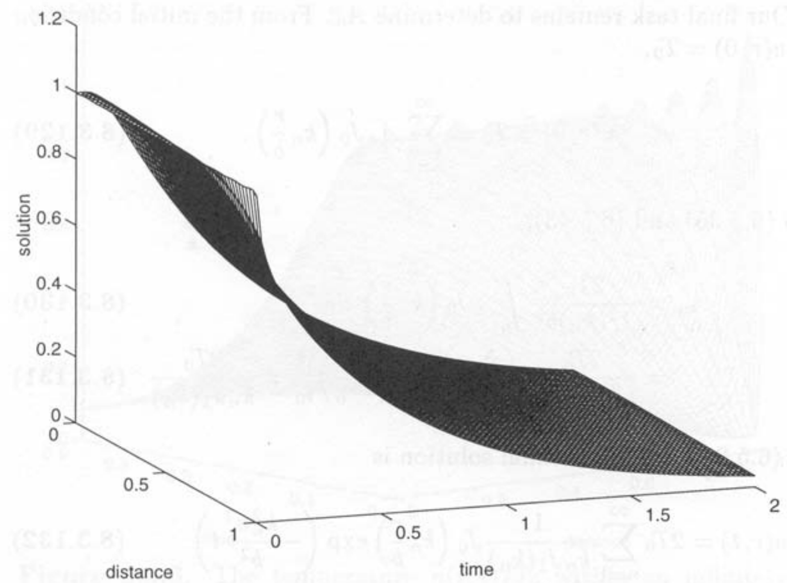

Figure 8.3.7: The temperature $u(r,t)/T_0$ within an infinitely long cylinder at various positions r/b and times a^2t/b^2 that we initially heated to the temperature T_0 and then allowed to radiatively cool with $hb = 1$.

The radiative boundary condition may be expressed as

$$\frac{\partial u(b,t)}{\partial r} + hu(b,t) = T(t)\left[\frac{dR(b)}{dr} + hR(b)\right] = 0. \qquad (8.3.135)$$

Because $T(t) \neq 0$,

$$kJ_0'(k) + hbJ_0(k) = -kJ_1(k) + hbJ_0(k) = 0, \qquad (8.3.136)$$

where the product hb is the Biot number. The solution of the transcendental equation (8.3.136) yields an infinite number of distinct k_n's. For each of these k_n's, the temporal part equals the solution of

$$\frac{dT_n}{dt} + \frac{k_n^2a^2}{b^2}T_n = 0, \qquad (8.3.137)$$

or

$$T_n(t) = A_n \exp\left(-\frac{k_n^2a^2}{b^2}t\right). \qquad (8.3.138)$$

The product solution is, therefore,

$$u_n(r,t) = A_n J_0\left(k_n\frac{r}{b}\right)\exp\left(-\frac{k_n^2a^2}{b^2}t\right) \qquad (8.3.139)$$

and the most general solution is a sum of these product solutions

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0\left(k_n\frac{r}{b}\right)\exp\left(-\frac{k_n^2a^2}{b^2}t\right). \qquad (8.3.140)$$

Finally, we must determine A_n. From the initial condition that $u(r,0) = T_0$,

$$u(r,0) = T_0 = \sum_{n=1}^{\infty} A_n J_0\left(k_n \frac{r}{b}\right), \tag{8.3.141}$$

where

$$A_n = \frac{2k_n^2 T_0}{b^2[k_n^2 + b^2h^2]J_0^2(k_n)} \int_0^b r J_0\left(k_n \frac{r}{b}\right)\, dr \tag{8.3.142}$$

$$= \frac{2k_n^2 T_0}{[k_n^2 + b^2h^2]J_0^2(k_n)} \left(\frac{k_n r}{b}\right) J_1\left(k_n \frac{r}{b}\right)\bigg|_0^b \tag{8.3.143}$$

$$= \frac{2k_n T_0 J_1(k_n)}{[k_n^2 + b^2h^2]J_0^2(k_n)} = \frac{2k_n T_0 J_1(k_n)}{k_n^2 J_0^2(k_n) + b^2h^2 J_0^2(k_n)} \tag{8.3.144}$$

$$= \frac{2k_n T_0 J_1(k_n)}{k_n^2 J_0^2(k_n) + k_n^2 J_1^2(k_n)} = \frac{2T_0 J_1(k_n)}{k_n[J_0^2(k_n) + J_1^2(k_n)]}, \tag{8.3.145}$$

which follows from (6.5.25), (6.5.35), (6.5.45), and (8.3.136). Consequently, the final solution is

$$u(r,t) = 2T_0 \sum_{n=1}^{\infty} \frac{J_1(k_n)}{k_n[J_0^2(k_n) + J_1^2(k_n)]} J_0\left(k_n \frac{r}{b}\right) \exp\left(-\frac{k_n^2 a^2}{b^2} t\right). \tag{8.3.146}$$

Figure 8.3.7 illustrates the solution (8.3.146) for various Fourier numbers a^2t/b^2 with $hb = 1$. It is similar to Example 8.3.5 except that we are in cylindrical coordinates. Heat flows from the interior and is removed at the cylinder's surface where it radiates to space at the temperature zero. Note that we do *not* suffer from Gibbs phenomena in this case because there is no initial jump in the temperature distribution.

• Example 8.3.9: Temperature within an electrical cable

In the design of cable installations we need the temperature reached within an electrical cable as a function of current and other parameters. To this end,[8] let us solve the nonhomogeneous heat equation in cylindrical coordinates with a radiation boundary condition.

The derivation of the heat equation follows from the conservation of energy:

heat generated = heat dissipated + heat stored

[8] Iskenderian, H. P. and Horvath, W. J., 1946: Determination of the temperature rise and the maximum safe current through multiconductor electric cables. *J. Appl. Phys.*, **17**, 255–262.

or

$$I^2RN\,dt = -\kappa\left[2\pi r\left.\frac{\partial u}{\partial r}\right|_r - 2\pi(r+\Delta r)\left.\frac{\partial u}{\partial r}\right|_{r+\Delta r}\right]dt + 2\pi r\Delta r c\rho\,du, \tag{8.3.147}$$

where I is the current through each wire, R is the resistance of each conductor, N is the number of conductors in the shell between radii r and $r+\Delta r = 2\pi m r\Delta r/(\pi b^2)$, b is the radius of the cable, m is the total number of conductors in the cable, κ is the thermal conductivity, ρ is the density, c is the average specific heat, and u is the temperature. In the limit of $\Delta r \to 0$, (8.3.147) becomes

$$\frac{\partial u}{\partial t} = A + a^2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \quad 0 \le r < b, 0 < t, \tag{8.3.148}$$

where $A = I^2Rm/(\pi b^2 c\rho)$ and $a^2 = \kappa/(\rho c)$.

Equation (8.3.148) is the nonhomogeneous heat equation for an infinitely long, axisymmetric cylinder. From Example 8.3.3, we know that we must write the temperature as the sum of a steady-state and transient solution: $u(r,t) = w(r) + v(r,t)$. The steady-state solution $w(r)$ satisfies

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) = -\frac{A}{a^2} \tag{8.3.149}$$

or

$$w(r) = T_c - \frac{Ar^2}{4a^2}, \tag{8.3.150}$$

where T_c is the (yet unknown) temperature in the center of the cable.

The transient solution $v(r,t)$ is govern by

$$\frac{\partial v}{\partial t} = a^2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right), \quad 0 \le r < b, 0 < t \tag{8.3.151}$$

with the initial condition that $u(r,0) = T_c - Ar^2/(4a^2) + v(0,t) = 0$. At the surface $r = b$ heat radiates to free space so that the boundary condition is $u_r = -hu$, where h is the surface conductance. Because the steady-state temperature must be true when all transient effects die away, it must satisfy this radiation boundary condition regardless of the transient solution. This requires that

$$T_c = \frac{A}{a^2}\left(\frac{b^2}{4} + \frac{b}{2h}\right). \tag{8.3.152}$$

Therefore, $v(r,t)$ must satisfy $v_r(b,t) = -hv(b,t)$ at $r = b$.

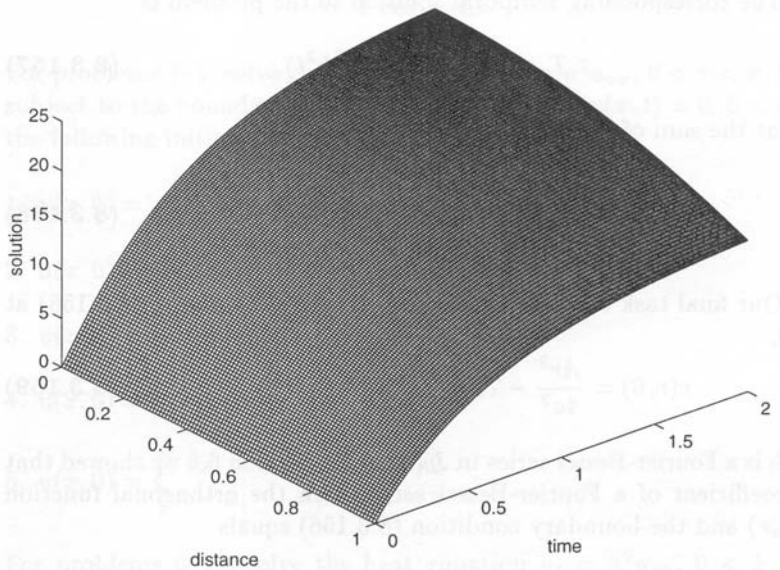

Figure 8.3.8: The temperature field (in degrees Celsius) within an electric copper cable containing 37 wires and a current of 22 amperes at various positions r/b and times a^2t/b^2. Initially the temperature was zero and then we allow the cable to cool radiatively as it is heated. The parameters are $hb = 1$ and the radius of the cable $b = 4$ cm.

We find the transient solution $v(r,t)$ by separation of variables $v(r,t) = R(r)T(t)$. Substituting into (8.3.151),

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \frac{1}{a^2T}\frac{dT}{dt} = -k^2 \qquad \textbf{(8.3.153)}$$

or

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + k^2rR = 0 \qquad \textbf{(8.3.154)}$$

and

$$\frac{dT}{dt} + k^2a^2T = 0, \qquad \textbf{(8.3.155)}$$

with $R'(b) = -hR(b)$. The only solution of (8.3.154) which remains finite at $r = 0$ and satisfies the boundary condition is $R(r) = J_0(kr)$, where J_0 is the zero-order Bessel function of the first kind. Substituting $J_0(kr)$ into the boundary condition, the transcendental equation is

$$kbJ_1(kb) - hbJ_0(kb) = 0. \qquad \textbf{(8.3.156)}$$

For a given value of h and b, (8.3.156) yields an infinite number of unique zeros k_n.

The corresponding temporal solution to the problem is

$$T_n(t) = A_n \exp(-a^2 k_n^2 t), \tag{8.3.157}$$

so that the sum of the product solutions is

$$v(r,t) = \sum_{n=1}^{\infty} A_n J_0(k_n r) \exp(-a^2 k_n^2 t). \tag{8.3.158}$$

Our final task remains to compute A_n. By evaluating (8.3.158) at $t = 0$,

$$v(r,0) = \frac{Ar^2}{4a^2} - T_c = \sum_{n=1}^{\infty} A_n J_0(k_n r), \tag{8.3.159}$$

which is a Fourier-Bessel series in $J_0(k_n r)$. In Section 6.5 we showed that the coefficient of a Fourier-Bessel series with the orthogonal function $J_0(k_n r)$ and the boundary condition (8.3.156) equals

$$A_n = \frac{2k_n^2}{(k_n^2 b^2 + h^2 b^2) J_0^2(k_n b)} \int_0^b r \left(\frac{Ar^2}{4a^2} - T_c \right) J_0(k_n r)\, dr \tag{8.3.160}$$

from (6.5.35) and (6.5.45). Carrying out the indicated integrations,

$$A_n = \frac{2}{(k_n^2 + h^2) J_0^2(k_n b)} \times \left[\left(\frac{A k_n b}{4a^2} - \frac{A}{k_n b a^2} - \frac{T_c k_n}{b} \right) J_1(k_n b) + \frac{A}{2a^2} J_0(k_n b) \right]. \tag{8.3.161}$$

We obtained (8.3.161) by using (6.5.25) and integrating by parts in a similar manner as was done in Example 6.5.5.

To illustrate this solution, let us compute it for the typical parameters $b = 4$ cm, $hb = 1$, $a^2 = 1.14$ cm^2/s, $A = 2.2747$ °C/s, and $T_c = 23.94$°C. The value of A corresponds to 37 wires of #6 AWG copper wire within a cable carrying a current of 22 amp.

Figure 8.3.8 illustrates the solution as a function of radius at various times. From an initial temperature of zero, the temperature rises due to the constant electrical heating. After a short period of time, it reaches its steady-state distribution given by (8.3.150). The cable is coolest at the surface where heat is radiating away. Heat flows from the interior to replace the heat lost by radiation.

Problems

For problems 1–5, solve the heat equation $u_t = a^2 u_{xx}$, $0 < x < \pi$, $0 < t$ subject to the boundary conditions that $u(0,t) = u(\pi,t) = 0$, $0 < t$ and the following initial conditions for $0 < x < \pi$:

1. $u(x,0) = A$, a constant

2. $u(x,0) = \sin^3(x) = [3\sin(x) - \sin(3x)]/4$

3. $u(x,0) = x$

4. $u(x,0) = \pi - x$

5. $u(x,0) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 < x < \pi \end{cases}$

For problems 6–10, solve the heat equation $u_t = a^2 u_{xx}$, $0 < x < \pi$, $0 < t$ subject to the boundary conditions that $u_x(0,t) = u_x(\pi,t) = 0$, $0 < t$ and the following initial conditions for $0 < x < \pi$:

6. $u(x,0) = 1$

7. $u(x,0) = x$

8. $u(x,0) = \cos^2(x) = [1 + \cos(2x)]/2$

9. $u(x,0) = \pi - x$

10. $u(x,0) = \begin{cases} T_0, & 0 < x < \pi/2 \\ T_1, & \pi/2 < x < \pi \end{cases}$

For problems 11–17, solve the heat equation $u_t = a^2 u_{xx}$, $0 < x < \pi$, $0 < t$ subject to the following boundary conditions and initial condition:

11. $u_x(0,t) = u(\pi,t) = 0$, $0 < t$; $u(x,0) = x^2 - \pi^2$, $0 < x < \pi$

12. $u(0,t) = u(\pi,t) = T_0$, $0 < t$; $u(x,0) = T_1 \neq T_0$, $0 < x < \pi$

13. $u(0,t) = 0$, $u_x(\pi,t) = 0$, $0 < t$; $u(x,0) = 1$, $0 < x < \pi$

14. $u(0,t) = 0$, $u_x(\pi,t) = 0$, $0 < t$; $u(x,0) = x$, $0 < x < \pi$

15. $u(0,t) = 0$, $u_x(\pi,t) = 0$, $0 < t$; $u(x,0) = \pi - x$, $0 < x < \pi$

16. $u(0,t) = T_0$, $u_x(\pi,t) = 0$, $0 < t$; $u(x,0) = T_1 \neq T_0$, $0 < x < \pi$

17. $u(0,t) = 0, u(\pi,t) = T_0$, $0 < t$; $u(x,0) = T_0$, $0 < x < \pi$

18. It is well known that a room with masonry walls is often very difficult to heat. Consider a wall of thickness L, conductivity κ, and diffusivity a^2 which we heat at a constant rate H. The temperature of the outside (out-of-doors) face of the wall remains constant at T_0 and the entire wall initially has the uniform temperature T_0. Let us find the temperature of the inside face as a function of time.[9]

We begin by solving the heat conduction problem

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, 0 < t$$

subject to the boundary conditions that

$$\frac{\partial u(0,t)}{\partial x} = -\frac{H}{\kappa} \qquad \text{and} \qquad u(L,t) = T_0$$

and the initial condition that $u(x,0) = T_0$. Show that the temperature field equals

$$u(x,t) = T_0 + \frac{HL}{\kappa}\left\{1 - \frac{x}{L} - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left[\frac{(2n-1)\pi x}{2L}\right] \times \exp\left[-\frac{(2n-1)^2\pi^2 a^2 t}{4L^2}\right]\right\}.$$

Therefore, the rise of temperature at the interior wall $x = 0$ is

$$\frac{HL}{\kappa}\left\{1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\exp\left[-\frac{(2n-1)^2\pi^2 a^2 t}{4L^2}\right]\right\}$$

or

$$\frac{8HL}{\kappa\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left\{1 - \exp\left[-\frac{(2n-1)^2\pi^2 a^2 t}{4L^2}\right]\right\}.$$

For $a^2t/L^2 \leq 1$ this last expression can be approximated[10] by $4Hat^{1/2}/\pi^{1/2}\kappa$. We thus see that the temperature will initially rise as the square

[9] Reproduced with acknowledgement to Taylor and Francis, Publishers, from Dufton, A. F., 1927: The warming of walls. *Philos. Mag., Ser. 7*, **4**, 888–889.

[10] Let us define the function

$$f(t) = \sum_{n=1}^{\infty}\frac{1 - \exp[-(2n-1)^2\pi^2 a^2 t/L^2]}{(2n-1)^2}.$$

root of time and diffusivity and inversely with conductivity. For an average rock $\kappa = 0.0042$ g/cm-s and $a^2 = 0.0118$ cm^2/s while for wood (Spruce) $\kappa = 0.0003$ g/cm-s and $a^2 = 0.0024$ cm^2/s.

The same set of equations applies to heat transfer within a transistor operating at low frequencies.[11] At the junction ($x = 0$) heat is produced at the rate of H and flows to the transistor's supports ($x = \pm L$) where it is removed. The supports are maintained at the temperature T_0 which is also the initial temperature of the transistor.

19. The linearized Boussinesq equation[12]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, 0 < t$$

governs the height of the water table $u(x,t)$ above some reference point, where a^2 is the product of the storage coefficient times the hydraulic coefficient divided by the aquifer thickness. A typical value of a^2 is 10 m^2/min. Consider the problem of a strip of land of width L that separates two reservoirs of depth h_1. Initially the height of the water table would be h_1. Suddenly we lower the reservoir on the right $x = L$ to a depth h_2 [$u(0,t) = h_1$, $u(L,t) = h_2$, and $u(x,0) = h_1$]. Find the

---

Then

$$f'(t) = \frac{a^2\pi^2}{L^2}\sum_{n=1}^{\infty} \exp[-(2n-1)^2\pi^2a^2t/L^2].$$

Consider now the integral

$$\int_0^\infty \exp\left(-\frac{a^2\pi^2 t}{L^2}x^2\right)\,dx = \frac{L}{2a\sqrt{\pi t}}.$$

If we approximate this integral by using the trapezoidal rule with $\Delta x = 2$, then

$$\int_0^\infty \exp\left(-\frac{a^2\pi^2 t}{L^2}x^2\right)\,dx \approx 2\sum_{n=1}^{\infty} \exp[-(2n-1)^2\pi^2a^2t/L^2]$$

and $f'(t) \approx a\pi^{3/2}/(4Lt^{1/2})$. Integrating and using $f(0) = 0$, we finally have $f(t) \approx a\pi^{3/2}t^{1/2}/(2L)$. The smaller a^2t/L^2 is, the smaller the error will be. For example, if $t = L^2/a^2$, then the error is 2.4 %.

[11] Mortenson, K. E., 1957: Transistor junction temperature as a function of time. *Proc. IRE*, **45**, 504–513. Eq. (2a) should read $T_x = -F/k$.

[12] See, for example, Van Schilfgaarde, J., 1970: Theory of flow to drains in *Advances in Hydroscience*, Academic Press, New York, pp. 81–85.

height of the water table at any position x within the aquifer and any time $t > 0$.

20. The equation (see Problem 19)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, 0 < t$$

governs the height of the water table $u(x,t)$. Consider the problem[13] of a piece of land that suddenly has two drains placed at the points $x = 0$ and $x = L$ so that $u(0,t) = u(L,t) = 0$. If the water table initially has the profile:

$$u(x,0) = 8H(L^3x - 3L^2x^2 + 4Lx^3 - 2x^4)/L^4,$$

find the height of the water table at any point within the aquifer and any time $t > 0$.

21. We want to find the rise of the water table of an aquifer which we sandwich between a canal and impervious rocks if we suddenly raise the water level in the canal h_0 units above its initial elevation and then maintain the canal at this level. The linearized Boussinesq equation (see Problem 19)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, 0 < t$$

governs the level of the water table with the boundary conditions $u(0,t) = h_0$ and $u_x(L,t) = 0$ and the initial condition $u(x,0) = 0$. Find the height of the water table at any point in the aquifer and any time $t > 0$.

22. Solve the nonhomogeneous heat equation

$$\frac{\partial u}{\partial t} - a^2\frac{\partial^2 u}{\partial x^2} = e^{-x}, \qquad 0 < x < \pi, 0 < t$$

subject to the boundary conditions $u(0,t) = u_x(\pi,t) = 0$, $0 < t$, and the initial condition $u(x,0) = f(x)$, $0 < x < \pi$.

23. Solve the nonhomogeneous heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = -1, \qquad 0 < x < 1, 0 < t$$

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[13] For a similar problem, see Dumm, L. D., 1954: New formula for determining depth and spacing of subsurface drains in irrigated lands. *Agric. Eng.*, **35**, 726–730.

subject to the boundary conditions $u_x(0,t) = u_x(1,t) = 0$, $0 < t$, and the initial condition $u(x,0) = \frac{1}{2}(1-x^2)$, $0 < x < 1$. [Hint: Note that any function of time satisfies the boundary conditions.]

24. Solve the nonhomogeneous heat equation

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = A\cos(\omega t), \qquad 0 < x < \pi, 0 < t$$

subject to the boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$, $0 < t$, and the initial condition $u(x,0) = f(x)$, $0 < x < \pi$. [Hint: Note that any function of time satisfies the boundary conditions.]

25. Solve the nonhomogeneous heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \begin{cases} x, & 0 < x \leq \pi/2 \\ \pi - x, & \pi/2 \leq x < \pi, \end{cases} \qquad 0 < x < \pi, 0 < t$$

subject to the boundary conditions $u(0,t) = u(\pi,t) = 0$, $0 < t$, and the initial condition $u(x,0) = 0$, $0 < x < \pi$. [Hint: Represent the forcing function as a half-range Fourier sine expansion over the interval $(0,\pi)$.]

26. A uniform, conducting rod of length L and thermometric diffusivity a^2 is initially at temperature zero. We supply heat uniformly throughout the rod so that the heat conduction equation is

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} - P, \qquad 0 < x < L, 0 < t,$$

where P is the rate at which the temperature would rise if there was no conduction. If we maintain the ends of the rod at the temperature of zero, find the temperature at any position and subsequent time.

27. Solve the nonhomogeneous heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + \frac{A_0}{c\rho} \qquad 0 < x < L, 0 < t,$$

where $a^2 = \kappa/c\rho$ with the boundary conditions that

$$\frac{\partial u(0,t)}{\partial x} = 0 \quad \text{and} \quad \kappa \frac{\partial u(L,t)}{\partial x} + hu(L,t) = 0, \qquad 0 < t$$

and the initial condition that $u(x,0) = 0$, $0 < x < L$.

28. Find the solution of

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u, \qquad 0 < x < L, 0 < t$$

with the boundary conditions $u(0,t) = 1$ and $u(L,t) = 0$, $0 < t$, and the initial condition $u(x,0) = 0$, $0 < x < L$.

29. Solve the heat equation in spherical coordinates

$$\frac{\partial u}{\partial t} = \frac{a^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right), \qquad 0 \le r < 1, 0 < t$$

subject to the boundary conditions $\lim_{r\to 0}|u(r,t)| < \infty$ and $u(1,t) = 0$, $0 < t$, and the initial condition $u(r,0) = 1$, $0 \le r < 1$.

30. Solve the heat equation in cylindrical coordinates

$$\frac{\partial u}{\partial t} = \frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \qquad 0 \le r < b, 0 < t$$

subject to the boundary conditions $\lim_{r\to 0}|u(r,t)| < \infty$ and $u(b,t) = \theta$, $0 < t$, and the initial condition $u(r,0) = 1$, $0 \le r < b$.

31. The equation[14]

$$\frac{\partial u}{\partial t} = \frac{G}{\rho} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \qquad 0 \le r < b, 0 < t$$

governs the velocity $u(r,t)$ of an incompressible fluid of density ρ and kinematic viscosity ν flowing in a long circular pipe of radius b with an imposed, constant pressure gradient $-G$. If the fluid is initially at rest $u(r,0) = 0$, $0 \le r < b$, and there is no slip at the wall $u(b,t) = 0$, $0 < t$, find the velocity at any subsequent time and position.

32. Solve the heat equation in cylindrical coordinates

$$\frac{\partial u}{\partial t} = \frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \qquad 0 \le r < b, 0 < t$$

subject to the boundary conditions $\lim_{r\to 0}|u(r,t)| < \infty$ and $u_r(b,t) = -hu(b,t)$, $0 < t$, and the initial condition $u(r,0) = b^2 - r^2$, $0 \le r < b$.

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[14] From Szymanski, P., 1932: Quelques solutions exactes des équations de l'hydrodynamique du fluide visqueux dans le cas d'un tube cylindrique. *J. math. pures appl., Ser. 9*, **11**, 67–107. ©Gauthier-Villars

33. In their study of heat conduction within a thermocouple through which a steady current flows, Reich and Madigan[15] solved the following nonhomogeneous heat conduction problem:

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = J - P\,\delta(x-b), \qquad 0 < x < L, 0 < t, 0 < b < L,$$

where J represents the Joule heating generated by the steady current and the P term represents the heat loss from Peltier cooling.[16] Find $u(x,t)$ if both ends are kept at zero $[u(0,t) = u(L,t) = 0]$ and initially the temperature is zero $[u(x,0) = 0]$. The interesting aspect of this problem is the presence of the delta function.

Step 1: Assuming that $u(x,t)$ equals the sum of a steady-state solution $w(x)$ and a transient solution $v(x,t)$, show that the steady-state solution is governed by

$$a^2 \frac{d^2 w}{dx^2} = P\,\delta(x-b) - J, \qquad w(0) = w(L) = 0.$$

Step 2: Show that the steady-state solution is

$$w(x) = \begin{cases} Jx(L-x)/2a^2 + Ax, & 0 < x < b \\ Jx(L-x)/2a^2 + B(L-x), & b < x < L. \end{cases}$$

Step 3: The temperature must be continuous at $x = b$; otherwise, we would have infinite heat conduction there. Use this condition to show that $Ab = B(L-b)$.

Step 4: To find a second relationship between A and B, integrate the steady-state differential equation across the interface at $x = b$ and show that

$$\lim_{\epsilon \to 0} a^2 \left. \frac{dw}{dx} \right|_{b-\epsilon}^{b+\epsilon} = P.$$

Step 5: Using the result from Step 4, show that $A + B = -P/a^2$ and

$$w(x) = \begin{cases} Jx(L-x)/2a^2 - Px(L-b)/a^2 L, & 0 < x < b \\ Jx(L-x)/2a^2 - Pb(L-x)/a^2 L, & b < x < L. \end{cases}$$

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[15] Reich, A. D. and Madigan, J. R., 1961: Transient response of a thermocouple circuit under steady currents. *J. Appl. Phys.*, **32**, 294–301.

[16] In 1834 Jean Charles Athanase Peltier (1785–1845) discovered that there is a heating or cooling effect, quite apart from ordinary resistance heating, whenever an electric current flows through the junction between two different metals.

Step 6: Reexpress $w(x)$ as a half-range Fourier sine expansion and show that

$$w(x) = \frac{4JL^2}{a^2\pi^3}\sum_{m=1}^{\infty}\frac{\sin[(2m-1)\pi x/L]}{(2m-1)^3} - \frac{2LP}{a^2\pi^2}\sum_{n=1}^{\infty}\frac{\sin(n\pi b/L)\sin(n\pi x/L)}{n^2}.$$

Step 7: Use separation of variables to find the transient solution by solving

$$\frac{\partial v}{\partial t} = a^2\frac{\partial^2 v}{\partial x^2}, \qquad 0 < x < L, 0 < t$$

subject to the boundary conditions $v(0,t) = v(L,t) = 0$, $0 < t$, and the initial condition $v(x,0) = -w(x), 0 < x < L$.

Step 8: Add the steady-state and transient solutions together and show that

$$u(x,t) = \frac{4JL^2}{a^2\pi^3}\sum_{m=1}^{\infty}\frac{\sin[(2m-1)\pi x/L]}{(2m-1)^3}\left[1 - e^{-a^2(2m-1)^2\pi^2 t/L^2}\right] - \frac{2LP}{a^2\pi^2}\sum_{n=1}^{\infty}\frac{\sin(n\pi b/L)\sin(n\pi x/L)}{n^2}\left[1 - e^{-a^2n^2\pi^2 t/L^2}\right].$$

8.4 THE LAPLACE TRANSFORM METHOD

In the previous chapter we showed that we may solve the wave equation by the method of Laplace transforms. This is also true for the heat equation. Once again, we take the Laplace transform with respect to time. From the definition of Laplace transforms,

$$\mathcal{L}[u(x,t)] = U(x,s), \tag{8.4.1}$$

$$\mathcal{L}[u_t(x,t)] = sU(x,s) - u(x,0) \tag{8.4.2}$$

and

$$\mathcal{L}[u_{xx}(x,t)] = \frac{d^2U(x,s)}{dx^2}. \tag{8.4.3}$$

We next solve the resulting ordinary differential equation, known as the *auxiliary equation*, along with the corresponding Laplace transformed boundary conditions. The initial condition gives us the value of $u(x,0)$. The final step is the inversion of the Laplace transform $U(x,s)$. We typically use the inversion integral.

• Example 8.4.1

To illustrate these concepts, we solve a heat conduction problem[17] in a plane slab of thickness $2L$. Initially the slab has a constant temperature of unity. For $0 < t$ we allow both faces of the slab to radiatively cool in a medium which has a temperature of zero.

If $u(x,t)$ denotes the temperature, a^2 is the thermal diffusivity, h is the relative emissivity, t is the time, and x is the distance perpendicular to the face of the slab and measured from the middle of the slab, then the governing equation is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad -L < x < L, 0 < t \tag{8.4.4}$$

with the initial condition

$$u(x,0) = 1, \qquad -L < x < L \tag{8.4.5}$$

and boundary conditions

$$\frac{\partial u(L,t)}{\partial x} + hu(L,t) = 0 \quad \text{and} \quad \frac{\partial u(-L,t)}{\partial x} + hu(-L,t) = 0, \quad 0 < t. \tag{8.4.6}$$

Taking the Laplace transform of (8.4.4) and substituting the initial condition,

$$a^2 \frac{d^2 U(x,s)}{dx^2} - sU(x,s) = -1. \tag{8.4.7}$$

If we write $s = a^2 q^2$, (8.4.7) becomes

$$\frac{d^2 U(x,s)}{dx^2} - q^2 U(x,s) = -\frac{1}{a^2}. \tag{8.4.8}$$

From the boundary conditions $U(x,s)$ is an even function in x and we may conveniently write the solution as

$$U(x,s) = \frac{1}{s} + A\cosh(qx). \tag{8.4.9}$$

From (8.4.6),

$$qA\sinh(qL) + \frac{h}{s} + hA\cosh(qL) = 0 \tag{8.4.10}$$

and

$$U(x,s) = \frac{1}{s} - \frac{h\cosh(qx)}{s[q\sinh(qL) + h\cosh(qL)]}. \tag{8.4.11}$$

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[17] Goldstein, S., 1932: The application of Heaviside's operational method to the solution of a problem in heat conduction. *Zeit. Angew. Math. Mech.*, **12**, 234–243.

The inverse of $U(x,s)$ consists of two terms. The inverse of the first term is simply unity. We will invert the second term by contour integration.

We begin by examining the nature and location of the singularities in the second term. Using the product formulas for the hyperbolic cosine and sine functions, the second term equals

$$\frac{h\left(1+\frac{4q^2x^2}{\pi^2}\right)\left(1+\frac{4q^2x^2}{9\pi^2}\right)\cdots}{s\left[q^2L\left(1+\frac{q^2L^2}{\pi^2}\right)\left(1+\frac{q^2L^2}{4\pi^2}\right)\cdots+h\left(1+\frac{4q^2L^2}{\pi^2}\right)\left(1+\frac{4q^2L^2}{9\pi^2}\right)\cdots\right]}. \tag{8.4.12}$$

Because $q^2 = s/a^2$, (8.4.12) shows that we do not have any $\sqrt{s}$ in the transform and we need not concern ourselves with branch points and cuts. Furthermore, we have only simple poles: one located at $s = 0$ and the others where

$$q\sinh(qL) + h\cosh(qL) = 0. \tag{8.4.13}$$

If we set $q = i\lambda$, (8.4.13) becomes

$$h\cos(\lambda L) - \lambda\sin(\lambda L) = 0 \tag{8.4.14}$$

or

$$\lambda L\tan(\lambda L) = hL. \tag{8.4.15}$$

From Bromwich's integral,

$$\mathcal{L}^{-1}\left\{\frac{h\cosh(qx)}{s[q\sinh(qL)+h\cosh(qL)]}\right\}$$
$$= \frac{1}{2\pi i}\oint_C \frac{h\cosh(qx)e^{tz}}{z[q\sinh(qL)+h\cosh(qL)]}\,dz, \tag{8.4.16}$$

where $q = z^{1/2}/a$ and the closed contour C consists of Bromwich's contour plus a semicircle of infinite radius in the left half of the z-plane. The residue at $z = 0$ is 1 while at $z_n = -a^2\lambda_n^2$,

$$\text{Res}\left\{\frac{h\cosh(qx)e^{tz}}{z[q\sinh(qL)+h\cosh(qL)]}; z_n\right\}$$
$$= \lim_{z\to z_n}\frac{h(z+a^2\lambda_n^2)\cosh(qx)e^{tz}}{z[q\sinh(qL)+h\cosh(qL)]} \tag{8.4.17}$$
$$= \lim_{z\to z_n}\frac{h\cosh(qx)e^{tz}}{z[(1+hL)\sinh(qL)+h\cosh(qL)]/(2a^2q)} \tag{8.4.18}$$
$$= \frac{2ha^2\lambda_n i\cosh(i\lambda_n x)\exp(-\lambda_n^2a^2t)}{(-a^2\lambda_n^2)[(1+hL)i\sin(\lambda_n L)+i\lambda_n L\cos(\lambda_n L)]} \tag{8.4.19}$$
$$= -\frac{2h\cos(\lambda_n x)\exp(-a^2\lambda_n^2t)}{\lambda_n[(1+hL)\sin(\lambda_n L)+\lambda_n L\cos(\lambda_n L)]}. \tag{8.4.20}$$

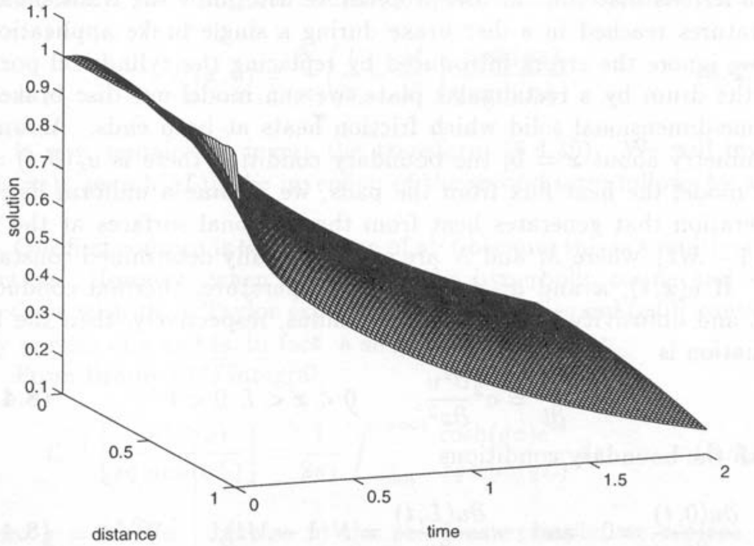

Figure 8.4.1: The temperature within the portion of a slab $0 < x/L < 1$ at various times a^2t/L^2 if the faces of the slab radiate to free space at temperature zero and the slab initially has the temperature 1. The parameter $hL = 1$.

Therefore, the inversion of $U(x,s)$ is

$$u(x,t) = 1 - \left\{1 - 2h\sum_{n=1}^{\infty}\frac{\cos(\lambda_n x)\exp(-a^2\lambda_n^2 t)}{\lambda_n[(1+hL)\sin(\lambda_n L) + \lambda_n L\cos(\lambda_n L)]}\right\} \tag{8.4.21}$$

or

$$u(x,t) = 2h\sum_{n=1}^{\infty}\frac{\cos(\lambda_n x)\exp(-a^2\lambda_n^2 t)}{\lambda_n[(1+hL)\sin(\lambda_n L) + \lambda_n L\cos(\lambda_n L)]}. \tag{8.4.22}$$

We can further simplify (8.4.22) by using $h/\lambda_n = \tan(\lambda_n L)$ and $hL = \lambda_n L\tan(\lambda_n L)$. Substituting these relationships into (8.4.22) and simplifying,

$$u(x,t) = 2\sum_{n=1}^{\infty}\frac{\sin(\lambda_n L)\cos(\lambda_n x)\exp(-a^2\lambda_n^2 t)}{\lambda_n L + \sin(\lambda_n L)\cos(\lambda_n L)}. \tag{8.4.23}$$

• **Example 8.4.2: Heat dissipation in disc brakes**

Disc brakes consist of two blocks of frictional material known as pads which press against each side of a rotating annulus, usually made of a ferrous material. In this problem we determine the transient temperatures reached in a disc brake during a single brake application.[18] If we ignore the errors introduced by replacing the cylindrical portion of the drum by a rectangular plate, we can model our disc brakes as a one-dimensional solid which friction heats at both ends. Assuming symmetry about $x = 0$, the boundary condition there is $u_x(0,t) = 0$. To model the heat flux from the pads, we assume a uniform disc deceleration that generates heat from the frictional surfaces at the rate $N(1 - Mt)$, where M and N are experimentally determined constants.

If $u(x,t)$, κ and a^2 denote the temperature, thermal conductivity, and diffusivity of the rotating annulus, respectively, then the heat equation is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, 0 < t \tag{8.4.24}$$

with the boundary conditions

$$\frac{\partial u(0,t)}{\partial x} = 0 \quad \text{and} \quad \kappa \frac{\partial u(L,t)}{\partial x} = N(1 - Mt), \qquad 0 < t. \tag{8.4.25}$$

The boundary condition at $x = L$ gives the frictional heating of the disc pads.

Introducing the Laplace transform of $u(x,t)$, defined as

$$U(x,s) = \int_0^\infty u(x,t)e^{-st}\,dt, \tag{8.4.26}$$

the equation to be solved becomes

$$\frac{d^2U}{dx^2} - \frac{s}{a^2}U = 0, \tag{8.4.27}$$

subject to the boundary conditions that

$$\frac{dU(0,s)}{dx} = 0 \quad \text{and} \quad \frac{dU(L,s)}{dx} = \frac{N}{\kappa}\left(\frac{1}{s} - \frac{M}{s^2}\right). \tag{8.4.28}$$

The solution of (8.4.27) is

$$U(x,s) = A\cosh(qx) + B\sinh(qx), \tag{8.4.29}$$

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[18] From Newcomb, T. P., 1958: The flow of heat in a parallel-faced infinite solid. *Br. J. Appl. Phys.*, **9**, 370–372. See also Newcomb, T. P., 1958/59: Transient temperatures in brake drums and linings. *Proc. Inst. Mech. Eng., Auto. Div.*, 227–237; Newcomb, T. P., 1959: Transient temperatures attained in disk brakes. *Br. J. Appl. Phys.*, **10**, 339–340.

where $q = s^{1/2}/a$. Using the boundary conditions, the solution becomes

$$U(x,s) = \frac{N}{\kappa}\left(\frac{1}{s} - \frac{M}{s^2}\right)\frac{\cosh(qx)}{q\sinh(qL)}. \tag{8.4.30}$$

It now remains to invert the transform (8.4.30). We will invert $\cosh(qx)/[sq\sinh(qL)]$; the inversion of the second term follows by analog.

Our first concern is the presence of $s^{1/2}$ because this is a multivalued function. However, when we replace the hyperbolic cosine and sine functions with their Taylor expansions, $\cosh(qx)/[sq\sinh(qL)]$ contains only powers of s and is, in fact, a single-valued function.

From Bromwich's integral,

$$\mathcal{L}^{-1}\left[\frac{\cosh(qx)}{sq\sinh(qL)}\right] = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{\cosh(qx)e^{tz}}{zq\sinh(qL)}\,dz, \tag{8.4.31}$$

where $q = z^{1/2}/a$. Just as in the previous example, we replace the hyperbolic cosine and sine with their product expansion and find that $z = 0$ is a second-order pole. The remaining poles are located where $z_n^{1/2}L/a = n\pi i$ or $z_n = -n^2\pi^2a^2/L^2$, where $n = 1, 2, 3, \ldots$. We have chosen the positive sign because $z^{1/2}$ must be single-valued; if we had chosen the negative sign the answer would have been the same. Our expansion also shows that the poles are simple.

Having classified the poles, we now close Bromwich's contour, which lies slightly to the right of the imaginary axis, with an infinite semicircle in the left half-plane, and use the residue theorem. The values of the residues are

$$\operatorname{Res}\left[\frac{\cosh(qx)e^{tz}}{zq\sinh(qL)}; 0\right]$$

$$= \frac{1}{1!}\lim_{z\to0}\frac{d}{dz}\left\{\frac{(z-0)^2\cosh(qx)e^{tz}}{zq\sinh(qL)}\right\} \tag{8.4.32}$$

$$= \lim_{z\to0}\frac{d}{dz}\left\{\frac{z\cosh(qx)e^{tz}}{q\sinh(qL)}\right\} \tag{8.4.33}$$

$$= \frac{a^2}{L}\lim_{z\to0}\frac{d}{dz}\left\{\frac{z\left[1+\frac{zx^2}{2!a^2}+\cdots\right]\left[1+tz+\frac{t^2z^2}{2!}+\cdots\right]}{z+\frac{L^2z^2}{3!a^2}+\cdots}\right\} \tag{8.4.34}$$

$$= \frac{a^2}{L}\lim_{z\to0}\frac{d}{dz}\left\{1+tz+\frac{zx^2}{2a^2}-\frac{zL^2}{3!a^2}+\cdots\right\} \tag{8.4.35}$$

$$= \frac{a^2}{L}\left\{t+\frac{x^2}{2a^2}-\frac{L^2}{6a^2}\right\} \tag{8.4.36}$$

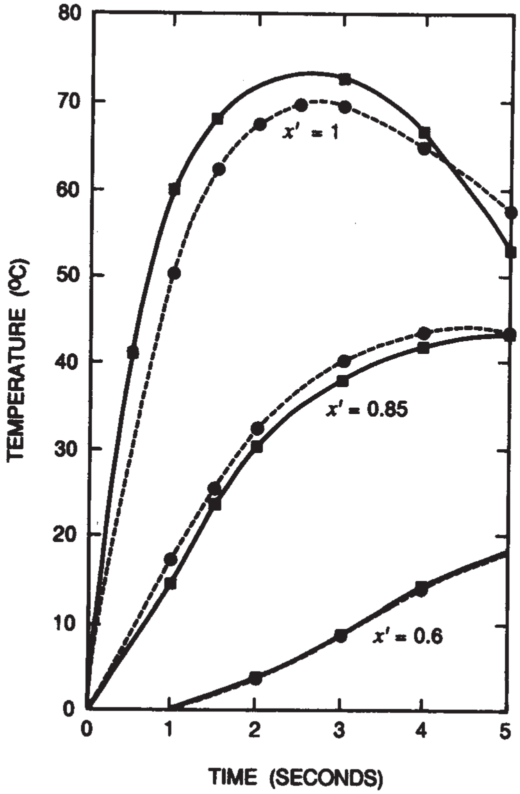

Figure 8.4.2: Typical curves of transient temperature at different locations in a brake lining. Circles denote computed values while squares are experimental measurements. (From Newcomb, T. P., 1958: The flow of heat in a parallel-faced infinite solid. *Br. J. Appl. Phys.*, **9**, 372 with permission.)

and

$$\operatorname{Res}\left[\frac{\cosh(qx)e^{tz}}{zq\sinh(qL)}; z_n\right] = \left[\lim_{z\to z_n} \frac{\cosh(qx)}{zq} e^{tz}\right]\left[\lim_{z\to z_n} \frac{z - z_n}{\sinh(qL)}\right] \tag{8.4.37}$$

$$= \lim_{z\to z_n} \frac{\cosh(qx)e^{tz}}{zq\cosh(qL)L/(2a^2q)} \tag{8.4.38}$$

$$= \frac{\cosh(n\pi xi/L)\exp(-n^2\pi^2a^2t/L^2)}{(-n^2\pi^2a^2/L^2)\cosh(n\pi i)L/(2a^2)} \tag{8.4.39}$$

$$= -\frac{2L(-1)^n}{n^2\pi^2}\cos(n\pi x/L)e^{-n^2\pi^2a^2t/L^2}. \tag{8.4.40}$$

When we sum all of the residues from both inversions, the solution is

$$
\begin{aligned}
u(x,t) = {} & \frac{a^2 N}{\kappa L}\left\{t + \frac{x^2}{2a^2} - \frac{L^2}{6a^2}\right\} \\
& - \frac{2LN}{\kappa\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x/L)e^{-n^2\pi^2a^2t/L^2} \\
& - \frac{a^2NM}{\kappa L}\left\{\frac{t^2}{2} + \frac{tx^2}{2a^2} - \frac{tL^2}{6a^2} + \frac{x^4}{24a^4} - \frac{x^2L^2}{12a^4} + \frac{7L^4}{360a^4}\right\} \\
& - \frac{2L^3NM}{a^2\kappa\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\cos(n\pi x/L)e^{-n^2\pi^2a^2t/L^2}. \qquad (8.4.41)
\end{aligned}
$$

Figure 8.4.2 shows the temperature in the brake lining at various places within the lining [$x' = x/L$] if $a^2 = 3.3 \times 10^{-3}$ cm^2/sec, $\kappa = 1.8 \times 10^{-3}$ cal/(cm sec°C), $L = 0.48$ cm and $N = 1.96$ cal/(cm^2 sec). Initially the frictional heating results in an increase in the disc brake's temperature. As time increases, the heating rate decreases and radiative cooling becomes sufficiently large that the temperature begins to fall.

• Example 8.4.3

In the previous example we showed that Laplace transforms are particularly useful when the boundary conditions are time dependent. Consider now the case when one of the boundaries is moving.

We wish to solve the heat equation

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}, \qquad \beta t < x < \infty, 0 < t \qquad (8.4.42)$$

subject to the boundary conditions

$$u(x,t)\big|_{x=\beta t} = f(t) \quad \text{and} \quad \lim_{x\to\infty}|u(x,t)| < \infty, \qquad 0 < t \qquad (8.4.43)$$

and the initial condition

$$u(x,0) = 0, \qquad 0 < x < \infty. \qquad (8.4.44)$$

This type of problems arises in combustion problems where the boundary moves due to the burning of the fuel.

We begin by introducing the coordinate $\eta = x - \beta t$. Then the problem can be reformulated as

$$\frac{\partial u}{\partial t} - \beta\frac{\partial u}{\partial \eta} = a^2\frac{\partial^2 u}{\partial \eta^2}, \qquad 0 < \eta < \infty, 0 < t \qquad (8.4.45)$$

subject to the boundary conditions

$$u(0,t) = f(t) \quad \text{and} \quad \lim_{\eta\to\infty} |u(\eta,t)| < \infty, \qquad 0 < t \tag{8.4.46}$$

and the initial condition

$$u(\eta,0) = 0, \qquad 0 < \eta < \infty. \tag{8.4.47}$$

Taking the Laplace transform of (8.4.45), we have that

$$\frac{d^2U(\eta,s)}{d\eta^2} + \frac{\beta}{a^2}\frac{dU(\eta,s)}{d\eta} - \frac{s}{a^2}U(\eta,s) = 0 \tag{8.4.48}$$

with

$$U(0,s) = F(s) \quad \text{and} \quad \lim_{\eta\to\infty} |U(\eta,s)| < \infty. \tag{8.4.49}$$

The solution to (8.4.48)–(8.4.49) is

$$U(\eta,s) = F(s)\exp\left(-\frac{\beta\eta}{2a^2} - \frac{\eta}{a}\sqrt{s+\frac{\beta^2}{4a^2}}\right). \tag{8.4.50}$$

Because

$$\mathcal{L}\left[\Phi(\eta,t)\right] = \exp\left(-\frac{\eta}{a}\sqrt{s+\frac{\beta^2}{4a^2}}\right), \tag{8.4.51}$$

where

$$\begin{aligned}\Phi(\eta,t) = \tfrac{1}{2}\bigg[&e^{-\beta\eta/2a^2}\operatorname{erfc}\left(\frac{\eta}{2a\sqrt{t}} - \frac{\beta\sqrt{t}}{2a}\right)\\ &+ e^{\beta\eta/2a^2}\operatorname{erfc}\left(\frac{\eta}{2a\sqrt{t}} + \frac{\beta\sqrt{t}}{2a}\right)\bigg]\end{aligned} \tag{8.4.52}$$

and

$$\operatorname{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-\eta^2}\,d\eta, \tag{8.4.53}$$

we have by the convolution theorem that

$$u(\eta,t) = e^{-\beta\eta/2a^2}\int_0^t f(t-\tau)\Phi(\eta,\tau)\,d\tau \tag{8.4.54}$$

or

$$u(x,t) = e^{-\beta(x-\beta t)/2a^2}\int_0^t f(t-\tau)\Phi(x-\beta\tau,\tau)\,d\tau. \tag{8.4.55}$$

Problems

1. Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - a^2(u - T_0), \qquad 0 < x < 1, 0 < t$$

subject to the boundary conditions

$$\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(1,t)}{\partial x} = 0, \qquad 0 < t$$

and the initial condition

$$u(x,0) = 0, \qquad 0 < x < 1.$$

2. Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, 0 < t$$

subject to the boundary conditions

$$\frac{\partial u(0,t)}{\partial x} = 0, \quad u(1,t) = t, \quad 0 < t$$

and the initial condition

$$u(x,0) = 0, \qquad 0 < x < 1.$$

3. Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, 0 < t$$

subject to the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 1, \quad 0 < t$$

and the initial condition

$$u(x,0) = 0, \qquad 0 < x < 1.$$

4. Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad -\tfrac{1}{2} < x < \tfrac{1}{2}, 0 < t$$

subject to the boundary conditions

$$u_x\left(-\tfrac{1}{2}, t\right) = 0, \quad u_x\left(\tfrac{1}{2}, t\right) = \delta(t), \qquad 0 < t$$

and the initial condition

$$u(x,0) = 0, \qquad -\tfrac{1}{2} < x < \tfrac{1}{2}.$$

5. Solve

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 1, \qquad 0 < x < 1, 0 < t$$

subject to the boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad 0 < t$$

and the initial condition

$$u(x,0) = 0, \qquad 0 < x < 1.$$

6. Solve[19]

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \infty, 0 < t$$

subject to the boundary conditions

$$u(0,t) = 1, \quad \lim_{x\to\infty} |u(x,t)| < \infty, \quad 0 < t$$

and the initial condition

$$u(x,0) = 0, \qquad 0 < x < \infty.$$

[Hint: Use tables to invert the Laplace transform.]

7. Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \infty, 0 < t$$

subject to the boundary conditions

$$\frac{\partial u(0,t)}{\partial x} = 1, \quad \lim_{x\to\infty} |u(x,t)| < \infty, \quad 0 < t$$

and the initial condition

$$u(x,0) = 0, \qquad 0 < x < \infty.$$

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[19] If $u(x,t)$ denotes the Eulerian velocity of a viscous fluid in the half space $x > 0$ and parallel to the wall located at $x = 0$, then this problem was first solved by Stokes, G. G., 1850: On the effect of the internal friction of fluids on the motions of pendulums. *Proc. Cambridge Philos. Soc.*, **9, Part II**, [8]–[106].

[Hint: Use tables to invert the Laplace transform.]

8. Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \infty, 0 < t$$

subject to the boundary conditions

$$u(0,t) = 1, \quad \lim_{x \to \infty} |u(x,t)| < \infty, \quad 0 < t$$

and the initial condition

$$u(x,0) = e^{-x}, \qquad 0 < x < \infty.$$

[Hint: Use tables to invert the Laplace transform.]

9. Solve

$$\frac{\partial u}{\partial t} = a^2 \left[\frac{\partial^2 u}{\partial x^2} + (1+\delta)\frac{\partial u}{\partial x} + \delta u \right], \qquad 0 < x < \infty, 0 < t,$$

where δ is a constant, subject to the boundary conditions

$$u(0,t) = u_0, \quad \lim_{x \to \infty} |u(x,t)| < \infty, \quad 0 < t$$

and the initial condition

$$u(x,0) = 0, \qquad 0 < x < \infty.$$

Note that

$$\mathcal{L}^{-1}\left[\frac{1}{s}\exp\left(-2\alpha\sqrt{s+\beta^2}\,\right)\right] = \tfrac{1}{2}e^{2\alpha\beta}\operatorname{erfc}\left(\frac{\alpha}{\sqrt{t}} + \beta\sqrt{t}\right) + \tfrac{1}{2}e^{-2\alpha\beta}\operatorname{erfc}\left(\frac{\alpha}{\sqrt{t}} - \beta\sqrt{t}\right),$$

where erfc is the complementary error function.

10. Solve

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} + Ae^{-kx}, \qquad 0 < x < \infty, 0 < t$$

subject to the boundary conditions

$$\frac{\partial u(0,t)}{\partial x} = 0, \quad \lim_{x \to \infty} u(x,t) = u_0, \quad 0 < t$$

and the initial condition

$$u(x,0) = u_0, \qquad 0 < x < \infty.$$

11. Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - P, \qquad 0 < x < L, 0 < t$$

subject to the boundary conditions

$$u(0,t) = t, \quad u(L,t) = 0, \quad 0 < t$$

and the initial condition

$$u(x,0) = 0, \qquad 0 < x < L.$$

12. An electric fuse protects electrical devices by using resistance heating to melt an enclosed wire when excessive current passes through it. A knowledge of the distribution of temperature along the wire is important in the design of the fuse. If the temperature rises to the melting point only over a small interval of the element, the melt will produce a small gap, resulting in an unnecessary prolongation of the fault and a considerable release of energy. Therefore, the desirable temperature distribution should melt most of the wire. For this reason, Guile and Carne[20] solved the heat conduction equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + q(1 + \alpha u), \qquad -L < x < L, 0 < t$$

to understand the temperature structure within the fuse just before meltdown. The second term on the right side of the heat conduction equation gives the resistance heating which is assumed to vary linearly with temperature. If the terminals at $x = \pm L$ remain at a constant temperature, which we can take to be zero, the boundary conditions are

$$u(-L,t) = u(L,t) = 0, \qquad 0 < t.$$

The initial condition is

$$u(x,0) = 0, \qquad -L < x < L.$$

[20] From Guile, A. E. and Carne, E. B., 1954: An analysis of an analogue solution applied to the heat conduction problem in a cartridge fuse. *AIEE Trans., Part I*, **72**, 861–868. ©AIEE (now IEEE).

Find the temperature field as a function of the parameters a, q, and α.

13. Solve[21]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}, \quad 0 \le r < 1, 0 < t$$

subject to the boundary conditions

$$\lim_{r \to 0} |u(r,t)| < \infty, \quad \frac{\partial u(1,t)}{\partial r} = 1, \quad 0 < t$$

and the initial condition

$$u(r,0) = 0, \qquad 0 \le r < 1.$$

[Hint: Use the new dependent variable $v(r,t) = ru(r,t)$.]

14. Consider[22] a viscous fluid located between two fixed walls $x = \pm L$. At $x = 0$ we introduce a thin, infinitely long rigid barrier of mass m per unit area and let it fall under the force of gravity which points in the direction of positive x. We wish to find the velocity of the fluid $u(x,t)$. The fluid is governed by the partial differential equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, 0 < t$$

subject to the boundary conditions

$$u(L,t) = 0 \quad \text{and} \quad \frac{\partial u(0,t)}{\partial t} - \frac{2\mu}{m}\frac{\partial u(0,t)}{\partial x} = g, \qquad 0 < t$$

and the initial condition

$$u(x,0) = 0, \qquad 0 < x < L.$$

15. Consider[23] a viscous fluid located between two fixed walls $x = \pm L$. At $x = 0$ we introduce a thin, infinitely long rigid barrier of mass m per

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[21] From Reismann, H., 1962: Temperature distribution in a spinning sphere during atmospheric entry. *J. Aerosp. Sci.*, **29**, 151–159 with permission.

[22] Reproduced with acknowledgement to Taylor and Francis, Publishers, from Havelock, T. H., 1921: The solution of an integral equation occurring in certain problems of viscous fluid motion. *Philos. Mag., Ser. 6*, **42**, 620–628.

[23] Reproduced with acknowledgement to Taylor and Francis, Publishers, from Havelock, T. H., 1921: On the decay of oscillation of a solid body in a viscous fluid. *Philos. Mag., Ser. 6*, **42**, 628–634.

unit area. The barrier is acted upon an elastic force in such a manner that it would vibrate with a frequency ω if the liquid were absent. We wish to find the barrier's deviation from equilibrium, $y(t)$. The fluid is governed by the partial differential equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, 0 < t.$$

The boundary conditions are

$$u(L,t) = m\frac{d^2 y}{dt^2} - 2\mu \frac{\partial u(0,t)}{\partial x} + m\omega^2 y = 0 \text{ and } \frac{dy}{dt} = u(0,t), \quad 0 < t$$

and the initial conditions are

$$u(x,0) = 0, \ 0 < x < L \qquad \text{and} \qquad y(0) = A, \ y'(0) = 0.$$

16. Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x < 1, 0 < t$$

subject to the boundary conditions

$$u(0,t) = 0, \qquad 3a\left[\frac{\partial u(1,t)}{\partial x} - u(1,t)\right] + \frac{\partial u(1,t)}{\partial t} = \delta(t), \qquad 0 < t$$

and the initial condition

$$u(x,0) = 0, \qquad 0 \le x < 1.$$

17. Solve[24] the partial differential equation

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, 0 < t,$$

where V is a constant, subject to the boundary conditions

$$u(0,t) = 1 \qquad \text{and} \qquad u_x(1,t) = 0, \qquad 0 < t$$

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[24] Reprinted from *Solar Energy*, **56**, Yoo, H., and E.-T. Pak, Analytical solutions to a one-dimensional finite-domain model for stratified thermal storage tanks, 315–322, ©1996, with kind permission from Elsevier Science Ltd, The Boulevard, Langford Lane, Kidlington OX5 1GB, UK.

and the initial condition

$$u(x,0) = 0, \qquad 0 < x < 1.$$

18. Solve

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{\partial u}{\partial t} = \delta(t), \qquad 0 \le r < a, 0 < t$$

subject to the boundary conditions

$$\lim_{r\to 0} |u(r,t)| < \infty, \qquad u(a,t) = 0, \qquad 0 < t$$

and the initial condition

$$u(r,0) = 0, \qquad 0 \le r < a.$$

Note that $J_n(iz) = i^n I_n(z)$ and $I_n(iz) = i^n J_n(z)$ for all complex z.

19. Solve

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + H(t), \qquad 0 \le r < a, 0 < t$$

subject to the boundary conditions

$$\lim_{r\to 0} |u(r,t)| < \infty, \qquad u(a,t) = 0, \qquad 0 < t$$

and the initial condition

$$u(r,0) = 0, \qquad 0 \le r < a.$$

Note that $J_n(iz) = i^n I_n(z)$ and $I_n(iz) = i^n J_n(z)$ for all complex z.

20. Solve

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \qquad 0 \le r < a, 0 < t$$

subject to the boundary conditions

$$\lim_{r\to 0} |u(r,t)| < \infty, \qquad u(a,t) = e^{-t/\tau_0}, \qquad 0 < t$$

and the initial condition

$$u(r,0) = 1, \qquad 0 \le r < a.$$

Note that $J_n(iz) = i^n I_n(z)$ and $I_n(iz) = i^n J_n(z)$ for all complex z.

21. Solve the nonhomogeneous heat equation for the spherical shell[25]

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{A}{r^4} \right), \qquad \alpha < r < \beta, 0 < t$$

subject to the boundary conditions

$$\frac{\partial u(\alpha, t)}{\partial r} = u(\beta, t) = 0, \qquad 0 < t$$

and the initial condition

$$u(r, 0) = 0, \qquad \alpha < r < \beta.$$

Step 1: By introducing $v(r,t) = r\,u(r,t)$, show that the problem simplifies to

$$\frac{\partial v}{\partial t} = a^2 \left(\frac{\partial^2 v}{\partial r^2} + \frac{A}{r^3} \right), \qquad \alpha < r < \beta, 0 < t$$

subject to the boundary conditions

$$\frac{\partial v(\alpha, t)}{\partial r} - \frac{v(\alpha, t)}{\alpha} = v(\beta, t) = 0, \qquad 0 < t$$

and the initial condition

$$v(r, 0) = 0, \qquad \alpha < r < \beta.$$

Step 2: Using Laplace transforms and variation of parameters, show that the Laplace transform of $u(r,t)$ is

$$U(r,s) = \frac{A}{srq} \left\{ \frac{\sinh[q(\beta - r)]}{\alpha q \cosh(q\ell) + \sinh(q\ell)} \int_0^\ell \frac{\alpha q \cosh(q\eta) + \sinh(q\eta)}{(\alpha + \eta)^3}\, d\eta - \int_0^{\beta - r} \frac{\sinh(q\eta)}{(r + \eta)^3}\, d\eta \right\},$$

where $q = \sqrt{s}/a$ and $\ell = \beta - \alpha$.

Step 3: Take the inverse of $U(r,s)$ and show that

$$u(r,t) = A \left\{ \left(\frac{1}{r} - \frac{1}{\beta} \right) \left[\frac{1}{\alpha} - \frac{1}{2} \left(\frac{1}{r} + \frac{1}{\beta} \right) \right] - \frac{2\alpha^2}{r\ell^2} \sum_{n=0}^{\infty} \frac{\sin[\gamma_n(\beta - r)] \exp(-a^2\gamma_n^2 t)}{\sin^2(\gamma_n \ell)(\beta + \alpha^2 \ell \gamma_n^2)} \int_0^1 \frac{\sin(\gamma_n \ell \eta)}{(\delta - \eta)^3}\, d\eta \right\},$$

where γ_n is the nth root of $\alpha\gamma + \tan(\ell\gamma) = 0$ and $\delta = 1 + \alpha/\ell$.

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[25] Abstracted with permission from Malkovich, R. Sh., 1977: Heating of a spherical shell by a radial current. *Sov. Phys. Tech. Phys.*, **22**, 636. ©1977 American Institute of Physics.

8.5 THE FOURIER TRANSFORM METHOD

We now consider the problem of one-dimensional heat flow in a rod of infinite length with insulated sides. Although there are no boundary conditions because the slab is of infinite dimension, we do require that the solution remains bounded as we go to either positive or negative infinity. The initial temperature within the rod is $u(x,0) = f(x)$.

Employing the product solution technique of Section 8.3, $u(x,t) = X(x)T(t)$ with

$$T' + a^2\lambda T = 0 \tag{8.5.1}$$

and

$$X'' + \lambda X = 0. \tag{8.5.2}$$

Solutions to (8.5.1)–(8.5.2) which remain finite over the entire x-domain are

$$X(x) = E\cos(kx) + F\sin(kx) \tag{8.5.3}$$

and

$$T(t) = C\exp(-k^2a^2t). \tag{8.5.4}$$

Because we do not have any boundary conditions, we must include *all* possible values of k. Thus, when we sum all of the product solutions according to the principle of linear superposition, we obtain the integral

$$u(x,t) = \int_0^\infty [A(k)\cos(kx) + B(k)\sin(kx)]\exp(-k^2a^2t)\,dk. \tag{8.5.5}$$

We can satisfy the initial condition by choosing

$$A(k) = \frac{1}{\pi}\int_{-\infty}^\infty f(x)\cos(kx)\,dx \tag{8.5.6}$$

and

$$B(k) = \frac{1}{\pi}\int_{-\infty}^\infty f(x)\sin(kx)\,dx, \tag{8.5.7}$$

because the initial condition has the form of a Fourier integral

$$f(x) = \int_0^\infty [A(k)\cos(kx) + B(k)\sin(kx)]\,dk, \tag{8.5.8}$$

when $t = 0$.

Several important results follow by rewriting (8.5.8) as

$$u(x,t) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(\xi)\cos(k\xi)\cos(kx)\,d\xi + \int_{-\infty}^\infty f(\xi)\sin(k\xi)\sin(kx)\,d\xi\right]\exp(-k^2a^2t)\,dk. \tag{8.5.9}$$

Combining terms,

$$u(x,t) = \frac{1}{\pi}\int_0^\infty \left\{\int_{-\infty}^\infty f(\xi)[\cos(k\xi)\cos(kx) + \sin(k\xi)\sin(kx)]\,d\xi\right\}e^{-k^2a^2t}\,dk \tag{8.5.10}$$

$$= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(\xi)\cos[k(\xi - x)]\,d\xi\right]e^{-k^2a^2t}\,dk. \tag{8.5.11}$$

Reversing the order of integration,

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^\infty f(\xi)\left[\int_0^\infty \cos[k(\xi - x)]\exp(-k^2a^2t)\,dk\right]d\xi. \tag{8.5.12}$$

The inner integral is called the *source function*. We may compute its value through an integration on the complex plane; it equals

$$\int_0^\infty \cos[k(\xi - x)]\exp(-k^2a^2t)\,dk = \left(\frac{\pi}{4a^2t}\right)^{1/2}\exp\left[-\frac{(\xi - x)^2}{4a^2t}\right], \tag{8.5.13}$$

if $0 < t$. This gives the final form for the temperature distribution:

$$u(x,t) = \frac{1}{\sqrt{4a^2\pi t}}\int_{-\infty}^\infty f(\xi)\exp\left[-\frac{(\xi - x)^2}{4a^2t}\right]d\xi. \tag{8.5.14}$$

• **Example 8.5.1**

Let us find the temperature field if the initial distribution is

$$u(x,0) = \begin{cases} T_0, & x > 0 \\ -T_0, & x < 0. \end{cases} \tag{8.5.15}$$

Then

$$u(x,t) = -\frac{T_0}{\sqrt{4a^2\pi t}}\int_{-\infty}^0 \exp\left[-\frac{(\xi - x)^2}{4a^2t}\right]d\xi + \frac{T_0}{\sqrt{4a^2\pi t}}\int_0^\infty \exp\left[-\frac{(\xi - x)^2}{4a^2t}\right]d\xi \tag{8.5.16}$$

$$= \frac{T_0}{\sqrt{\pi}}\left[\int_{-x/2a\sqrt{t}}^\infty e^{-\tau^2}\,d\tau - \int_{x/2a\sqrt{t}}^\infty e^{-\tau^2}\,d\tau\right] \tag{8.5.17}$$

$$= \frac{T_0}{\sqrt{\pi}}\int_{-x/2a\sqrt{t}}^{x/2a\sqrt{t}} e^{-\tau^2}\,d\tau = \frac{2T_0}{\sqrt{\pi}}\int_0^{x/2a\sqrt{t}} e^{-\tau^2}\,d\tau \tag{8.5.18}$$

$$= T_0\,\mathrm{erf}\left(\frac{x}{2a\sqrt{t}}\right), \tag{8.5.19}$$

where erf is the error function.

• Example 8.5.2: Kelvin's estimate of the age of the earth

In the middle of the nineteenth century Lord Kelvin[26] estimated the age of the earth using the observed vertical temperature gradient at the earth's surface. He hypothesized that the earth was initially formed at a uniform high temperature T_0 and that its surface was subsequently maintained at the lower temperature of T_S. Assuming that most of the heat conduction occurred near the earth's surface, he reasoned that he could neglect the curvature of the earth, consider the earth's surface planar, and employ our one-dimensional heat conduction model in the vertical direction to compute the observed heat flux.

Following Kelvin, we model the earth's surface as a flat plane with an infinitely deep earth below ($z > 0$). Initially the earth has the temperature T_0. Suddenly we drop the temperature at the surface to T_S. We wish to find the heat flux across the boundary at $z = 0$ from the earth into an infinitely deep atmosphere.

The first step is to redefine our temperature scale $v(z,t) = u(z,t) + T_S$, where $v(z,t)$ is the observed temperature so that $u(0,t) = 0$ at the surface. Next, in order to use (8.5.14), we must define our initial state for $z < 0$. To maintain the temperature $u(0,t) = 0$, $f(z)$ must be an odd function or

$$f(z) = \begin{cases} T_0 - T_S, & z > 0 \\ T_S - T_0, & z < 0. \end{cases} \tag{8.5.20}$$

From (8.5.14)

$$u(z,t) = -\frac{T_0 - T_S}{\sqrt{4a^2\pi t}} \int_{-\infty}^{0} \exp\left[-\frac{(\xi - z)^2}{4a^2 t}\right] d\xi + \frac{T_0 - T_S}{\sqrt{4a^2\pi t}} \int_{0}^{\infty} \exp\left[-\frac{(\xi - z)^2}{4a^2 t}\right] d\xi \tag{8.5.21}$$

$$= (T_0 - T_S) \operatorname{erf}\left(\frac{z}{2a\sqrt{t}}\right), \tag{8.5.22}$$

following the work in the previous example.

The heat flux q at the surface $z = 0$ is obtained by differentiating (8.5.22) according to Fourier's law and evaluating the result at $z = 0$:

$$q = -\kappa \left.\frac{\partial v}{\partial z}\right|_{z=0} = \frac{\kappa(T_S - T_0)}{a\sqrt{\pi t}}. \tag{8.5.23}$$

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26 Thomson, W., 1863: On the secular cooling of the earth. *Philos. Mag., Ser. 4*, **25**, 157–170.

The surface heat flux is infinite at $t = 0$ because of the sudden application of the temperature T_S at $t = 0$. After that time, the heat flux decreases with time. Consequently, the time t at which we have the temperature gradient $\partial v(0,t)/\partial z$ is

$$t = \frac{(T_0 - T_S)^2}{\pi a^2 [\partial v(0,t)/\partial z]^2}. \tag{8.5.24}$$

For the present near-surface thermal gradient of 25 K/km, $T_0 - T_S = 2000$ K and $a^2 = 1$ mm^2/s, the age of the earth from (8.5.24) is 65 million years.

Although Kelvin realized that this was a very rough estimate, his calculation showed that the earth had a finite age. This was a direct frontal assault on the contemporary geological principle of *uniformitarianism* that the earth's surface and upper crust had remained unchanged in temperature and other physical quantities for millions and millions of years. This debate would rage throughout the latter half of the nineteenth century and feature such luminaries as Kelvin, Charles Darwin, Thomas Huxley, and Oliver Heaviside.[27] Eventually Kelvin's arguments would prevail and uniformitarianism would fade into history.

Today, Kelvin's estimate is of academic interest because of the discovery of radioactivity at the turn of the twentieth century. The radioactivity was assumed to be uniformly distributed around the globe and restricted to the upper few tens of kilometers of the crust. Then geologists would use observed heat fluxes to discover the distribution of radioactivity within the solid earth.[28] Today we know that the interior of the earth is quite dynamic; the oceans and continents are mobile and interconnected according to the theory of plate tectonics. However, geophysicists still use measured surface heat fluxes to infer the interior[29] of the earth.

Problems

For problems 1–4, find the solution of the heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, 0 < t$$

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[27] See Burchfield, J. D., 1975: *Lord Kelvin and the Age of the Earth*, Science History Publ., 260 pp.

[28] See Slichter, L. B., 1941: Cooling of the earth. *Bull. Geol. Soc. Am.*, **52**, 561–600.

[29] Sclater, J. G., Jaupart, C., and Galson, D., 1980: The heat flow through oceanic and continental crust and the heat loss of the earth. *Rev. Geophys. Space Phys.*, **18**, 269–311.

subject to the stated initial conditions.

1.

$$u(x,0) = \begin{cases} 1, & |x| < b \\ 0, & |x| > b \end{cases}$$

Lovering[30] has applied this solution to problems involving the cooling of lava.

2.

$$u(x,0) = e^{-b|x|}$$

3.

$$u(x,0) = \begin{cases} 0, & -\infty < x < 0 \\ T_0, & 0 < x < b \\ 0, & b < x < \infty \end{cases}$$

4.

$$u(x,0) = \delta(x)$$

5. Solve the spherically symmetric equation of diffusion,[31]

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right), \qquad 0 \le r < \infty, 0 < t$$

with $u(r,0) = u_0(r)$.

Step 1: Assuming $v(r,t) = r\,u(r,t)$, show that the problem can be recast as

$$\frac{\partial v}{\partial t} = a^2\frac{\partial^2 v}{\partial r^2} \qquad 0 \le r < \infty, 0 < t$$

with $v(r,0) = r\,u_0(r)$.

Step 2: Using (8.5.14), show that the general solution is

$$u(r,t) = \frac{1}{2ar\sqrt{\pi t}}\int_0^\infty u_0(\rho)\left\{\exp\left[-\left(\frac{r-\rho}{2a\sqrt{t}}\right)^2\right]\right.$$
$$\left. - \exp\left[-\left(\frac{r+\rho}{2a\sqrt{t}}\right)^2\right]\right\}\rho\,d\rho.$$

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[30] Lovering, T. S., 1935: Theory of heat conduction applied to geological problems. *Bull. Geol. Soc. Am.*, **46**, 69–94.

[31] From Shklovskii, I. S. and Kurt, V. G., 1960: Determination of atmospheric density at a height of 430 km by means of the diffusion of sodium vapors. *ARS J.*, **30**, 662–667 with permission.

Hint: What is the constraint on (8.5.14) so that the solution remains radially symmetric.

Step 3: For the initial concentration of

$$u_0(r) = \begin{cases} N_0, & 0 \le r < r_0 \\ 0, & r > r_0, \end{cases}$$

show that

$$u(r,t) = \tfrac{1}{2}N_0\left[\operatorname{erf}\left(\frac{r_0 - r}{2a\sqrt{t}}\right) + \operatorname{erf}\left(\frac{r_0 + r}{2a\sqrt{t}}\right) + \frac{2a\sqrt{t}}{r\sqrt{\pi}}\left\{\exp\left[-\left(\frac{r_0 + r}{2a\sqrt{t}}\right)^2\right] - \exp\left[-\left(\frac{r_0 - r}{2a\sqrt{t}}\right)^2\right]\right\}\right],$$

where erf is the error function.

8.6 THE SUPERPOSITION INTEGRAL

In our study of Laplace transforms, we showed that we may construct solutions to ordinary differential equations with a general forcing $f(t)$ by first finding the solution to a similar problem where the forcing equals Heaviside's step function. Then we can write the general solution in terms of a superposition integral according to Duhamel's theorem. In this section we show that similar considerations hold in solving the heat equation with time-dependent boundary conditions or forcings.

Let us solve the heat condition problem

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, 0 < t \tag{8.6.1}$$

with the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = f(t), \quad 0 < t \tag{8.6.2}$$

and the initial condition

$$u(x,0) = 0, \quad 0 < x < L. \tag{8.6.3}$$

The solution of (8.6.1)–(8.6.3) is difficult because of the time-dependent boundary condition. Instead of solving this system directly, let us solve the easier problem

$$\frac{\partial A}{\partial t} = a^2 \frac{\partial^2 A}{\partial x^2}, \quad 0 < x < L, 0 < t \tag{8.6.4}$$

with the boundary conditions

$$A(0,t) = 0, \quad A(L,t) = 1, \quad 0 < t \tag{8.6.5}$$

and the initial condition

$$A(x,0) = 0, \quad 0 < x < L. \tag{8.6.6}$$

Separation of variables yields the solution

$$A(x,t) = \frac{x}{L} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{a^2n^2\pi^2t}{L^2}\right). \tag{8.6.7}$$

Consider the following case. Suppose that we maintain the temperature at zero at the end $x = L$ until $t = \tau_1$ and then raise it to the value of unity. The resulting temperature distribution equals zero everywhere when $t < \tau_1$ and equals $A(x, t - \tau_1)$ for $t > \tau_1$. We have merely shifted our time axis so that the initial condition occurs at $t = \tau_1$.

Consider an analogous, but more complicated, situation of the temperature at the end position $x = L$ held at $f(0)$ from $t = 0$ to $t = \tau_1$ at which time we abruptly change it by the amount $f(\tau_1) - f(0)$ to the value $f(\tau_1)$. This temperature remains until $t = \tau_2$ when we again abruptly change it by an amount $f(\tau_2) - f(\tau_1)$. We can imagine this process continuing up to the instant $t = \tau_n$. Because of linear superposition, the temperature distribution at any given time equals the sum of these temperature increments:

$$\begin{aligned} u(x,t) &= f(0)A(x,t) + [f(\tau_1) - f(0)]A(x, t-\tau_1) \\ &\quad + [f(\tau_2) - f(\tau_1)]A(x,t-\tau_2) + \cdots \\ &\quad + [f(\tau_n) - f(\tau_{n-1})]A(x,t-\tau_n), \end{aligned} \tag{8.6.8}$$

where τ_n is the time of the most recent temperature change. If we write

$$\Delta f_k = f(\tau_k) - f(\tau_{k-1}) \quad \text{and} \quad \Delta\tau_k = \tau_k - \tau_{k-1}, \tag{8.6.9}$$

(8.6.8) becomes

$$u(x,t) = f(0)A(x,t) + \sum_{k=1}^{n} A(x, t-\tau_k)\frac{\Delta f_k}{\Delta \tau_k}\Delta\tau_k. \tag{8.6.10}$$

Consequently, in the limit of $\Delta\tau_k \to 0$, (8.6.10) becomes

$$u(x,t) = f(0)A(x,t) + \int_0^t A(x,t-\tau)f'(\tau)\,d\tau, \tag{8.6.11}$$

assuming that $f(t)$ is differentiable. Equation (8.6.11) is the *superposition integral.* We can obtain an alternative form by integration by parts:

$$u(x,t) = f(t)A(x,0) - \int_0^t f(\tau)\frac{\partial A(x,t-\tau)}{\partial \tau}\,d\tau \tag{8.6.12}$$

or

$$u(x,t) = f(t)A(x,0) + \int_0^t f(\tau)\frac{\partial A(x,t-\tau)}{\partial t}\,d\tau, \tag{8.6.13}$$

because

$$\frac{\partial A(x,t-\tau)}{\partial \tau} = -\frac{\partial A(x,t-\tau)}{\partial t}. \tag{8.6.14}$$

To illustrate the superposition integral, suppose $f(t) = t$. Then, by (8.6.11),

$$u(x,t) = \int_0^t \left\{\frac{x}{L} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{L}\right)\exp\left[-\frac{a^2n^2\pi^2}{L^2}(t-\tau)\right]\right\}d\tau \tag{8.6.15}$$

$$= \frac{xt}{L} - \frac{2L^2}{a^2\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\left(\frac{n\pi x}{L}\right)\left[1-\exp\left(-\frac{a^2n^2\pi^2t}{L^2}\right)\right]. \tag{8.6.16}$$

• Example 8.6.1: Temperature oscillations in a wall heated by an alternating current

In addition to finding solutions to heat conduction problems with time-dependent boundary conditions, we may also apply the superposition integral to the nonhomogeneous heat equation when the source is time dependent. Jeglic[32] used this technique in obtaining the temperature distribution within a slab heated by alternating electric current. If we assume that the flat plate has a surface area A and depth L, then the heat equation for the plate when electrically heated by an alternating current of frequency ω is

$$\frac{\partial u}{\partial t} - a^2\frac{\partial^2 u}{\partial x^2} = \frac{2q}{\rho C_p AL}\sin^2\omega t, \quad 0 < x < L, 0 < t, \tag{8.6.17}$$

[32] Jeglic, F. A., 1962: An analytical determination of temperature oscillations in a wall heated by alternating current. *NASA Tech. Note No. D-1286.*

where q is the average heat rate caused by the current, ρ is the density, C_p is the specific heat at constant pressure, and a^2 is the diffusivity of the slab. We will assume that we have insulated the inner wall so that

$$\frac{\partial u(0,t)}{\partial x} = 0, \qquad 0 < t, \tag{8.6.18}$$

while we allow the outer wall to radiatively cool to free space at the temperature of zero

$$\kappa \frac{\partial u(L,t)}{\partial x} + hu(L,t) = 0, \qquad 0 < t, \tag{8.6.19}$$

where κ is the thermal conductivity and h is the heat transfer coefficient. The slab is initially at the temperature of zero

$$u(x,0) = 0, \qquad 0 < x < L. \tag{8.6.20}$$

To solve the heat equation, we first solve the simpler problem of

$$\frac{\partial A}{\partial t} - a^2 \frac{\partial^2 A}{\partial x^2} = 1, \quad 0 < x < L, 0 < t \tag{8.6.21}$$

with the boundary conditions

$$\frac{\partial A(0,t)}{\partial x} = 0 \quad \text{and} \quad \kappa \frac{\partial A(L,t)}{\partial x} + hA(L,t) = 0, \quad 0 < t \tag{8.6.22}$$

and the initial condition

$$A(x,0) = 0, \qquad 0 < x < L. \tag{8.6.23}$$

The solution $A(x,t)$ is the *indicial admittance* because it is the response of a system to forcing by the step function $H(t)$.

We will solve (8.6.21)–(8.6.23) by separation of variables. We begin by assuming that $A(x,t)$ consists of a steady-state solution $w(x)$ plus a transient solution $v(x,t)$, where

$$a^2 w''(x) = -1, \quad w'(0) = 0, \quad \kappa w'(L) + hw(L) = 0, \tag{8.6.24}$$

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}, \quad \frac{\partial v(0,t)}{\partial x} = 0, \quad \kappa \frac{\partial v(L,t)}{\partial x} + hv(L,t) = 0 \tag{8.6.25}$$

and

$$v(x,0) = -w(x). \tag{8.6.26}$$

Solving (8.6.24),

$$w(x) = \frac{L^2 - x^2}{2a^2} + \frac{\kappa L}{ha^2}. \tag{8.6.27}$$

Table 8.6.1: The First Six Roots of the Equation $k_n \tan(k_n) = h^*$.

h^*	k_1	k_2	k_3	k_4	k_5	k_6
0.001	0.03162	3.14191	6.28334	9.42488	12.56645	15.70803
0.002	0.04471	3.14223	6.28350	9.42499	12.56653	15.70809
0.005	0.07065	3.14318	6.28398	9.42531	12.56677	15.70828
0.010	0.09830	3.14477	6.28478	9.42584	12.56717	15.70860
0.020	0.14095	3.14795	6.28637	9.42690	12.56796	15.70924
0.050	0.22176	3.15743	6.29113	9.43008	12.57035	15.71115
0.100	0.31105	3.17310	6.29906	9.43538	12.57432	15.71433
0.200	0.43284	3.20393	6.31485	9.44595	12.58226	15.72068
0.500	0.65327	3.29231	6.36162	9.47748	12.60601	15.73972
1.000	0.86033	3.42562	6.43730	9.52933	12.64529	15.77128
2.000	1.07687	3.64360	6.57833	9.62956	12.72230	15.83361
5.000	1.31384	4.03357	6.90960	9.89275	12.93522	16.01066
10.000	1.42887	4.30580	7.22811	10.20026	13.21418	16.25336
20.000	1.49613	4.49148	7.49541	10.51167	13.54198	16.58640
∞	1.57080	4.71239	7.85399	10.99557	14.13717	17.27876

Turning to the transient solution $v(x,t)$, we use separation of variables and find that

$$v(x,t) = \sum_{n=1}^{\infty} C_n \cos\left(\frac{k_n x}{L}\right) \exp\left(-\frac{a^2 k_n^2 t}{L^2}\right), \tag{8.6.28}$$

where k_n is the nth root of the transcendental equation:

$$k_n \tan(k_n) = hL/\kappa = h^*. \tag{8.6.29}$$

Table 8.6.1 gives the first six roots for various values of hL/κ.

Our final task is to compute C_n. After substituting $t = 0$ into (8.6.28), we are left with a orthogonal expansion of $-w(x)$ using the eigenfunctions $\cos(k_n x/L)$. From (6.3.4),

$$C_n = \frac{\int_0^L -w(x)\cos(k_n x/L)\,dx}{\int_0^L \cos^2(k_n x/L)\,dx} = \frac{-L^3 \sin(k_n)/(a^2 k_n^3)}{L[k_n + \sin(2k_n)/2]/(2k_n)} \tag{8.6.30}$$

$$= -\frac{2L^2 \sin(k_n)}{a^2 k_n^2 [k_n + \sin(2k_n)/2]}. \tag{8.6.31}$$

Combining (8.6.28) and (8.6.31),

$$v(x,t) = -\frac{2L^2}{a^2} \sum_{n=1}^{\infty} \frac{\sin(k_n)}{k_n^2 [k_n + \sin(2k_n)/2]} \cos\left(\frac{k_n x}{L}\right) \exp\left(-\frac{a^2 k_n^2 t}{L^2}\right). \tag{8.6.32}$$

Consequently, $A(x,t)$ equals

$$A(x,t) = \frac{L^2 - x^2}{2a^2} + \frac{\kappa L}{ha^2} - \frac{2L^2}{a^2} \sum_{n=1}^{\infty} \frac{\sin(k_n)}{k_n^2[k_n + \sin(2k_n)/2]} \cos\left(\frac{k_n x}{L}\right) \exp\left(-\frac{a^2 k_n^2 t}{L^2}\right). \tag{8.6.33}$$

We now wish to use the solution (8.6.33) to find the temperature distribution within the slab when it is heated by a time-dependent source $f(t)$. As in the case of time-dependent boundary conditions, we imagine that we can break the process into an infinite number of small changes to the heating which occur at the times $t = \tau_1$, $t = \tau_2$, etc. Consequently, the temperature distribution at the time t following the change at $t = \tau_n$ and before the change at $t = \tau_{n+1}$ is

$$u(x,t) = f(0)A(x,t) + \sum_{k=1}^{n} A(x, t-\tau_k) \frac{\Delta f_k}{\Delta \tau_k} \Delta \tau_k, \tag{8.6.34}$$

where

$$\Delta f_k = f(\tau_k) - f(\tau_{k-1}) \qquad \text{and} \qquad \Delta \tau_k = \tau_k - \tau_{k-1}. \tag{8.6.35}$$

In the limit of $\Delta \tau_k \to 0$,

$$u(x,t) = f(0)A(x,t) + \int_0^t A(x, t-\tau) f'(\tau)\, d\tau \tag{8.6.36}$$

$$= f(t)A(x,0) + \int_0^t f(\tau) \frac{\partial A(x, t-\tau)}{\partial \tau}\, d\tau. \tag{8.6.37}$$

In our present problem,

$$f(t) = \frac{2q}{\rho C_p AL} \sin^2(\omega t), \qquad f'(t) = \frac{2q\omega}{\rho C_p AL} \sin(2\omega t). \tag{8.6.38}$$

Therefore,

$$u(x,t) = \frac{2q\omega}{\rho C_p AL} \int_0^t \sin(2\omega\tau) \left\{ \frac{L^2 - x^2}{2a^2} + \frac{\kappa L}{ha^2} - \frac{2L^2}{a^2} \sum_{n=1}^{\infty} \frac{\sin(k_n)}{k_n^2[k_n + \sin(2k_n)/2]} \cos\left(\frac{k_n x}{L}\right) \times \exp\left[-\frac{a^2 k_n^2 (t-\tau)}{L^2}\right] \right\} d\tau \tag{8.6.39}$$

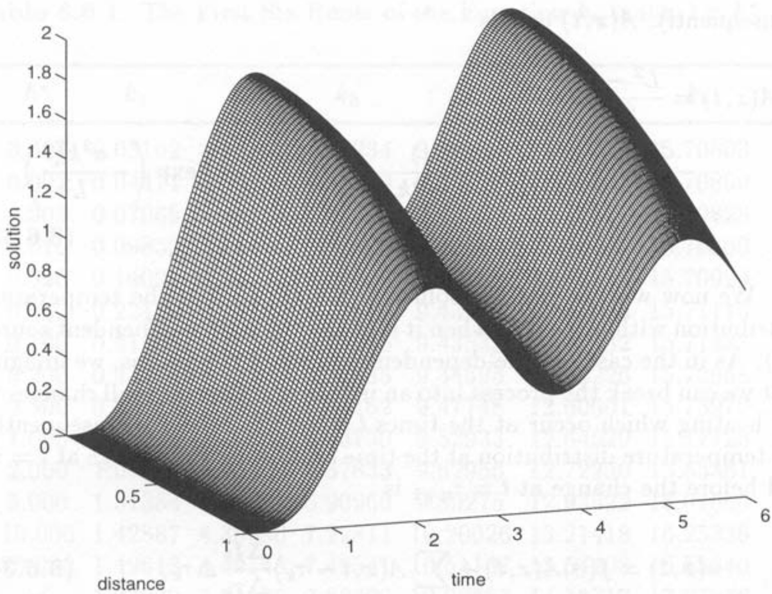

Figure 8.6.1: The nondimensional temperature $a^2 A\rho C_p u(x,t)/qL$ within a slab that we heat by alternating electric current as a function of position x/L and time a^2t/L^2 when we insulate the $x = 0$ end and let the $x = L$ end radiate to free space at temperature zero. The initial temperature is zero, $hL/\kappa = 1$, and $a^2/(L^2\omega) = 1$.

$$\begin{aligned} u(x,t) &= -\frac{q}{\rho C_p AL}\left(\frac{L^2-x^2}{2a^2}+\frac{\kappa L}{ha^2}\right)\cos(2\omega\tau)\Big|_0^t \\ &\quad -\frac{4L^2 q\omega}{a^2\rho C_p AL}\sum_{n=1}^{\infty}\frac{\sin(k_n)\exp(-a^2k_n^2t/L^2)}{k_n^2[k_n+\sin(2k_n)/2]}\cos\left(\frac{k_n x}{L}\right) \\ &\quad \times\int_0^t \sin(2\omega\tau)\exp\left(\frac{a^2k_n^2\tau}{L^2}\right)d\tau \qquad (8.6.40) \\ &= \frac{qL}{a^2 A\rho C_p}\left\{\left[\frac{L^2-x^2}{2L^2}+\frac{\kappa}{hL}\right][1-\cos(2\omega t)]\right. \\ &\quad -\sum_{n=1}^{\infty}\frac{4\sin(k_n)\cos(k_n x/L)}{k_n^2[k_n+\sin(2k_n)/2][4+a^4k_n^4/(L^4\omega^2)]} \\ &\quad \left.\times\left[\frac{a^2k_n^2}{\omega L^2}\sin(2\omega t)-2\cos(2\omega t)+2\exp\left(-\frac{a^2k_n^2t}{L^2}\right)\right]\right\}. \qquad (8.6.41) \end{aligned}$$

Figure 8.6.1 illustrates (8.6.41) for $hL/\kappa = 1$ and $a^2/(L^2\omega) = 1$. The oscillating solution, reflecting the periodic heating by the alternating current, rapidly reaches equilibrium. Because heat is radiated to space at $x = L$, the temperature is maximum at $x = 0$ at any given instant as heat flows from $x = 0$ to $x = L$.

Problems

1. Solve the heat equation[33]

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, 0 < t$$

subject to the boundary conditions $u(0,t) = u(L,t) = f(t)$, $0 < t$ and the initial condition $u(x,0) = 0$, $0 < x < L$.

Step 1: First solve the heat conduction problem

$$\frac{\partial A}{\partial t} = a^2 \frac{\partial^2 A}{\partial x^2}, \quad 0 < x < L, 0 < t$$

subject to the boundary conditions $A(0,t) = A(L,t) = 1$, $0 < t$ and the initial condition $A(x,0) = 0$, $0 < x < L$. Show that

$$A(x,t) = 1 - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin[(2n-1)\pi x/L]}{2n-1} e^{-a^2(2n-1)^2\pi^2 t/L^2}.$$

Step 2: Use Duhamel's theorem and show that

$$u(x,t) = \frac{4\pi a^2}{L^2} \sum_{n=1}^{\infty} (2n-1) \sin\left[\frac{(2n-1)\pi x}{L}\right] e^{-a^2(2n-1)^2\pi^2 t/L^2}$$
$$\times \int_0^t f(\tau) e^{a^2(2n-1)^2\pi^2 \tau/L^2} \, d\tau.$$

2. A thermometer measures temperature by the thermal expansion of a liquid (usually mercury or alcohol) stored in a bulb into a glass stem containing an empty cylindrical channel. Under normal conditions, temperature changes occur sufficiently slow so that the temperature within the liquid is uniform. However, for rapid temperature changes (such as those that would occur during the rapid ascension of an airplane or meteorological balloon), significant errors could occur. In such situations the recorded temperature would lag behind the actual temperature because of the time needed for the heat to conduct in or out of the bulb.

[33] From Tao, L. N., 1960: Magnetohydrodynamic effects on the formation of Couette flow. *J. Aerosp. Sci.*, **27**, 334–338 with permission.

During his investigation of this question, McLeod[34] solved

$$\frac{\partial u}{\partial t} = a^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \qquad 0 \le r < b, 0 < t$$

subject to the boundary conditions $\lim_{r\to 0} |u(r,t)| < \infty$ and $u(b,t) = \varphi(t)$, $0 < t$ and the initial condition $u(r,0) = 0$, $0 \le r < b$. The analysis was as follows:

Step 1: First solve the heat conduction problem

$$\frac{\partial A}{\partial t} = a^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right), \qquad 0 \le r < b, 0 < t$$

subject to the boundary conditions $\lim_{r\to 0} |A(r,t)| < \infty$ and $A(b,t) = 1$, $0 < t$ and the initial condition $A(r,0) = 0$, $0 \le r < b$. Show that

$$A(r,t) = 1 - 2\sum_{n=1}^{\infty} \frac{J_0(k_n r/b)}{k_n\, J_1(k_n)} e^{-a^2 k_n^2 t/b^2},$$

where $J_0(k_n) = 0$.

Step 2: Use Duhamel's theorem and show that

$$u(r,t) = \frac{2a^2}{b^2}\sum_{n=1}^{\infty} \frac{k_n\, J_0(k_n r/b)}{J_1(k_n)} \int_0^t \varphi(\tau) e^{-a^2 k_n^2 (t-\tau)/b^2}\, d\tau.$$

Step 3: If $\varphi(t) = Gt$, show that

$$u(r,t) = 2G\sum_{n=1}^{\infty} \frac{J_0(k_n r/b)}{k_n\, J_1(k_n)} \left[t + \frac{b^2}{a^2 k_n^2}\left(e^{-a^2 k_n^2 t/b^2} - 1\right)\right].$$

McLeod found that for a mercury thermometer of 10-cm length a lag of 0.01°C would occur for a warming rate of 0.032°C s^{-1} (a warming gradient of 1.9°C per thousand feet and a descent of one thousand feet per minute). Although this is a very small number, when he included

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[34] Reproduced with acknowledgement to Taylor and Francis, Publishers, from McLeod, A. R., 1919: On the lags of thermometers with spherical and cylindrical bulbs in a medium whose temperature is changing at a constant rate. *Philos. Mag., Ser. 6*, **37**, 134–144. See also Bromwich, T. J. I'A., 1919: Examples of operational methods in mathematical physics. *Philos. Mag., Ser. 6*, **37**, 407–419; McLeod, A. R., 1922: On the lags of thermometers. *Philos. Mag., Ser. 6*, **43**, 49–70.

the surface conductance of the glass tube, the lag increased to 0.85°C. Similar problems plague bimetal thermometers[35] and thermistors[36] used in radiosondes (meteorological sounding balloons).

3. A classic problem[37] in fluid mechanics is the motion of a semi-infinite viscous fluid that results from the sudden movement of the adjacent wall starting at $t = 0$. Initially the fluid is at rest. If we denote the velocity of the fluid parallel to the wall by $u(x,t)$, the governing equation is

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \infty, 0 < t$$

with the boundary conditions

$$u(0,t) = V(t), \qquad \lim_{x\to\infty} u(x,t) \to 0, \qquad 0 < t$$

and the initial condition $u(x,0) = 0$, $0 < x < \infty$.

Step 1: Find the step response by solving

$$\frac{\partial A}{\partial t} = \nu \frac{\partial^2 A}{\partial x^2}, \quad 0 < x < \infty, 0 < t$$

subject to the boundary conditions

$$A(0,t) = 1 \quad \text{and} \quad \lim_{x\to\infty} A(x,t) \to 0, \qquad 0 < t$$

and the initial condition $A(x,0) = 0$, $0 < x < \infty$. Show that

$$A(x,t) = \operatorname{erfc}\left(\frac{x}{2\sqrt{\nu t}}\right) = \frac{2}{\sqrt{\pi}} \int_{x/2\sqrt{\nu t}}^{\infty} e^{-\eta^2}\, d\eta,$$

where erf is the error function. Hint: Use Laplace transforms.

Step 2: Use Duhamel's theorem and show that the solution is

$$\begin{aligned} u(x,t) &= \int_0^t V(t-\tau) \frac{x \exp(-x^2/4\nu\tau)}{2\sqrt{\pi\nu\tau^3}}\, d\tau \\ &= \frac{2}{\pi} \int_{x/\sqrt{4\nu t}}^{\infty} V\left(t - \frac{x^2}{4\nu\eta^2}\right) e^{-\eta^2}\, d\eta. \end{aligned}$$

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[35] Mitra, H. and Datta, M. B., 1954: Lag coefficient of bimetal thermometer of chronometric radiosonde. *Indian J. Meteorol. Geophys.*, **5**, 257–261.

[36] Badgley, F. I., 1957: Response of radiosonde thermistors. *Rev. Sci. Instrum.*, **28**, 1079–1084.

[37] This problem was first posed and partially solved by Stokes, G. G., 1850: On the effect of the internal friction of fluids on the motions of pendulums. *Proc. Cambridge Philos. Soc.*, **9, Part II**, [8]–[106].

8.7 NUMERICAL SOLUTION OF THE HEAT EQUATION

In the previous chapter we showed how we may use finite difference techniques to solve the wave equation. In this section we show that similar considerations hold for the heat equation.

Starting with the heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \tag{8.7.1}$$

we must first replace the exact derivatives with finite differences. Drawing upon our work in Section 7.6,

$$\frac{\partial u(x_m, t_n)}{\partial t} = \frac{u_m^{n+1} - u_m^n}{\Delta t} + O(\Delta t) \tag{8.7.2}$$

and

$$\frac{\partial^2 u(x_m, t_n)}{\partial x^2} = \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{(\Delta x)^2} + O[(\Delta x)^2], \tag{8.7.3}$$

where the notation u_m^n denotes $u(x_m, t_n)$. Figure 8.7.1 illustrates our numerical scheme when we hold both ends at the temperature of zero. Substituting (8.7.2)–(8.7.3) into (8.7.1) and rearranging,

$$u_m^{n+1} = u_m^n + \frac{a^2 \Delta t}{(\Delta x)^2} \left(u_{m+1}^n - 2u_m^n + u_{m-1}^n \right). \tag{8.7.4}$$

The numerical integration begins with $n = 0$ and the value of u_{m+1}^0, u_m^0 and u_{m-1}^0 are given by $f(m\Delta x)$.

Once again we must check the *convergence*, *stability* and *consistency* of our scheme. We begin by writing u_{m+1}^n, u_{m-1}^n and u_m^{n+1} in terms of the exact solution u and its derivatives evaluated at the point $x_m = m\Delta x$ and $t_n = n\Delta t$. By Taylor's expansion,

$$u_{m+1}^n = u_m^n + \Delta x \left.\frac{\partial u}{\partial x}\right|_n^m + \tfrac{1}{2}(\Delta x)^2 \left.\frac{\partial^2 u}{\partial x^2}\right|_n^m + \tfrac{1}{6}(\Delta x)^3 \left.\frac{\partial^3 u}{\partial x^3}\right|_n^m + \cdots, \tag{8.7.5}$$

$$u_{m-1}^n = u_m^n - \Delta x \left.\frac{\partial u}{\partial x}\right|_n^m + \tfrac{1}{2}(\Delta x)^2 \left.\frac{\partial^2 u}{\partial x^2}\right|_n^m - \tfrac{1}{6}(\Delta x)^3 \left.\frac{\partial^3 u}{\partial x^3}\right|_n^m + \cdots \tag{8.7.6}$$

and

$$u_m^{n+1} = u_m^n + \Delta t \left.\frac{\partial u}{\partial t}\right|_n^m + \tfrac{1}{2}(\Delta t)^2 \left.\frac{\partial^2 u}{\partial t^2}\right|_n^m + \tfrac{1}{6}(\Delta t)^3 \left.\frac{\partial^3 u}{\partial t^3}\right|_n^m + \cdots \tag{8.7.7}$$

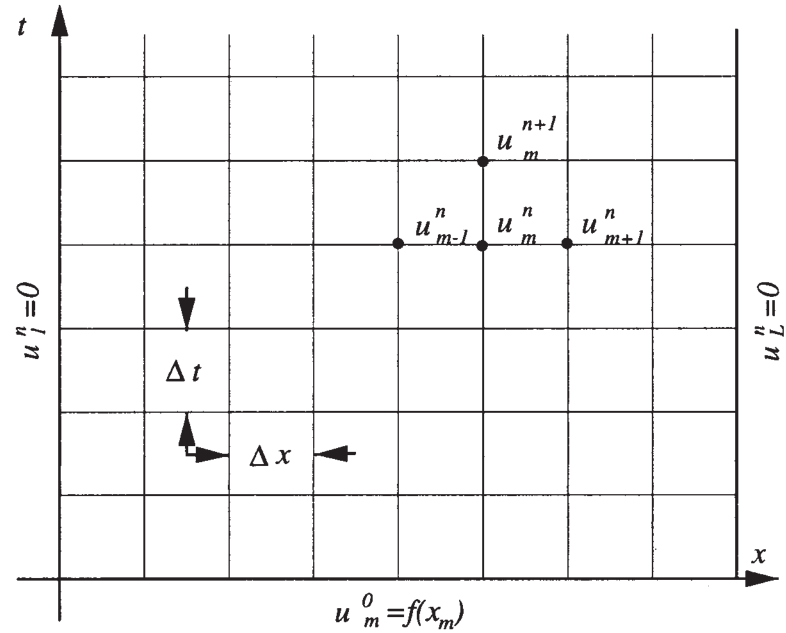

Figure 8.7.1: Schematic of the numerical solution of the heat equation when we hold both ends at a temperature of zero.

Substituting into (8.7.4), we obtain

$$\frac{u_m^{n+1}-u_m^n}{\Delta t} - a^2\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{(\Delta x)^2} = \left(\frac{\partial u}{\partial t} - a^2\frac{\partial^2 u}{\partial x^2}\right)\Bigg|_n^m + \tfrac{1}{2}\Delta t\frac{\partial^2 u}{\partial t^2}\Bigg|_n^m - \tfrac{1}{12}(a\Delta x)^2\frac{\partial^4 u}{\partial x^4}\Bigg|_n^m + \cdots \tag{8.7.8}$$

The first term on the right side of (8.7.8) vanishes because $u(x,t)$ satisfies the heat equation. Thus, in the limit of $\Delta x \to 0$, $\Delta t \to 0$, the right side of (8.7.8) vanishes and the scheme is *consistent*.

To determine the *stability* of the explicit scheme, we again use the Fourier method. Assuming a solution of the form:

$$u_n^m = e^{im\theta}e^{in\lambda}, \tag{8.7.9}$$

we substitute (8.7.9) into (8.7.4) and find that

$$\frac{e^{i\lambda}-1}{\Delta t} = a^2\frac{e^{i\theta}-2+e^{-i\theta}}{(\Delta x)^2} \tag{8.7.10}$$

or

$$e^{i\lambda} = 1 - 4\frac{a^2\Delta t}{(\Delta x)^2}\sin^2\left(\frac{\theta}{2}\right). \tag{8.7.11}$$

The quantity $e^{i\lambda}$ will grow exponentially unless

$$-1 \le 1 - 4\frac{a^2\Delta t}{(\Delta x)^2}\sin^2\left(\frac{\theta}{2}\right) < 1. \tag{8.7.12}$$

The right inequality is trivially satisfied if $a^2\Delta t/(\Delta x)^2 > 0$ while the left inequality yields

$$\frac{a^2\Delta t}{(\Delta x)^2} \le \frac{1}{2\sin^2(\theta/2)}, \tag{8.7.13}$$

leading to the stability condition $0 < a^2\Delta t/(\Delta x)^2 \le \frac{1}{2}$. This is a rather restrictive condition because doubling the resolution (halfing Δx) requires that we reduce the time step by a quarter. Thus, for many calculations the required time step may be unacceptably small. For this reason, many use an implicit form of the finite differencing (Crank-Nicolson implicit method[38]):

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} = \frac{a^2}{2}\left[\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{(\Delta x)^2} + \frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{(\Delta x)^2}\right], \tag{8.7.14}$$

although it requires the solution of a simultaneous set of linear equations. However, there are several efficient methods for their solution.

Finally we must check and see if our explicit scheme *converges* to the true solution. If we let e_m^n denote the difference between the exact and our finite differenced solution to the heat equation, we can use (8.7.8) to derive the equation governing e_m^n and find that

$$e_m^{n+1} = e_m^n + \frac{a^2\Delta t}{(\Delta x)^2}\left(e_{m+1}^n - 2e_m^n + e_{m-1}^n\right) + O[(\Delta t)^2 + \Delta t(\Delta x)^2], \tag{8.7.15}$$

for $m = 1, 2, \ldots, M$. Assuming that $a^2\Delta t/(\Delta x)^2 \le \frac{1}{2}$, then

$$\left|e_m^{n+1}\right| \le \frac{a^2\Delta t}{(\Delta x)^2}\left|e_{m-1}^n\right| + \left[1 - 2\frac{a^2\Delta t}{(\Delta x)^2}\right]\left|e_m^n\right| + \frac{a^2\Delta t}{(\Delta x)^2}\left|e_{m+1}^n\right|$$

$$+ A[(\Delta t)^2 + \Delta t(\Delta x)^2] \tag{8.7.16}$$

$$\le ||e_n|| + A[(\Delta t)^2 + \Delta t(\Delta x)^2], \tag{8.7.17}$$

where $||e_n|| = \max_{m=0,1,\ldots,M} |e_m^n|$. Consequently,

$$||e_{n+1}|| \le ||e_n|| + A[(\Delta t)^2 + \Delta t(\Delta x)^2]. \tag{8.7.18}$$

[38] Crank, J. and Nicolson, P., 1947: A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type. *Proc. Cambridge. Philos. Soc.*, **43**, 50–67.

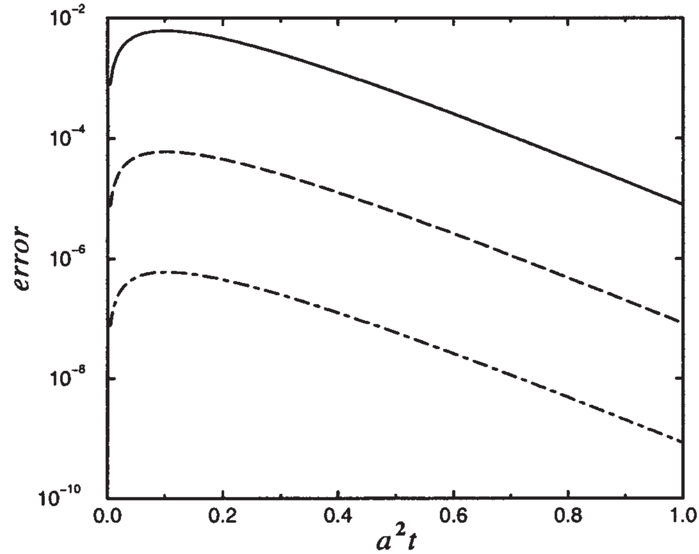

Figure 8.7.2: The growth of error $||e_n||$ as a function of a^2t for various resolutions. For the top line, $\Delta x = 0.1$; for the middle line, $\Delta x = 0.01$; and for the bottom line, $\Delta x = 0.001$.

Because $||e_0|| = 0$ and $n\Delta t \leq t_n$, we find that

$$||e_{n+1}|| \leq An[(\Delta t)^2 + \Delta t(\Delta x)^2] \leq At_n[\Delta t + (\Delta x)^2]. \quad \textbf{(8.7.19)}$$

As $\Delta x \to 0$, $\Delta t \to 0$, the errors tend to zero and we have convergence. We have illustrated (8.7.19) in Figure 8.7.2 by using the finite difference equation (8.7.4) to compute $||e_n||$ during a numerical experiment that used $a^2\Delta t/(\Delta x)^2 = 0.5$ and $f(x) = \sin(\pi x)$. Note how each increase of resolution by 10 results in a drop in the error by 100.

The following examples illustrate the use of numerical methods.

- **Example 8.7.1**

For our first example, we redo Example 8.3.1 with $a^2\Delta t/(\Delta x)^2 = 0.499$ and 0.501. As Figure 8.7.3 shows, the solution with $a^2\Delta t/(\Delta x)^2 < 1/2$ performs well while small-scale, growing disturbances occur for $a^2\Delta t/(\Delta x)^2 > 1/2$. This is best seen at $t' = 0.2$. It should be noted that for the reasonable $\Delta x = L/100$, it takes approximately *20,000* time steps before we reach $a^2t/L^2 = 1$.

- **Example 8.7.2**

In this example, we redo the previous example with an insulated end at $x = L$. Using the centered differencing formula,

$$u^n_{L+1} - u^n_{L-1} = 0, \quad \textbf{(8.7.20)}$$

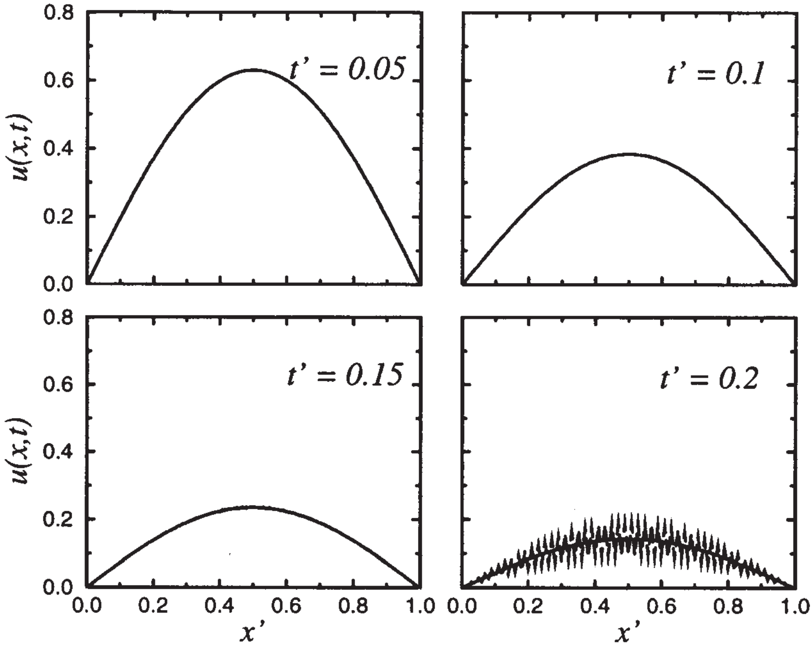

Figure 8.7.3: The numerical solution $u(x,t)$ of the heat equation with $a^2\Delta t/(\Delta x)^2 = 0.499$ (solid line) and 0.501 (jagged line) at various positions $x' = x/L$ and times $t' = a^2t/L^2$ using (8.7.4). The initial temperature $u(x,0)$ equals $4x'(1-x')$ and we hold both ends at a temperature of zero.

because $u_x(0,t) = 0$. Also, at $i = L$,

$$u_L^{n+1} = u_L^n + \frac{a^2\Delta t}{(\Delta x)^2}\left(u_{L+1}^n - 2u_L^n + u_{L-1}^n\right). \tag{8.7.21}$$

Eliminating u_{L+1}^n between the two equations,

$$u_L^{n+1} = u_L^n + \frac{a^2\Delta t}{(\Delta x)^2}\left(2u_{L-1}^n - 2u_L^n\right). \tag{8.7.22}$$

Figure 8.7.4 illustrates our numerical solution at various positions and times.

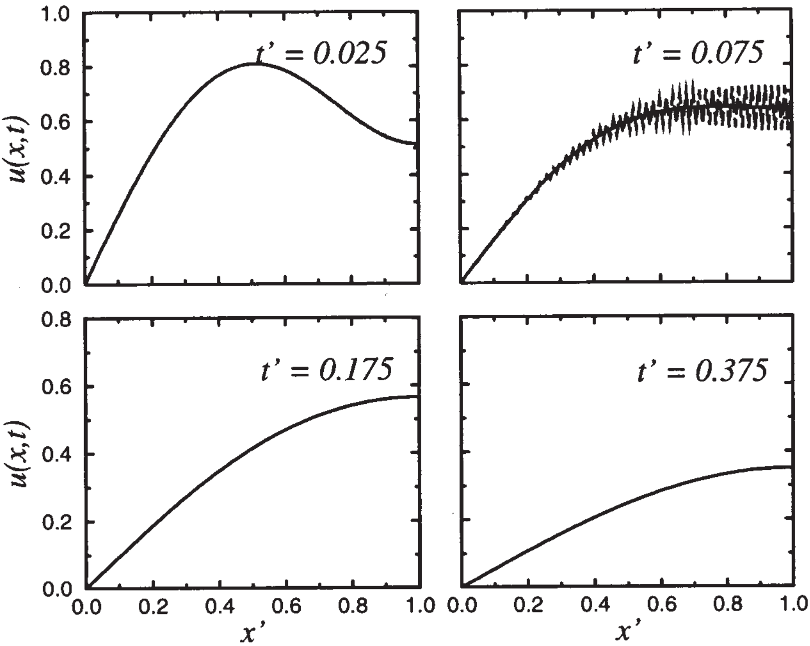

Figure 8.7.4: Same as Figure 8.7.3 except we have an insulated end at $x = L$. We have not plotted the jagged line in the bottom two frames because the solution has grown very large.

Project: Implicit Numerical Integration of the Heat Equation

The difficulty in using explicit time differencing to solve the heat equation is the very small time step that must be taken at moderate spatial resolutions to ensure stability. This small time step translates into an unacceptably long execution time. In this project you will investigate the Crank-Nicolson implicit scheme which allows for a much more reasonable time step.

Step 1: Develop code to use the Crank-Nicolson equation (8.7.14) to numerically integrate the heat equation. To do this, you will need a tridiagonal solver to find u_m^{n+1}. This is explained at the end of Section 11.1. However, many numerical methods books[39] actually have code already developed for your use. You might as well use this code.

Step 2: Test out your code by solving the heat equation given the initial condition $u(x, 0) = \sin(\pi x)$ and the boundary conditions $u(0, t) =$

[39] For example, Press, W. H., Flannery, B. P., Teukolsky, S. A., and Vetterling, W. T., 1986: *Numerical Recipes: The Art of Scientific Computing*, Cambridge University Press, Cambridge, Section 2.6.

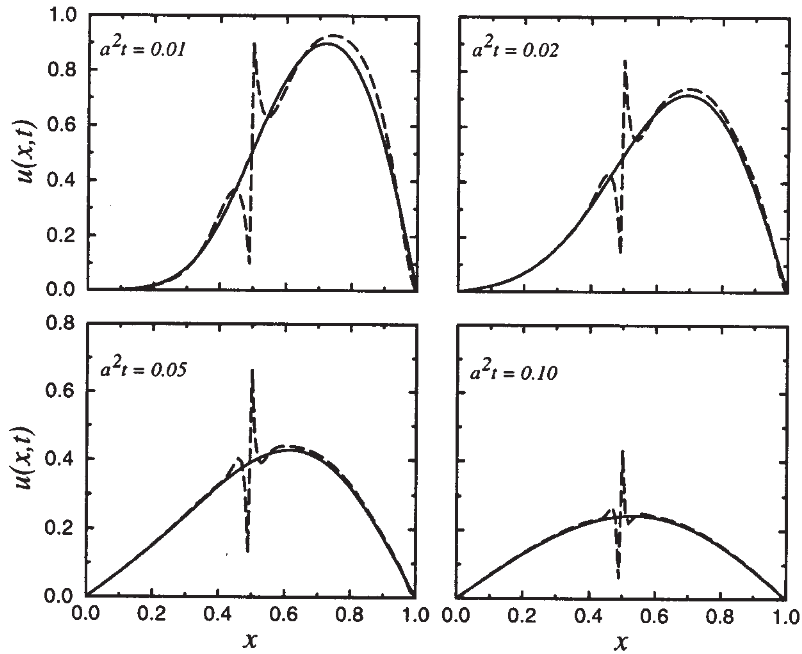

Figure 8.7.5: The numerical solution $u(x,t)$ of the heat equation $u_t = a^2 u_{xx}$ using the Crank-Nicolson method. The solid line gives the numerical solution with $a^2\Delta t = 0.0005$ while the dashed line gives the solution for $a^2\Delta t = 0.005$. Both ends are held at zero with an initial condition of $u(x,0) = 0$ for $0 \leq x < \frac{1}{2}$ and $u(x,0) = 1$ for $\frac{1}{2} < x \leq 1$.

$u(1,t) = 0$. Find the solution for various Δt's with $\Delta x = 0.01$. Compare this numerical solution against the exact solution which you can find. How does the error (between the numerical and exact solutions) change with Δt? For small Δt, the errors should be small. If not, then you have a mistake in your code.

Step 3: Once you have confidence in your code, discuss the behavior of the scheme for various values of Δx and Δt for the initial condition $u(x,0) = 0$ for $0 \leq x < \frac{1}{2}$ and $u(x,0) = 1$ for $\frac{1}{2} < x \leq 1$ with the boundary conditions $u(0,t) = u(1,t) = 0$. Although you can take quite a large Δt, what happens? Did a similar problem arise in Step 2? Explain your results.

Chapter 9

Laplace's Equation

In the previous chapter we solved the one-dimensional heat equation. Quite often we found that the transient solution died away, leaving a steady state. The partial differential equation that describes the steady state for two-dimensional heat conduction is Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{9.0.1}$$

In general, this equation governs physical processes where *equilibrium* has been reached. It also serves as the prototype for a wider class of *elliptic equations*:

$$a(x,t)\frac{\partial^2 u}{\partial x^2} + b(x,t)\frac{\partial^2 u}{\partial x \partial t} + c(x,t)\frac{\partial^2 u}{\partial t^2} = f\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right), \tag{9.0.2}$$

where $b^2 < 4ac$. Unlike the heat and wave equations, there are no initial conditions and the boundary conditions completely specify the solution. In this chapter we present some of the common techniques for solving this equation.

9.1 DERIVATION OF LAPLACE'S EQUATION

Let us imagine a thin, flat plate of heat-conducting material between two sheets of insulation. A sufficient time has passed so that the temperature depends only on the spatial coordinates x and y. We now apply the law of conservation of energy (in rate form) to a small rectangle with sides Δx and Δy.

Let $q_x(x,y)$ and $q_y(x,y)$ denote the heat flow rates in the x- and y-direction, respectively. Conservation of energy requires that the heat flow into the slab must equal the heat flow out of the slab if there is no storage or generation of heat. Now

$$\text{rate in} = q_x(x, y+\Delta y/2)\Delta y + q_y(x+\Delta x/2, y)\Delta x \tag{9.1.1}$$

and

$$\text{rate out} = q_x(x+\Delta x, y+\Delta y/2)\Delta y + q_y(x+\Delta x/2, y+\Delta y)\Delta x. \tag{9.1.2}$$

If the plate has unit thickness,

$$\begin{aligned}&[q_x(x, y+\Delta y/2) - q_x(x+\Delta x, y+\Delta y/2)]\Delta y\\ &\quad + [q_y(x+\Delta x/2, y) - q_y(x+\Delta x/2, y+\Delta y)]\Delta x = 0.\end{aligned} \tag{9.1.3}$$

Upon dividing through by $\Delta x \Delta y$, we obtain two differences quotients on the left side of (9.1.3). In the limit as Δx and Δy tend to zero, they become partial derivatives, giving

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0 \tag{9.1.4}$$

for any point (x, y).

We now employ Fourier's law to eliminate the rates q_x and q_y, yielding

$$\frac{\partial}{\partial x}\left(a^2 \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(a^2 \frac{\partial u}{\partial y}\right) = 0, \tag{9.1.5}$$

if we have an isotropic (same in all directions) material. Finally, if a^2 is constant, (9.1.5) reduces to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{9.1.6}$$

which is the two-dimensional, steady-state heat equation (i.e., $u_t \approx 0$ as $t \to \infty$).

Solutions of Laplace's equation (called harmonic functions) differ fundamentally from those encountered with the heat and wave equations. These latter two equations describe the evolution of some phenomena. Laplace's equation, on the other hand, describes things at

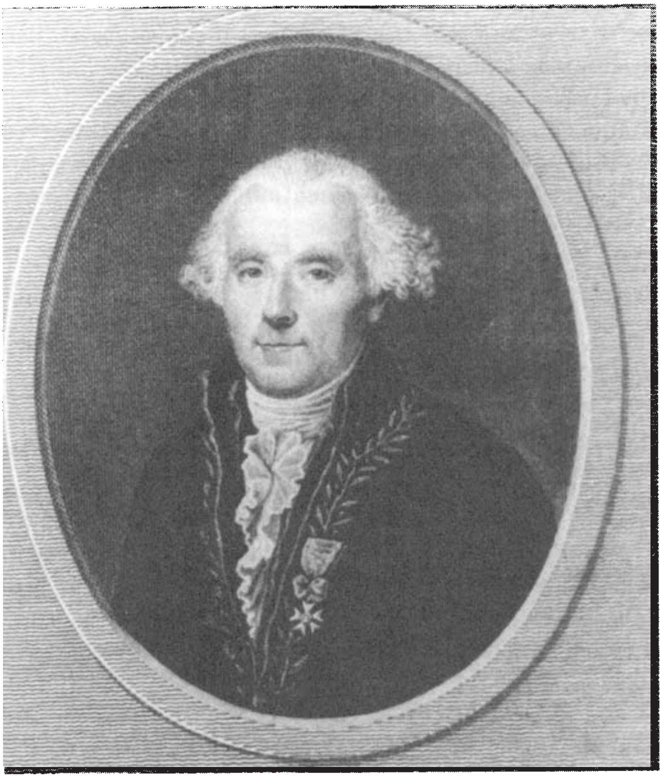

Figure 9.1.1: Today we best remember Pierre-Simon Laplace (1749–1827) for his work in celestial mechanics and probability. In his five volumes *Traité de Mécanique céleste* (1799–1825), he accounted for the theoretical orbits of the planets and their satellites. Laplace's equation arose during this study of gravitational attraction. (Portrait courtesy of the Archives de l'Académie des sciences, Paris.)

equilibrium. Consequently, any change in the boundary conditions will affect to some degree the *entire* domain because a change to any one point will cause its neighbors to change in order to reestablish the equilibrium. Those points will, in turn, affect others. Because all of these points are in equilibrium, this modification must occur instantaneously.

Further insight follows from the *maximum principle*. If Laplace's equation governs a region, then its solution cannot have a relative maximum or minimum *inside* the region unless the solution is constant.[1] If

[1] See Courant, R. and Hilbert, D., 1962: *Methods of Mathematical Physics, Vol. II: Partial Differential Equations*, Interscience, New York,

we think of the solution as a steady-state temperature distribution, this principle is clearly true because at any one point the temperature cannot be greater than at all other nearby points. If that were so, heat would flow away from the hot point to cooler points nearby, thus eliminating the hot spot when equilibrium was once again restored.

It is often useful to consider the two-dimensional Laplace's equation in other coordinate systems. In polar coordinates, where $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $z = z$, Laplace's equation becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \tag{9.1.7}$$

if the problem possesses axisymmetry. On the other hand, if the solution is independent of z, Laplace's equation becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{9.1.8}$$

In spherical coordinates, $x = r\cos(\varphi)\sin(\theta)$, $y = r\sin(\varphi)\sin(\theta)$, and $z = r\cos(\theta)$, where $r^2 = x^2 + y^2 + z^2$, θ is the angle measured *down* to the point from the z-axis (colatitude) and φ is the angle made between the x-axis and the projection of the point on the xy plane. In the case of axisymmetry (no φ dependence), Laplace's equation becomes

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin(\theta)}\frac{\partial}{\partial \theta}\left[\sin(\theta)\frac{\partial u}{\partial \theta}\right] = 0. \tag{9.1.9}$$

9.2 BOUNDARY CONDITIONS

Because Laplace's equation involves time-independent phenomena, we must only specify boundary conditions. As we discussed in Section 8.2, we may classify these boundary conditions as follows:

1. Dirichlet condition: u given
2. Neumann condition: $\dfrac{\partial u}{\partial n}$ given, where n is the unit normal direction
3. Robin condition: $u + \alpha\dfrac{\partial u}{\partial n}$ given

along any section of the boundary. In the case of Laplace's equation, if all of the boundaries have Neumann conditions, then the solution is

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326–331 for the proof.

not unique. This follows from the fact that if $u(x, y)$ is a solution, so is $u(x, y) + c$, where c is any constant.

Finally we note that we must specify the boundary conditions along each side of the boundary. These sides may be at infinity as in problems with semi-infinite domains. We must specify values along the entire boundary because we could not have an equilibrium solution if any portion of the domain was undetermined.

9.3 SEPARATION OF VARIABLES

As in the case of the heat and wave equations, separation of variables is the most popular technique for solving Laplace's equation. Although the same general procedure carries over from the previous two chapters, the following examples fill out the details.

• Example 9.3.1: Groundwater flow in a valley

Over a century ago, a French hydraulic engineer named Henri-Philibert-Gaspard Darcy (1803–1858) published the results of a laboratory experiment on the flow of water through sand. He showed that the *apparent* fluid velocity $\mathbf{q}$ relative to the sand grains is directly proportional to the gradient of the hydraulic potential $-k\nabla\varphi$, where the hydraulic potential φ equals the sum of the elevation of the point of measurement plus the pressure potential $(p/\rho g)$. In the case of steady flow, the combination of Darcy's law with conservation of mass $\nabla \cdot \mathbf{q} = 0$ yields Laplace's equation $\nabla^2\varphi = 0$ if the aquifer is isotropic (same in all directions) and homogeneous.

To illustrate how separation of variables may be used to solve Laplace's equation, we shall determine the hydraulic potential within a small drainage basin that lies in a shallow valley. See Figure 9.3.1. Following Tóth,[2] the governing equation is the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < L, 0 < y < z_0 \tag{9.3.1}$$

along with the boundary conditions

$$u(x, z_0) = gz_0 + gcx, \tag{9.3.2}$$

$$u_x(0, y) = u_x(L, y) = 0 \qquad \text{and} \qquad u_y(x, 0) = 0, \tag{9.3.3}$$

where $u(x, y)$ is the hydraulic potential, g is the acceleration due to gravity, and c gives the slope of the topography. The conditions $u_x(L, y) = 0$

[2] Tóth, J., *J. Geophys. Res.*, **67**, 4375–4387, 1962, copyright by the American Geophysical Union.

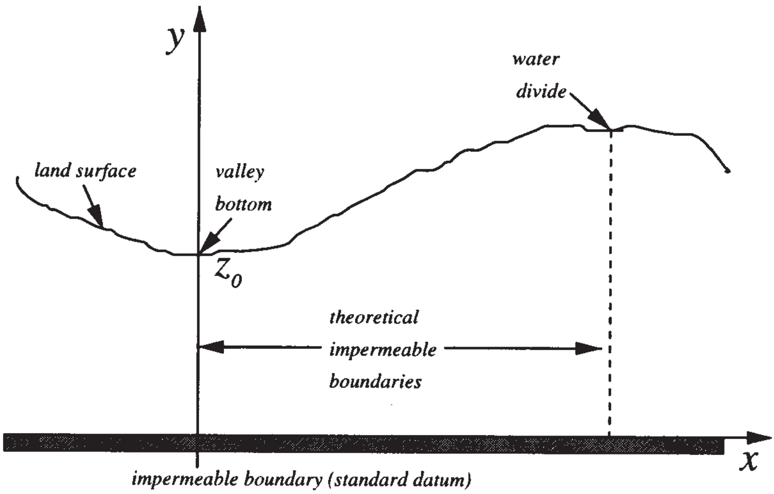

Figure 9.3.1: Cross section of a valley.

and $u_y(x, 0) = 0$ specify a no-flow condition through the bottom and sides of the aquifer. The condition $u_x(0, y) = 0$ ensures symmetry about the $x = 0$ line. Equation (9.3.2) gives the fluid potential at the water table, where z_0 is the elevation of the water table above the standard datum. The term gcx in (9.3.2) expresses the increase of the potential from the valley bottom toward the water divide. On average it closely follows the topography.

Following the pattern set in the previous two chapters, we assume that $u(x, y) = X(x)Y(y)$. Then (9.3.1) becomes

$$X''Y + XY'' = 0. \tag{9.3.4}$$

Separating the variables yields

$$\frac{X''}{X} = -\frac{Y''}{Y}. \tag{9.3.5}$$

Both sides of (9.3.5) must be constant, but the sign of that constant is not obvious. From previous experience we anticipate that the ordinary differential equation in the x-direction will lead to a Sturm-Liouville problem because it possesses homogeneous boundary conditions. Proceeding along this line of reasoning, we consider three separation constants.

Trying a positive constant (say, m^2), (9.3.5) separates into the two ordinary differential equations

$$X'' - m^2X = 0 \qquad \text{and} \qquad Y'' + m^2Y = 0, \tag{9.3.6}$$

which have the solutions

$$X(x) = A\cosh(mx) + B\sinh(mx) \tag{9.3.7}$$

and

$$Y(y) = C\cos(my) + D\sin(my). \tag{9.3.8}$$

Because the boundary conditions (9.3.3) imply $X'(0) = X'(L) = 0$, both A and B must be zero, leading to the trivial solution $u(x, y) = 0$.

When the separation constant equals zero, we find a nontrivial solution given by the eigenfunction $X_0(x) = 1$ and $Y_0(y) = \frac{1}{2}A_0 + B_0 y$. However, because $Y_0'(0) = 0$ from (9.3.3), $B_0 = 0$. Thus, the particular solution for a zero separation constant is $u_0(x, y) = A_0/2$.

Finally, taking both sides of (9.3.5) equal to $-k^2$,

$$X'' + k^2 X = 0 \qquad \text{and} \qquad Y'' - k^2 Y = 0. \tag{9.3.9}$$

The first of these equations, along with the boundary conditions $X'(0) = X'(L) = 0$, gives the eigenfunction $X_n(x) = \cos(k_n x)$, with $k_n = n\pi/L$, $n = 1, 2, 3, \ldots$. The function $Y_n(y)$ for the same separation constant is

$$Y_n(y) = A_n\cosh(k_n y) + B_n\sinh(k_n y). \tag{9.3.10}$$

We must take $B_n = 0$ because $Y_n'(0) = 0$.

We now have the product solution $X_n(x)Y_n(y)$, which satisfies Laplace's equation and all of the boundary conditions except (9.3.2). By the principle of superposition, the general solution is

$$u(x, y) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi y}{L}\right). \tag{9.3.11}$$

Applying (9.3.2), we find that

$$u(x, z_0) = gz_0 + gcx = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi z_0}{L}\right), \tag{9.3.12}$$

which we recognize as a Fourier half-range cosine series such that

$$A_0 = \frac{2}{L}\int_0^L (gz_0 + gcx)\, dx \tag{9.3.13}$$

and

$$\cosh\left(\frac{n\pi z_0}{L}\right) A_n = \frac{2}{L}\int_0^L (gz_0 + gcx)\cos\left(\frac{n\pi x}{L}\right) dx. \tag{9.3.14}$$

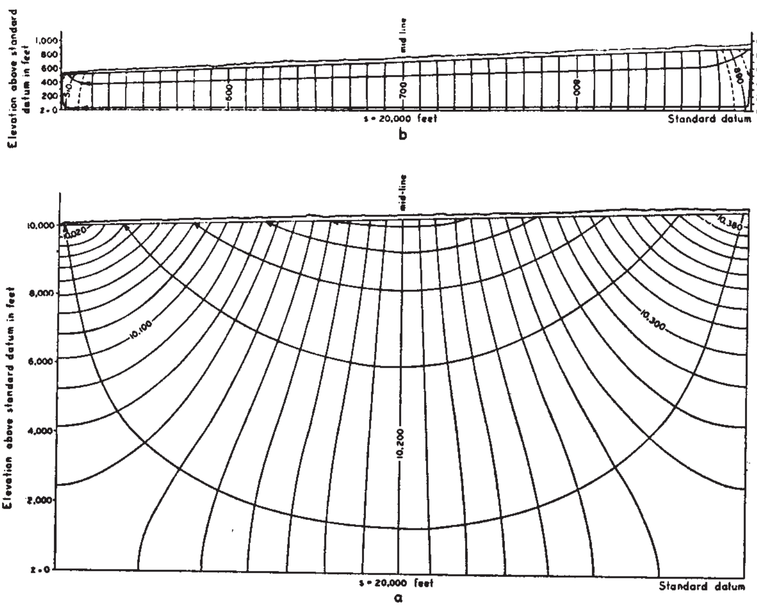

Figure 9.3.2: Two-dimensional potential distribution and flow patterns for different depths of the horizontally impermeable boundary.

Performing the integrations,

$$A_0 = 2gz_0 + gcL \tag{9.3.15}$$

and

$$A_n = -\frac{2gcL[1-(-1)^n]}{n^2\pi^2\cosh(n\pi z_0/L)}. \tag{9.3.16}$$

Finally, the final solution is

$$u(x,y) = gz_0 + \frac{gcL}{2} - \frac{4gcL}{\pi^2}\sum_{m=1}^{\infty}\frac{\cos[(2m-1)\pi x/L]\cosh[(2m-1)\pi y/L]}{(2m-1)^2\cosh[(2m-1)\pi z_0/L]}. \tag{9.3.17}$$

Figure 9.3.2 presents two graphs by Tóth for two different aquifers. We see that the solution satisfies the boundary condition at the bottom and side boundaries. Water flows from the elevated land (on the right) into the valley (on the left), from regions of high to low hydraulic potential.

• **Example 9.3.2**

In the previous example, we had the advantage of homogeneous boundary conditions along $x = 0$ and $x = L$. In a different hydraulic problem, Kirkham[3] solved the more difficult problem of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x < L, 0 < y < h \tag{9.3.18}$$

subject to the Dirichlet boundary conditions

$$u(x,0) = Rx, \qquad u(x,h) = RL, \qquad u(L,y) = RL \tag{9.3.19}$$

and

$$u(0,y) = \begin{cases} 0, & 0 < y < a \\ \frac{RL}{b-a}(y-a), & a < y < b \\ RL, & b < y < h. \end{cases} \tag{9.3.20}$$

This problem arises in finding the steady flow within an aquifer resulting from the introduction of water at the top due to a steady rainfall and its removal along the sides by drains. The parameter L equals half of the distance between the drains, h is the depth of the aquifer, and R is the rate of rainfall.

The point of this example is *We need homogeneous boundary conditions along either the x or y boundaries for separation of variables to work.* We achieve this by breaking the original problem into two parts, namely

$$u(x,y) = v(x,y) + w(x,y) + RL, \tag{9.3.21}$$

where

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad 0 < x < L, 0 < y < h \tag{9.3.22}$$

with

$$v(0,y) = v(L,y) = 0, \qquad v(x,h) = 0 \tag{9.3.23}$$

and

$$v(x,0) = R(x-L); \tag{9.3.24}$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0, \qquad 0 < x < L, 0 < y < h \tag{9.3.25}$$

with

$$w(x,0) = w(x,h) = 0, \qquad w(L,y) = 0 \tag{9.3.26}$$

and

$$w(0,y) = \begin{cases} -RL, & 0 < y < a \\ \frac{RL}{b-a}(y-a) - RL, & a < y < b \\ 0, & b < y < h. \end{cases} \tag{9.3.27}$$

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[3] Kirkham, D., *Trans. Am. Geophys. Union*, **39**, 892–908, 1958, copyright by the American Geophysical Union.

Employing the same technique as in Example 9.3.1, we find that

$$v(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \frac{\sinh[n\pi(h-y)/L]}{\sinh(n\pi h/L)}, \tag{9.3.28}$$

where

$$A_n = \frac{2}{L}\int_0^L R(x-L)\sin\left(\frac{n\pi x}{L}\right)\,dx = -\frac{2RL}{n\pi}. \tag{9.3.29}$$

Similarly, the solution to $w(x,y)$ is found to be

$$w(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi y}{h}\right) \frac{\sinh[n\pi(L-x)/h]}{\sinh(n\pi L/h)}, \tag{9.3.30}$$

where

$$B_n = \frac{2}{h}\left[-RL\int_0^a \sin\left(\frac{n\pi y}{h}\right)dy + RL\int_a^b \left(\frac{y-a}{b-a}-1\right)\sin\left(\frac{n\pi y}{h}\right)dy\right] \tag{9.3.31}$$

$$= \frac{2RL}{\pi}\left\{\frac{h}{(b-a)n^2\pi}\left[\sin\left(\frac{n\pi b}{h}\right) - \sin\left(\frac{n\pi a}{h}\right)\right] - \frac{1}{n}\right\}. \tag{9.3.32}$$

The final answer consists of substituting (9.3.28) and (9.3.30) into (9.3. 21).

• Example 9.3.3

The *electrostatic potential* is defined as the amount of work which must be done against electric forces to bring a unit charge from a reference point to a given point. It is readily shown[4] that the electrostatic potential is described by Laplace's equation if there is no charge within the domain. Let us find the electrostatic potential $u(r,z)$ inside a closed cylinder of length L and radius a. The base and lateral surfaces have the potential 0 while the upper surface has the potential V.

Because the potential varies in only r and z, Laplace's equation in cylindrical coordinates reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} = 0, \qquad 0 \le r < a, 0 < z < L \tag{9.3.33}$$

[4] For static fields, $\nabla \times \mathbf{E} = \mathbf{0}$, where $\mathbf{E}$ is the electric force. From Section 10.4, we can introduce a potential φ such that $\mathbf{E} = \nabla\varphi$. From Gauss' law, $\nabla \cdot \mathbf{E} = \nabla^2\varphi = 0$.

subject to the boundary conditions

$$u(a, z) = u(r, 0) = 0 \quad \text{and} \quad u(r, L) = V. \tag{9.3.34}$$

To solve this problem by separation of variables, let $u(r, z) = R(r)Z(z)$ and

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{Z}\frac{d^2 Z}{dz^2} = -\frac{k^2}{a^2}. \tag{9.3.35}$$

Only a negative separation constant yields nontrivial solutions in the radial direction. In that case, we have that

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{k^2}{a^2}R = 0. \tag{9.3.36}$$

The solutions of (9.3.36) are the Bessel functions $J_0(kr/a)$ and $Y_0(kr/a)$. Because $Y_0(kr/a)$ becomes infinite at $r = 0$, the only permissible solution is $J_0(kr/a)$. The condition that $u(a, z) = R(a)Z(z) = 0$ forces us to choose k's such that $J_0(k) = 0$. Therefore, the solution in the radial direction is $J_0(k_n r/a)$, where k_n is the nth root of $J_0(k) = 0$.

In the z direction,

$$\frac{d^2 Z_n}{dz^2} + \frac{k_n^2}{a^2}Z_n = 0. \tag{9.3.37}$$

The general solution to (9.3.37) is

$$Z_n(z) = A_n \sinh\left(\frac{k_n z}{a}\right) + B_n \cosh\left(\frac{k_n z}{a}\right). \tag{9.3.38}$$

Because $u(r, 0) = R(r)Z(0) = 0$ and $\cosh(0) = 1$, B_n must equal zero. Therefore, the general product solution is

$$u(r, z) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{k_n r}{a}\right) \sinh\left(\frac{k_n z}{a}\right). \tag{9.3.39}$$

The condition that $u(r, L) = V$ determines the arbitrary constant A_n. Along $z = L$,

$$u(r, L) = V = \sum_{n=1}^{\infty} A_n J_0\left(\frac{k_n r}{a}\right) \sinh\left(\frac{k_n L}{a}\right), \tag{9.3.40}$$

where

$$\sinh\left(\frac{k_n L}{a}\right) A_n = \frac{2V}{a^2 J_1^2(k_n)} \int_0^L r\, J_0\left(\frac{k_n r}{a}\right) dr \tag{9.3.41}$$

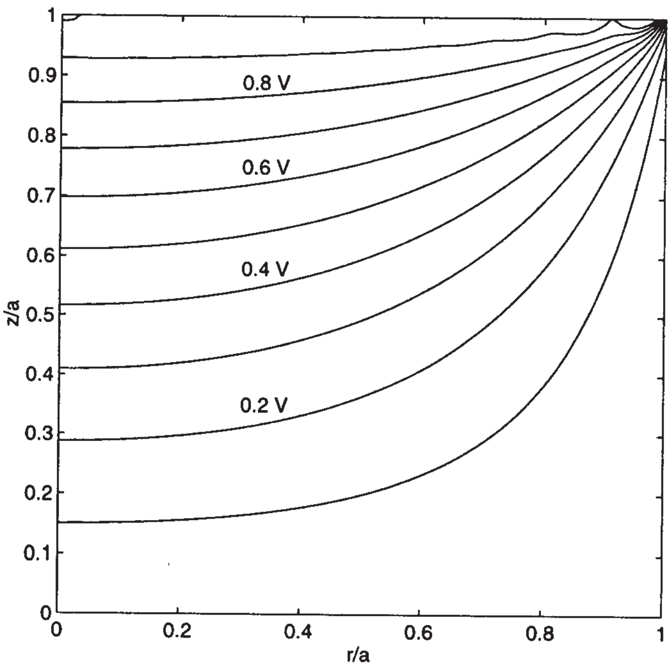

Figure 9.3.3: The steady-state potential within a cylinder of equal radius and height a when the top has the potential V while the lateral side and bottom are at potential 0.

from (6.5.35) and (6.5.43). Thus,

$$\sinh\left(\frac{k_n L}{a}\right) A_n = \frac{2V}{k_n^2 J_1^2(k_n)} \left(\frac{k_n r}{a}\right) J_1\left(\frac{k_n r}{a}\right)\bigg|_0^a = \frac{2V}{k_n J_1(k_n)}. \tag{9.3.42}$$

The solution is then

$$u(r,z) = 2V \sum_{n=1}^{\infty} \frac{J_0(k_n r/a)}{k_n J_1(k_n)} \frac{\sinh(k_n z/a)}{\sinh(k_n L/a)}. \tag{9.3.43}$$

Figure 9.3.3 illustrates (9.3.43) for the case when $L = a$ where we have included the first 20 terms of the series. Of particular interest is the convergence of the isolines in the upper right corner. At that point, the solution must jump from 0 along the line $r = a$ to V along the line $z = a$. For that reason our solution suffers from Gibbs phenomena near the top boundary. Outside of that region the electrostatic potential varies smoothly.

• Example 9.3.4

Let us now consider a similar, but slightly different, version of example 9.3.3, where the ends are held at zero potential while the lateral side has the value V. Once again, the governing equation is (9.3.33) with the boundary conditions

$$u(r,0) = u(r,L) = 0 \quad \text{and} \quad u(a,z) = V. \tag{9.3.44}$$

Separation of variables yields

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{Z}\frac{d^2Z}{dz^2} = \frac{k^2}{L^2} \tag{9.3.45}$$

with $Z(0) = Z(L) = 0$. We have chosen a positive separation constant because a negative constant would give hyperbolic functions in z which cannot satisfy the boundary conditions. A separation constant of zero would give a straight line for $Z(z)$. Applying the boundary conditions gives a trivial solution. Consequently, the only solution in the z direction which satisfies the boundary conditions is $Z_n(z) = \sin(n\pi z/L)$.

In the radial direction, the differential equation is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR_n}{dr}\right) - \frac{n^2\pi^2}{L^2}R_n = 0. \tag{9.3.46}$$

As we showed in Section 6.5, the general solution is

$$R_n(r) = A_n I_0\left(\frac{n\pi r}{L}\right) + B_n K_0\left(\frac{n\pi r}{L}\right), \tag{9.3.47}$$

where I_0 and K_0 are modified Bessel functions of the first and second kind, respectively, of order zero. Because $K_0(x)$ behaves as $-\ln(x)$ as $x \to 0$, we must discard it and our solution in the radial direction becomes $R_n(r) = I_0(n\pi r/L)$. Hence, the product solution is

$$u_n(r,z) = A_n I_0\left(\frac{n\pi r}{L}\right)\sin\left(\frac{n\pi z}{L}\right) \tag{9.3.48}$$

and the general solution is a sum of these particular solutions, namely

$$u(r,z) = \sum_{n=1}^{\infty} A_n I_0\left(\frac{n\pi r}{L}\right)\sin\left(\frac{n\pi z}{L}\right). \tag{9.3.49}$$

Finally, we use the boundary conditions that $u(a,z) = V$ to compute A_n. This condition gives

$$u(a,z) = V = \sum_{n=1}^{\infty} A_n I_0\left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi z}{L}\right) \tag{9.3.50}$$

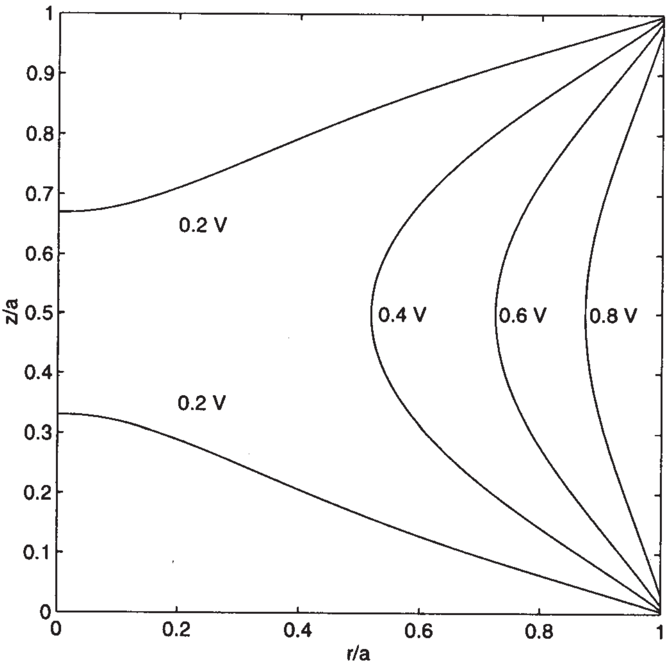

Figure 9.3.4: Potential within a conducting cylinder when the top and bottom have a potential 0 while the lateral side have a potential V.

so that

$$I_0\left(\frac{n\pi a}{L}\right)A_n = \frac{2}{L}\int_0^L V\sin\left(\frac{n\pi z}{L}\right)\,dz = \frac{2V[1-(-1)^n]}{n\pi}. \qquad (9.3.51)$$

Therefore, the final answer is

$$u(r,z) = \frac{4V}{\pi}\sum_{m=1}^{\infty}\frac{I_0[(2m-1)\pi r/L]\sin[(2m-1)\pi z/L]}{(2m-1)I_0[(2m-1)\pi a/L]}. \qquad (9.3.52)$$

Figure 9.3.4 illustrates the solution (9.3.52) for the case when $L = a$. Once again, there is a convergence of equipotentials at the corners along the right side. If we had plotted more contours, we would have observed Gibbs phenomena in the solution along the top and bottom of the cylinder.

• Example 9.3.5

Let us find the potential at any point P within a conducting sphere of radius a. At the surface, the potential is held at V_0 in the hemisphere $0 < \theta < \pi/2$ and $-V_0$ for $\pi/2 < \theta < \pi$.

Laplace's equation in spherical coordinates is

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left[\sin(\theta)\frac{\partial u}{\partial\theta}\right]=0, \qquad 0\le r<a, 0<\theta<\pi. \tag{9.3.53}$$

To solve (9.3.53) we use the separation of variables $u(r,\theta)=R(r)\Theta(\theta)$. Substituting into (9.3.53), we have that

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=-\frac{1}{\sin(\theta)\Theta}\frac{d}{d\theta}\left[\sin(\theta)\frac{d\Theta}{d\theta}\right]=k^2 \tag{9.3.54}$$

or

$$r^2R''+2rR'-k^2R=0 \tag{9.3.55}$$

and

$$\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left[\sin(\theta)\frac{d\Theta}{d\theta}\right]+k^2\Theta=0. \tag{9.3.56}$$

A common substitution replaces θ with $\mu=\cos(\theta)$. Then, as θ varies from 0 to π, μ varies from 1 to -1. With this substitution (9.3.56) becomes

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}\right]+k^2\Theta=0. \tag{9.3.57}$$

This is Legendre's equation which we examined in Section 6.4. Consequently, because the solution must remain finite at the poles, $k^2=n(n+1)$ and

$$\Theta_n(\theta)=P_n(\mu)=P_n[\cos(\theta)], \tag{9.3.58}$$

where $n=0,1,2,3,\ldots$

Turning to (9.3.55), this equation is the equidimensional or Euler-Cauchy linear differential equation. One method of solving this equation consists of introducing a new independent variable s so that $r=e^s$ or $s=\ln(r)$. Because

$$\frac{d}{dr}=\frac{ds}{dr}\frac{d}{ds}=e^{-s}\frac{d}{ds}, \tag{9.3.59}$$

it follows that

$$\frac{d^2}{dr^2}=\frac{d}{dr}\left(e^{-s}\frac{d}{ds}\right)=e^{-s}\frac{d}{ds}\left(e^{-s}\frac{d}{ds}\right)=e^{-2s}\left(\frac{d^2}{ds^2}-\frac{d}{ds}\right). \tag{9.3.60}$$

Substituting into (9.3.55),

$$\frac{d^2R_n}{ds^2}+\frac{dR_n}{ds}-n(n+1)R_n=0. \tag{9.3.61}$$

Equation (9.3.61) is a second-order, constant coefficient ordinary differential equation which has the solution

$$R_n(s) = C_n e^{ns} + D_n e^{-(n+1)s} \tag{9.3.62}$$

$$R_n(r) = C_n \exp[n\ln(r)] + D_n \exp[-(n+1)\ln(r)] \tag{9.3.63}$$

$$= C_n \exp[\ln(r^n)] + D_n \exp[\ln(r^{-1-n})] \tag{9.3.64}$$

$$= C_n r^n + D_n r^{-1-n}. \tag{9.3.65}$$

A more convenient form of the solution is

$$R_n(r) = A_n \left(\frac{r}{a}\right)^n + B_n \left(\frac{r}{a}\right)^{-1-n}, \tag{9.3.66}$$

where $A_n = a^n C_n$ and $B_n = D_n/a^{n+1}$. We introduced the constant a, the radius of the sphere, to simplify future calculations.

Using the results from (9.3.58) and (9.3.66), the solution to Laplace's equation in axisymmetric problems is

$$u(r,\theta) = \sum_{n=0}^{\infty} \left[A_n \left(\frac{r}{a}\right)^n + B_n \left(\frac{r}{a}\right)^{-1-n}\right] P_n[\cos(\theta)]. \tag{9.3.67}$$

In our particular problem we must take $B_n = 0$ because the solution becomes infinite at $r = 0$ otherwise. If the problem had involved the domain $a < r < \infty$, then $A_n = 0$ because the potential must remain finite as $r \to \infty$.

Finally, we must evaluate A_n. Finding the potential at the surface,

$$u(a,\mu) = \sum_{n=0}^{\infty} A_n P_n(\mu) = \begin{cases} V_0, & 0 < \mu \le 1 \\ -V_0, & -1 \le \mu < 0. \end{cases} \tag{9.3.68}$$

Upon examing (9.3.68), it is merely an expansion in Legendre polynomials of the function

$$f(\mu) = \begin{cases} V_0, & 0 < \mu \le 1 \\ -V_0, & -1 \le \mu < 0. \end{cases} \tag{9.3.69}$$

Consequently, from (9.3.69),

$$A_n = \frac{2n+1}{2} \int_{-1}^{1} f(\mu) P_n(\mu)\, d\mu. \tag{9.3.70}$$

Because $f(\mu)$ is an odd function, $A_n = 0$ if n is even. When n is odd, however,

$$A_n = (2n+1) \int_0^1 V_0 P_n(\mu)\, d\mu. \tag{9.3.71}$$

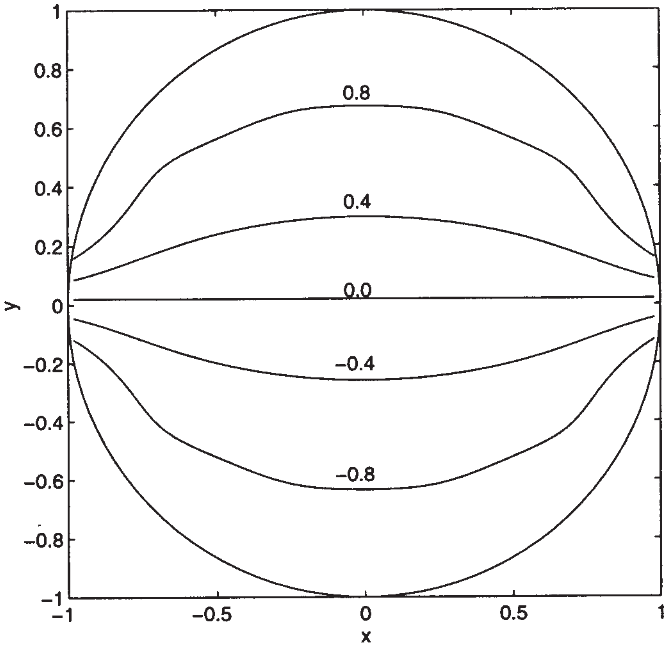

Figure 9.3.5: Electrostatic potential within a conducting sphere when the upper hemispheric surface has the potential 1 and the lower surface has the potential −1.

We can further simplify (9.3.71) by using the relationship that

$$\int_x^1 P_n(t)\,dt = \frac{1}{2n+1}\left[P_{n-1}(x) - P_{n+1}(x)\right], \tag{9.3.72}$$

where $n \geq 1$. In our problem, then,

$$A_n = \begin{cases} V_0[P_{n-1}(0) - P_{n+1}(0)], & n \text{ odd} \\ 0, & n \text{ even.} \end{cases} \tag{9.3.73}$$

This first few terms are $A_1 = 3V_0/2$, $A_3 = -7V_0/8$, and $A_5 = 11V_0/16$.

Figure 9.3.5 illustrates our solution. Here we have the convergence of the equipotentials along the equator and at the surface. The slow rate at which the coefficients are approaching zero suggests that the solution will suffer from Gibbs phenomena along the surface.

• Example 9.3.6

We now find the steady-state temperature field within a metallic sphere of radius a, which we place in direct sunlight and allow to radiatively cool. This classic problem, first solved by Rayleigh,[5] requires the use of spherical coordinates with its origin at the center of sphere and its z-axis pointing toward the sun. With this choice for the coordinate system, the incident sunlight is

$$D(\theta) = \begin{cases} D(0)\cos(\theta), & 0 \le \theta \le \pi/2 \\ 0, & \pi/2 \le \theta \le \pi. \end{cases} \tag{9.3.74}$$

If the heat dissipation takes place at the surface $r = a$ according to Newton's law of cooling and the temperature of the surrounding medium is zero, the solar heat absorbed by the surface dA must balance the Newtonian cooling at the surface plus the energy absorbed into the sphere's interior. This physical relationship is

$$(1-\rho)D(\theta)\,dA = \epsilon u(a,\theta)\,dA + \kappa \frac{\partial u(a,\theta)}{\partial r}\,dA, \tag{9.3.75}$$

where ρ is the reflectance of the surface (the albedo), ϵ is the surface conductance or coefficient of surface heat transfer, and κ is the thermal conductivity. Simplifying (9.3.75), we have that

$$\frac{\partial u(a,\theta)}{\partial r} = \frac{1-\rho}{\kappa} D(\theta) - \frac{\epsilon}{\kappa} u(a,\theta) \tag{9.3.76}$$

for $r = a$.

If the sphere has reached thermal equilibrium, Laplace's equation describes the temperature field within the sphere. In the previous example, we showed that the solution to Laplace's equation in axisymmetric problems is

$$u(r,\theta) = \sum_{n=0}^{\infty} \left[A_n \left(\frac{r}{a}\right)^n + B_n \left(\frac{r}{a}\right)^{-1-n} \right] P_n[\cos(\theta)]. \tag{9.3.77}$$

In this problem, $B_n = 0$ because the solution would become infinite at $r = 0$ otherwise. Therefore,

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n \left(\frac{r}{a}\right)^n P_n[\cos(\theta)]. \tag{9.3.78}$$

Differentiation gives

$$\frac{\partial u}{\partial r} = \sum_{n=0}^{\infty} A_n \frac{n r^{n-1}}{a^n} P_n[\cos(\theta)]. \tag{9.3.79}$$

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[5] Rayleigh, J. W., 1870: On the values of the integral $\int_0^1 Q_n Q_{n'}\,d\mu$, Q_n, $Q_{n'}$ being Laplace's coefficients of the orders n, n', with application to the theory of radiation. *Philos. Trans. R. Soc., London*, **160**, 579–590.

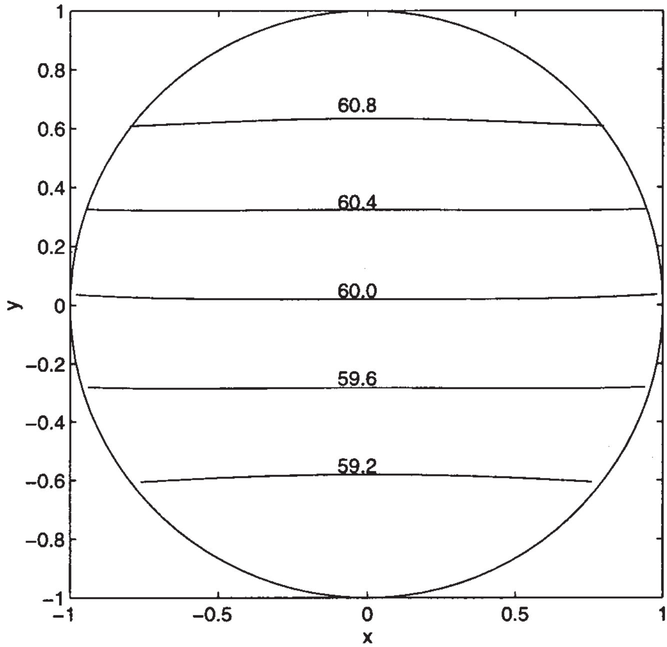

Figure 9.3.6: The difference (in °C) between the temperature field within a blackened iron surface of radius 0.1 m and the surrounding medium when we heat the surface by sunlight and allow it to radiatively cool.

Substituting into the boundary conditions leads to

$$\sum_{n=0}^{\infty} A_n \left(\frac{n}{a} + \frac{\epsilon}{\kappa}\right) P_n[\cos(\theta)] = \left(\frac{1-\rho}{\kappa}\right) D(\theta) \tag{9.3.80}$$

or

$$D(\mu) = \sum_{n=0}^{\infty} \left[\frac{n\kappa + \epsilon a}{a(1-\rho)}\right] A_n P_n(\mu) = \sum_{n=0}^{\infty} C_n P_n(\mu), \tag{9.3.81}$$

where

$$C_n = \left[\frac{n\kappa + \epsilon a}{a(1-\rho)}\right] A_n \quad \text{and} \quad \mu = \cos(\theta). \tag{9.3.82}$$

We determine the coefficients by

$$C_n = \frac{2n+1}{2} \int_{-1}^{1} D(\mu) P_n(\mu)\, d\mu = \frac{2n+1}{2} D(0) \int_{-1}^{1} \mu P_n(\mu)\, d\mu. \tag{9.3.83}$$

Evaluation of the first few coefficients gives

$$A_0 = \frac{(1-\rho)D(0)}{4\epsilon}, \quad A_1 = \frac{a(1-\rho)D(0)}{2(\kappa+\epsilon a)}, \quad A_2 = \frac{5a(1-\rho)D(0)}{16(2\kappa+\epsilon a)}, \tag{9.3.84}$$

$$A_3 = 0 \quad \text{and} \quad A_4 = -\frac{3a(1-\rho)D(0)}{32(4\kappa+\epsilon a)}. \tag{9.3.85}$$

Figure 9.3.6 illustrates the temperature field within the interior of the sphere with $D(0) = 1200$ W/m^2, $\kappa = 45$ W/m K, $\epsilon = 5$ W/m^2 K, $\rho = 0$, and $a = 0.1$ m. This corresponds to a cast iron sphere with blackened surface in sunlight. The temperature is quite warm with the highest temperature located at the position where the solar radiation is largest; the coolest temperatures are located in the shadow region.

• Example 9.3.7

In this example we will find the potential at any point P which results from a point charge $+q$ placed at $z = a$ on the z-axis when we introduce a conducting, grounded sphere at $z = 0$. See Figure 9.3.7. From the principle of linear superposition, the total potential $u(r,\theta)$ equals the sum of the potential from the point charge and the potential $v(r,\theta)$ due to the induced charge on the sphere

$$u(r,\theta) = \frac{q}{s} + v(r,\theta). \tag{9.3.86}$$

In common with the first term q/s, $v(r,\theta)$ must be a solution of Laplace's equation. In Example 9.3.5 we showed that the general solution to Laplace's equation in axisymmetric problems is

$$v(r,\theta) = \sum_{n=0}^{\infty} \left[A_n \left(\frac{r}{r_0}\right)^n + B_n \left(\frac{r}{r_0}\right)^{-1-n} \right] P_n[\cos(\theta)]. \tag{9.3.87}$$

Because the solutions must be valid *anywhere* outside of the sphere, $A_n = 0$; otherwise, the solution would not remain finite as $r \to \infty$. Hence,

$$v(r,\theta) = \sum_{n=0}^{\infty} B_n \left(\frac{r}{r_0}\right)^{-1-n} P_n[\cos(\theta)]. \tag{9.3.88}$$

We determine the coefficient B_n by the condition that $u(r_0,\theta) = 0$ or

$$\left.\frac{q}{s}\right|_{\text{on sphere}} + \sum_{n=0}^{\infty} B_n P_n[\cos(\theta)] = 0. \tag{9.3.89}$$

We need to expand the first term on the left side of (9.3.89) in terms of Legendre polynomials. From the law of cosines,

$$s = \sqrt{r^2 + a^2 - 2ar\cos(\theta)}. \tag{9.3.90}$$

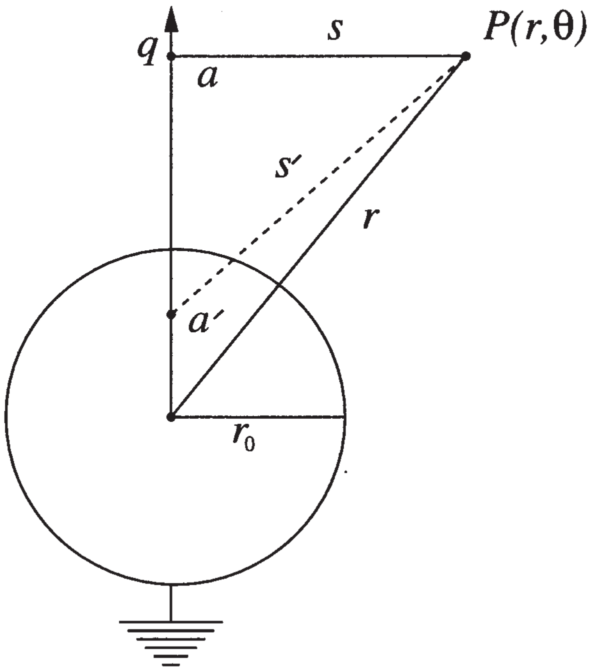

Figure 9.3.7: Point charge $+q$ in the presence of a grounded conducting sphere.

Consequently, if $a > r$, then

$$\frac{1}{s} = \frac{1}{a}\left[1 - 2\cos(\theta)\frac{r}{a} + \left(\frac{r}{a}\right)^2\right]^{-1/2} \tag{9.3.91}$$

In Section 6.4, we showed that

$$(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)z^n. \tag{9.3.92}$$

Therefore,

$$\frac{1}{s} = \frac{1}{a}\sum_{n=0}^{\infty} P_n[\cos(\theta)]\left(\frac{r}{a}\right)^n. \tag{9.3.93}$$

From (9.3.89),

$$\sum_{n=0}^{\infty}\left[\frac{q}{a}\left(\frac{r_0}{a}\right)^n + B_n\right] P_n[\cos(\theta)] = 0. \tag{9.3.94}$$

We can only satisfy (9.3.94) if the square-bracketed term vanishes identically so that

$$B_n = -\frac{q}{a}\left(\frac{r_0}{a}\right)^n \tag{9.3.95}$$

On substituting (9.3.95) back into (9.3.88),

$$v(r,\theta) = -\frac{qr_0}{ra}\sum_{n=0}^{\infty}\left(\frac{r_0^2}{ar}\right)^n P_n[\cos(\theta)]. \tag{9.3.96}$$

The physical interpretation of (9.3.96) is as follows. Consider a point, such as a' (see Figure 9.3.7) on the z-axis. If $r > a'$, the expression of $1/s'$ is

$$\frac{1}{s'} = \frac{1}{r}\sum_{n=0}^{\infty} P_n[\cos(\theta)]\left(\frac{a'}{r}\right)^n, \quad r > a'. \tag{9.3.97}$$

Using (9.3.97), we can rewrite (9.3.96) as

$$v(r,\theta) = -\frac{qr_0}{as'}, \tag{9.3.98}$$

if we set $a' = r_0^2/a$. Our final result is then

$$u(r,\theta) = \frac{q}{s} - \frac{q'}{s'}, \tag{9.3.99}$$

provided that q' equals $r_0 q/a$. In other words, when we place a grounded conducting sphere near a point charge $+q$, it changes the potential in the same manner as would a point charge of the opposite sign and magnitude $q' = r_0 q/a$, placed at the point $a' = r_0^2/a$. The charge q' is the *image* of q.

Figure 9.3.8 illustrates the solution (9.3.96). Because the charge is located above the sphere for any fixed r, the electrostatic potential is largest at the point $\theta = 0$ and weakest at $\theta = \pi$.

• **Example 9.3.8: Poisson's integral formula**

In this example we find the solution to Laplace's equation within a unit disc. The problem may be posed as

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = 0, \quad 0 \le r < 1, 0 \le \varphi \le 2\pi \tag{9.3.100}$$

with the boundary condition $u(1,\varphi) = f(\varphi)$.

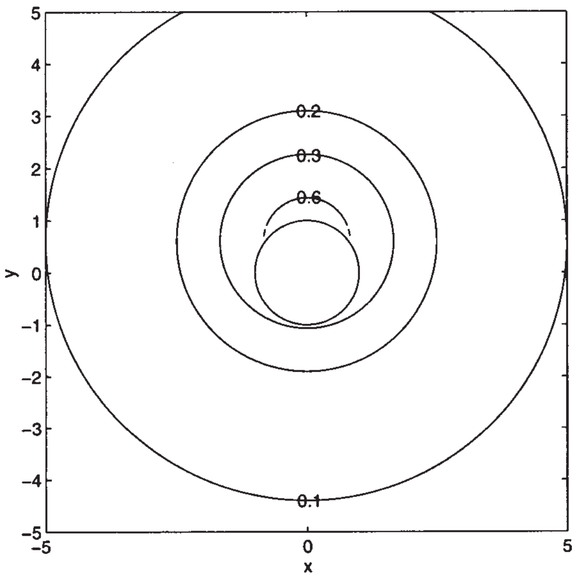

Figure 9.3.8: Electrostatic potential outside of a grounded conducting sphere in the presence of a point charge located at $a/r_0 = 2$. Contours are in units of q/r_0.

We begin by assuming the separable solution $u(r,\varphi) = R(r)\Phi(\varphi)$ so that

$$\frac{r^2R'' + rR'}{R} = -\frac{\Phi''}{\Phi} = k^2. \tag{9.3.101}$$

The solution to $\Phi'' + k^2\Phi = 0$ is

$$\Phi(\varphi) = A\cos(k\varphi) + B\sin(k\varphi). \tag{9.3.102}$$

The solution to $R(r)$ is

$$R(r) = Cr^k + Dr^{-k}. \tag{9.3.103}$$

Because the solution must be bounded for all r and periodic is φ, we must take $D = 0$ and $k = n$, where $n = 0, 1, 2, 3, \ldots$. Then, the most general solution is

$$u(r,\varphi) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\varphi) + b_n\sin(n\varphi)\right]r^n, \tag{9.3.104}$$

where a_n and b_n are chosen to satisfy

$$u(1,\varphi) = f(\varphi) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(n\varphi) + b_n\sin(n\varphi). \tag{9.3.105}$$

Because

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\varphi)\cos(n\varphi)\,d\varphi, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\varphi)\sin(n\varphi)\,d\varphi, \tag{9.3.106}$$

we may write $u(r,\varphi)$ as

$$u(r,\varphi) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\varphi)\left\{\tfrac{1}{2} + \sum_{n=1}^{\infty} r^n \cos[n(\theta-\varphi)]\right\} d\theta. \tag{9.3.107}$$

If we let $\alpha = \theta - \varphi$ and $z = r[\cos(\alpha) + i\sin(\alpha)]$, then

$$\sum_{n=0}^{\infty} r^n \cos(n\alpha) = \mathrm{Re}\left(\sum_{n=0}^{\infty} z^n\right) = \mathrm{Re}\left(\frac{1}{1-z}\right) \tag{9.3.108}$$

$$= \mathrm{Re}\left[\frac{1}{1 - r\cos(\alpha) - ir\sin(\alpha)}\right] \tag{9.3.109}$$

$$= \mathrm{Re}\left[\frac{1 - r\cos(\alpha) + ir\sin(\alpha)}{1 - 2r\cos(\alpha) + r^2}\right] \tag{9.3.110}$$

for all r such that $|r| < 1$. Consequently,

$$\sum_{n=0}^{\infty} r^n \cos(n\alpha) = \frac{1 - r\cos(\alpha)}{1 - 2r\cos(\alpha) + r^2} \tag{9.3.111}$$

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos(n\alpha) = \frac{1 - r\cos(\alpha)}{1 - 2r\cos(\alpha) + r^2} - \frac{1}{2} \tag{9.3.112}$$

$$= \frac{1}{2}\frac{1 - r^2}{1 - 2r\cos(\alpha) + r^2}. \tag{9.3.113}$$

Substituting (9.3.113) into (9.3.107), we finally have that

$$u(r,\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\frac{1 - r^2}{1 - 2r\cos(\theta-\varphi) + r^2}\,d\theta. \tag{9.3.114}$$

This solution to Laplace's equation within the unit circle is referred to as *Poisson's integral formula.*[6]

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[6] Poisson, S. D., 1820: Mémoire sur la manière d'exprimer les fonctions par des séries de quantités périodiques, et sur l'usage de cette transformation dans la résolution de différens problèmes. *J. École Polytech.*, **18**, 417–489.

Problems

Solve Laplace's equation over the rectangular region $0 < x < a$, $0 < y < b$ with the following boundary conditions:

1. $u(x,0) = u(x,b) = u(a,y) = 0, u(0,y) = 1$

2. $u(x,0) = u(0,y) = u(a,y) = 0, u(x,b) = x$

3. $u(x,0) = u(0,y) = u(a,y) = 0, u(x,b) = x - a$

4. $u(x,0) = u(0,y) = u(a,y) = 0,$
$$u(x,b) = \begin{cases} 2x/a, & 0 < x < a/2 \\ 2(a-x)/a, & a/2 < x < a \end{cases}$$

5. $u_x(0,y) = u(a,y) = u(x,0) = 0, u(x,b) = 1$

6. $u_y(x,0) = u(x,b) = u(a,y) = 0, u(0,y) = 1$

7. $u_y(x,0) = u_y(x,b) = 0, u(0,y) = u(a,y) = 1$

8. $u_x(a,y) = u_y(x,b) = 0, u(0,y) = u(x,0) = 1$

9. $u_y(x,0) = u(x,b) = 0, u(0,y) = u(a,y) = 1$

10. $u(a,y) = u(x,b) = 0, u(0,y) = u(x,0) = 1$

11. $u_x(0,y) = 0, u(a,y) = u(x,0) = u(x,b) = 1$

12. $u_x(0,y) = u_x(a,y) = 0, u(x,b) = u_1,$
$$u(x,0) = \begin{cases} f(x), & 0 < x < \alpha \\ 0, & \alpha < x < a \end{cases}$$

13. Variations in the earth's surface temperature can arise as a result of topographic undulations and the altitude dependence of the atmospheric temperature. These variations, in turn, affect the temperature within the solid earth. To show this, solve Laplace's equation with the surface boundary condition that

$$u(x,0) = T_0 + \Delta T \cos(2\pi x/\lambda),$$

where λ is the wavelength of the spatial temperature variation. What must be the condition on $u(x,y)$ as we go towards the center of the earth (i.e., $y \to \infty$)?

14. Tóth[7] generalized his earlier analysis of groundwater in an aquifer when the water table follows the topography. Find the groundwater potential if it varies as

$$u(x, z_0) = g[z_0 + cx + a\sin(bx)]$$

at the surface $y = z_0$ while $u_x(0, y) = u_x(L, y) = u_y(x, 0) = 0$, where g is the acceleration due to gravity. Assume that $bL \neq n\pi$, where $n = 1, 2, 3, \ldots$.

15. Solve

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 \le r < a, -L < z < L$$

with

$$u(a, z) = 0 \qquad \text{and} \qquad \frac{\partial u(r, -L)}{\partial z} = \frac{\partial u(r, L)}{\partial z} = 1.$$

16. Solve

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 \le r < a, 0 < z < h$$

with

$$\frac{\partial u(a, z)}{\partial r} = u(r, h) = 0$$

and

$$\frac{\partial u(r, 0)}{\partial z} = \begin{cases} 1, & 0 \le r < r_0 \\ 0, & r_0 < r < a. \end{cases}$$

17. Solve

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 \le r < 1, 0 < z < d$$

with

$$\frac{\partial u(1, z)}{\partial r} = \frac{\partial u(r, 0)}{\partial z} = 0$$

and

$$u(r, d) = \begin{cases} -1, & 0 \le r < a, \quad b < r < 1 \\ 1/(b^2 - a^2) - 1, & a < r < b. \end{cases}$$

[7] Tóth, J., *J. Geophys. Res.*, **68**, 4795–4812, 1963, copyright by the American Geophysical Union.

18. Solve

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 \le r < a, 0 < z < h$$

with

$$u(r,0) = u(a,z) = 0 \qquad \text{and} \qquad \frac{\partial u(r,h)}{\partial z} = Ar.$$

19. Solve

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 \le r < a, 0 < z < 1$$

with

$$u(r,0) = u(r,1) = 0 \qquad \text{and} \qquad u(a,z) = z.$$

20. Solve

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 \le r < a, 0 < z < h$$

with

$$\frac{\partial u(a,z)}{\partial r} = u(r,0) = 0 \qquad \text{and} \qquad \frac{\partial u(r,h)}{\partial z} = r.$$

21. Solve[8]

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial u}{\partial z} = 0, \quad 0 \le r < 1, 0 < z < \infty$$

with the boundary conditions

$$\lim_{r\to 0} |u(r,z)| < \infty, \qquad \frac{\partial u(1,z)}{\partial r} = -Bu(1,z), \qquad z > 0,$$

and

$$u(r,0) = 1, \qquad \lim_{z\to\infty} |u(r,z)| < \infty, \qquad 0 \le r < 1,$$

where B is a constant.

[8] Reprinted from *Int. J. Heat Mass Transfer*, **19**, Kern, J., and J. O. Hansen, Transient heat conduction in cylindrical systems with an axially moving boundary, 707–714, ©1976, with kind permission from Elsevier Science Ltd., The Boulevard, Langford Lane, Kidlington OX5 1GB, UK.

22. Find the steady-state temperature within a sphere of radius a if the temperature along its surface is maintained at the temperature $u(a, \theta) = 100[\cos(\theta) - \cos^5(\theta)]$.

23. Find the steady-state temperature within a sphere if the upper half of the exterior surface at radius a is maintained at the temperature 100 while the lower half is maintained at the temperature 0.

24. The surface of a sphere of radius a has a temperature of zero everywhere except in a spherical cap at the north pole (defined by the cone $\theta = \alpha$) where it equals T_0. Find the steady-state temperature within the sphere.

25. Using the relationship

$$\int_0^{2\pi} \frac{d\varphi}{1 - b\cos(\varphi)} = \frac{2\pi}{\sqrt{1-b^2}}, \qquad |b| < 1$$

and Poisson's integral formula, find the solution to Laplace's equation within a unit disc if $u(1, \varphi) = f(\varphi) = T_0$, a constant.

9.4 THE SOLUTION OF LAPLACE'S EQUATION ON THE UPPER HALF-PLANE

In this section we shall use Fourier integrals and convolution to find the solution of Laplace's equation on the upper half-plane $y > 0$. We require that the solution remains bounded over the entire domain and specify it along the x-axis, $u(x, 0) = f(x)$. Under these conditions, we can take the Fourier transform of Laplace's equation and find that

$$\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{-i\omega x}\, dx + \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial y^2} e^{-i\omega x}\, dx = 0. \tag{9.4.1}$$

If everything is sufficiently differentiable, we may successively integrate by parts the first integral in (9.4.1) which yields

$$\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{-i\omega x}\, dx = \left.\frac{\partial u}{\partial x} e^{-i\omega x}\right|_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{-i\omega x}\, dx \tag{9.4.2}$$

$$= i\omega\, u(x, y)e^{-i\omega x}\Big|_{-\infty}^{\infty} - \omega^2 \int_{-\infty}^{\infty} u(x, y) e^{-i\omega x}\, dx \tag{9.4.3}$$

$$= -\omega^2 \mathcal{U}(\omega, y), \tag{9.4.4}$$

where

$$\mathcal{U}(\omega, y) = \int_{-\infty}^{\infty} u(x, y) e^{-i\omega x}\, dx. \tag{9.4.5}$$

The second integral becomes

$$\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial y^2} e^{-i\omega x}\, dx = \frac{d^2}{dy^2}\left[\int_{-\infty}^{\infty} u(x,y)e^{-i\omega x}\, dx\right] = \frac{d^2\mathcal{U}(\omega,y)}{dy^2} \quad (9.4.6)$$

along with the boundary condition that

$$F(\omega) = \mathcal{U}(\omega,0) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\, dx. \quad (9.4.7)$$

Consequently we have reduced Laplace's equation, a partial differential equation, to an ordinary differential equation in y, where ω is merely a parameter:

$$\frac{d^2\mathcal{U}(\omega,y)}{dy^2} - \omega^2\mathcal{U}(\omega,y) = 0, \quad (9.4.8)$$

with the boundary condition $\mathcal{U}(\omega,0) = F(\omega)$. The solution to (9.4.8) is

$$\mathcal{U}(\omega,y) = A(\omega)e^{|\omega|y} + B(\omega)e^{-|\omega|y}, \quad y \ge 0. \quad (9.4.9)$$

We must discard the $e^{|\omega|y}$ term because it becomes unbounded as we go to infinity along the y-axis. The boundary condition results in $B(\omega) = F(\omega)$. Consequently,

$$\mathcal{U}(\omega,y) = F(\omega)e^{-|\omega|y}. \quad (9.4.10)$$

The inverse of the Fourier transform $e^{-|\omega|y}$ equals

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|\omega|y}e^{i\omega x}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{0} e^{\omega y}e^{i\omega x}\, d\omega + \frac{1}{2\pi}\int_{0}^{\infty} e^{-\omega y}e^{i\omega x}\, d\omega \quad (9.4.11)$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} e^{-\omega y}e^{-i\omega x}\, d\omega + \frac{1}{2\pi}\int_{0}^{\infty} e^{-\omega y}e^{i\omega x}\, d\omega \quad (9.4.12)$$

$$= \frac{1}{\pi}\int_{0}^{\infty} e^{-\omega y}\cos(\omega x)\, d\omega \quad (9.4.13)$$

$$= \frac{1}{\pi}\left\{\frac{\exp(-\omega y)}{x^2+y^2}\left[-y\cos(\omega x) + x\sin(\omega x)\right]\right\}\Bigg|_{0}^{\infty} \quad (9.4.14)$$

$$= \frac{1}{\pi}\frac{y}{x^2+y^2}. \quad (9.4.15)$$

Furthermore, because (9.4.10) is a convolution of two Fourier transforms, its inverse is

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yf(t)}{(x-t)^2+y^2}\, dt. \quad (9.4.16)$$

Equation (9.4.16) is *Poisson's integral formula*[9] for the half-plane $y > 0$ or *Schwarz' integral formula.*[10]

• Example 9.4.1

As an example, let $u(x,0) = 1$ if $|x| < 1$ and $u(x,0) = 0$ otherwise. Then,

$$u(x,y) = \frac{1}{\pi}\int_{-1}^{1} \frac{y}{(x-t)^2 + y^2}\, dt \tag{9.4.17}$$

$$= \frac{1}{\pi}\left[\tan^{-1}\left(\frac{1-x}{y}\right) + \tan^{-1}\left(\frac{1+x}{y}\right)\right]. \tag{9.4.18}$$

Problems

Find the solution to Laplace's equation in the upper half-plane for the following boundary conditions:

1.

$$u(x,0) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

2.

$$u(x,0) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

3.

$$u(x,0) = \begin{cases} T_0, & x < 0 \\ 0, & x > 0 \end{cases}$$

4.

$$u(x,0) = \begin{cases} 2T_0, & x < -1 \\ T_0, & -1 < x < 1 \\ 0, & x > 1 \end{cases}$$

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[9] Poisson, S. D., 1823: Suite du mémoire sur les intégrales définies et sur la sommation des séries. *J. École Polytech.*, **19**, 404–509. See pg. 462.

[10] Schwarz, H. A., 1870: Über die Integration der partiellen Differentialgleichung $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ für die Fläche eines Kreises, *Vierteljahrsschr. Naturforsch. Ges. Zürich*, **15**, 113–128.

5.
$$u(x,0) = \begin{cases} T_0, & -1 < x < 0 \\ T_0 + (T_1 - T_0)x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

6.
$$u(x,0) = \begin{cases} T_0, & x < a_1 \\ T_1, & a_1 < x < a_2 \\ T_2, & a_2 < x < a_3 \\ \vdots & \vdots \\ T_n, & a_n < x \end{cases}$$

9.5 POISSON'S EQUATION ON A RECTANGLE

Poisson's equation[11] is Laplace's equation with a source term:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y). \tag{9.5.1}$$

It arises in such diverse areas as groundwater flow, electromagnetism, and potential theory. Let us solve it if $u(0,y) = u(a,y) = u(x,0) = u(x,b) = 0$.

We begin by solving a similar partial differential equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \lambda u, \qquad 0 < x < a, 0 < y < b \tag{9.5.2}$$

by separation of variables. If $u(x,y) = X(x)Y(y)$, then

$$\frac{X''}{X} + \frac{Y''}{Y} = \lambda. \tag{9.5.3}$$

Because we must satisfy the boundary conditions that $X(0) = X(a) = Y(0) = Y(b) = 0$, we have the following eigenfunction solutions:

$$X_n(x) = \sin\left(\frac{n\pi x}{a}\right), \qquad Y_m(x) = \sin\left(\frac{m\pi y}{b}\right) \tag{9.5.4}$$

with $\lambda_{nm} = -n^2\pi^2/a^2 - m^2\pi^2/b^2$; otherwise, we would only have trivial solutions. The corresponding particular solutions are

$$u_{nm} = A_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right), \tag{9.5.5}$$

[11] Poisson, S. D., 1813: Remarques sur une équation qui se présente dans la théorie des attractions des sphéroïdes. *Nouv. Bull. Soc. Philomath. Paris*, **3**, 388–392.

Figure 9.5.1: Siméon-Denis Poisson (1781–1840) was a product as well as a member of the French scientific establishment of his day. Educated at the École Polytechnique, he devoted his life to teaching, both in the classroom and with administrative duties, and to scientific research. Poisson's equation dates from 1813 when Poisson sought to extend Laplace's work on gravitational attraction. (Portrait courtesy of the Archives de l'Académie des sciences, Paris.)

where $n = 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots$.

For a fixed y, we can expand $f(x, y)$ in the half-range Fourier sine series:

$$f(x, y) = \sum_{n=1}^{\infty} A_n(y) \sin\left(\frac{n\pi x}{a}\right), \tag{9.5.6}$$

where

$$A_n(y) = \frac{2}{a} \int_0^a f(x, y) \sin\left(\frac{n\pi x}{a}\right) dx. \tag{9.5.7}$$

However, we can also expand $A_n(y)$ in a half-range Fourier sine series:

$$A_n(y) = \sum_{m=1}^{\infty} a_{nm} \sin\left(\frac{m\pi y}{b}\right), \tag{9.5.8}$$

where

$$a_{nm} = \frac{2}{b}\int_0^b A_n(y) \sin\left(\frac{m\pi y}{b}\right) dy \tag{9.5.9}$$

$$= \frac{4}{ab}\int_0^b \int_0^a f(x,y) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) dx\, dy \tag{9.5.10}$$

and

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right). \tag{9.5.11}$$

In other words, we have reexpressed $f(x,y)$ in terms of a *double Fourier series*.

Because (9.5.2) must hold for each particular solution,

$$\frac{\partial^2 u_{nm}}{\partial x^2} + \frac{\partial^2 u_{nm}}{\partial y^2} = \lambda_{nm} u_{nm} = a_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right), \tag{9.5.12}$$

if we now associate (9.5.1) with (9.5.2). Therefore, the solution to Poisson's equation on a rectangle where the boundaries are held at zero is the double Fourier series:

$$u(x,y) = -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_{nm}}{n^2\pi^2/a^2 + m^2\pi^2/b^2} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right). \tag{9.5.13}$$

Problems

1. The equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{R}{T}, \qquad -a < x < a, -b < y < b$$

describes the hydraulic potential (elevation of the water table) $u(x,y)$ within a rectangular island on which a recharging well is located at $(0,0)$. Here R is the rate of recharging and T is the product of the hydraulic conductivity and aquifer thickness. If the water table is at sea level around the island so that $u(-a,y) = u(a,y) = u(x,-b) = u(x,b) = 0$, find $u(x,y)$ everywhere in the island. [Hint: Use symmetry and redo the above analysis with the boundary conditions: $u_x(0,y) = u(a,y) = u_y(x,0) = u(x,b) = 0$.]

2. Let us apply the same approach that we used to find the solution of Poisson's equation on a rectangle to solve the axisymmetric Poisson equation inside a circular cylinder

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} = f(r,z), \qquad 0 \le r < a, -b < z < b$$

subject to the boundary conditions

$$\lim_{r\to 0} |u(r,z)| < \infty, \qquad u(a,z) = 0, \qquad -b < z < b$$

and

$$u(r,-b) = u(r,b) = 0, \qquad 0 \le r < a.$$

Step 1. Replace the original problem with

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} = \lambda u, \qquad 0 \le r < a, -b < z < b$$

subject to the same boundary conditions. Use separation of variables to show that the solution to this new problem is

$$u_{nm}(r,z) = A_{nm} J_0\left(k_n \frac{r}{a}\right) \cos\left[\frac{\left(m+\frac{1}{2}\right)\pi z}{b}\right],$$

where k_n is the nth zero of $J_0(k) = 0$, $n = 1, 2, 3, \ldots$ and $m = 0, 1, 2, \ldots$.

Step 2. Show that $f(r,z)$ can be expressed as

$$f(r,z) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_{nm} J_0\left(k_n \frac{r}{a}\right) \cos\left[\frac{\left(m+\frac{1}{2}\right)\pi z}{b}\right],$$

where

$$a_{nm} = \frac{2}{a^2 b J_1^2(k_n)} \int_{-b}^{b}\int_0^a f(r,z) J_0\left(k_n \frac{r}{a}\right) \cos\left[\frac{\left(m+\frac{1}{2}\right)\pi z}{b}\right] r\,dr\,dz.$$

Step 3. Show that the general solution is

$$u(r,z) = -\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_{nm} \frac{J_0(k_n r/a)\cos\left[\left(m+\frac{1}{2}\right)\pi z/b\right]}{(k_n/a)^2 + \left[\left(m+\frac{1}{2}\right)\pi/b\right]^2}.$$

9.6 THE LAPLACE TRANSFORM METHOD

Laplace transforms are useful in solving Laplace's or Poisson's equation over a semi-infinite strip. The following problem illustrates this technique.

Let us solve Poisson's equation within an semi-infinite circular cylinder

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} = \frac{2}{b}n(z)\delta(r-b) \quad 0 \le r < a, 0 < z < \infty \qquad (9.6.1)$$

subject to the boundary conditions

$$u(r,0) = 0 \quad \text{and} \quad \lim_{z\to\infty} |u(r,z)| < \infty, \quad 0 \le r < a \qquad (9.6.2)$$

and

$$u(a,z) = 0, \qquad 0 < z < \infty, \qquad (9.6.3)$$

where $0 < b < a$. This problem gives the electrostatic potential within a semi-infinite cylinder of radius a that is grounded and has the charge density of $n(z)$ within an infinitesimally thin shell located at $r = b$.

Because the domain is semi-infinite in the z direction, we introduce the Laplace transform

$$U(r,s) = \int_0^\infty u(r,z)\,e^{-sz}\,dz. \qquad (9.6.4)$$

Thus, taking the Laplace transform of (9.6.1), we have that

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dU(r,s)}{dr}\right] + s^2U(r,s) - su(r,0) - u_z(r,0) = \frac{2}{b}N(s)\delta(r-b). \qquad (9.6.5)$$

Although $u(r,0) = 0$, $u_z(r,0)$ is unknown and we denote its value by $f(r)$. Therefore, (9.6.5) becomes

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dU(r,s)}{dr}\right] + s^2U(r,s) = f(r) + \frac{2}{b}N(s)\delta(r-b), \quad 0 \le r < a \qquad (9.6.6)$$

with $\lim_{r\to 0}|U(r,s)| < \infty$ and $U(a,s) = 0$.

To solve (9.6.6) we first assume that we can rewrite $f(r)$ as the Fourier-Bessel series:

$$f(r) = \sum_{n=1}^{\infty} A_n J_0(k_n r/a), \qquad (9.6.7)$$

where k_n is the nth root of the $J_0(k) = 0$ and

$$A_n = \frac{2}{a^2 J_1^2(k_n)} \int_0^a f(r)\, J_0(k_n r/a)\, r\, dr. \tag{9.6.8}$$

Similarly, the expansion for the delta function is

$$\delta(r-b) = \frac{2b}{a^2} \sum_{n=1}^{\infty} \frac{J_0(k_n b/a) J_0(k_n r/a)}{J_1^2(k_n)}, \tag{9.6.9}$$

because

$$\int_0^a \delta(r-b) J_0(k_n r/a)\, r\, dr = b\, J_0(k_n b/a). \tag{9.6.10}$$

Why we have chosen this particular expansion will become apparent shortly.

Thus, (9.6.6) may be rewritten as

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dU(r,s)}{dr}\right] + s^2 U(r,s) = \frac{2}{a^2} \sum_{n=1}^{\infty} \frac{2N(s)J_0(k_n b/a) + a_k}{J_1^2(k_n)} J_0(k_n r/a), \tag{9.6.11}$$

where $a_k = \int_0^a f(r)\, J_0(k_n r/a)\, r\, dr$.

The form of the right side of (9.6.11) suggests that we seek solutions of the form

$$U(r,s) = \sum_{n=1}^{\infty} B_n J_0(k_n r/a), \qquad 0 \le r < a. \tag{9.6.12}$$

We now understand why we rewrote the right side of (9.6.6) as a Fourier-Bessel series; the solution $U(r,s)$ automatically satisfies the boundary condition $U(a,s) = 0$. Substituting (9.6.12) into (9.6.11), we find that

$$U(r,s) = \frac{2}{a^2} \sum_{n=1}^{\infty} \frac{2N(s)J_0(k_n b/a) + a_k}{(s^2 - k_n^2/a^2)J_1^2(k_n)} J_0(k_n r/a), \qquad 0 \le r < a. \tag{9.6.13}$$

We have not yet determined a_k. Note, however, that in order for the inverse of (9.6.13) *not* to grow as $e^{k_n z/a}$, the numerator must vanish when $s = k_n/a$. Thus, $a_k = -2N(k_n/a)J_0(k_n b/a)$ and

$$U(r,s) = \frac{4}{a^2} \sum_{n=1}^{\infty} \frac{[N(s) - N(k_n/a)]J_0(k_n b/a)}{(s^2 - k_n^2/a^2)J_1^2(k_n)} J_0(k_n r/a), \quad 0 \le r < a. \tag{9.6.14}$$

The inverse of $U(r,s)$ then follows directly from simple inversions, the convolution theorem, and the definition of the Laplace transform. The final solution is

$$u(r,z) = \frac{2}{a}\sum_{n=1}^{\infty} \frac{J_0(k_n b/a)J_0(k_n r/a)}{k_n\, J_1^2(k_n)} \times \left[\int_0^z n(\tau)e^{k_n(z-\tau)/a}\,d\tau - \int_0^z n(\tau)e^{-k_n(z-\tau)/a}\,d\tau - \int_0^\infty n(\tau)e^{-k_n\tau/a}e^{k_n z/a}\,d\tau + \int_0^\infty n(\tau)e^{-k_n\tau/a}e^{-k_n z/a}\,d\tau\right] \tag{9.6.15}$$

$$= \frac{2}{a}\sum_{n=1}^{\infty} \frac{J_0(k_n b/a)J_0(k_n r/a)}{k_n\, J_1^2(k_n)} \times \left[\int_0^\infty n(\tau)e^{-k_n(z+\tau)/a}\,d\tau - \int_0^z n(\tau)e^{-k_n(z-\tau)/a}\,d\tau - \int_z^\infty n(\tau)e^{-k_n(\tau-z)/a}\,d\tau\right]. \tag{9.6.16}$$

Problems

1. Use Laplace transforms to solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < \infty, 0 < y < a$$

subject to the boundary conditions

$$u(0,y) = 1, \qquad \lim_{x\to\infty} |u(x,y)| < \infty, \qquad 0 < y < a$$

and

$$u(x,0) = u(x,a) = 0, \qquad 0 < x < \infty.$$

2. Use Laplace transforms to solve

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 \le r < a, 0 < z < \infty$$

subject to the boundary conditions

$$u(r,0) = 1, \qquad \lim_{z\to\infty} |u(r,z)| < \infty, \qquad 0 < r < a$$

and

$$\lim_{r \to 0} |u(r, z)| < \infty \quad \text{and} \quad u(a, z) = 0, \qquad 0 < z < \infty.$$

9.7 NUMERICAL SOLUTION OF LAPLACE'S EQUATION

As in the case of the heat and wave equations, numerical methods can be used to solve elliptic partial differential equations when analytic techniques fail or are too cumbersome. They are also employed when the domain differs from simple geometries.

The numerical analysis of an elliptic partial differential equation begins by replacing the continuous partial derivatives by finite-difference formulas. Employing centered differencing,

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{(\Delta x)^2} + O[(\Delta x)^2] \tag{9.7.1}$$

and

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{m,n+1} - 2u_{m,n} + u_{m,n-1}}{(\Delta y)^2} + O[(\Delta y)^2], \tag{9.7.2}$$

where $u_{m,n}$ denote the solution value at the grid point m, n. If $\Delta x = \Delta y$, Laplace's equation becomes the difference equation

$$u_{m+1,n} + u_{m-1,n} + u_{m,n+1} + u_{m,n-1} - 4u_{m,n} = 0. \tag{9.7.3}$$

Thus, we must now solve a set of simultaneous linear equations that yield the value of the solution at each grid point.

The solution of (9.7.3) is best done using techniques developed by algebraist. Later on, in Chapter 11, we will show that a very popular method for directly solving systems of linear equations is Gaussian elimination. However, for many grids at a reasonable resolution, the number of equations are generally in the tens of thousands. Because most of the coefficients in the equations are zero, Gaussian elimination is unsuitable, both from the point of view of computational expense and accuracy. For this reason alternative methods have been developed that generally use successive corrections or iterations. The most common of these point iterative methods are the Jacobi method, unextrapolated Liebmann or Gauss-Seidel method, and extrapolated Liebmann or successive over-relaxation (SOR). None of these approaches is completely satisfactory because of questions involving convergence and efficiency. Because of its simplicity we will focus on the Gauss-Seidel method.

We may illustrate the Gauss-Seidel method by considering the system:

$$10x + y + z = 39 \tag{9.7.4}$$

$$2x + 10y + z = 51 \tag{9.7.5}$$

$$2x + 2y + 10z = 64. \tag{9.7.6}$$

An important aspect of this system is the dominance of the coefficient of x in the first equation of the set and that the coefficients of y and z are dominant in the second and third equations, respectively.

The Gauss-Seidel method may be outlined as follow:

- Assign an initial value for each unknown variable. If possible, make a good first guess. If not, any arbitrarily selected values may be chosen. The initial value will not affect the convergence but will affect the number of iterations until convergence.

- Starting with (9.7.4), solve that equation for a new value of the unknown which has the largest coefficient in that equation, using the assumed values for the other unknowns.

- Go to (9.7.5) and employ the same technique used in the previous step to compute the unknown that has the largest coefficient in that equation. Where possible, use the latest values.

- Proceed to the remaining equations, always solving for the unknown having the largest coefficient in the particular equation and always using the *most recently* calculated values for the other unknowns in the equation. When the last equation (9.7.6) has been solved, you have completed a single iteration.

- Iterate until the value of each unknown does not change within a predetermined value.

Usually a compromise must be struck between the accuracy of the solution and the desired rate of convergence. The more accurate the solution is, the longer it will take for the solution to converge.

To illustrate this method, let us solve our system (9.7.4)–(9.7.6) with the initial guess $x = y = z = 0$. The first iteration yields $x = 3.9$, $y = 4.32$, and $z = 4.756$. The second iteration yields $x = 2.9924$, $y = 4.02592$, and $z = 4.996336$. As can be readily seen, the solution is converging to the correction solution of $x = 3$, $y = 4$, and $z = 5$.

Applying these techniques to (9.7.3),

$$u_{m,n}^{k+1} = \frac{1}{4}\left(u_{m+1,n}^{k} + u_{m-1,n}^{k+1} + u_{m,n+1}^{k} + u_{m,n-1}^{k+1}\right), \tag{9.7.7}$$

where we have assumed that the calculations occur in order of increasing m and n.

• **Example 9.7.1**

To illustrate the numerical solution of Laplace's equation, let us redo Example 9.3.1 with the boundary condition along $y = H$ simplified to $u(x, H) = 1 + x/L$.

We begin by finite-differencing the boundary conditions. The condition $u_x(0, y) = u_x(L, y) = 0$ leads to $u_{1,n} = u_{-1,n}$ and $u_{L+1,n} = u_{L-1,n}$ if we employ centered differences at $m = 0$ and $m = L$. Substituting these values in (9.7.7), we have the following equations for the left and right boundaries:

$$u_{0,n}^{k+1} = \tfrac{1}{4}\left(2u_{1,n}^{k} + u_{0,n+1}^{k} + u_{0,n-1}^{k+1}\right) \qquad \textbf{(9.7.8)}$$

and

$$u_{L,n}^{k+1} = \tfrac{1}{4}\left(2u_{L-1,n}^{k+1} + u_{L,n+1}^{k} + u_{L,n-1}^{k+1}\right). \qquad \textbf{(9.7.9)}$$

On the other hand, $u_y(x, 0) = 0$ yields $u_{m,1} = u_{m,-1}$ and

$$u_{m,0}^{k+1} = \tfrac{1}{4}\left(u_{m+1,0}^{k} + u_{m-1,0}^{k+1} + 2u_{m,1}^{k}\right). \qquad \textbf{(9.7.10)}$$

At the bottom corners, (9.7.8)–(9.7.10) simplify to

$$u_{0,0}^{k+1} = \tfrac{1}{2}\left(u_{1,0}^{k} + u_{0,1}^{k}\right) \qquad \textbf{(9.7.11)}$$

and

$$u_{L,0}^{k+1} = \tfrac{1}{2}\left(u_{L-1,0}^{k+1} + u_{L,1}^{k}\right). \qquad \textbf{(9.7.12)}$$

These equations along with (9.7.7) were solved using the Gauss-Seidel method. The initial guess everywhere except along the top boundary was zero. In Figure 9.7.1 we illustrate the numerical solution after 100 and 300 iterations where we have taken 101 grid points in the x and y directions.

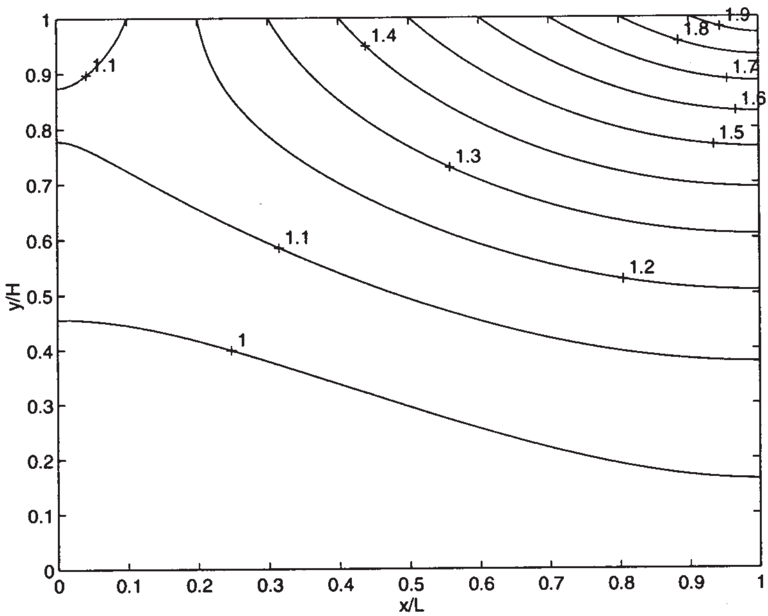
Project: Successive Over-Relaxation

The fundamental difficulty with relaxation methods used in solving Laplace's equation is the rate of convergence. Assuming $\Delta x = \Delta y$, the most popular method for accelerating convergence of these techniques is *successive over-relaxation*:

$$u_{m,n}^{k+1} = u_{m,n}^{k} + \omega R_{m,n},$$

where

$$R_{m,n} = \tfrac{1}{4}\left(u_{m+1,n}^{k} + u_{m-1,n}^{k+1} + u_{m,n+1}^{k} + u_{m,n-1}^{k+1}\right).$$



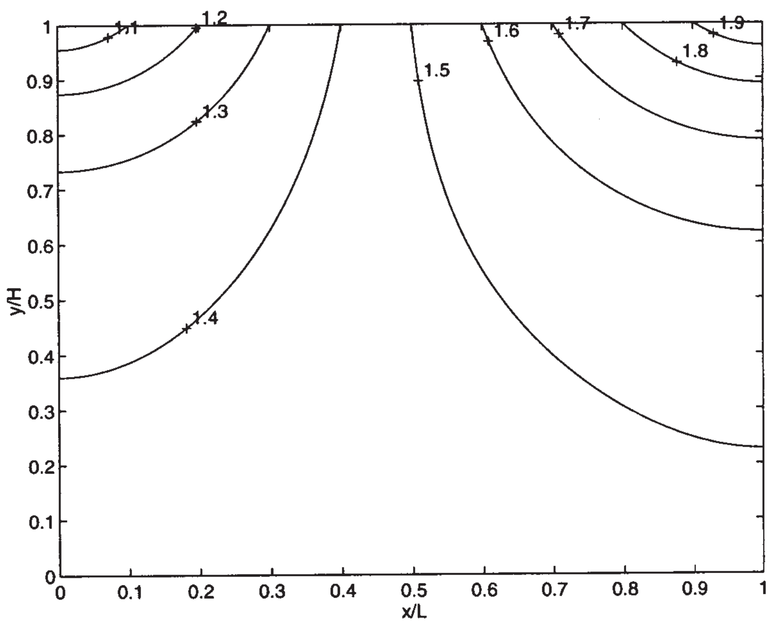


Figure 9.7.1: The solution to Laplace's equation by the Gauss-Seidel method after 100 (top) and 300 (bottom) iterations. The boundary conditions are $u_x(0, y) = u_x(L, y) = u_y(x, 0) = 0$ and $u(x, H) = 1+x/L$.

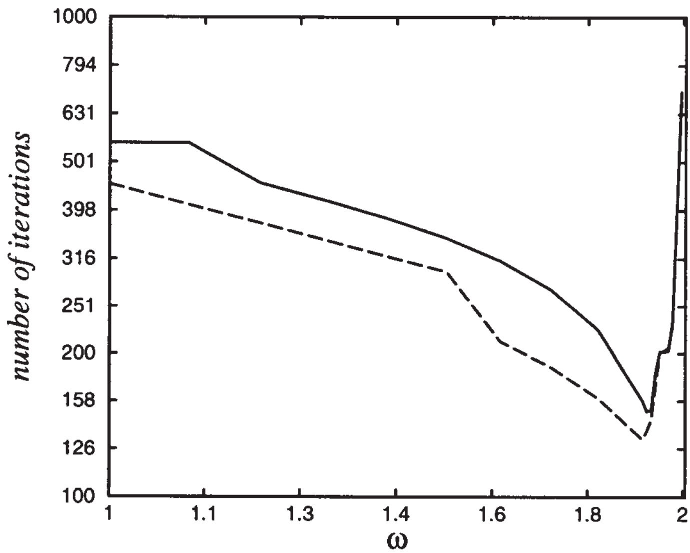


Figure 9.7.2: The number of iterations required so that $|R_{m,n}| \le 10^{-3}$ as a function of ω during the iterative solution of the problem posed in the project. We used $\Delta x = \Delta y = 0.01$ and $L = z_0 = 1$. The iteration count for the boundary conditions stated in Step 1 are given by the solid line while the iteration count for the boundary conditions given in Step 2 are shown by the dotted line. The initial guess equaled zero.

Most numerical methods dealing with partial differential equations will discuss the theoretical reasons behind this technique;[12] the optimum value always lies between one and two.

Step 1: Solve Laplace's equation numerically $0 \le x \le L$, $0 \le y \le z_0$ with the following boundary conditions:

$$u(x,0) = 0, u(x,z_0) = 1 + x/L, u(0,y) = y/z_0, \text{ and } u(L,y) = 2y/z_0.$$

Count the number of iterations until $|R_{m,n}| \le 10^{-3}$ for *all* m and n. Plot this number of iterations as a function of ω. How does the curve change with resolution Δx?

Step 2: Redo Step 1 with the exception of $u(0,y) = u(L,y) = 0$. How has the convergence rate changed? Can you explain why? How sensitive are your results to the first guess?

[12] For example, Young, D. M., 1971: *Iterative Solution of Large Linear Systems*, Academic Press, New York.

Chapter 10
Vector Calculus

Physicists invented vectors and vector operations to facilitate their mathematical expression of such diverse topics as mechanics and electromagnetism. In this chapter we focus on multivariable differentiations and integrations of vector fields, such as the velocity of a fluid, where the vector field is solely a function of its position.

10.1 REVIEW

The physical sciences and engineering abound with vectors and scalars. *Scalars* are physical quantities which only possess magnitude. Examples include mass, temperature, density, and pressure. *Vectors* are physical quantities that possess both magnitude and direction. Examples include velocity, acceleration, and force. We shall denote vectors by boldface letters.

Two vectors are equal if they have the same magnitude and direction. From the limitless number of possible vectors, two special cases are the *zero vector* $\mathbf{0}$ which has no magnitude and unspecified direction and the *unit vector* which has unit magnitude.

The most convenient method for expressing a vector analytically is in terms of its components. A vector $\mathbf{a}$ in three-dimensional real space is any order triplet of real numbers (*components*) a_1, a_2, and a_3 such that $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, where $a_1\mathbf{i}$, $a_2\mathbf{j}$, and $a_3\mathbf{k}$ are vectors which lie along the coordinate axes and have their origin at a common initial point. The *magnitude*, *length*, or *norm* of a vector $\mathbf{a}$, $|\mathbf{a}|$, equals $\sqrt{a_1^2 + a_2^2 + a_3^2}$. A particularly important vector is the *position vector*, defined by $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

As in the case of scalars, certain arithmetic rules hold. Addition and subtraction are very similar to their scalar counterparts:

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k} \tag{10.1.1}$$

and

$$\mathbf{a} - \mathbf{b} = (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j} + (a_3 - b_3)\mathbf{k}. \tag{10.1.2}$$

In contrast to its scalar counterpart, there are two types of multiplication. The *dot product* is defined as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta) = a_1b_1 + a_2b_2 + a_3b_3, \tag{10.1.3}$$

where θ is the angle between the vector such that $0 \leq \theta \leq \pi$. The dot product yields a scalar answer. A particularly important case is $\mathbf{a} \cdot \mathbf{b} = 0$ with $|\mathbf{a}| \neq 0$ and $|\mathbf{b}| \neq 0$. In this case the vectors are orthogonal (perpendicular) to each other.

The other form of multiplication is the *cross product* which is defined by $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin(\theta)\mathbf{n}$, where θ is the angle between the vectors such that $0 \leq \theta \leq \pi$ and $\mathbf{n}$ is a unit vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ with the direction given by the right-hand rule. A convenient method for computing the cross product from the scalar components of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}. \tag{10.1.4}$$

Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are *parallel* if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Most of the vectors that we will use are vector-valued functions. These functions are vectors that vary either with a single parametric variable t or multiple variables, say x, y, and z.

The most commonly encountered example of a vector-valued function which varies with a single independent variable involves the trajectory of particles. If a *space curve* is parameterized by the equations $x = f(t)$, $y = g(t)$, and $z = h(t)$ with $a \leq t \leq b$, the position vector $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ gives the location of a point P as it moves from its initial position to its final position. Furthermore, because the increment quotient $\Delta\mathbf{r}/\Delta t$ is in the direction of a secant line, then the limit of this quotient as $\Delta t \to 0$, $\mathbf{r}'(t)$, gives the tangent to the curve at P.

• **Example 10.1.1: Foucault pendulum**

One of the great experiments of mid-nineteenth century physics was the demonstration by J. B. L. Foucault (1819–1868) in 1851 of the earth's rotation by designing a (spherical) pendulum, supported by a long wire, that essentially swings in an nonaccelerating coordinate system. This problem demonstrates many of the fundamental concepts of vector calculus.

The total force[1] acting on the bob of the pendulum is $\mathbf{F} = \mathbf{T} + m\mathbf{G}$, where $\mathbf{T}$ is the tension in the pendulum and $\mathbf{G}$ is the gravitational attraction per unit mass. Using Newton's second law,

$$\left.\frac{d^2\mathbf{r}}{dt^2}\right|_{\text{inertial}} = \frac{\mathbf{T}}{m} + \mathbf{G}, \tag{10.1.5}$$

where $\mathbf{r}$ is the position vector from a fixed point in an inertial coordinate system to the bob. This system is inconvenient because we live in a rotating coordinate system. Employing the conventional geographic coordinate system,[2] (10.1.5) becomes

$$\frac{d^2\mathbf{r}}{dt^2} + 2\mathbf{\Omega} \times \frac{d\mathbf{r}}{dt} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = \frac{\mathbf{T}}{m} + \mathbf{G}, \tag{10.1.6}$$

where $\mathbf{\Omega}$ is the angular rotation vector of the earth and $\mathbf{r}$ now denotes a position vector in the rotating reference system with its origin at the center of the earth and terminal point at the bob. If we define the gravity vector $\mathbf{g} = \mathbf{G} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$, then the dynamical equation is

$$\frac{d^2\mathbf{r}}{dt^2} + 2\mathbf{\Omega} \times \frac{d\mathbf{r}}{dt} = \frac{\mathbf{T}}{m} + \mathbf{g}, \tag{10.1.7}$$

where the second term on the left side of (10.1.7) is called the *Coriolis force*.

Because the equation is *linear*, let us break the position vector $\mathbf{r}$ into two separate vectors: $\mathbf{r}_0$ and $\mathbf{r}_1$, where $\mathbf{r} = \mathbf{r}_0 + \mathbf{r}_1$. The vector $\mathbf{r}_0$ extends from the center of the earth to the pendulum's point of support and $\mathbf{r}_1$ extends from the support point to the bob. Because $\mathbf{r}_0$ is a constant in the geographic system,

$$\frac{d^2\mathbf{r}_1}{dt^2} + 2\mathbf{\Omega} \times \frac{d\mathbf{r}_1}{dt} = \frac{\mathbf{T}}{m} + \mathbf{g}. \tag{10.1.8}$$

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[1] From Broxmeyer, C., 1960: Foucault pendulum effect in a Schuler-tuned system. *J. Aerosp. Sci.*, **27**, 343–347 with permission.

[2] For the derivation, see Marion, J. B., 1965: *Classical Dynamics of Particles and Systems*, Academic Press, New York, Sections 12.2–12.3.

If the length of the pendulum is L, then for small oscillations $\mathbf{r}_1 \approx x\mathbf{i} + y\mathbf{j} + L\mathbf{k}$ and the equations of motion are

$$\frac{d^2x}{dt^2} + 2\Omega\sin(\lambda)\frac{dy}{dt} = \frac{T_x}{m}, \tag{10.1.9}$$

$$\frac{d^2y}{dt^2} - 2\Omega\sin(\lambda)\frac{dx}{dt} = \frac{T_y}{m} \tag{10.1.10}$$

and

$$2\Omega\cos(\lambda)\frac{dy}{dt} - g = \frac{T_z}{m}, \tag{10.1.11}$$

where λ denotes the latitude of the point and Ω is the rotation rate of the earth. The relationships between the components of tension are $T_x = xT_z/L$ and $T_y = yT_z/L$. From (10.1.11),

$$\frac{T_z}{m} + g = 2\Omega\cos(\lambda)\frac{dy}{dt} \approx 0. \tag{10.1.12}$$

Substituting the definitions of T_x, T_y and (10.1.12) into (10.1.9) and (10.1.10),

$$\frac{d^2x}{dt^2} + \frac{g}{L}x + 2\Omega\sin(\lambda)\frac{dy}{dt} = 0 \tag{10.1.13}$$

and

$$\frac{d^2y}{dt^2} + \frac{g}{L}y - 2\Omega\sin(\lambda)\frac{dx}{dt} = 0. \tag{10.1.14}$$

The approximate solution to these coupled differential equations is

$$x(t) = A_0\cos[\Omega\sin(\lambda)t]\sin\left(\sqrt{g/L}\,t\right) \tag{10.1.15}$$

and

$$y(t) = A_0\sin[\Omega\sin(\lambda)t]\sin\left(\sqrt{g/L}\,t\right) \tag{10.1.16}$$

if $\Omega^2 \ll g/L$. Thus, we have a pendulum that swings with an angular frequency $\sqrt{g/L}$. However, depending upon the *latitude* λ, the direction in which the pendulum swings changes counterclockwise with time, completing a full cycle in $2\pi/[\Omega\sin(\lambda)]$. This result is most clearly seen when $\lambda = \pi/2$ and we are at the North Pole. There the earth is turning underneath the pendulum. If initially we set the pendulum swinging along the 0° longitude, the pendulum will shift with time to longitudes east of the Greenwich median. Eventually, after 24 hours, the process will repeat itself.

Consider now vector-valued functions that vary with several variables. A *vector function of position* assigns a vector value for every value

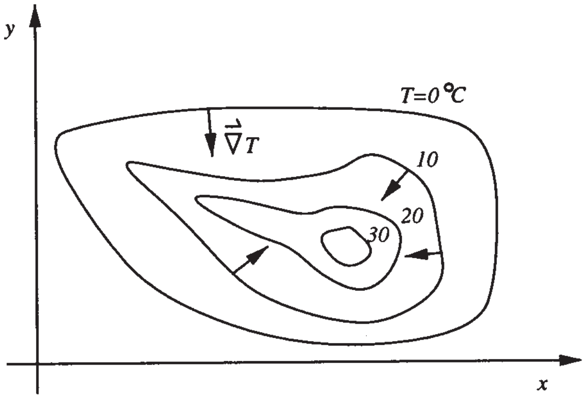

Figure 10.1.1: A graphical example of the gradient: A vector that is perpendicular to the isotherms $T(x,y)$ = constant and points in the direction of most rapidly increasing temperatures.

of x, y, and z within some domain. Examples include the velocity field of a fluid at a given instant:

$$\mathbf{v} = u(x,y,z)\mathbf{i} + v(x,y,z)\mathbf{j} + w(x,y,z)\mathbf{k}. \tag{10.1.17}$$

Another example arises in electromagnetism where electric and magnetic fields often vary as a function of the space coordinates. For us, however, probably the most useful example involves the vector differential operator, *del* or *nabla*,

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \tag{10.1.18}$$

which we apply to the multivariable differentiable scalar function $F(x,y,z)$ to give the *gradient* ∇F.

An important geometric interpretation of the gradient – one which we shall use frequently – is the fact that ∇f is perpendicular (normal) to the level surface at a given point P. To prove this, let the equation $F(x,y,z) = c$ describe a three-dimensional surface. If the differentiable functions $x = f(t)$, $y = g(t)$, and $z = h(t)$ are the parametric equations of a curve on the surface, then the derivative of $F[f(t), g(t), h(t)] = c$ is

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0 \tag{10.1.19}$$

or

$$\nabla F \cdot \mathbf{r}' = 0. \tag{10.1.20}$$

When $\mathbf{r}' \neq \mathbf{0}$, the vector ∇F is orthogonal to the tangent vector. Because our argument holds for any differentiable curve that passes through the arbitrary point (x,y,z), then ∇F is normal to the level surface at that point.

Figure 10.1.1 gives a common application of the gradient. Consider a two-dimensional temperature field $T(x, y)$. The level curves $T(x, y) =$ constant are lines that connect points where the temperature is the same (isotherms). The gradient in this case ∇T is a vector that is perpendicular or normal to these isotherms and points in the direction of most rapidly increasing temperature.

• **Example 10.1.2**

Let us find the gradient of the function $f(x, y, z) = x^2z^2\sin(4y)$. Using the definition of gradient,

$$\nabla f = \frac{\partial[x^2z^2\sin(4y)]}{\partial x}\mathbf{i} + \frac{\partial[x^2z^2\sin(4y)]}{\partial y}\mathbf{j} + \frac{\partial[x^2z^2\sin(4y)]}{\partial z}\mathbf{k} \quad (10.1.21)$$

$$= 2xz^2\sin(4y)\mathbf{i} + 4x^2z^2\cos(4y)\mathbf{j} + 2x^2z\sin(4y)\mathbf{k}. \quad (10.1.22)$$

• **Example 10.1.3**

Let us find the unit normal to the unit sphere at any arbitrary point (x, y, z).

The surface of a unit sphere is defined by the equation $f(x, y, z) = x^2 + y^2 + z^2 = 1$. Therefore, the normal is given by the gradient

$$\mathbf{N} = \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \quad (10.1.23)$$

and the unit normal

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \quad (10.1.24)$$

because $x^2 + y^2 + z^2 = 1$.

A popular method for visualizing a vector field $\mathbf{F}$ is to draw space curves which are tangent to the vector field at each x, y, z. In fluid mechanics these lines are called *streamlines* while in physics they are generally called *lines of force* or *flux lines* for an electric, magnetic, or gravitational field. For a fluid with a velocity field that does not vary with time, the streamlines give the paths along which small parcels of the fluid move.

To find the streamlines of a given vector field $\mathbf{F}$ with components $P(x, y, z)$, $Q(x, y, z)$, and $R(x, y, z)$, we assume that we can parameterize the streamlines in the form $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then the tangent line is $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$. Because the streamline must be parallel to the vector field at any t, $\mathbf{r}'(t) = \lambda\mathbf{F}$ or

$$\frac{dx}{dt} = \lambda P(x, y, z), \frac{dy}{dt} = \lambda Q(x, y, z) \quad \text{and} \quad \frac{dz}{dt} = \lambda R(x, y, z) \quad (10.1.25)$$

or

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}. \tag{10.1.26}$$

The solution of this system of differential equations yields the streamlines.

• Example 10.1.4

Let us find the streamlines for the vector field $\mathbf{F} = \sec(x)\mathbf{i} - \cot(y)\mathbf{j} + \mathbf{k}$ that passes through the point $(\pi/4, \pi, 1)$. In this particular example, $\mathbf{F}$ represents a measured or computed fluid's velocity at a particular instant.

From (10.1.26),

$$\frac{dx}{\sec(x)} = -\frac{dy}{\cot(y)} = \frac{dz}{1}. \tag{10.1.27}$$

This yields two differential equations:

$$\cos(x)\,dx = -\frac{\sin(y)}{\cos(y)}\,dy \quad \text{and} \quad dz = -\frac{\sin(y)}{\cos(y)}\,dy. \tag{10.1.28}$$

Integrating these equations yields

$$\sin(x) = \ln|\cos(y)| + c_1 \quad \text{and} \quad z = \ln|\cos(y)| + c_2. \tag{10.1.29}$$

Substituting for the given point, we finally have that

$$\sin(x) = \ln|\cos(y)| + \sqrt{2}/2 \quad \text{and} \quad z = \ln|\cos(y)| + 1. \tag{10.1.30}$$

• Example 10.1.5

Let us find the streamlines for the vector field $\mathbf{F} = \sin(z)\mathbf{j} + e^y\mathbf{k}$ that passes through the point $(2, 0, 0)$.

From (10.1.26),

$$\frac{dx}{0} = \frac{dy}{\sin(z)} = \frac{dz}{e^y}. \tag{10.1.31}$$

This yields two differential equations:

$$dx = 0 \qquad \text{and} \qquad \sin(z)\,dz = e^y\,dy. \tag{10.1.32}$$

Integrating these equations gives

$$x = c_1 \qquad \text{and} \qquad e^y = -\cos(z) + c_2. \tag{10.1.33}$$

Substituting for the given point, we finally have that

$$x = 2 \qquad \text{and} \qquad e^y = 2 - \cos(z). \tag{10.1.34}$$

Note that (10.1.34) only applies for a certain strip in the yz-plane.

Problems

Given the following vectors $\mathbf{a}$ and $\mathbf{b}$, verify that $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ and $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$:

1. $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}, \quad \mathbf{b} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$

2. $\mathbf{a} = \mathbf{i} - 3\mathbf{j} + \mathbf{k}, \quad \mathbf{b} = 2\mathbf{i} + 4\mathbf{k}$

3. $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \quad \mathbf{b} = -5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

4. $\mathbf{a} = 8\mathbf{i} + \mathbf{j} - 6\mathbf{k}, \quad \mathbf{b} = \mathbf{i} - 2\mathbf{j} + 10\mathbf{k}$

5. $\mathbf{a} = 2\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}, \quad \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.

6. Prove $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

7. Prove $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$.

Find the gradient of the following functions:

8. $f(x, y, z) = xy^2/z^3$

9. $f(x, y, z) = xy\cos(yz)$

10. $f(x, y, z) = \ln(x^2 + y^2 + z^2)$

11. $f(x, y, z) = x^2y^2(2z + 1)^2$

12. $f(x, y, z) = 2x - y^2 + z^2$.

Sketch the following surfaces. For each of these surfaces, find a mathematical expression for the unit normal and then sketch it.

13. $z = 3$

14. $x^2 + y^2 = 4$

15. $z = x^2 + y^2$

16. $z = \sqrt{x^2 + y^2}$

17. $z = y$

18. $x + y + z = 1$

19. $z = x^2$.

Find the streamlines for the following vector fields that pass through the specified point:

20. $\mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k}$; $(0, 1, 1)$

21. $\mathbf{F} = 2\mathbf{i} - y^2\mathbf{j} + z\mathbf{k}$; $(1, 1, 1)$

22. $\mathbf{F} = 3x^2\mathbf{i} - y^2\mathbf{j} + z^2\mathbf{k}$; $(2, 1, 3)$

23. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} - z^3\mathbf{k}$; $(1, 1, 1)$

24. $\mathbf{F} = (1/x)\mathbf{i} + e^y\mathbf{j} - \mathbf{k}$; $(2, 0, 4)$

25. Solve the differential equations (10.1.13)–(10.1.14) with the initial conditions $x(0) = y(0) = y'(0) = 0$ and $x'(0) = A_0\sqrt{g/L}$ assuming that $\Omega^2 \ll g/L$.

26. If a fluid is bounded by a fixed surface $f(x, y, z) = c$, show that the fluid must satisfy the boundary condition $\mathbf{v} \cdot \nabla f = 0$, where $\mathbf{v}$ is the velocity of the fluid.

27. A sphere of radius a is moving in a fluid with the constant velocity $\mathbf{u}$. Show that the fluid satisfies the boundary condition $(\mathbf{v} - \mathbf{u}) \cdot (\mathbf{r} - \mathbf{u}t) = 0$ at the surface of the sphere, if the center of the sphere coincides with the origin at $t = 0$ and $\mathbf{v}$ denotes the velocity of the fluid.

10.2 DIVERGENCE AND CURL

Consider a vector field $\mathbf{v}$ defined in some region of three-dimensional space. The function $\mathbf{v}(\mathbf{r})$ can be resolved into components along the $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ directions or

$$\mathbf{v}(\mathbf{r}) = u(x, y, z)\mathbf{i} + v(x, y, z)\mathbf{j} + w(x, y, z)\mathbf{k}. \tag{10.2.1}$$

If $\mathbf{v}$ is a fluid's velocity field, then we can compute the flow rate through a small (differential) rectangular box defined by increments $(\Delta x, \Delta y, \Delta z)$ centered at the point (x, y, z). See Figure 10.2.1. The flow out from the box through the face with the outwardly pointing normal $\mathbf{n} = -\mathbf{j}$ is

$$\mathbf{v} \cdot (-\mathbf{j}) = -v(x, y - \Delta y/2, z)\Delta x \Delta z \tag{10.2.2}$$

and the flow through the face with the outwardly pointing normal $\mathbf{n} = \mathbf{j}$ is

$$\mathbf{v} \cdot \mathbf{j} = v(x, y + \Delta y/2, z)\Delta x \Delta z. \tag{10.2.3}$$

The net flow through the two faces is

$$[v(x, y + \Delta y/2, z) - v(x, y - \Delta y/2, z)]\Delta x \Delta z \approx v_y(x, y, z)\Delta x \Delta y \Delta z. \tag{10.2.4}$$

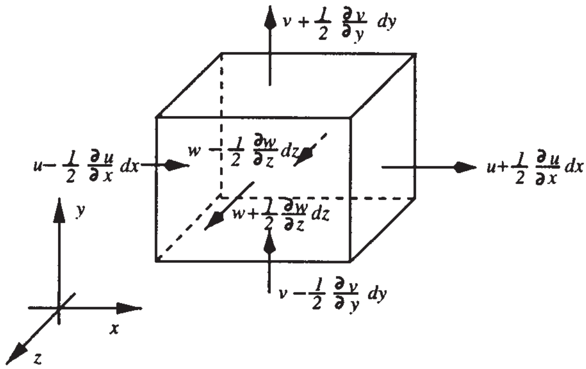

Figure 10.2.1: Divergence of a vector function $\mathbf{v}(x, y, z)$.

A similar analysis of the other faces and combination of the results give the approximate total flow from the box as

$$[u_x(x, y, z) + v_y(x, y, z) + w_z(x, y, z)]\Delta x \Delta y \Delta z. \tag{10.2.5}$$

Dividing by the volume $\Delta x\Delta y\Delta z$ and taking the limit as the dimensions of the box tend to zero yield $u_x + v_y + w_z$ as the flow out from (x, y, z) per unit volume per unit time. This scalar quantity is called the *divergence* of the vector $\mathbf{v}$:

$$\operatorname{div}(\mathbf{v}) = \nabla \cdot \mathbf{v} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) = u_x + v_y + w_z. \tag{10.2.6}$$

Thus, if the divergence is positive, either the fluid is expanding and its density at the point is falling with time, or the point is a *source* at which fluid is entering the field. When the divergence is negative, either the fluid is contracting and its density is rising at the point, or the point is a negative source or *sink* at which fluid is leaving the field.

If the divergence of a vector field is zero everywhere within a domain, then the flux entering any element of space exactly equals that leaving it and the vector field is called *nondivergent* or *solenoidal* (from a Greek word meaning a tube). For a fluid, if there are no sources or sinks, then its density cannot change.

Some useful properties of the divergence operator are

$$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}, \tag{10.2.7}$$

$$\nabla \cdot (\varphi \mathbf{F}) = \varphi \nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla \varphi \tag{10.2.8}$$

and

$$\nabla^2 \varphi = \nabla \cdot \nabla \varphi = \varphi_{xx} + \varphi_{yy} + \varphi_{zz}. \tag{10.2.9}$$

The expression (10.2.9) is very important in physics and is given the special name of the *Laplacian*.[3]

[3] Some mathematicians write Δ instead of ∇^2.

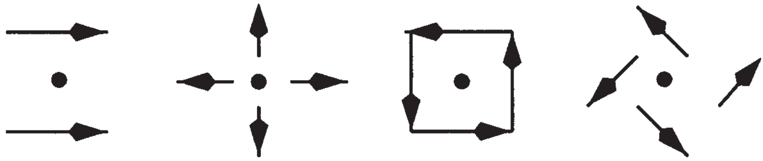

Figure 10.2.2: Examples of vector fields with and without divergence and curl.

• **Example 10.2.1**

If $\mathbf{F} = x^2z\mathbf{i} - 2y^3z^2\mathbf{j} + xy^2z\mathbf{k}$, compute the divergence of $\mathbf{F}$.

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(-2y^3z^2) + \frac{\partial}{\partial z}(xy^2z) \quad (10.2.10)$$

$$= 2xz - 6y^2z^2 + xy^2. \quad (10.2.11)$$

• **Example 10.2.2**

If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, show that $\mathbf{r}/|\mathbf{r}|^3$ is nondivergent.

$$\nabla \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|^3}\right) = \frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{\partial}{\partial y}\left[\frac{y}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{\partial}{\partial z}\left[\frac{z}{(x^2+y^2+z^2)^{3/2}}\right] \quad (10.2.12)$$

$$= \frac{3}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2+3y^2+3z^2}{(x^2+y^2+z^2)^{5/2}} = 0. \quad (10.2.13)$$

Another important vector function involving the vector field $\mathbf{v}$ is the curl of $\mathbf{v}$, written curl($\mathbf{v}$) or rot($\mathbf{v}$) in some older textbooks. In fluid flow problems it is proportional to the instantaneous angular velocity of a fluid element. In rectangular coordinates,

$$\text{curl}(\mathbf{v}) = \nabla \times \mathbf{v} = (w_y - v_z)\mathbf{i} + (u_z - w_x)\mathbf{j} + (v_x - u_y)\mathbf{k}, \quad (10.2.14)$$

where $\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ as before. However, it is best remembered in the mnemonic form:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = (w_y - v_z)\mathbf{i} + (u_z - w_x)\mathbf{j} + (v_x - u_y)\mathbf{k}. \quad (10.2.15)$$

If the curl of a vector field is zero everywhere within a region, then the field is *irrotational*.

Figure 10.2.2 illustrates graphically some vector fields that do and do not possess divergence and curl. Let the vectors that are illustrated represent the motion of fluid particles. In the case of divergence only, fluid is streaming from the point, at which the density is falling. Alternatively the point could be a source. In the case where there is only curl, the fluid rotates about the point and the fluid is incompressible. Finally, the point that possesses both divergence and curl is a compressible fluid with rotation.

Some useful computational formulas exist for both the divergence and curl operations:

$$\nabla \times (\mathbf{F}+\mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}, \tag{10.2.16}$$

$$\nabla \times \nabla\varphi = \mathbf{0}, \tag{10.2.17}$$

$$\nabla \cdot \nabla \times \mathbf{F} = 0, \tag{10.2.18}$$

$$\nabla \times (\varphi\mathbf{F}) = \varphi\nabla \times \mathbf{F} + \nabla\varphi \times \mathbf{F}, \tag{10.2.19}$$

$$\nabla(\mathbf{F}\cdot\mathbf{G}) = (\mathbf{F}\cdot\nabla)\mathbf{G}+(\mathbf{G}\cdot\nabla)\mathbf{F}+\mathbf{F}\times(\nabla\times\mathbf{G})+\mathbf{G}\times(\nabla\times\mathbf{F}), \tag{10.2.20}$$

$$\nabla \times (\mathbf{F}\times\mathbf{G}) = (\mathbf{G}\cdot\nabla)\mathbf{F}-(\mathbf{F}\cdot\nabla)\mathbf{G}+\mathbf{F}(\nabla\cdot\mathbf{G})-\mathbf{G}(\nabla\cdot\mathbf{F}), \tag{10.2.21}$$

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - (\nabla\cdot\nabla)\mathbf{F} \tag{10.2.22}$$

and

$$\nabla \cdot (\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot\nabla\times\mathbf{F} - \mathbf{F}\cdot\nabla\times\mathbf{G}. \tag{10.2.23}$$

In this book the operation $\nabla\mathbf{F}$ is undefined.

• Example 10.2.3

If $\mathbf{F} = xz^3\mathbf{i} - 2x^2yz\mathbf{j} + 2yz^4\mathbf{k}$, compute the curl of $\mathbf{F}$ and verify that $\nabla \cdot \nabla \times \mathbf{F} = 0$.

From the definition of curl,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix} \tag{10.2.24}$$

$$= \left[\tfrac{\partial}{\partial y}\left(2yz^4\right) - \tfrac{\partial}{\partial z}\left(-2x^2yz\right)\right]\mathbf{i} - \left[\tfrac{\partial}{\partial x}\left(2yz^4\right) - \tfrac{\partial}{\partial z}\left(xz^3\right)\right]\mathbf{j}$$

$$+ \left[\tfrac{\partial}{\partial x}\left(-2x^2yz\right) - \tfrac{\partial}{\partial y}\left(xz^3\right)\right]\mathbf{k} \tag{10.2.25}$$

$$= (2z^4 + 2x^2y)\mathbf{i} - (0 - 3xz^2)\mathbf{j} + (-4xyz - 0)\mathbf{k} \tag{10.2.26}$$

$$= (2z^4 + 2x^2y)\mathbf{i} + 3xz^2\mathbf{j} - 4xyz\mathbf{k}. \tag{10.2.27}$$

From the definition of divergence and (10.2.27),

$$\nabla\cdot\nabla\times\mathbf{F} = \frac{\partial}{\partial x}(2z^4+2x^2y)+\frac{\partial}{\partial y}(3xz^2)+\frac{\partial}{\partial z}(-4xyz) = 4xy+0-4xy = 0. \tag{10.2.28}$$

• **Example 10.2.4: Potential flow theory**

One of the topics in most elementary fluid mechanics courses is the study of irrotational and nondivergent fluid flows. Because the fluid is irrotational, the velocity vector field $\mathbf{v}$ satisfies $\nabla \times \mathbf{v} = \mathbf{0}$. From (10.2.17) we can introduce a potential φ such that $\mathbf{v} = \nabla\varphi$. Because the flow field is nondivergent, $\nabla \cdot \mathbf{v} = \nabla^2\varphi = 0$. Thus, the fluid flow can be completely described in terms of solutions to Laplace's equation. This area of fluid mechanics is called *potential flow theory.*

Problems

Compute $\nabla \cdot \mathbf{F}, \nabla \times \mathbf{F}, \nabla \cdot (\nabla \times \mathbf{F})$ and $\nabla(\nabla \cdot \mathbf{F})$ for the following vector fields:

1. $\mathbf{F} = x^2z\mathbf{i} + yz^2\mathbf{j} + xy^2\mathbf{k}$

2. $\mathbf{F} = 4x^2y^2\mathbf{i} + (2x + 2yz)\mathbf{j} + (3z + y^2)\mathbf{k}$

3. $\mathbf{F} = (x - y)^2\mathbf{i} + e^{-xy}\mathbf{j} + xze^{2y}\mathbf{k}$

4. $\mathbf{F} = 3xy\mathbf{i} + 2xz^2\mathbf{j} + y^3\mathbf{k}$

5. $\mathbf{F} = 5yz\mathbf{i} + x^2z\mathbf{j} + 3x^3\mathbf{k}$

6. $\mathbf{F} = y^3\mathbf{i} + (x^3y^2 - xy)\mathbf{j} - (x^3yz - xz)\mathbf{k}$

7. $\mathbf{F} = xe^{-y}\mathbf{i} + yz^2\mathbf{j} + 3e^{-z}\mathbf{k}$

8. $\mathbf{F} = y\ln(x)\mathbf{i} + (2 - 3yz)\mathbf{j} + xyz^3\mathbf{k}$

9. $\mathbf{F} = xyz\mathbf{i} + x^3yze^z\mathbf{j} + xye^z\mathbf{k}$

10. $\mathbf{F} = (xy^3 - z^4)\mathbf{i} + 4x^4y^2z\mathbf{j} - y^4z^5\mathbf{k}$.

11. $\mathbf{F} = xy^2\mathbf{i} + xyz^2\mathbf{j} + xy\cos(z)\mathbf{k}$

12. $\mathbf{F} = xy^2\mathbf{i} + xyz^2\mathbf{j} + xy\sin(z)\mathbf{k}$

13. $\mathbf{F} = xy^2\mathbf{i} + xyz\mathbf{j} + xy\cos(z)\mathbf{k}$.

14. (a) Assuming continuity of all partial derivatives, show that

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}.$$

(b) Using $\mathbf{F} = 3xy\mathbf{i} + 4yz\mathbf{j} + 2xz\mathbf{k}$, verify the results in part (a).

15. If $\mathbf{E} = \mathbf{E}(x, y, z, t)$ and $\mathbf{B} = \mathbf{B}(x, y, z, t)$ represent the electric and magnetic fields in a vacuum, Maxwell's field equations are

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t},$$

where c is the speed of light. Using the results from Problem 14, show that $\mathbf{E}$ and $\mathbf{B}$ satisfy

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \quad \text{and} \quad \nabla^2 \mathbf{B} = \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

16. If f and g are continuously differentiable scalar fields, show that $\nabla f \times \nabla g$ is solenoidal. Hint: Show that $\nabla f \times \nabla g = \nabla \times (f \nabla g)$.

17. An inviscid (frictionless) fluid in equilibrium obeys the relationship $\nabla p = \rho \mathbf{F}$, where ρ denotes the density of the fluid, p denotes the pressure, and $\mathbf{F}$ denotes the body forces (such as gravity). Show that $\mathbf{F} \cdot \nabla \times \mathbf{F} = 0$.

10.3 LINE INTEGRALS

Line integrals are ubiquitous in physics. In mechanics they are used to compute work. In electricity and magnetism, they provide simple methods for computing the electric and magnetic fields for simple geometries.

The line integral most frequently encountered is an *oriented* one in which the path C is directed and the integrand is the dot product between the vector function $\mathbf{F}(\mathbf{r})$ and the tangent of the path $d\mathbf{r}$. It is usually written in the economical form

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz, \qquad \mathbf{(10.3.1)}$$

where $\mathbf{F} = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$. If the starting and terminal points are the same so that the contour is closed, then this *closed contour integral* will be denoted by $\oint_C$. In the following examples we show how to evaluate the line integrals along various types of curves.

• Example 10.3.1

If $\mathbf{F} = (3x^2 + 6y)\mathbf{i} - 14yz\mathbf{j} + 20xz^2\mathbf{k}$, let us evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the parametric curves $x(t) = t$, $y(t) = t^2$, and $z(t) = t^3$ from the point $(0, 0, 0)$ to $(1, 1, 1)$. See Figure 10.3.1.

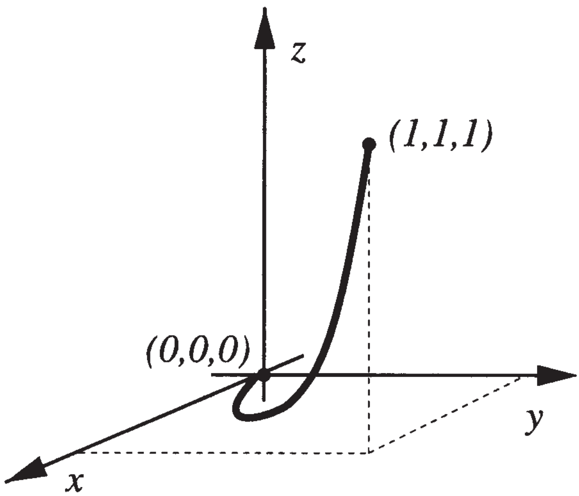

Figure 10.3.1: Diagram for the line integration in Example 10.3.1.

We begin by finding the values of t which give the corresponding end points. A quick check shows that $t = 0$ gives $(0, 0, 0)$ while $t = 1$ yields $(1, 1, 1)$. It should be noted that the same value of t must give the correct coordinates in each direction. Failure to do so suggests an error in the parameterization. Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (3t^2 + 6t^2)\, dt - 14t^2(t^3)\, d(t^2) + 20t(t^3)^2 d(t^3) \quad \textbf{(10.3.2)}$$

$$= \int_0^1 9t^2\, dt - 28t^6\, dt + 60t^9\, dt \quad \textbf{(10.3.3)}$$

$$= \left(3t^3 - 4t^7 + 6t^{10}\right)\Big|_0^1 = 5. \quad \textbf{(10.3.4)}$$

• **Example 10.3.2**

Let us redo the previous example with a contour that consists of three "dog legs", namely straight lines from $(0, 0, 0)$ to $(1, 0, 0)$, from $(1, 0, 0)$ to $(1, 1, 0)$, and from $(1, 1, 0)$ to $(1, 1, 1)$. See Figure 10.3.2.

In this particular problem we break the integration down into three distinct integrals:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r}. \quad \textbf{(10.3.5)}$$

For C_1, $y = z = dy = dz = 0$ and

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (3x^2 + 6 \cdot 0)\, dx - 14 \cdot 0 \cdot 0 \cdot 0 + 20x \cdot 0^2 \cdot 0 = \int_0^1 3x^2\, dx = 1. \quad \textbf{(10.3.6)}$$

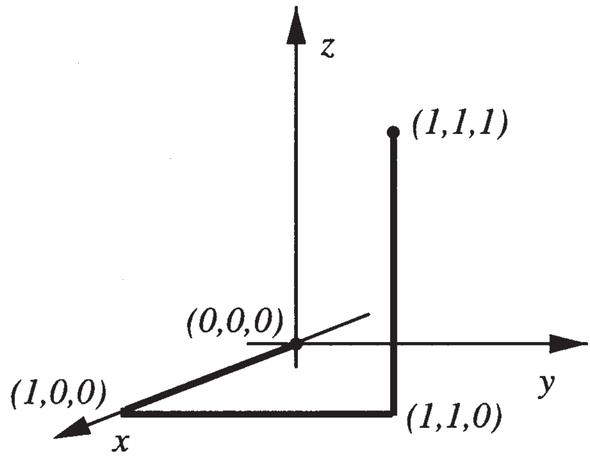

Figure 10.3.2: Diagram for the line integration in Example 10.3.2.

For C_2, $x = 1$ and $z = dx = dz = 0$ so that

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (3 \cdot 1^2 + 6y) \cdot 0 - 14y \cdot 0 \cdot dy + 20 \cdot 1 \cdot 0^2 \cdot 0 = 0. \quad \textbf{(10.3.7)}$$

For C_3, $x = y = 1$ and $dx = dy = 0$ so that

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (3 \cdot 1^2 + 6 \cdot 1) \cdot 0 - 14 \cdot 1 \cdot z \cdot 0 + 20 \cdot 1 \cdot z^2 \, dz = \int_0^1 20z^2 \, dz = \tfrac{20}{3}. \quad \textbf{(10.3.8)}$$

Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \tfrac{23}{3}. \quad \textbf{(10.3.9)}$$

• **Example 10.3.3**

For our third calculation, we redo the first example where the contour is a straight line. The parameterization in this case is $x = y = z = t$ with $0 \le t \le 1$. See Figure 10.3.3. Then,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (3t^2 + 6t) \, dt - 14(t)(t) \, dt + 20t(t)^2 \, dt \quad \textbf{(10.3.10)}$$

$$= \int_0^1 (3t^2 + 6t - 14t^2 + 20t^3) \, dt = \tfrac{13}{3}. \quad \textbf{(10.3.11)}$$

An interesting aspect of these three examples is that, although we used a common vector field and moved from $(0, 0, 0)$ to $(1, 1, 1)$ in each

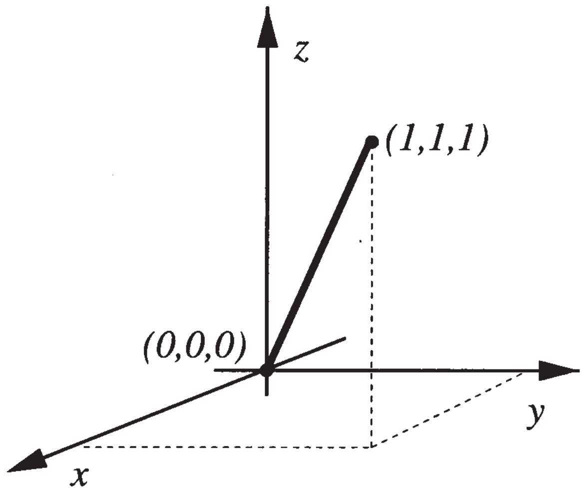

Figure 10.3.3: Diagram for the line integration in Example 10.3.3.

case, we obtained a different answer in each case. Thus, for this vector field, the line integral is *path dependent.* This is generally true. In the next section we will meet *conservative vector fields* where the results will be path independent.

• **Example 10.3.4**

If $\mathbf{F} = (x^2+y^2)\mathbf{i} - 2xy\mathbf{j} + x\mathbf{k}$, let us evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ if the contour is that portion of the circle $x^2 + y^2 = a^2$ from the point $(a, 0, 3)$ to $(-a, 0, 3)$. See Figure 10.3.4.

The parametric equations for this example are $x = a\cos(\theta)$, $y = a\sin(\theta)$, $z = 3$ with $0 \le \theta \le \pi$. Therefore,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi [a^2\cos^2(\theta) + a^2\sin^2(\theta)][-a\sin(\theta)\,d\theta] - 2a^2\cos(\theta)\sin(\theta)[a\cos(\theta)\,d\theta] + a\cos(\theta)\cdot 0 \quad \textbf{(10.3.12)}$$

$$= -a^3\int_0^\pi \sin(\theta)\,d\theta - 2a^3\int_0^\pi \cos^2(\theta)\sin(\theta)\,d\theta \quad \textbf{(10.3.13)}$$

$$= a^3\cos(\theta)\Big|_0^\pi + \tfrac{2}{3}a^3\cos^3(\theta)\Big|_0^\pi \quad \textbf{(10.3.14)}$$

$$= -2a^3 - \tfrac{4}{3}a^3 = -\tfrac{10}{3}a^3. \quad \textbf{(10.3.15)}$$

• **Example 10.3.5: Circulation**

Let $\mathbf{v}(x, y, z)$ denote the velocity at the point (x, y, z) in a moving fluid. If it varies with time, this is the velocity at a particular instant

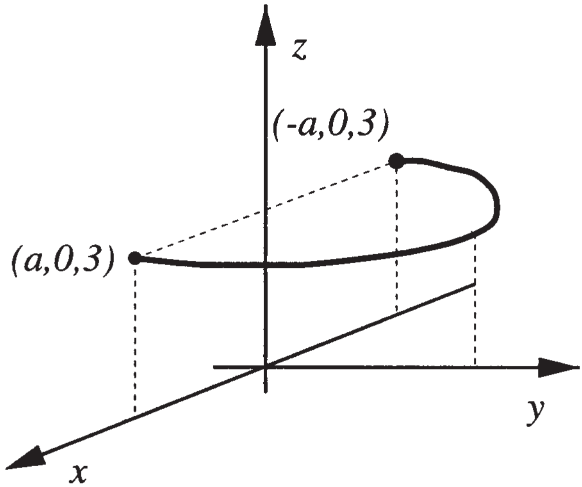

Figure 10.3.4: Diagram for the line integration in Example 10.3.4.

of time. The integral $\oint_C \mathbf{v} \cdot d\mathbf{r}$ around a closed path C is called the *circulation* around that path. The average component of velocity along the path is

$$\overline{v}_s = \frac{\oint_C v_s\, ds}{s} = \frac{\oint_C \mathbf{v} \cdot d\mathbf{r}}{s}, \qquad \textbf{(10.3.16)}$$

where s is the total length of the path. The circulation is thus $\oint_C \mathbf{v} \cdot d\mathbf{r} = \overline{u}_s s$, the product of the length of the path and the average velocity along the path. When the circulation is positive, the flow is more in the direction of integration than opposite to it. Circulation is thus an indication and to some extent a measure of motion around the path.

Problems

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the following vector fields and curves:

1. $\mathbf{F} = y\sin(\pi z)\mathbf{i} + x^2 e^y \mathbf{j} + 3xz\mathbf{k}$ and C is the curve $x = t$, $y = t^2$ and $z = t^3$ from $(0,0,0)$ to $(1,1,1)$.

2. $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ and C consists of the line segments $(0,0,0)$ to $(2,3,0)$ and from $(2,3,0)$ to $(2,3,4)$.

3. $\mathbf{F} = e^x\mathbf{i} + xe^{xy}\mathbf{j} + xye^{xyz}\mathbf{k}$ and C is the curve $x = t$, $y = t^2$ and $z = t^3$ with $0 \le t \le 2$.

4. $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and C is the curve $x = t^3$, $y = t^2$ and $z = t$ with $1 \le t \le 2$.

5. $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + 3xy\mathbf{k}$ and C consists of the semicircle $x^2 + y^2 = 4$, $z = 0$, $y > 0$ and the line segment from $(-2, 0, 0)$ to $(2, 0, 0)$.

6. $\mathbf{F} = (x + 2y)\mathbf{i} + (6y - 2x)\mathbf{j}$ and C consists of the sides of the triangle with vertices at $(0, 0, 0)$, $(1, 1, 1)$ and $(1, 1, 0)$. Proceed from $(0, 0, 0)$ to $(1, 1, 1)$ to $(1, 1, 0)$ and back to $(0, 0, 0)$.

7. $\mathbf{F} = 2xz\mathbf{i} + 4y^2\mathbf{j} + x^2\mathbf{k}$ and C is taken counterclockwise around the ellipse $x^2/4 + y^2/9 = 1$, $z = 1$.

8. $\mathbf{F} = 2x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and C is the contour $x = t$, $y = \sin(t)$ and $z = \cos(t) + \sin(t)$ with $0 \le t \le 2\pi$.

9. $\mathbf{F} = (2y^2 + z)\mathbf{i} + 4xy\mathbf{j} + x\mathbf{k}$ and C is the spiral $x = \cos(t)$, $y = \sin(t)$ and $z = t$ with $0 \le t \le 2\pi$ between the points $(1, 0, 0)$ and $(1, 0, 2\pi)$.

10. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + (z^2 + 2xy)\mathbf{k}$ and C consists of the edges of the triangle with vertices at $(0, 0, 0)$, $(1, 1, 0)$, and $(0, 1, 0)$. Proceed from $(0, 0, 0)$ to $(1, 1, 0)$ to $(0, 1, 0)$ and back to $(0, 0, 0)$.

10.4 THE POTENTIAL FUNCTION

In Section 10.2 we showed that the curl operation applied to a gradient produces the zero vector: $\nabla \times \nabla\varphi = \mathbf{0}$. Consequently, if we have a vector field $\mathbf{F}$ such that $\nabla \times \mathbf{F} \equiv \mathbf{0}$ everywhere, then that vector field is called a *conservative* field and we may compute a potential φ such that $\mathbf{F} = \nabla\varphi$.

• Example 10.4.1

Let us show that the vector field $\mathbf{F} = ye^{xy}\cos(z)\mathbf{i} + xe^{xy}\cos(z)\mathbf{j} - e^{xy}\sin(z)\mathbf{k}$ is conservative and then find the corresponding potential function.

To show that the field is conservative, we compute the curl of $\mathbf{F}$ or

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^{xy}\cos(z) & xe^{xy}\cos(z) & -e^{xy}\sin(z) \end{vmatrix} = \mathbf{0}. \tag{10.4.1}$$

To find the potential we must solve three partial differential equations:

$$\varphi_x = ye^{xy}\cos(z) = \mathbf{F}\cdot\mathbf{i}, \tag{10.4.2}$$

$$\varphi_y = xe^{xy}\cos(z) = \mathbf{F}\cdot\mathbf{j} \tag{10.4.3}$$

and

$$\varphi_z = -e^{xy}\sin(z) = \mathbf{F}\cdot\mathbf{k}. \tag{10.4.4}$$

We begin by integrating any one of these three equations. Choosing (10.4.2),

$$\varphi(x,y,z) = e^{xy}\cos(z) + f(y,z). \tag{10.4.5}$$

To find $f(y,z)$ we differentiate (10.4.5) with respect to y and find that

$$\varphi_y = xe^{xy}\cos(z) + f_y(y,z) = xe^{xy}\cos(z) \tag{10.4.6}$$

from (10.4.3). Thus, $f_y = 0$ and $f(y,z)$ can only be a function of z, say $g(z)$. Then,

$$\varphi(x,y,z) = e^{xy}\cos(z) + g(z). \tag{10.4.7}$$

Finally,

$$\varphi_z = -e^{xy}\sin(z) + g'(z) = -e^{xy}\sin(z) \tag{10.4.8}$$

from (10.4.4) and $g'(z) = 0$. Therefore, the potential is

$$\varphi(x,y,z) = e^{xy}\cos(z) + \text{constant}. \tag{10.4.9}$$

Potentials can be very useful in computing line integrals because

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \varphi_x\,dx + \varphi_y\,dy + \varphi_z\,dz = \int_C d\varphi = \varphi(B) - \varphi(A), \tag{10.4.10}$$

where the point B is the terminal point of the integration while the point A is the starting point. Thus, any path integration between any two points is *path independent*.

Finally, if we close the path so that A and B coincide, then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = 0. \tag{10.4.11}$$

It should be noted that the converse is *not* true. Just because $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$, we do not necessarily have a conservative field $\mathbf{F}$.

In summary then, an irrotational vector in a given region has three fundamental properties: (1) its integral around every simply connected circuit is zero, (2) its curl equals zero, (3) it is the gradient of a scalar function. For continuously differentiable vectors these properties are equivalent. For vectors which are only piece-wise differentiable, this is not true. Generally the first property is the most fundamental and taken as the definition of irrotationality.

• Example 10.4.2

Using the potential found in Example 10.4.1, let us find the value of the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ from the point $(0,0,0)$ to $(-1,2,\pi)$.

From (10.4.9),

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \left[e^{xy}\cos(z) + \text{constant}\right]\Big|_{(0,0,0)}^{(-1,2,\pi)} = -1 - e^{-2}. \quad \textbf{(10.4.12)}$$

Problems

Verify that the following vector fields are conservative and then find the corresponding potential:

1. $\mathbf{F} = 2xy\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 4)\mathbf{k}$

2. $\mathbf{F} = (2x + 2ze^{2x})\mathbf{i} + (2y - 1)\mathbf{j} + e^{2x}\mathbf{k}$

3. $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

4. $\mathbf{F} = 2x\mathbf{i} + 3y^2\mathbf{j} + 4z^3\mathbf{k}$

5. $\mathbf{F} = [2x\sin(y) + e^{3z}]\mathbf{i} + x^2\cos(y)\mathbf{j} + (3xe^{3z} + 4)\mathbf{k}$

6. $\mathbf{F} = (2x + 5)\mathbf{i} + 3y^2\mathbf{j} + (1/z)\mathbf{k}$

7. $\mathbf{F} = e^{2z}\mathbf{i} + 3y^2\mathbf{j} + 2xe^{2z}\mathbf{k}$

8. $\mathbf{F} = y\mathbf{i} + (x + z)\mathbf{j} + y\mathbf{k}$

9. $\mathbf{F} = (z + y)\mathbf{i} + x\mathbf{j} + x\mathbf{k}$.

10.5 SURFACE INTEGRALS

Surface integrals appear in such diverse fields as electromagnetism and fluid mechanics. For example, if we were oceanographers we might be interested in the rate of volume of seawater through an instrument which has the curved surface S. The volume rate equals $\iint_S \mathbf{v} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{v}$ is the velocity and $\mathbf{n}\, d\sigma$ is an infinitesimally small element on the surface of the instrument. The surface element $\mathbf{n}\, d\sigma$ must have an orientation (given by $\mathbf{n}$) because it makes a considerable difference whether the flow is directly through the surface or at right angles. More generally, if the surface encloses a three-dimensional volume, then we have a *closed surface integral.*

To illustrate the concept of computing a surface integral, we will do three examples with simple geometries. Later we will show how to use surface coordinates to do more complicated geometries.

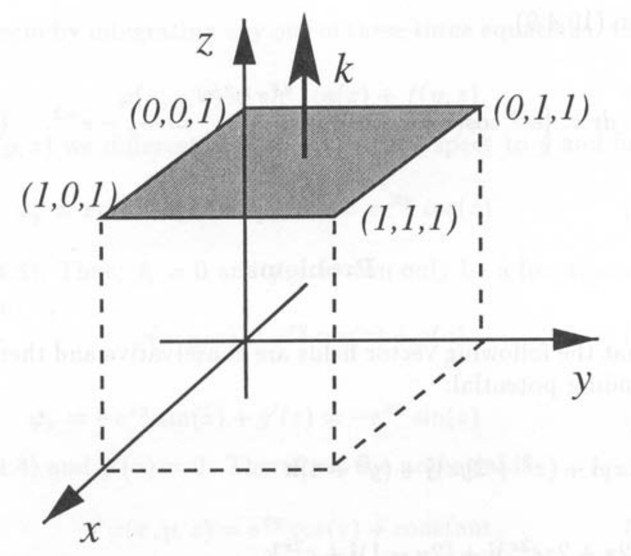

Figure 10.5.1: Diagram for the surface integration in Example 10.5.1.

• **Example 10.5.1**

Let us find the flux out the top of a unit cube if the vector field is $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. See Figure 10.5.1.

The top of a unit cube consists of the surface $z = 1$ with $0 \le x \le 1$ and $0 \le y \le 1$. By inspection the unit normal to this surface is $\mathbf{n} = \mathbf{k}$ or $\mathbf{n} = -\mathbf{k}$. Because we are interested in the flux *out* of the unit cube, $\mathbf{n} = \mathbf{k}$, and

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^1 \int_0^1 (x\mathbf{i} + y\mathbf{j} + \mathbf{k}) \cdot \mathbf{k}\, dx\, dy = 1 \qquad \mathbf{(10.5.1)}$$

because $z = 1$.

• **Example 10.5.2**

Let us find the flux out of that portion of the cylinder $y^2 + z^2 = 4$ in the first octant bounded by $x = 0$, $x = 3$, $y = 0$, and $z = 0$. The vector field is $\mathbf{F} = x\mathbf{i} + 2z\mathbf{j} + y\mathbf{k}$. See Figure 10.5.2.

Because we are dealing with a cylinder, cylindrical coordinates are appropriate. Let $y = 2\cos(\theta)$, $z = 2\sin(\theta)$, and $x = x$ with $0 \le \theta \le \pi/2$. To find $\mathbf{n}$, we use the gradient in conjunction with the definition of the surface of the cylinder $f(x, y, z) = y^2 + z^2 = 4$. Then

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{4y^2 + 4z^2}} = \frac{y}{2}\mathbf{j} + \frac{z}{2}\mathbf{k} \qquad \mathbf{(10.5.2)}$$

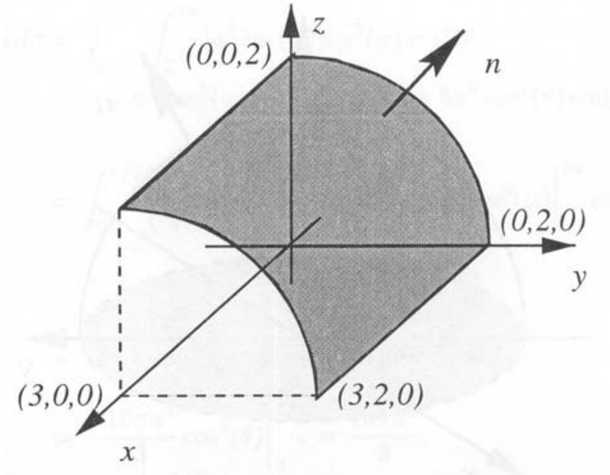

Figure 10.5.2: Diagram for the surface integration in Example 10.5.2.

because $y^2 + z^2 = 4$ along the surface. Because we want the flux *out* of the surface, then $\mathbf{n} = y\mathbf{j}/2 + z\mathbf{k}/2$ whereas the flux *into* the surface would require $\mathbf{n} = -y\mathbf{j}/2 - z\mathbf{k}/2$. Therefore,

$$\mathbf{F} \cdot \mathbf{n} = (x\mathbf{i} + 2z\mathbf{j} + y\mathbf{k}) \cdot \left(\frac{y}{2}\mathbf{j} + \frac{z}{2}\mathbf{k}\right) = \frac{3yz}{2} = 6\cos(\theta)\sin(\theta). \quad \mathbf{(10.5.3)}$$

What is $d\sigma$? Our infinitesimal surface area has a side in the x direction of length dx and a side in the θ direction of length $2\,d\theta$ because the radius equals 2. Therefore, $d\sigma = 2\,dx\,d\theta$.

Bringing all of these elements together,

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^3 \int_0^{\pi/2} 12\cos(\theta)\sin(\theta)\, d\theta\, dx \quad \mathbf{(10.5.4)}$$

$$= 6\int_0^3 \left[\sin^2(\theta)\Big|_0^{\pi/2}\right] dx = 6\int_0^3 dx = 18. \quad \mathbf{(10.5.5)}$$

As counterpoint to this example, let us find the flux out of the pie-shaped surface at $x = 3$. In this case, $y = r\cos(\theta)$ and $z = r\sin(\theta)$ and

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{\pi/2} \int_0^2 [3\mathbf{i} + 2r\sin(\theta)\mathbf{j} + r\cos(\theta)\mathbf{k}] \cdot \mathbf{i}\, r\, dr\, d\theta \quad \mathbf{(10.5.6)}$$

$$= 3\int_0^{\pi/2} \int_0^2 r\, dr\, d\theta = 3\pi. \quad \mathbf{(10.5.7)}$$

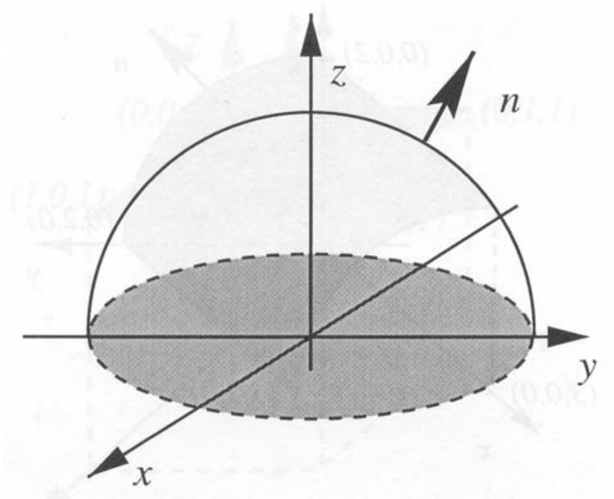

Figure 10.5.3: Diagram for the surface integration in Example 10.5.3.

• **Example 10.5.3**

Let us find the flux of the vector field $\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j} + 5z\mathbf{k}$ out of the hemispheric surface $x^2 + y^2 + z^2 = a^2$, $z > 0$. See Figure 10.5.3.

We begin by finding the outwardly pointing normal. Because the surface is defined by $f(x, y, z) = x^2 + y^2 + z^2 = a^2$,

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x}{a}\mathbf{i} + \frac{y}{a}\mathbf{j} + \frac{z}{a}\mathbf{k} \tag{10.5.8}$$

because $x^2 + y^2 + z^2 = a^2$. This is also the outwardly pointing normal because $\mathbf{n} = \mathbf{r}/a$, where $\mathbf{r}$ is the radial vector.

Using spherical coordinates, $x = a\cos(\varphi)\sin(\theta)$, $y = a\sin(\varphi)\sin(\theta)$, and $z = a\cos(\theta)$, where φ is the angle made by the projection of the point onto the equatorial plane, measured from the x-axis, and θ is the colatitude or "cone angle" measured from the z-axis. To compute $d\sigma$, the infinitesimal length in the θ direction is $a\,d\theta$ while in the φ direction it is $a\sin(\theta)\,d\varphi$, where the $\sin(\theta)$ factor takes into account the convergence of the meridians. Therefore, $d\sigma = a^2\sin(\theta)\,d\theta\,d\varphi$ and

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^{\pi/2} (y^2\mathbf{i} + x^2\mathbf{j} + 5z\mathbf{k}) \cdot \left(\frac{x}{a}\mathbf{i} + \frac{y}{a}\mathbf{j} + \frac{z}{a}\mathbf{k}\right) a^2\sin(\theta)\,d\theta\,d\varphi \tag{10.5.9}$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \left(\frac{xy^2}{a} + \frac{x^2y}{a} + \frac{5z^2}{a}\right) a^2\sin(\theta)\,d\theta\,d\varphi \tag{10.5.10}$$

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{\pi/2}\int_0^{2\pi} \big[a^4\cos(\varphi)\sin^2(\varphi)\sin^4(\theta) + a^4\cos^2(\varphi)\sin(\varphi)\sin^4(\theta) + 5a^3\cos^2(\theta)\sin(\theta)\big]\,d\varphi\,d\theta \tag{10.5.11}$$

$$= \int_0^{\pi/2}\left[\frac{a^4}{3}\sin^3(\varphi)\Big|_0^{2\pi}\sin^4(\theta) - \frac{a^4}{3}\cos^3(\varphi)\Big|_0^{2\pi}\sin^4(\theta) + 5a^3\cos^2(\theta)\sin(\theta)\varphi\Big|_0^{2\pi}\right]d\theta \tag{10.5.12}$$

$$= 10\pi a^3\int_0^{\pi/2}\cos^2(\theta)\sin(\theta)\,d\theta \tag{10.5.13}$$

$$= -\frac{10\pi a^3}{3}\cos^3(\theta)\Big|_0^{\pi/2} = \frac{10\pi a^3}{3}. \tag{10.5.14}$$

Although these techniques apply for simple geometries such as a cylinder or sphere, we would like a *general* method for treating any arbitrary surface. We begin by noting that a surface is an aggregate of points whose coordinates are functions of two variables. For example, in the previous example, the surface was described by the coordinates φ and θ. Let us denote these surface coordinates in general by u and v. Consequently, on any surface we can reexpress x, y, and z in terms of u and v: $x = x(u,v)$, $y = y(u,v)$, and $z = z(u,v)$.

Next, we must find an infinitesimal element of area. The position vector to the surface is $\mathbf{r} = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$. Therefore, the tangent vectors along $v =$ constant, $\mathbf{r}_u$, and along $u =$ constant, $\mathbf{r}_v$, equal

$$\mathbf{r}_u = x_u\mathbf{i} + y_u\mathbf{j} + z_u\mathbf{k} \tag{10.5.15}$$

and

$$\mathbf{r}_v = x_v\mathbf{i} + y_v\mathbf{j} + z_v\mathbf{k}. \tag{10.5.16}$$

Consequently, the sides of the infinitesimal area is $\mathbf{r}_u\,du$ and $\mathbf{r}_v\,dv$. Therefore, the vectorial area of the parallelogram that these vectors form is

$$\mathbf{n}\,d\sigma = \mathbf{r}_u \times \mathbf{r}_v\,du\,dv \tag{10.5.17}$$

and is called the *vector element of area* on the surface. Thus, we may convert $\mathbf{F}\cdot\mathbf{n}\,d\sigma$ into an expression involving only u and v and then evaluate the surface integral by integrating over the appropriate domain in the uv-plane. Of course, we are in trouble if $\mathbf{r}_u\times\mathbf{r}_v = \mathbf{0}$. Therefore, we only treat regular points where $\mathbf{r}_u\times\mathbf{r}_v \neq \mathbf{0}$. In the next few examples, we show how to use these surface coordinates to evaluate surface integrals.

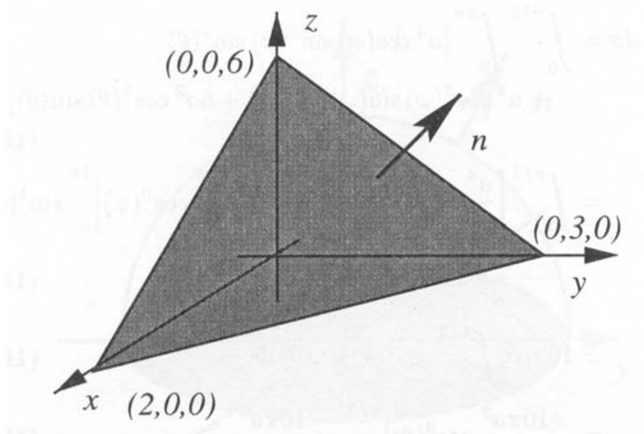

Figure 10.5.5: Diagram for the surface integration in Example 10.5.4.

• **Example 10.5.4**

Let us find the flux of the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through the top of the plane $3x + 2y + z = 6$ which lies in the first octant. See Figure 10.5.5.

Our parametric equations are $x = u$, $y = v$, and $z = 6 - 3u - 2v$. Therefore,

$$\mathbf{r} = u\mathbf{i} + v\mathbf{j} + (6 - 3u - 2v)\mathbf{k} \tag{10.5.18}$$

so that

$$\mathbf{r}_u = \mathbf{i} - 3\mathbf{k}, \quad \mathbf{r}_v = \mathbf{j} - 2\mathbf{k} \tag{10.5.19}$$

and

$$\mathbf{r}_u \times \mathbf{r}_v = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}. \tag{10.5.20}$$

Bring all of these elements together,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^2 \int_0^{3-3u/2} (3u + 2v + 6 - 3u - 2v)\,dv\,du \tag{10.5.21}$$

$$= 6\int_0^2 \int_0^{3-3u/2} dv\,du = 6\int_0^2 (3 - 3u/2)\,du \tag{10.5.22}$$

$$= 6\left(3u - \tfrac{3}{4}u^2\right)\Big|_0^2 = 18. \tag{10.5.23}$$

To set up the limits of integration, we note that the area in u, v space corresponds to the xy-plane. On the xy-plane, $z = 0$ and $3u + 2v = 6$, along with boundaries $u = v = 0$.

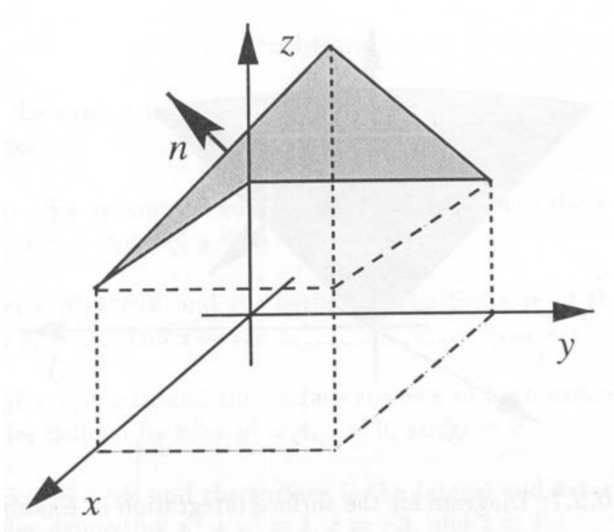

Figure 10.5.6: Diagram for the surface integration in Example 10.5.5.

• **Example 10.5.5**

Let us find the flux of the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through the top of the surface $z = xy + 1$ which covers the square $0 \le x \le 1$, $0 \le y \le 1$ in the xy-plane. See Figure 10.5.6.

Our parametric equations are $x = u$, $y = v$, and $z = uv + 1$ with $0 \le u \le 1$ and $0 \le v \le 1$. Therefore,

$$\mathbf{r} = u\mathbf{i} + v\mathbf{j} + (uv + 1)\mathbf{k} \tag{10.5.24}$$

so that

$$\mathbf{r}_u = \mathbf{i} + v\mathbf{k}, \qquad \mathbf{r}_v = \mathbf{j} + u\mathbf{k} \tag{10.5.25}$$

and

$$\mathbf{r}_u \times \mathbf{r}_v = -v\mathbf{i} - u\mathbf{j} + \mathbf{k}. \tag{10.5.26}$$

Bring all of these elements together,

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^1 \int_0^1 [u\mathbf{i} + v\mathbf{j} + (uv + 1)\mathbf{k}] \cdot (-v\mathbf{i} - u\mathbf{j} + \mathbf{k})\, du\, dv \tag{10.5.27}$$

$$= \int_0^1 \int_0^1 (1 - uv)\, du\, dv = \int_0^1 \left(u - \tfrac{1}{2}u^2 v\right)\Big|_0^1 dv \tag{10.5.28}$$

$$= \int_0^1 \left(1 - \tfrac{1}{2}v\right) dv = \left(v - \tfrac{1}{4}v^2\right)\Big|_0^1 = \tfrac{3}{4}. \tag{10.5.29}$$

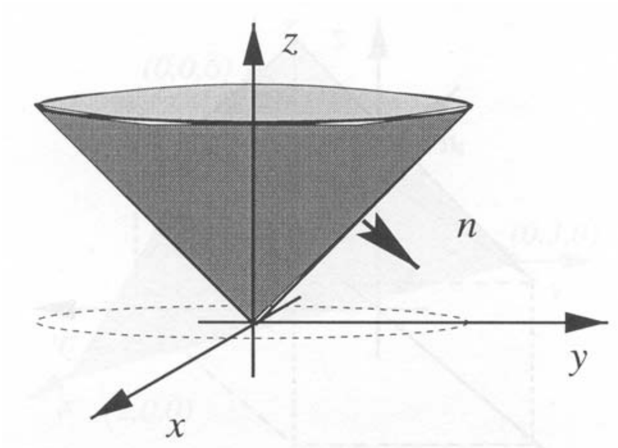

Figure 10.5.7: Diagram for the surface integration in Example 10.5.6.

- **Example 10.5.6**

Let us find the flux of the vector field $\mathbf{F} = 4xz\mathbf{i}+xyz^2\mathbf{j}+3z\mathbf{k}$ through the exterior surface of the cone $z^2 = x^2 + y^2$ above the xy-plane and below $z = 4$. See Figure 10.5.7.

A natural choice for the surface coordinates is polar coordinates r and θ. Because $x = r\cos(\theta)$ and $y = r\sin(\theta)$, $z = r$. Then

$$\mathbf{r} = r\cos(\theta)\mathbf{i} + r\sin(\theta)\mathbf{j} + r\mathbf{k} \qquad (\mathbf{10.5.30})$$

with $0 \le r \le 4$ and $0 \le \theta \le 2\pi$ so that

$$\mathbf{r}_r = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} + \mathbf{k}, \mathbf{r}_\theta = -r\sin(\theta)\mathbf{i} + r\cos(\theta)\mathbf{j} \qquad (\mathbf{10.5.31})$$

and

$$\mathbf{r}_r \times \mathbf{r}_\theta = -r\cos(\theta)\mathbf{i} - r\sin(\theta)\mathbf{j} + r\mathbf{k}. \qquad (\mathbf{10.5.32})$$

This is the unit area *inside* the cone. Because we want the exterior surface, we must take the negative of (10.5.32). Bring all of these elements together,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^4\int_0^{2\pi} \{[4r\cos(\theta)]r[r\cos(\theta)] + [r^2\sin(\theta)\cos(\theta)]r^2[r\sin(\theta)] - 3r^2\}\,d\theta\,dr \qquad (\mathbf{10.5.33})$$

$$= \int_0^4 \left\{2r^3\left[\theta + \tfrac{1}{2}\sin(2\theta)\right]\Big|_0^{2\pi} + r^5\tfrac{1}{3}\sin^3(\theta)\Big|_0^{2\pi} - 3r^2\theta\Big|_0^{2\pi}\right\}dr \qquad (\mathbf{10.5.34})$$

$$= \int_0^4 \left(4\pi r^3 - 6\pi r^2\right)\,dr = \left(\pi r^4 - 2\pi r^3\right)\Big|_0^4 = 128\pi. \qquad (\mathbf{10.5.35})$$

Problems

Compute the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ for the following vector fields and surfaces:

1. $\mathbf{F} = x\mathbf{i} - z\mathbf{j} + y\mathbf{k}$ and the surface is the top portion of the $z = 1$ plane where $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

2. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + xz\mathbf{k}$ and the surface is the top side of the cylinder $x^2 + y^2 = 9$, $z = 0$, and $z = 1$.

3. $\mathbf{F} = xy\mathbf{i} + z\mathbf{j} + xz\mathbf{k}$ and the surface consists of both exterior *ends* of the cylinder defined by $x^2 + y^2 = 4$, $z = 0$, and $z = 2$.

4. $\mathbf{F} = x\mathbf{i} + z\mathbf{j} + y\mathbf{k}$ and the surface is the lateral and exterior side of the cylinder defined by $x^2 + y^2 = 4$, $z = -3$, and $z = 3$.

5. $\mathbf{F} = xy\mathbf{i} + z^2\mathbf{j} + y\mathbf{k}$ and the surface is the curved exterior side of the cylinder $y^2 + z^2 = 9$ in the first octant bounded by $x = 0$, $x = 1$, $y = 0$, and $z = 0$.

6. $\mathbf{F} = y\mathbf{j} + z^2\mathbf{k}$ and the surface is the exterior of the semicircular cylinder $y^2 + z^2 = 4$, $z \geq 0$ cut by the planes $x = 0$ and $x = 1$.

7. $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ and the surface is the curved exterior side of the cylinder $x^2 + y^2 = 4$ in the first octant cut by the planes $z = 1$ and $z = 2$.

8. $\mathbf{F} = x^2\mathbf{i} - z^2\mathbf{j} + yz\mathbf{k}$ and the surface is the exterior of the hemispheric surface of $x^2 + y^2 + z^2 = 16$ above the plane $z = 2$.

9. $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ and the surface is the top of the surface $z = x + 1$ where $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.

10. $\mathbf{F} = z\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ and the surface is the top of the plane $x + y + z = 2a$ that lies above the square $0 \leq x \leq a$, $0 \leq y \leq a$ in the xy-plane.

11. $\mathbf{F} = (y^2 + z^2)\mathbf{i} + (x^2 + z^2)\mathbf{j} + (x^2 + y^2)\mathbf{k}$ and the surface is the top of the surface $z = 1 - x^2$ with $-1 \leq x \leq 1$ and $-2 \leq y \leq 2$.

12. $\mathbf{F} = y^2\mathbf{i} + xz\mathbf{j} - \mathbf{k}$ and the surface is the cone $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq 1$ with the normal pointing away from the z-axis.

13. $\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j} + 5z\mathbf{k}$ and the surface is the top of the plane $z = y + 1$ where $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.

14. $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ and the surface is the exterior or bottom of the paraboloid $z = x^2 + y^2$ where $0 \leq z \leq 1$.

15. $\mathbf{F} = -y\mathbf{i}+x\mathbf{j}+6z^2\mathbf{k}$ and the surface is the exterior of the paraboloids $z = 4 - x^2 - y^2$ and $z = x^2 + y^2$.

10.6 GREEN'S LEMMA

Consider a rectangle in the xy-plane which is bounded by the lines $x = a$, $x = b$, $y = c$, and $y = d$. We assume that the boundary of the rectangle is a piece-wise smooth curve which we denote by C. If we have a continuously differentiable vector function $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ at each point of enclosed region R, then

$$\iint_R \frac{\partial Q}{\partial x}\, dA = \int_c^d \left[\int_a^b \frac{\partial Q}{\partial x}\, dx\right] dy \tag{10.6.1}$$

$$= \int_c^d Q(b,y)\, dy - \int_c^d Q(a,y)\, dy \tag{10.6.2}$$

$$= \oint_C Q(x,y)\, dy, \tag{10.6.3}$$

where the last integral is a closed line integral counterclockwise around the rectangle because the horizontal sides vanish since $dy = 0$. By similar arguments,

$$\iint_R \frac{\partial P}{\partial y}\, dA = -\oint_C P(x,y)\, dx \tag{10.6.4}$$

so that

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_C P(x,y)\, dx + Q(x,y)\, dy. \tag{10.6.5}$$

This result, often known as *Green's lemma*, may be expressed in vector form as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA. \tag{10.6.6}$$

Although this proof was for a rectangular area, it can be generalized to *any* simply closed region on the xy-plane as follows. Consider

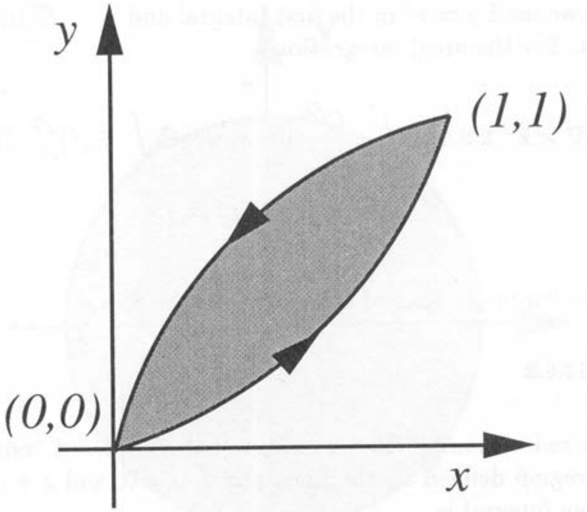

Figure 10.6.1: Diagram for the verification of Green's lemma in Example 10.6.1.

an area which is surrounded by simply closed curves. Within the closed contour we can divide the area into an infinite number of infinitesimally small rectangles and apply (10.6.6) to each rectangle. When we sum up all of these rectangles, we find $\iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA$, where the integration is over the entire surface area. On the other hand, away from the boundary, the line integral along any one edge of a rectangle cancels the line integral along the same edge in a contiguous rectangle. Thus, the only nonvanishing contribution from the line integrals arises from the outside boundary of the domain $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

• **Example 10.6.1**

Let us *verify* Green's lemma using the vector field $\mathbf{F} = (3x^2 - 8y^2)\mathbf{i} + (4y - 6xy)\mathbf{j}$ and the enclosed area lies between the curves $y = \sqrt{x}$ and $y = x^2$. The two curves intersect at $x = 0$ and $x = 1$. See Figure 10.6.1.

We begin with the line integral:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (3x^2 - 8x^4)\, dx + (4x^2 - 6x^3)(2x\, dx)$$
$$+ \int_1^0 (3x^2 - 8x)\, dx + (4x^{1/2} - 6x^{3/2})(\tfrac{1}{2}x^{-1/2}\, dx) \tag{10.6.7}$$
$$= \int_0^1 (-20x^4 + 8x^3 + 11x - 2)\, dx = \tfrac{3}{2}. \tag{10.6.8}$$

In (10.6.7) we used $y = x^2$ in the first integral and $y = \sqrt{x}$ in our return integration. For the areal integration,

$$\iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA = \int_0^1 \int_{x^2}^{\sqrt{x}} 10y\, dy\, dx = \int_0^1 5y^2\Big|_{x^2}^{\sqrt{x}}\, dx \quad (10.6.9)$$

$$= 5\int_0^1 (x - x^4)\, dx = \tfrac{3}{2} \quad (10.6.10)$$

and Green's lemma is verified in this particular case.

• Example 10.6.2

Let us redo Example 10.6.1 except that the closed contour is the triangular region defined by the lines $x = 0$, $y = 0$, and $x + y = 1$.

The line integral is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (3x^2 - 8 \cdot 0^2)dx + (4 \cdot 0 - 6x \cdot 0) \cdot 0$$

$$+ \int_0^1 [3(1-y)^2 - 8y^2](-dy) + [4y - 6(1-y)y]\, dy$$

$$+ \int_1^0 (3 \cdot 0^2 - 8y^2) \cdot 0 + (4y - 6 \cdot 0 \cdot y)\, dy \quad (10.6.11)$$

$$= \int_0^1 3x^2\, dx - \int_0^1 4y\, dy + \int_0^1 (-3 + 4y + 11y^2)\, dy \quad (10.6.12)$$

$$= x^3\Big|_0^1 - 2y^2\Big|_0^1 + \left(-3y + 2y^2 + \tfrac{11}{3}y^3\right)\Big|_0^1 = \tfrac{5}{3}. \quad (10.6.13)$$

On the other hand, the areal integration is

$$\iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA = \int_0^1 \int_0^{1-x} 10y\, dy\, dx = \int_0^1 5y^2\Big|_0^{1-x}\, dx \quad (10.6.14)$$

$$= 5\int_0^1 (1-x)^2\, dx = -\tfrac{5}{3}(1-x)^3\Big|_0^1 = \tfrac{5}{3} \quad (10.6.15)$$

and Green's lemma is verified in this particular case.

• Example 10.6.3

Let us verify Green's lemma using the vector field $\mathbf{F} = (3x + 4y)\mathbf{i} + (2x - 3y)\mathbf{j}$ and the closed contour is a circle of radius two centered at the origin of the xy-plane. See Figure 10.6.2.

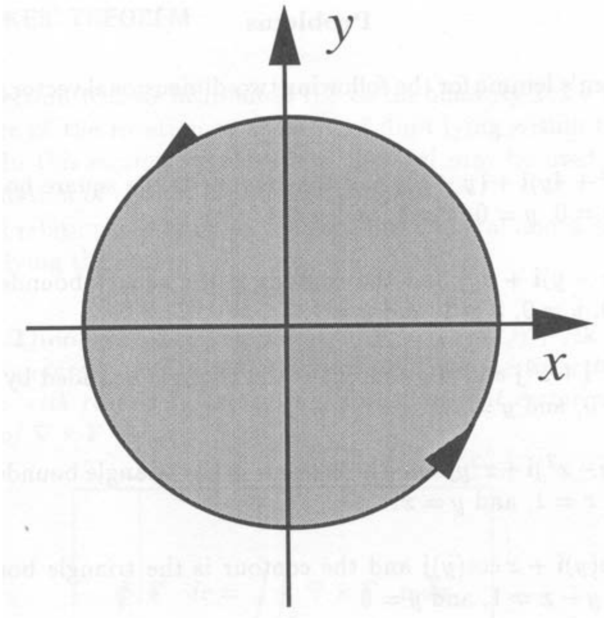

Figure 10.6.2: Diagram for the verification of Green's lemma in Example 10.6.3.

Beginning with the line integration,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} [6\cos(\theta) + 8\sin(\theta)][-2\sin(\theta)\,d\theta] + [4\cos(\theta) - 6\sin(\theta)][2\cos(\theta)\,d\theta] \tag{10.6.16}$$

$$= \int_0^{2\pi} [-24\cos(\theta)\sin(\theta) - 16\sin^2(\theta) + 8\cos^2(\theta)]\,d\theta \tag{10.6.17}$$

$$= 12\cos^2(\theta)\Big|_0^{2\pi} - 8\left[\theta - \tfrac{1}{2}\sin(2\theta)\right]\Big|_0^{2\pi} + 4\left[\theta + \tfrac{1}{2}\sin(2\theta)\right]\Big|_0^{2\pi} \tag{10.6.18}$$

$$= -8\pi. \tag{10.6.19}$$

For the areal integration,

$$\iint_R \nabla\times\mathbf{F}\cdot\mathbf{k}\,dA = \int_0^2\int_0^{2\pi} -2\,r\,d\theta\,dr = -8\pi \tag{10.6.20}$$

and Green's lemma is verified in the special case.

Problems

Verify Green's lemma for the following two-dimensional vector fields and contours:

1. $\mathbf{F} = (x^2 + 4y)\mathbf{i} + (y - x)\mathbf{j}$ and the contour is the square bounded by the lines $x = 0$, $y = 0$, $x = 1$, and $y = 1$.

2. $\mathbf{F} = (x - y)\mathbf{i} + xy\mathbf{j}$ and the contour is the square bounded by the lines $x = 0$, $y = 0$, $x = 1$, and $y = 1$.

3. $\mathbf{F} = -y^2\mathbf{i} + x^2\mathbf{j}$ and the contour is the triangle bounded by the lines $x = 1$, $y = 0$, and $y = x$.

4. $\mathbf{F} = (xy - x^2)\mathbf{i} + x^2y\mathbf{j}$ and the contour is the triangle bounded by the line $y = 0$, $x = 1$, and $y = x$.

5. $\mathbf{F} = \sin(y)\mathbf{i} + x\cos(y)\mathbf{j}$ and the contour is the triangle bounded by $x + y = 1$, $y - x = 1$, and $y = 0$.

6. $\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j}$ and the contour is the same contour used in problem 4.

7. $\mathbf{F} = -y^2\mathbf{i} + x^2\mathbf{j}$ and the contour is the circle $x^2 + y^2 = 4$.

8. $\mathbf{F} = -x^2\mathbf{i} + xy^2\mathbf{j}$ and the contour is the closed circle of radius a.

9. $\mathbf{F} = (6y + x)\mathbf{i} + (y + 2x)\mathbf{j}$ and the contour is the circle $(x - 1)^2 + (y - 2)^2 = 4$.

10. $\mathbf{F} = (x + y)\mathbf{i} + (2x^2 - y^2)\mathbf{j}$ and the contour is the boundary of the region determined by the graphs of $y = x^2$ and $y = 4$.

11. $\mathbf{F} = 3y\mathbf{i} + 2x\mathbf{j}$ and the contour is the boundary of the region determined by the graphs of $y = 0$ and $y = \sin(x)$ with $0 \leq x \leq \pi$.

12. $\mathbf{F} = -16y\mathbf{i} + (4e^y + 3x^2)\mathbf{j}$ and the contour is the pie wedge defined by the lines $y = x$, $y = -x$, $x^2 + y^2 = 4$, and $y > 0$.

10.7 STOKES' THEOREM

In Section 10.2 we introduced the vector quantity $\nabla \times \mathbf{v}$ which gives a measure of the rotation of a parcel of fluid lying within the velocity field $\mathbf{v}$. In this section we show how the curl may be used to simplify the calculation of certain closed line integrals.

This relationship between a closed line integral and a surface integral involving the curl is

Stokes' Theorem: *The circulation of* $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ *around the closed boundary* C *of an oriented surface* S *in the direction counterclockwise with respect to the surface's unit normal vector* $\mathbf{n}$ *equals the integral of* $\nabla \times \mathbf{F} \cdot \mathbf{n}$ *over* S *or*

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma. \tag{10.7.1}$$

Stokes' theorem requires that all of the functions and derivatives be continuous.

The proof of Stokes' theorem is as follows: Consider a finite surface S whose boundary is the loop C. We divide this surface into a number of small elements $\mathbf{n}\, d\sigma$ and compute the *circulation* $d\Gamma = \oint_L \mathbf{F} \cdot d\mathbf{r}$ around each element. When we add all of the circulations together, the contribution from an integration along a boundary line between two adjoining elements cancels out because the boundary is transversed once in each direction. For this reason, the only contributions that survive are those parts where the element boundaries form part of C. Thus, the sum of all circulations equals $\oint_C \mathbf{F} \cdot d\mathbf{r}$, the circulation around the edge of the whole surface.

Next, let us compute the circulation another way. We begin by finding the Taylor expansion for $P(x, y, z)$ about the arbitrary point (x_0, y_0, z_0):

$$P(x, y, z) = P(x_0, y_0, z_0) + (x - x_0)\frac{\partial P(x_0, y_0, z_0)}{\partial x} + (y - y_0)\frac{\partial P(x_0, y_0, z_0)}{\partial y} + (z - z_0)\frac{\partial P(x_0, y_0, z_0)}{\partial z} + \cdots \tag{10.7.2}$$

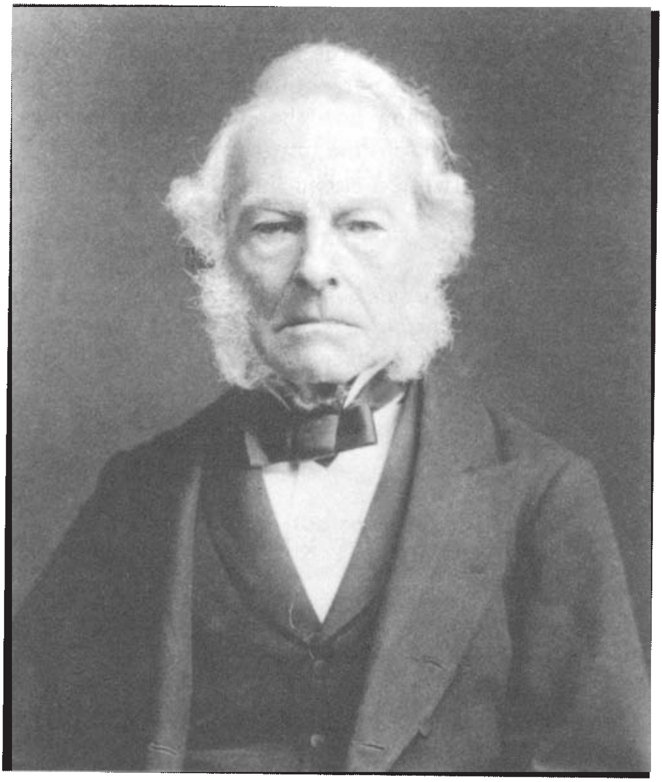

Figure 10.7.1: Sir George Gabriel Stokes (1819–1903) was Lucasian Professor of Mathematics at Cambridge University from 1849 until his death. Having learned of an integral theorem from his friend Lord Kelvin, Stokes included it a few years later among his questions on an examination that he wrote for the Smith prize. It is this integral theorem that we now call Stokes' theorem. (Portrait courtesy of the Royal Society of London.)

with similar expansions for $Q(x,y,z)$ and $R(x,y,z)$. Then

$$\begin{aligned} d\Gamma = \oint_L \mathbf{F}\cdot d\mathbf{r} = P(x_0,y_0,z_0)\oint_L dx + \frac{\partial P(x_0,y_0,z_0)}{\partial x}\oint_L (x-x_0)\,dx \\ + \frac{\partial P(x_0,y_0,z_0)}{\partial y}\oint_L (y-y_0)\,dy + \cdots \\ + \frac{\partial Q(x_0,y_0,z_0)}{\partial x}\oint_L (x-x_0)\,dy + \cdots, \end{aligned} \tag{10.7.3}$$

where L denotes some small loop located in the surface S. Note that integrals such as $\oint_L dx$ and $\oint_L (x-x_0)dx$ will vanish.

If we now require that the loop integrals be in the *clockwise* or *positive* sense so that we preserve the right-hand screw convention, then

$$\mathbf{n}\cdot\mathbf{k}\,\delta\sigma = \oint_L (x - x_0)\,dy = -\oint_L (y - y_0)\,dx, \tag{10.7.4}$$

$$\mathbf{n}\cdot\mathbf{j}\,\delta\sigma = \oint_L (z - z_0)\,dx = -\oint_L (x - x_0)\,dz, \tag{10.7.5}$$

$$\mathbf{n}\cdot\mathbf{i}\,\delta\sigma = \oint_L (y - y_0)\,dz = -\oint_L (z - z_0)\,dy \tag{10.7.6}$$

and

$$d\Gamma = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i}\cdot\mathbf{n}\,\delta\sigma + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j}\cdot\mathbf{n}\,\delta\sigma + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}\cdot\mathbf{n}\,\delta\sigma = \nabla\times\mathbf{F}\cdot\mathbf{n}\,\delta\sigma \tag{10.7.7}$$

Therefore, the sum of all circulations in the limit when all elements are made infinitesimally small becomes the surface integral $\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma$ and Stokes' theorem is proven. □

In the following examples we first apply Stokes' theorem to a few simple geometries. We then show how to apply this theorem to more complicated geometries.[4]

• Example 10.7.1

Let us verify Stokes' theorem using the vector field $\mathbf{F} = x^2\mathbf{i} + 2x\mathbf{j} + z^2\mathbf{k}$ and the closed curve is a square with vertices at $(0,0,3)$, $(1,0,3)$, $(1,1,3)$ and $(0,1,3)$. See Figure 10.7.2.

We begin with the line integral:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1}\mathbf{F}\cdot d\mathbf{r} + \int_{C_2}\mathbf{F}\cdot d\mathbf{r} + \int_{C_3}\mathbf{F}\cdot d\mathbf{r} + \int_{C_4}\mathbf{F}\cdot d\mathbf{r}, \tag{10.7.8}$$

where C_1, C_2, C_3, and C_4 represent the four sides of the square. Along C_1, x varies while $y = 0$ and $z = 3$. Therefore,

$$\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_0^1 x^2\,dx + 2x\cdot 0 + 9\cdot 0 = \tfrac{1}{3}, \tag{10.7.9}$$

because $dy = dz = 0$ and $z = 3$. Along C_2, y varies with $x = 1$ and $z = 3$. Therefore,

$$\int_{C_2}\mathbf{F}\cdot d\mathbf{r} = \int_0^1 1^2\cdot 0 + 2\cdot 1\cdot dy + 9\cdot 0 = 2. \tag{10.7.10}$$

[4] Thus, different Stokes for different folks.

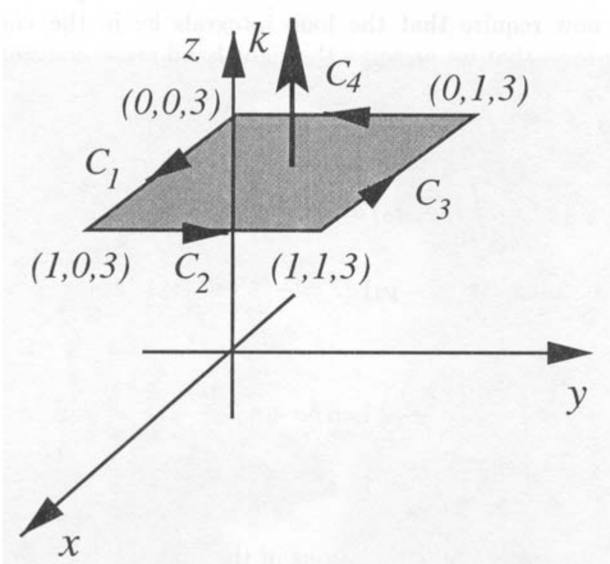

Figure 10.7.2: Diagram for the verification of Stokes' theorem in Example 10.7.1.

Along C_3, x again varies with $y = 1$ and $z = 3$, and so,

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_1^0 x^2\, dx + 2x \cdot 0 + 9 \cdot 0 = -\tfrac{1}{3}. \tag{10.7.11}$$

Note how the limits run from 1 to 0 because x is decreasing. Finally, for C_4, y again varies with $x = 0$ and $z = 3$. Hence,

$$\int_{C_4} \mathbf{F} \cdot d\mathbf{r} = \int_1^0 0^2 \cdot 0 + 2 \cdot 0 \cdot dy + 9 \cdot 0 = 0. \tag{10.7.12}$$

Hence,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 2. \tag{10.7.13}$$

Turning to the other side of the equation,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2x & z^2 \end{vmatrix} = 2\mathbf{k}. \tag{10.7.14}$$

Our line integral has been such that the normal vector must be $\mathbf{n} = \mathbf{k}$. Therefore,

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^1 \int_0^1 2\mathbf{k} \cdot \mathbf{k}\, dx\, dy = 2 \tag{10.7.15}$$

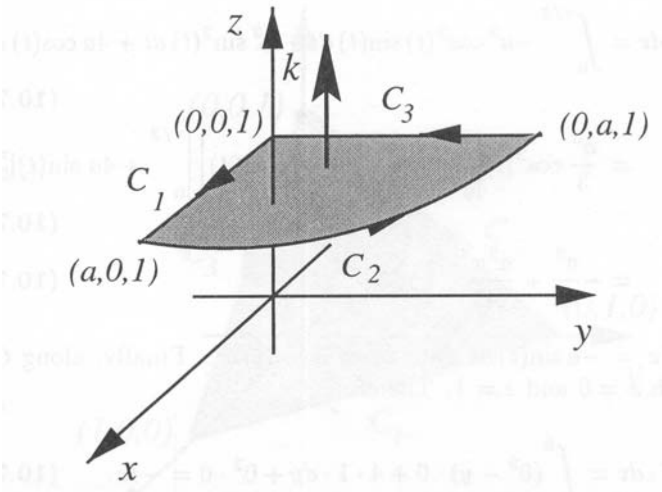

Figure 10.7.3: Diagram for the verification of Stokes' theorem in Example 10.7.2.

and Stokes' theorem is verified for this special case.

• Example 10.7.2

Let us verify Stokes' theorem using the vector field $\mathbf{F} = (x^2 - y)\mathbf{i} + 4z\mathbf{j} + x^2\mathbf{k}$, where the closed contour consists of the x and y coordinate axes and that portion of the circle $x^2 + y^2 = a^2$ that lies in the first quadrant with $z = 1$. See Figure 10.7.3.

The line integral consists of three parts:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r}. \quad (10.7.16)$$

Along C_1, x varies while $y = 0$ and $z = 1$. Therefore,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^a (x^2 - 0)\, dx + 4 \cdot 1 \cdot 0 + x^2 \cdot 0 = \frac{a^3}{3}. \quad (10.7.17)$$

Along the circle C_2, we use polar coordinates with $x = a\cos(t)$, $y = a\sin(t)$ and $z = 1$. Therefore,

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} [a^2\cos^2(t) - a\sin(t)][-a\sin(t)\, dt] + 4 \cdot 1 \cdot a\cos(t)\, dt + a^2\cos^2(t) \cdot 0 \quad (10.7.18)$$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} -a^3 \cos^2(t) \sin(t)\, dt + a^2 \sin^2(t)\, dt + 4a \cos(t)\, dt \tag{10.7.19}$$

$$= \frac{a^3}{3} \cos^3(t)\Big|_0^{\pi/2} + \frac{a^2}{2}\left[t - \frac{1}{2}\sin(2t)\right]\Big|_0^{\pi/2} + 4a \sin(t)|_0^{\pi/2} \tag{10.7.20}$$

$$= -\frac{a^3}{3} + \frac{a^2\pi}{4} + 4a, \tag{10.7.21}$$

because $dx = -a\sin(t)\,dt$ and $dy = a\cos(t)\,dt$. Finally, along C_3, y varies with $x = 0$ and $z = 1$. Therefore,

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_a^0 (0^2 - y) \cdot 0 + 4 \cdot 1 \cdot dy + 0^2 \cdot 0 = -4a. \tag{10.7.22}$$

so that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \frac{a^2\pi}{4}. \tag{10.7.23}$$

Turning to the other side of the equation,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y & 4z & x^2 \end{vmatrix} = -4\mathbf{i} - 2x\mathbf{j} + \mathbf{k}. \tag{10.7.24}$$

From the path of our line integral, our unit normal vector must be $\mathbf{n} = \mathbf{k}$. Then,

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^a \int_0^{\pi/2} [-4\mathbf{i} - 2r\cos(\theta)\mathbf{j} + \mathbf{k}] \cdot \mathbf{k}\, r\, d\theta\, dr = \frac{\pi a^2}{4} \tag{10.7.25}$$

and Stokes' theorem is verified for this case.

• Example 10.7.3

Let us verify Stokes' theorem using the vector field $\mathbf{F} = 2yz\mathbf{i} - (x + 3y - 2)\mathbf{j} + (x^2 + z)\mathbf{k}$, where the closed triangular region is that portion of the plane $x + y + z = 1$ that lies in the first octant.

As shown in Figure 10.7.4, the closed line integration consists of three line integrals:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r}. \tag{10.7.26}$$

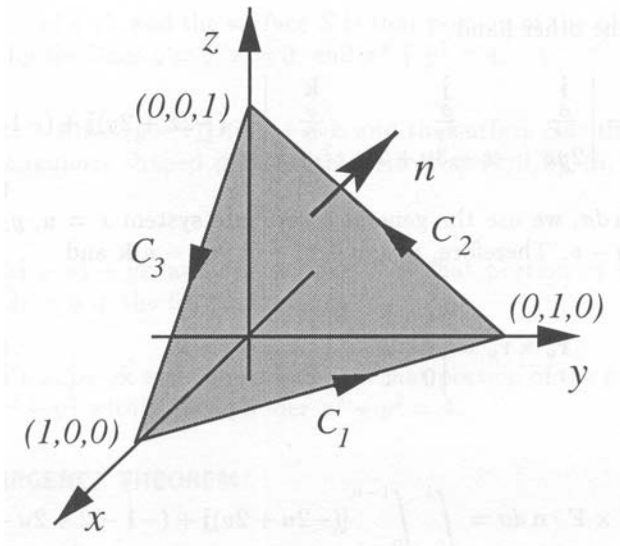

Figure 10.7.4: Diagram for the verification of Stokes' theorem in Example 10.7.3.

Along C_1, $z = 0$ and $y = 1 - x$. Therefore, using x as the independent variable,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_1^0 2(1-x) \cdot 0 \cdot dx - (x + 3 - 3x - 2)(-dx) + (x^2 + 0) \cdot 0$$
$$= -x^2\Big|_1^0 + x\Big|_1^0 = 0. \tag{10.7.27}$$

Along C_2, $x = 0$ and $y = 1 - z$. Thus,

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 2(1-z)z \cdot 0 - (0 + 3 - 3z - 2)(-dz) + (0^2 + z)\,dz$$
$$= -\tfrac{3}{2}z^2 + z + \tfrac{1}{2}z^2\Big|_0^1 = 0. \tag{10.7.28}$$

Finally, along C_3, $y = 0$ and $z = 1 - x$. Hence,

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 2 \cdot 0 \cdot (1-x)\,dx - (x + 0 - 2) \cdot 0 + (x^2 + 1 - x)(-dx)$$
$$= -\tfrac{1}{3}x^3 - x + \tfrac{1}{2}x^2\Big|_0^1 = -\tfrac{5}{6}. \tag{10.7.29}$$

Thus,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = -\tfrac{5}{6}. \tag{10.7.30}$$

On the other hand,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz & -x-3y+2 & x^2+z \end{vmatrix} = (-2x+2y)\mathbf{j} + (-1-2z)\mathbf{k}. \tag{10.7.31}$$

To find $\mathbf{n}\,d\sigma$, we use the general coordinate system $x = u$, $y = v$, and $z = 1 - u - v$. Therefore, $\mathbf{r} = u\mathbf{i} + v\mathbf{j} + (1 - u - v)\mathbf{k}$ and

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}. \tag{10.7.32}$$

Thus,

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\,d\sigma = \int_0^1 \int_0^{1-u} [(-2u+2v)\mathbf{j} + (-1-2+2u+2v)\mathbf{k}] \cdot [\mathbf{i}+\mathbf{j}+\mathbf{k}]\,dv\,du \tag{10.7.33}$$

$$= \int_0^1 \int_0^{1-u} (4v-3)\,dv\,du \tag{10.7.34}$$

$$= \int_0^1 [2(1-u)^2 - 3(1-u)]\,du \tag{10.7.35}$$

$$= \int_0^1 (-1 - u + 2u^2)\,du = -\tfrac{5}{6} \tag{10.7.36}$$

and Stokes' theorem is verified for this case.

Problems

Verify Stokes' theorem using the following vector fields and surfaces:

1. $\mathbf{F} = 5y\mathbf{i} - 5x\mathbf{j} + 3z\mathbf{k}$ and the surface S is that portion of the plane $z = 1$ with the square at the vertices $(0,0,1)$, $(1,0,1)$, $(1,1,1)$, and $(0,1,1)$.

2. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ and the surface S is the rectangular portion of the plane $z = 2$ defined by the corners $(0,0,2)$, $(2,0,2)$, $(2,1,2)$, and $(0,1,2)$.

3. $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ and the surface S is the triangular portion of the plane $z = 1$ defined by the vertices $(0,0,1)$, $(2,0,1)$, and $(0,2,1)$.

4. $\mathbf{F} = 2z\mathbf{i} - 3x\mathbf{j} + 4y\mathbf{k}$ and the surface S is that portion of the plane $z = 5$ within the cylinder $x^2 + y^2 = 4$.

5. $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ and the surface S is that portion of the plane $z = 3$ bounded by the lines $y = 0$, $x = 0$, and $x^2 + y^2 = 4$.

6. $\mathbf{F} = (2z + x)\mathbf{i} + (y - z)\mathbf{j} + (x + y)\mathbf{k}$ and the surface S is the interior of the triangularly shaped plane with vertices at $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

7. $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ and the surface S is that portion of the plane $2x + y + 2z = 6$ in the first octant.

8. $\mathbf{F} = x\mathbf{i} + xz\mathbf{j} + y\mathbf{k}$ and the surface S is that portion of the paraboloid $z = 9 - x^2 - y^2$ within the cylinder $x^2 + y^2 = 4$.

10.8 DIVERGENCE THEOREM

Although Stokes' theorem is useful in computing closed line integrals, it is usually very difficult to go the other way and convert a surface integral into a closed line integral because the integrand must have a very special form, namely $\nabla \times \mathbf{F} \cdot \mathbf{n}$. In this section we introduce a theorem that allows with equal facility the conversion of a closed surface integral into a volume integral and *vice versa*. Furthermore, if we can convert a given surface integral into a closed one by the introduction of a simple surface (for example, closing a hemispheric surface by adding an equatorial plate), it may be easier to use the divergence theorem and subtract off the contribution from the new surface integral rather than do the original problem.

This relationship between a closed surface integral and a volume integral involving the divergence operator is

The Divergence or Gauss' Theorem: *Let V be a closed and bounded region in three dimensional space with a piece-wise smooth boundary S that is oriented outward. Let $\mathbf{F} = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ be a vector field for which P, Q, and R are continuous and have continuous first partial derivatives in a region of three dimensional space containing V. Then*

$$\oiint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_V \nabla \cdot \mathbf{F}\, dV. \qquad \textbf{(10.8.1)}$$

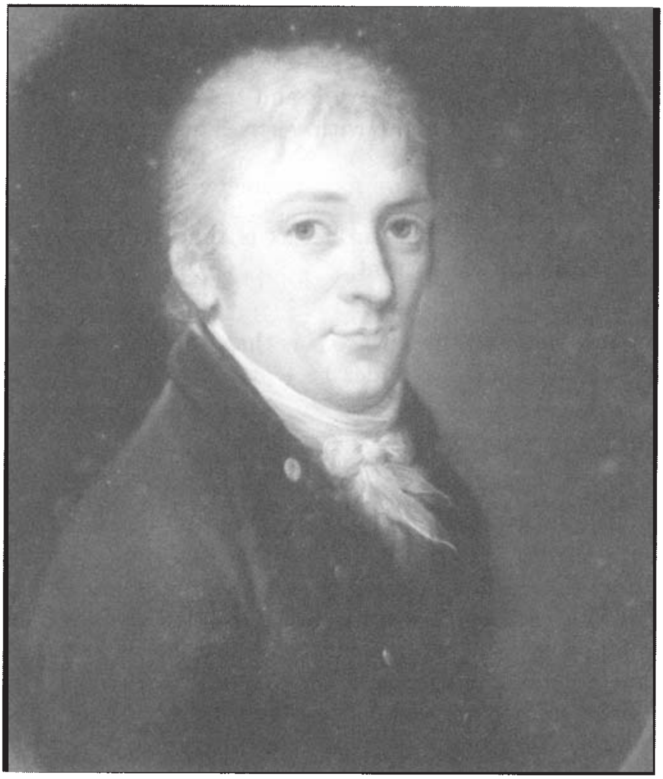

Figure 10.8.1: Carl Friedrich Gauss (1777–1855), the prince of mathematicians, must be on the list of the greatest mathematicians who ever lived. Gauss, a child prodigy, is almost as well known for what he did not publish during his lifetime as for what he did. This is true of Gauss' divergence theorem which he proved while working on the theory of gravitation. It was only when his notebooks were published in 1898 that his precedence over the published work of Ostrogradsky (1801–1862) was established. (Portrait courtesy of Photo AKG, London.)

Here, the circle on the double integral signs denotes a closed surface integral.

A nonrigorous proof of Gauss' theorem is as follows. Imagine that our volume V is broken down into small elements $d\tau$ of volume of any shape so long as they include all of the original volume. In general, the surfaces of these elements are composed of common interfaces between adjoining elements. However, for the elements at the periphery of V, part of their surface will be part of the surface S that encloses V. Now $d\Phi = \nabla \cdot \mathbf{F}\, d\tau$ is the net flux of the vector $\mathbf{F}$ out from the element $d\tau$.

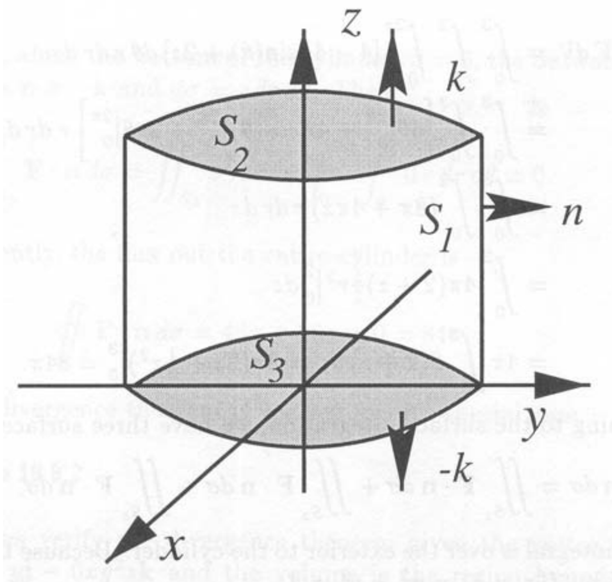

Figure 10.8.2: Diagram for the verification of the divergence theorem in Example 10.8.1.

At the common interface between elements, the flux *out* of one element equals the flux *into* its neighbor. Therefore, the sum of all such terms yields

$$\Phi = \iiint_V \nabla \cdot \mathbf{F}\, d\tau \tag{10.8.2}$$

and all the contributions from these common interfaces cancel; only the contribution from the parts on the outer surface S will be left. These contributions, when added together, give $\oiint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ over S and the proof is completed. □

• **Example 10.8.1**

Let us verify the divergence theorem using the vector field $\mathbf{F} = 4x\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k}$ and the enclosed surface is the cylinder $x^2 + y^2 = 4$, $z = 0$, and $z = 3$. See Figure 10.8.2.

We begin by computing the volume integration. Because

$$\nabla \cdot \mathbf{F} = \frac{\partial(4x)}{\partial x} + \frac{\partial(-2y^2)}{\partial y} + \frac{\partial(z^2)}{\partial z} = 4 - 4y + 2z, \tag{10.8.3}$$

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = \iiint_V (4 - 4y + 2z)\, dV \tag{10.8.4}$$

$$\iiint_V \nabla\cdot\mathbf{F}\,dV = \int_0^3\int_0^2\int_0^{2\pi} [4-4r\sin(\theta)+2z]\,d\theta\,r\,dr\,dz \qquad (10.8.5)$$

$$= \int_0^3\int_0^2 \left[4\theta\big|_0^{2\pi} + 4r\cos(\theta)\big|_0^{2\pi} + 2z\theta\big|_0^{2\pi}\right] r\,dr\,dz \qquad (10.8.6)$$

$$= \int_0^3\int_0^2 (8\pi+4\pi z)\,r\,dr\,dz \qquad (10.8.7)$$

$$= \int_0^3 4\pi(2+z)\tfrac{1}{2}r^2\big|_0^2\,dz \qquad (10.8.8)$$

$$= 4\pi\int_0^3 2(2+z)\,dz = 8\pi(2z+\tfrac{1}{2}z^2)\big|_0^3 = 84\pi. \qquad (10.8.9)$$

Turning to the surface integration, we have three surfaces:

$$\oiint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,d\sigma + \iint_{S_2}\mathbf{F}\cdot\mathbf{n}\,d\sigma + \iint_{S_3}\mathbf{F}\cdot\mathbf{n}\,d\sigma. \qquad (10.8.10)$$

The first integral is over the exterior to the cylinder. Because the surface is defined by $f(x,y,z)=x^2+y^2=4$,

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\mathbf{i}+2y\mathbf{j}}{\sqrt{4x^2+4y^2}} = \frac{x}{2}\mathbf{i}+\frac{y}{2}\mathbf{j}. \qquad (10.8.11)$$

Therefore,

$$\iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_{S_1}(2x^2-y^3)\,d\sigma \qquad (10.8.12)$$

$$= \int_0^3\int_0^{2\pi}\left\{2[2\cos(\theta)]^2-[2\sin(\theta)]^3\right\}\,2\,d\theta\,dz \qquad (10.8.13)$$

$$= 8\int_0^3\int_0^{2\pi}\left\{\tfrac{1}{2}[1+\cos(2\theta)]-\sin(\theta)+\cos^2(\theta)\sin(\theta)\right\}2\,d\theta\,dz \qquad (10.8.14)$$

$$= 16\int_0^3\left[\tfrac{1}{2}\theta+\tfrac{1}{4}\sin(2\theta)+\cos(\theta)-\tfrac{1}{3}\cos^3(\theta)\right]\Big|_0^{2\pi}\,dz \qquad (10.8.15)$$

$$= 16\pi\int_0^3 dz = 48\pi, \qquad (10.8.16)$$

because $x=2\cos(\theta)$, $y=2\sin(\theta)$ and $d\sigma=2\,d\theta\,dz$ in cylindrical coordinates.

Along the top of the cylinder, $z=3$, the outward pointing normal is $\mathbf{n}=\mathbf{k}$ and $d\sigma=r\,dr\,d\theta$. Then,

$$\iint_{S_2}\mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_{S_2} z^2\,d\sigma = \int_0^{2\pi}\int_0^2 9\,r\,dr\,d\theta = 2\pi\times 9\times 2 = 36\pi. \qquad (10.8.17)$$

However, along the bottom of the cylinder, $z = 0$, the outward pointing normal is $\mathbf{n} = -\mathbf{k}$ and $d\sigma = r\,dr\,d\theta$. Then,

$$\iint_{S_3} \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_{S_3} z^2\,d\sigma = \int_0^{2\pi}\int_0^2 0\,r\,dr\,d\theta = 0. \qquad (10.8.18)$$

Consequently, the flux out the entire cylinder is

$$\oiint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = 48\pi + 36\pi + 0 = 84\pi \qquad (10.8.19)$$

and the divergence theorem is verified for this special case.

• **Example 10.8.2**

Let us verify the divergence theorem given the vector field $\mathbf{F} = 3x^2y^2\mathbf{i} + y\mathbf{j} - 6xy^2z\mathbf{k}$ and the volume is the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 2y$. See Figure 10.8.3.

Computing the divergence,

$$\nabla\cdot\mathbf{F} = \frac{\partial(3x^2y^2)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(-6xy^2z)}{\partial z} = 6xy^2 + 1 - 6xy^2 = 1. \qquad (10.8.20)$$

Then,

$$\iiint_V \nabla\cdot\mathbf{F}\,dV = \iiint_V dV \qquad (10.8.21)$$

$$= \int_0^{\pi}\int_0^{2\sin(\theta)}\int_{r^2}^{2r\sin(\theta)} dz\,r\,dr\,d\theta \qquad (10.8.22)$$

$$= \int_0^{\pi}\int_0^{2\sin(\theta)} [2r\sin(\theta) - r^2]\,r\,dr\,d\theta \qquad (10.8.23)$$

$$= \int_0^{\pi}\left[\tfrac{2}{3}r^3\Big|_0^{2\sin(\theta)}\sin(\theta) - \tfrac{1}{4}r^4\Big|_0^{2\sin(\theta)}\right]d\theta \qquad (10.8.24)$$

$$= \int_0^{\pi}\left[\tfrac{16}{3}\sin^4(\theta) - 4\sin^4(\theta)\right]d\theta \qquad (10.8.25)$$

$$= \int_0^{\pi}\tfrac{4}{3}\sin^4(\theta)\,d\theta \qquad (10.8.26)$$

$$= \tfrac{1}{3}\int_0^{\pi}[1 - 2\cos(2\theta) + \cos^2(2\theta)]\,d\theta \qquad (10.8.27)$$

$$= \tfrac{1}{3}\left[\theta\Big|_0^{\pi} - \sin(2\theta)\Big|_0^{\pi} + \tfrac{1}{2}\theta\Big|_0^{\pi} + \tfrac{1}{8}\sin(4\theta)\Big|_0^{\pi}\right] = \tfrac{\pi}{2}. \qquad (10.8.28)$$

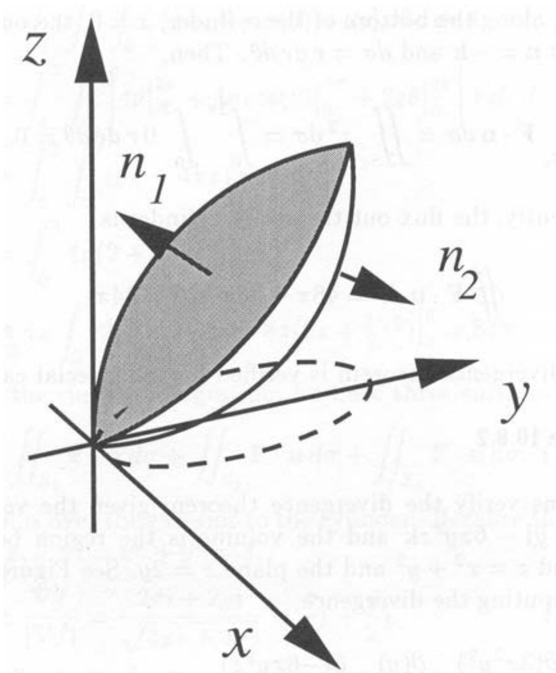

Figure 10.8.3: Diagram for the verification of the divergence theorem in Example 10.8.2. The curve $r = 2\sin(\theta)$ is denoted by a dashed line.

The limits in the radial direction are given by the intersection of the paraboloid and plane: $r^2 = 2r\sin(\theta)$ or $r = 2\sin(\theta)$ and y is greater than zero.

Turning to the surface integration, we have two surfaces:

$$\oiint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,d\sigma + \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,d\sigma, \tag{10.8.29}$$

where S_1 is the plane $z = 2y$ and S_2 is the paraboloid. For either surface, polar coordinates are best so that $x = r\cos(\theta)$, $y = r\sin(\theta)$. For the integration over the plane, $z = 2r\sin(\theta)$. Therefore,

$$\mathbf{r} = r\cos(\theta)\mathbf{i} + r\sin(\theta)\mathbf{j} + 2r\sin(\theta)\mathbf{k} \tag{10.8.30}$$

so that

$$\mathbf{r}_r = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} + 2\sin(\theta)\mathbf{k} \tag{10.8.31}$$

and

$$\mathbf{r}_\theta = -r\sin(\theta)\mathbf{i} + r\cos(\theta)\mathbf{j} + 2r\cos(\theta)\mathbf{k}. \tag{10.8.32}$$

Then,

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\theta) & \sin(\theta) & 2\sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) & 2r\cos(\theta) \end{vmatrix} = -2r\mathbf{j} + r\mathbf{k}. \tag{10.8.33}$$

This is an outwardly pointing normal so that we can immediately set up the surface integral:

$$\iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{\pi}\int_0^{2\sin(\theta)} \{3r^4\cos^2(\theta)\sin^2(\theta)\mathbf{i} + r\sin(\theta)\mathbf{j} - 6[2r\sin(\theta)][r\cos(\theta)][r^2\sin^2(\theta)]\mathbf{k}\}\cdot(-2r\mathbf{j}+r\mathbf{k})\,dr\,d\theta \tag{10.8.34}$$

$$= \int_0^{\pi}\int_0^{2\sin(\theta)} \left[-2r^2\sin(\theta) - 12r^5\sin^3(\theta)\cos(\theta)\right] dr\,d\theta \tag{10.8.35}$$

$$= \int_0^{\pi}\left[-\tfrac{2}{3}r^3\Big|_0^{2\sin(\theta)}\sin(\theta) - 2r^6\Big|_0^{2\sin(\theta)}\sin^3(\theta)\cos(\theta)\right] d\theta \tag{10.8.36}$$

$$= \int_0^{\pi}\left[-\tfrac{16}{3}\sin^4(\theta) - 128\sin^9(\theta)\cos(\theta)\right] d\theta \tag{10.8.37}$$

$$= -\tfrac{4}{3}\left[\theta\Big|_0^{\pi} - \sin(2\theta)\Big|_0^{\pi} + \tfrac{1}{2}\theta\Big|_0^{\pi} + \tfrac{1}{8}\sin(4\theta)\Big|_0^{\pi}\right] - \tfrac{64}{5}\sin^{10}(\theta)\Big|_0^{\pi} \tag{10.8.38}$$

$$= -2\pi. \tag{10.8.39}$$

For the surface of the paraboloid,

$$\mathbf{r} = r\cos(\theta)\mathbf{i} + r\sin(\theta)\mathbf{j} + r^2\mathbf{k} \tag{10.8.40}$$

so that

$$\mathbf{r}_r = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} + 2r\mathbf{k} \tag{10.8.41}$$

and

$$\mathbf{r}_\theta = -r\sin(\theta)\mathbf{i} + r\cos(\theta)\mathbf{j}. \tag{10.8.42}$$

Then,

$$\mathbf{r}_r\times\mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\theta) & \sin(\theta) & 2r \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{vmatrix} \tag{10.8.43}$$

$$= -2r^2\cos(\theta)\mathbf{i} - 2r^2\sin(\theta)\mathbf{j} + r\mathbf{k}. \tag{10.8.44}$$

This is an inwardly pointing normal so that we must take the negative of it before we do the surface integral. Then,

$$\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{\pi}\int_0^{2\sin(\theta)} \{3r^4\cos^2(\theta)\sin^2(\theta)\mathbf{i} + r\sin(\theta)\mathbf{j} - 6r^2[r\cos(\theta)][r^2\sin^2(\theta)]\mathbf{k}\}$$

$$\cdot \left[2r^2\cos(\theta)\mathbf{i} + 2r^2\sin(\theta)\mathbf{j} - r\mathbf{k}\right] dr\, d\theta \tag{10.8.45}$$

$$\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\, d\sigma = \int_0^{\pi}\int_0^{2\sin(\theta)} \left[6r^6\cos^3(\theta)\sin^2(\theta) + 2r^3\sin^2(\theta) + 6r^6\cos(\theta)\sin^2(\theta)\right] dr\, d\theta \tag{10.8.46}$$

$$= \int_0^{\pi} \left[\tfrac{6}{7}r^7\Big|_0^{2\sin(\theta)}\cos^3(\theta)\sin^2(\theta) + \tfrac{1}{2}r^4\Big|_0^{2\sin(\theta)}\sin^2(\theta) + \tfrac{6}{7}r^7\Big|_0^{2\sin(\theta)}\cos(\theta)\sin^2(\theta)\right] d\theta \tag{10.8.47}$$

$$= \int_0^{\pi} \left\{\tfrac{768}{7}\sin^9(\theta)[1-\sin^2(\theta)]\cos(\theta) + 8\sin^6(\theta) + \tfrac{768}{7}\sin^9(\theta)\cos(\theta)\right\} d\theta \tag{10.8.48}$$

$$= \tfrac{1536}{70}\sin^{10}(\theta)\Big|_0^{\pi} - \tfrac{64}{7}\sin^{12}(\theta)\Big|_0^{\pi} + \int_0^{\pi} [1-\cos(2\theta)]^3\, d\theta \tag{10.8.49}$$

$$= \int_0^{\pi}\left\{1 - 3\cos(2\theta) + 3\cos^2(2\theta) - \cos(2\theta)[1-\sin^2(2\theta)]\right\} d\theta \tag{10.8.50}$$

$$= \theta\Big|_0^{\pi} - \tfrac{3}{2}\sin(2\theta)\Big|_0^{\pi} + \tfrac{3}{2}[\theta + \tfrac{1}{4}\sin(4\theta)]\Big|_0^{\pi} - \tfrac{1}{2}\sin(2\theta)\Big|_0^{\pi} + \tfrac{1}{3}\sin^3(2\theta)\Big|_0^{\pi} \tag{10.8.51}$$

$$= \pi + \tfrac{3}{2}\pi = \tfrac{5}{2}\pi. \tag{10.8.52}$$

Consequently,

$$\oiint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma = -2\pi + \tfrac{5}{2}\pi = \tfrac{1}{2}\pi \tag{10.8.53}$$

and the divergence theorem is verified for this special case.

• Example 10.8.3: Archimedes' Principle

Consider a solid[5] of volume V and surface S that is immersed in a vessel filled with a fluid of density ρ. The pressure field p in the fluid is a function of the distance from the liquid/air interface and equals

$$p = p_0 - \rho g z, \tag{10.8.54}$$

[5] Adapted from Altintas, A., 1990: Archimedes' principle as an application of the divergence theorem. *IEEE Trans. Educ.*, **33**, 222. ©IEEE.

where g is the gravitational acceleration, z is the vertical distance measured from the interface (increasing in the $\mathbf{k}$ direction), and p_0 is the constant pressure along the liquid/air interface.

If we define $\mathbf{F} = -p\mathbf{k}$, then $\mathbf{F} \cdot \mathbf{n}\, d\sigma$ is the vertical component of the force on the surface due to the pressure and $\oint\!\!\oint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ is the total lift. Using the divergence theorem and noting that $\nabla \cdot \mathbf{F} = \rho g$, the total lift also equals

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = \rho g \iiint_V dV = \rho g V, \tag{10.8.55}$$

which is the weight of the displaced liquid. This is *Archimedes' principle*: the buoyant force on a solid immersed in a fluid of constant density equals the weight of the fluid displaced.

• Example 10.8.4: Conservation of charge

Let a charge of density ρ flow with an average velocity $\mathbf{v}$. Then the charge crossing the element $d\mathbf{S}$ per unit time is $\rho\mathbf{v} \cdot d\mathbf{S} = \mathbf{J} \cdot d\mathbf{S}$, where $\mathbf{J}$ is defined as the conduction current vector or current density vector. The current across any surface drawn in the medium is $\oint\!\!\oint_S \mathbf{J} \cdot d\mathbf{S}$.

The total charge inside the closed surface is $\iiint_V \rho\, dV$. If there are no sources or sinks inside the surface, the rate at which the charge is decreasing is $-\iiint_V \rho_t\, dV$. Because this change is due to the outward flow of charge,

$$-\iiint_V \frac{\partial \rho}{\partial t}\, dV = \oint\!\!\oint_S \mathbf{J} \cdot d\mathbf{S}. \tag{10.8.56}$$

Applying the divergence theorem,

$$\iiint_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} \right) dV = 0. \tag{10.8.57}$$

Because the result holds true for any arbitrary volume, the integrand must vanish identically and we have the equation of continuity or the *equation of conservation of charge*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \tag{10.8.58}$$

Problems

Verify the divergence theorem using the following vector fields and volumes:

1. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ and the volume V is the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$.

2. $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$ and the volume V is the cube bounded by $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$.

3. $\mathbf{F} = (y - x)\mathbf{i} + (z - y)\mathbf{j} + (y - x)\mathbf{k}$ and the volume V is the cube bounded by $-1 \le x \le 1$, $-1 \le y \le 1$, and $-1 \le z \le 1$.

4. $\mathbf{F} = x^2\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and the volume V is the cylinder defined by the surfaces $x^2 + y^2 = 1$, $z = 0$, and $z = 1$.

5. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ and the volume V is the cylinder defined by the surfaces $x^2 + y^2 = 4$, $z = 0$, and $z = 1$.

6. $\mathbf{F} = y^2\mathbf{i} + xz^3\mathbf{j} + (z - 1)^2\mathbf{k}$ and the volume V is the cylinder bounded by the surface $x^2 + y^2 = 4$ and the planes $z = 1$ and $z = 5$.

7. $\mathbf{F} = 6xy\mathbf{i} + 4yz\mathbf{j} + xe^{-y}\mathbf{k}$ and the volume V is that region created by the plane $x + y + z = 1$ and the three coordinate planes.

8. $\mathbf{F} = y\mathbf{i} + xy\mathbf{j} - z\mathbf{k}$ and the volume V is that solid created by the paraboloid $z = x^2 + y^2$ and plane $z = 1$.

Chapter 11
Linear Algebra

Linear algebra involves the systematic solving of linear algebraic or differential equations. These equations arise in a wide variety of situations. They usually involve some system, either electrical, mechanical, or even human, where two or more components are interacting with each other. In this chapter we present efficient techniques for expressing these systems and their solution.

11.1 FUNDAMENTALS OF LINEAR ALGEBRA

In this chapter we shall study the solution of m simultaneous linear equations in n unknowns $x_1, x_2, x_3, \ldots, x_n$ of the form:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m,
\end{aligned}
\tag{11.1.1}
$$

where the a's and b's are known real or complex numbers. *Matrix algebra* allows us to solve these systems. First, succinct notation is introduced so that we can replace (11.1.1) with rather simple expressions. Then a set of rules is used to manipulate these simple expressions. In this section we focus on developing these simple expressions.

The fundamental quantity in linear algebra is the *matrix*. A matrix is an ordered rectangular array of numbers or mathematical expressions. We shall use upper case letters to denote them. The $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{ij} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & a_{m3} & \cdot & \cdot & \cdot & a_{mn} \end{pmatrix} \tag{11.1.2}$$

has m *rows* and n *columns*. The *order* (or size) of a matrix is determined by the number of rows and columns; (11.1.2) is of order m by n. If $m = n$, the matrix is a *square* matrix; otherwise, A is *rectangular*. The numbers or expressions in the array a_{ij} are the *elements* of A and may be either real or complex. When all of the elements are real, A is a *real matrix*. If some or all of the elements are complex, then A is a *complex matrix*. For a square matrix, the diagonal from the top left corner to the bottom right corner is the *principal diagonal*.

From the limitless number of possible matrices, certain ones appear with sufficient regularity that they are given special names. A *zero* matrix (sometimes called a *null* matrix) has all of its elements equal to zero. It fulfills the role in matrix algebra that is analogous to that of zero in scalar algebra. The *unit* or *identity* matrix is a $n \times n$ matrix having 1's along its principal diagonal and zero everywhere else. The unit matrix serves essentially the same purpose in matrix algebra as does the number one in scalar algebra. A *symmetric* matrix is one where $a_{ij} = a_{ji}$ for all i and j.

• Example 11.1.1

Examples of zero, identity, and symmetric matrices are

$$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 1 & 0 \\ 4 & 0 & 5 \end{pmatrix}, \tag{11.1.3}$$

respectively.

A special class of matrices are *column vectors* and *row vectors*:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}. \tag{11.1.4}$$

We denote row and column vectors by lower case, boldface letters. The length or *norm* of the vector $\mathbf{x}$ of n elements is

$$||\mathbf{x}|| = \left(\sum_{k=1}^{n} x_k^2\right)^{1/2}. \tag{11.1.5}$$

Two matrices A and B are equal if and only if $a_{ij} = b_{ij}$ for all possible i and j and they have the same dimensions.

Having defined a matrix, let us explore some of its arithmetic properties. For two matrices A and B with the same dimensions (conformable for addition), the matrix $C = A + B$ contains the elements $c_{ij} = a_{ij} + b_{ij}$. Similarly, $C = A - B$ contains the elements $c_{ij} = a_{ij} - b_{ij}$. Because the order of addition does not matter, addition is *commutative*: $A + B = B + A$.

Consider now a scalar constant k. The product kA is formed by multiplying every element of A by k. Thus the matrix kA has elements ka_{ij}.

So far the rules for matrix arithmetic have conformed to their scalar counterparts. However, there are several possible ways of multiplying two matrices together. For example, we might simply multiply together the corresponding elements from each matrix. As we will see, the multiplication rule is designed to facilitate the solution of linear equations.

We begin by requiring that the dimensions of A be $m \times n$ while for B they are $n \times p$. That is, the number of columns in A must equal the number of rows in B. The matrices A and B are then said to be *conformable* for multiplication. If this is true, then $C = AB$ will be a matrix $m \times p$, where its elements equal

$$c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}. \tag{11.1.6}$$

The right side of (11.1.6) is referred to as an *inner product* of the ith row of A and the jth column of B. Although (11.1.6) is the method used with a computer, an easier method for human computation is as a running sum of the products given by successive elements of the ith row of A and the corresponding elements of the jth column of B.

The product AA is usually written A^2; the product AAA, A^3, and so forth.

• **Example 11.1.2**

If

$$A = \begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \tag{11.1.7}$$

then

$$AB = \begin{pmatrix} [(-1)(1)+(4)(3)] & [(-1)(2)+(4)(4)] \\ [(2)(1)+(-3)(3)] & [(2)(2)+(-3)(4)] \end{pmatrix} \tag{11.1.8}$$

$$= \begin{pmatrix} 11 & 14 \\ -7 & -8 \end{pmatrix}. \tag{11.1.9}$$

Matrix multiplication is associative and distributive with respect to addition:

$$(kA)B = k(AB) = A(kB), \tag{11.1.10}$$

$$A(BC) = (AB)C, \tag{11.1.11}$$

$$(A+B)C = AC + BC \tag{11.1.12}$$

and

$$C(A+B) = CA + CB. \tag{11.1.13}$$

On the other hand, matrix multiplication is *not commutative*. In general, $AB \neq BA$.

• **Example 11.1.3**

Does $AB = BA$ if

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}? \tag{11.1.14}$$

Because

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{11.1.15}$$

and

$$BA = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \tag{11.1.16}$$

$$AB \neq BA. \tag{11.1.17}$$

• **Example 11.1.4**

Given

$$A = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{11.1.18}$$

find the product AB.

Performing the calculation, we find that

$$AB = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{11.1.19}$$

The point here is that just because $AB = 0$, this does *not* imply that either A or B equals the zero matrix.

We cannot properly speak of division when we are dealing with matrices. Nevertheless, a matrix A is said to be *nonsingular* or *invertible* if there exists a matrix B such that $AB = BA = I$. This matrix B is the multiplicative inverse of A or simply the *inverse* of A, written A^{-1}. A $n \times n$ matrix is *singular* if it does not have a multiplicative inverse.

• **Example 11.1.5**

If

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix}, \tag{11.1.20}$$

let us verify that its inverse is

$$A^{-1} = \begin{pmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{pmatrix}. \tag{11.1.21}$$

We perform the check by finding AA^{-1} or $A^{-1}A$,

$$AA^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{11.1.22}$$

In a later section we will show how to compute the inverse, given A.

Another matrix operation is transposition. The *transpose* of a matrix A with dimensions $m \times n$ is another matrix, written A^T, where we have interchanged the rows and columns from A. Clearly, $(A^T)^T = A$ as well as $(A + B)^T = A^T + B^T$ and $(kA)^T = kA^T$. If A and B are conformable for multiplication, then $(AB)^T = B^T A^T$. Note the reversal of order between the two sides. To prove this last result, we first show that the results are true for two 3×3 matrices A and B and then generalize to larger matrices.

Having introduced some of the basic concepts of linear algebra, we are ready to rewrite (11.1.1) in a canonical form so that we may present techniques for its solution. We begin by writing (11.1.1) as a single column vector:

$$\begin{pmatrix} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n \\ \vdots & & \vdots & & \vdots & & \vdots \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_m \end{pmatrix}. \tag{11.1.23}$$

On the left side of (11.1.23) we can use the multiplication rule to write

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_m \end{pmatrix} \tag{11.1.24}$$

or

$$A\mathbf{x} = \mathbf{b}, \tag{11.1.25}$$

where $\mathbf{x}$ is the solution vector. If $\mathbf{b} = \mathbf{0}$, we have a *homogeneous* set of equations; otherwise, we have a *nonhomogeneous* set. In the next few sections, we will give a number of methods for finding $\mathbf{x}$.

• Example 11.1.6: Solution of a tridiagonal system

A common problem in linear algebra involves solving systems such as

$$b_1y_1 + c_1y_2 = d_1 \tag{11.1.26}$$

$$a_2y_1 + b_2y_2 + c_2y_3 = d_2 \tag{11.1.27}$$

$$\vdots$$

$$a_{N-1}y_{N-2} + b_{N-1}y_{N-1} + c_{N-1}y_N = d_{N-1} \tag{11.1.28}$$

$$b_Ny_{N-1} + c_Ny_N = d_N. \tag{11.1.29}$$

Such systems arise in the numerical solution of ordinary and partial differential equations.

We begin our analysis by rewriting (11.1.26)–(11.1.29) in the matrix notation:

$$\begin{pmatrix} b_1 & c_1 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & \cdots & 0 & 0 & 0 \\ 0 & a_3 & b_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{N-1} & b_{N-1} & c_{N-1} \\ 0 & 0 & 0 & \cdots & 0 & a_N & b_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-1} \\ y_N \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{N-1} \\ d_N \end{pmatrix}. \quad \textbf{(11.1.30)}$$

The matrix in (11.1.30) is an example of a *banded matrix*: a matrix where all of the elements in each row are zero except for the diagonal element and a limited number on either side of it. In our particular case, we have a *tridiagonal* matrix in which only the diagonal element and the elements immediately to its left and right in each row are nonzero.

Consider the nth equation. We can eliminate a_n by multiplying the $(n-1)$th equation by a_n/b_{n-1} and subtracting this new equation from the nth equation. The values of b_n and d_n become

$$b'_n = b_n - a_n c_{n-1}/b_{n-1} \quad \textbf{(11.1.31)}$$

and

$$d'_n = d_n - a_n d_{n-1}/b_{n-1} \quad \textbf{(11.1.32)}$$

for $n = 2, 3, \ldots, N$. The coefficient c_n is unaffected. Because elements a_1 and c_N are never involved, their values can be anything or they can be left undefined. The new system of equations may be written

$$\begin{pmatrix} b'_1 & c_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & b'_2 & c_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & b'_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & b'_{N-1} & c_{N-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & b'_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-1} \\ y_N \end{pmatrix} = \begin{pmatrix} d'_1 \\ d'_2 \\ d'_3 \\ \vdots \\ d'_{N-1} \\ d'_N \end{pmatrix}. \quad \textbf{(11.1.33)}$$

The matrix in (11.1.33) is in *upper triangular* form because all of the elements below the principal diagonal are zero. This is particularly useful because y_n may be computed by *back substitution*. That is, we first compute y_N. Next, we calculate y_{N-1} in terms of y_N. The solution y_{N-2} may then be computed in terms of y_N and y_{N-1}. We continue this process until we find y_1 in terms of $y_N, y_{N-1}, \ldots, y_2$. In the present case, we have the rather simple:

$$y_N = d'_N/b'_N \quad \textbf{(11.1.34)}$$

and

$$y_n = (d'_n - c_n d'_{n+1})/b'_n \quad \textbf{(11.1.35)}$$

for $n = N-1, N-2, \ldots, 2, 1$.

As we shall show shortly, this is an example of solving a system of linear equations by Gaussian elimination. For a tridiagonal case, we have the advantage that the solution can be expressed in terms of a recurrence relationship, a very convenient feature from a computational point of view. This algorithm is very robust, being stable[1] as long as $|a_i + c_i| < |b_i|$. By stability, we mean that if we change $\mathbf{b}$ by $\Delta\mathbf{b}$ so that $\mathbf{x}$ changes by $\Delta\mathbf{x}$, then $||\Delta\mathbf{x}|| < M\epsilon$, where $\epsilon \geq ||\Delta\mathbf{b}||$, $0 < M < \infty$, for *any* N.

Problems

Given $A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, find

1. $A + B$, $B + A$
2. $A - B$, $B - A$
3. $3A - 2B$, $3(2A - B)$
4. A^T,B^T,$(B^T)^T$
5. $(A + B)^T$, $A^T + B^T$
6. $B + B^T$, $B - B^T$
7. AB,A^TB,BA,B^TA
8. A^2, B^2
9. BB^T, B^TB
10. $A^2 - 3A + I$
11. $A^3 + 2A$
12. $A^4 - 4A^2 + 2I$

Can multiplication occur between the following matrices? If so, compute it.

13. $\begin{pmatrix} 3 & 5 & 1 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & 1 \\ 1 & 3 \end{pmatrix}$

14. $\begin{pmatrix} -2 & 4 \\ -4 & 6 \\ -6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

15. $\begin{pmatrix} 1 & 4 & 2 \\ 0 & 0 & 4 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$

16. $\begin{pmatrix} 4 & 6 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \end{pmatrix}$

17. $\begin{pmatrix} 6 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 1 & 4 \\ 2 & 0 & 6 \end{pmatrix}$

If $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 3 & 1 \end{pmatrix}$ verify that

18. $7A = 4A + 3A$,
19. $10A = 5(2A)$,
20. $(A^T)^T = A$.

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[1] Torii, T., 1966: Inversion of tridiagonal matrices and the stability of tridiagonal systems of linear systems. *Tech. Rep. Osaka Univ.*, **16**, 403–414.

If $A = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -2 \\ 4 & 0 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, verify that

21. $(A + B) + C = A + (B + C)$, 22. $(AB)C = A(BC)$,
23. $A(B + C) = AB + AC$, 24. $(A + B)C = AC + BC$.

Verify that the following A^{-1} are indeed the inverse of A:

25. $A = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$ $\qquad A^{-1} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$

26. $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\qquad A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Write the following linear systems of equations in matrix form: $A\mathbf{x} = \mathbf{b}$.

27.

$$x_1 - 2x_2 = 5$$
$$3x_1 + x_2 = 1$$

28.

$$2x_1 + x_2 + 4x_3 = 2$$
$$4x_1 + 2x_2 + 5x_3 = 6$$
$$6x_1 - 3x_2 + 5x_3 = 2$$

29.

$$x_2 + 2x_3 + 3x_4 = 2$$
$$3x_1 - 4x_3 - 4x_4 = 5$$
$$x_1 + x_2 + x_3 + x_4 = -3$$
$$2x_1 - 3x_2 + x_3 - 3x_4 = 7.$$

11.2 DETERMINANTS

Determinants appear naturally during the solution of simultaneous equations. Consider, for example, two simultaneous equations with two unknowns x_1 and x_2,

$$a_{11}x_1 + a_{12}x_2 = b_1 \tag{11.2.1}$$

and

$$a_{21}x_1 + a_{22}x_2 = b_2. \tag{11.2.2}$$

The solution to these equations for the value of x_1 and x_2 is

$$x_1 = \frac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}} \tag{11.2.3}$$

and

$$x_2 = \frac{b_2 a_{11} - a_{21} b_1}{a_{11} a_{22} - a_{12} a_{21}}. \tag{11.2.4}$$

Note that the denominator of (11.2.3) and (11.2.4) are the same. This term, which will always appear in the solution of 2 × 2 systems, is formally given the name of *determinant* and written

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \tag{11.2.5}$$

Although determinants have their origin in the solution of systems of equations, any square array of numbers or expressions possesses a unique determinant, independent of whether it is involved in a system of equations or not. This determinant is evaluated (or expanded) according to a formal rule known as *Laplace's expansion of cofactors.*[2] The process revolves around expanding the determinant using any arbitrary column *or* row of A. If the ith row or jth column is chosen, the determinant is given by

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \tag{11.2.6}$$

$$= a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}, \tag{11.2.7}$$

where A_{ij}, the *cofactor* of a_{ij}, equals $(-1)^{i+j}M_{ij}$. The minor M_{ij} is the determinant of the $(n-1) \times (n-1)$ submatrix obtained by deleting row i, column j of A. This rule, of course, was chosen so that determinants are still useful in solving systems of equations.

• **Example 11.2.1**

Let us evaluate

$$\begin{vmatrix} 2 & -1 & 2 \\ 1 & 3 & 2 \\ 5 & 1 & 6 \end{vmatrix}$$

by an expansion in cofactors.

Using the first column,

$$\begin{vmatrix} 2 & -1 & 2 \\ 1 & 3 & 2 \\ 5 & 1 & 6 \end{vmatrix} = 2(-1)^2 \begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} + 1(-1)^3 \begin{vmatrix} -1 & 2 \\ 1 & 6 \end{vmatrix} + 5(-1)^4 \begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} \tag{11.2.8}$$

$$= 2(16) - 1(-8) + 5(-8) = 0. \tag{11.2.9}$$

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[2] Laplace, P. S., 1772: Recherches sur le calcul intégral et sur le système du monde. *Hist. Acad. R. Sci., II^e Partie*, 267–376. *Œuvres*, **8**, pp. 369–501. See Muir, T., 1960: *The Theory of Determinants in the Historical Order of Development, Vol. I, Part 1, General Determinants Up to 1841*, Dover Publishers, Mineola, NY, pp. 24–33.

The greatest source of error is forgetting to take the factor $(-1)^{i+j}$ into account during the expansion.

Although Laplace's expansion does provide a method for calculating $\det(A)$, the number of calculations equals $(n!)$. Consequently, for hand calculations, an obvious strategy is to select the column or row that has the greatest number of zeros. An even better strategy would be to manipulate a determinant with the goal of introducing zeros into a particular column or row. In the remaining portion of section, we show some operations that may be performed on a determinant to introduce the desired zeros. Most of the properties follow from the expansion of determinants by cofactors.

- *Rule 1* : For every square matrix A, $\det(A^T) = \det(A)$.

The proof is left as an exercise.

- *Rule 2* : If any two rows or columns of A are identical, $\det(A) = 0$.

To see that this is true, consider the following 3×3 matrix:

$$\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} = c_1(b_2b_3 - b_3b_2) - c_2(b_1b_3 - b_3b_1) + c_3(b_1b_2 - b_2b_1) = 0. \tag{11.2.10}$$

- *Rule 3* : The determinant of a triangular matrix is equal to the product of its diagonal elements.

If A is lower triangular, successive expansions by elements in the first column give

$$\det(A) = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} \tag{11.2.11}$$

$$= \cdots = a_{11}a_{22}\cdots a_{nn}. \tag{11.2.12}$$

If A is upper triangular, successive expansions by elements of the first row proves the property.

- *Rule 4* : If a square matrix A has either a row or a column of all zeros, then $\det(A) = 0$.

The proof is left as an exercise.

• *Rule 5* : If each element in one row (column) of a determinant is multiplied by a number c, the value of the determinant is multiplied by c.

Suppose $|B|$ has been obtained from $|A|$ by multiplying row i (column j) of $|A|$ by c. Upon expanding $|B|$ in terms of row i (column j) each term in the expansion contains c as a factor. Factor out the common c, the result is just c times the expansion $|A|$ by the same row (column).

• *Rule 6* : If each element of a row (or a column) of a determinant can be expressed as a binomial, the determinant can be written as the sum of two determinants.

To understand this property, consider the following 3×3 determinant:

$$\begin{vmatrix} a_1+d_1 & b_1 & c_1 \\ a_2+d_2 & b_2 & c_2 \\ a_3+d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}. \quad \textbf{(11.2.13)}$$

The proof follows by expanding the determinant by the row (or column) that contains the binomials.

• *Rule 7* : If B is a matrix obtained by interchanging any two rows (columns) of a square matrix A, then $\det(B) = -\det(A)$.

The proof is by induction. It is easily shown for any 2×2 matrix. Assume that this rule holds of any $(n-1) \times (n-1)$ matrix. If A is $n \times n$, then let B be a matrix formed by interchanging rows i and j. Expanding $|B|$ and $|A|$ by a different row, say k, we have that

$$|B| = \sum_{s=1}^{n} (-1)^{k+s} b_{ks} M_{ks} \quad \text{and} \quad |A| = \sum_{s=1}^{n} (-1)^{k+s} a_{ks} N_{ks}, \quad \textbf{(11.2.14)}$$

where M_{ks} and N_{ks} are the minors formed by deleting row k, column s from $|B|$ and $|A|$, respectively. For $s = 1, 2, \ldots, n$, we obtain N_{ks} and M_{ks} by interchanging rows i and j. By the induction hypothesis and recalling that N_{ks} and M_{ks} are $(n-1) \times (n-1)$ determinants, $N_{ks} = -M_{ks}$ for $s = 1, 2, \ldots, n$. Hence, $|B| = -|A|$. Similar arguments hold if two columns are interchanged.

• *Rule 8* : If one row (column) of a square matrix A equals to a number c times some other row (column), then $\det(A) = 0$.

Suppose one row of a square matrix A is equal to c times some other row. If $c = 0$, then $|A| = 0$. If $c \neq 0$, then $|A| = c|B|$, where $|B| = 0$ because $|B|$ has two identical rows. A similar argument holds for two columns.

• Rule 9 : The value of $\det(A)$ is unchanged if any arbitrary multiple of any line (row or column) is added to any other line.

To see that this is true, consider the simple example:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} cb_1 & b_1 & c_1 \\ cb_2 & b_2 & c_2 \\ cb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + cb_1 & b_1 & c_1 \\ a_2 + cb_2 & b_2 & c_2 \\ a_3 + cb_3 & b_3 & c_3 \end{vmatrix}, \tag{11.2.15}$$

where $c \neq 0$. The first determinant on the left side is our original determinant. In the second determinant, we can again expand the first column and find that

$$\begin{vmatrix} cb_1 & b_1 & c_1 \\ cb_2 & b_2 & c_2 \\ cb_3 & b_3 & c_3 \end{vmatrix} = c \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} = 0. \tag{11.2.16}$$

• **Example 11.2.2**

Let us evaluate

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 2 & 3 \\ 1 & -1 & 1 & 2 \\ -1 & 1 & -1 & 5 \end{vmatrix}$$

using a combination of the properties stated above and expansion by cofactors.

By adding or subtracting the first row to the other rows, we have that

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 2 & 3 \\ 1 & -1 & 1 & 2 \\ -1 & 1 & -1 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 5 & 7 \\ 0 & -3 & -2 & -2 \\ 0 & 3 & 2 & 9 \end{vmatrix} \tag{11.2.17}$$

$$= \begin{vmatrix} 3 & 5 & 7 \\ -3 & -2 & -2 \\ 3 & 2 & 9 \end{vmatrix} = \begin{vmatrix} 3 & 5 & 7 \\ 0 & 3 & 5 \\ 0 & -3 & 2 \end{vmatrix} \tag{11.2.18}$$

$$= 3 \begin{vmatrix} 3 & 5 \\ -3 & 2 \end{vmatrix} = 3 \begin{vmatrix} 3 & 5 \\ 0 & 7 \end{vmatrix} = 63. \tag{11.2.19}$$

Problems

Evaluate the following determinants:

1. $\begin{vmatrix} 3 & 5 \\ -2 & -1 \end{vmatrix}$

2. $\begin{vmatrix} 5 & -1 \\ -8 & 4 \end{vmatrix}$

3. $\begin{vmatrix} 3 & 1 & 2 \\ 2 & 4 & 5 \\ 1 & 4 & 5 \end{vmatrix}$

4. $\begin{vmatrix} 4 & 3 & 0 \\ 3 & 2 & 2 \\ 5 & -2 & -4 \end{vmatrix}$

5. $\begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix}$

6. $\begin{vmatrix} 2 & -1 & 2 \\ 1 & 3 & 3 \\ 5 & 1 & 6 \end{vmatrix}$

7. $\begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 1 & 0 \\ 1 & 1 & -2 & 3 \end{vmatrix}$

8. $\begin{vmatrix} 2 & 1 & 2 & 1 \\ 3 & 0 & 2 & 2 \\ -1 & 2 & -1 & 1 \\ -3 & 2 & 3 & 1 \end{vmatrix}$

9. Using the properties of determinants, show that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = (b-a)(c-a)(d-a)(c-b)(d-b)(d-c).$$

This determinant is called *Vandermonde's determinant.*

10. Show that

$$\begin{vmatrix} a & b+c & 1 \\ b & a+c & 1 \\ c & a+b & 1 \end{vmatrix} = 0.$$

11. Show that if all of the elements of a row or column are zero, then $\det(A) = 0$.

12. Prove that $\det(A^T) = \det(A)$.

11.3 CRAMER'S RULE

One of the most popular methods for solving simple systems of linear equations is Cramer's rule.[3] It is very useful for 2×2 systems, acceptable for 3×3 systems, and of doubtful use for 4×4 or larger systems.

Let us have n equations with n unknowns, $A\mathbf{x} = \mathbf{b}$. Cramer's rule states that

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}, \tag{11.3.1}$$

where A_i is a matrix obtained from A by replacing the ith column with $\mathbf{b}$ and $n = 1, 2, 3, \ldots$. Obviously, $\det(A) \neq 0$ if Cramer's rule is to work.

To prove Cramer's rule, consider

$$x_1 \det(A) = \begin{vmatrix} a_{11}x_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21}x_1 & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31}x_1 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1}x_1 & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \tag{11.3.2}$$

by Rule 5 from the previous section. By adding x_2 times the second column to the first column,

$$x_1 \det(A) = \begin{vmatrix} a_{11}x_1 + a_{12}x_2 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21}x_1 + a_{22}x_2 & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31}x_1 + a_{32}x_2 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}. \tag{11.3.3}$$

Multiplying each of the columns by the corresponding x_i and adding it to the first column yields,

$$x_1 \det(A) = \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}. \tag{11.3.4}$$

[3] Cramer, G., 1750: *Introduction à l'analyse des lignes courbes algébriques*, Geneva, p. 657.

The first column of (11.3.4) equals $A\mathbf{x}$ and we replace it with $\mathbf{b}$. Thus,

$$x_1 \det(A) = \begin{vmatrix} b_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ b_2 & a_{22} & a_{23} & \cdots & a_{2n} \\ b_3 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_n & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = \det(A_1) \qquad (11.3.5)$$

or

$$x_1 = \frac{\det(A_1)}{\det(A)} \qquad (11.3.6)$$

provided $\det(A) \neq 0$. To complete the proof we do exactly the same procedure to the jth column. □

- **Example 11.3.1**

Let us solve the following system of equations by Cramer's rule:

$$2x_1 + x_2 + 2x_3 = -1, \qquad (11.3.7)$$

$$x_1 + x_3 = -1 \qquad (11.3.8)$$

and

$$-x_1 + 3x_2 - 2x_3 = 7. \qquad (11.3.9)$$

From the matrix form of the equations,

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ -1 & 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 7 \end{pmatrix}, \qquad (11.3.10)$$

we have that

$$\det(A) = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ -1 & 3 & -2 \end{vmatrix} = 1, \qquad (11.3.11)$$

$$\det(A_1) = \begin{vmatrix} -1 & 1 & 2 \\ -1 & 0 & 1 \\ 7 & 3 & -2 \end{vmatrix} = 2, \qquad (11.3.12)$$

$$\det(A_2) = \begin{vmatrix} 2 & -1 & 2 \\ 1 & -1 & 1 \\ -1 & 7 & -2 \end{vmatrix} = 1 \qquad (11.3.13)$$

and

$$\det(A_3) = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & 3 & 7 \end{vmatrix} = -3. \qquad (11.3.14)$$

Finally,

$$x_1 = \frac{2}{1} = 2, \quad x_2 = \frac{1}{1} = 1 \quad \text{and} \quad x_3 = \frac{-3}{1} = -3. \tag{11.3.15}$$

Problems

Solve the following systems of equations by Cramer's rule:

1. $x_1 + 2x_2 = 3, \quad 3x_1 + x_2 = 6$
2. $2x_1 + x_2 = -3, \; x_1 - x_2 = 1$
3. $x_1 + 2x_2 - 2x_3 = 4, \quad 2x_1 + x_2 + x_3 = -2, \; -x_1 + x_2 - x_3 = 2$
4. $2x_1 + 3x_2 - x_3 = -1, \; -x_1 - 2x_2 + x_3 = 5, \; 3x_1 - x_2 = -2.$

11.4 ROW ECHELON FORM AND GAUSSIAN ELIMINATION

So far, we have assumed that every system of equations has a unique solution. This is not necessary true as the following examples show.

• **Example 11.4.1**

Consider the system

$$x_1 + x_2 = 2 \tag{11.4.1}$$

and

$$2x_1 + 2x_2 = -1. \tag{11.4.2}$$

This system is inconsistent because the second equation does not follow after multiplying the first by 2. Geometrically (11.4.1) and (11.4.2) are parallel lines; they never intersect to give a unique x_1 and x_2.

• **Example 11.4.2**

Even if a system is consistent, it still may not have a unique solution. For example, the system

$$x_1 + x_2 = 2 \tag{11.4.3}$$

and

$$2x_1 + 2x_2 = 4 \tag{11.4.4}$$

is consistent, the second equation formed by multiplying the first by 2. However, there are an infinite number of solutions.

Our examples suggest the following:

Theorem: *A system of m linear equation in n unknowns may: (1) have no solution, in which case it is called an* inconsistent *system, or (2) have exactly one solution (called a unique solution), or (3) have an infinite number of solutions. In the latter two cases, the system is said to be* consistent.

Before we can prove this theorem at the end of this section, we need to introduce some new concepts.

The first one is equivalent systems. Two systems of equations involving the same variables are *equivalent* if they have the same solution set. Of course, the only reason for introducing equivalent systems is the possibility of transforming one system of linear systems into another which is easier to solve. But what operations are permissible? Also what is the ultimate goal of our transformation?

From a complete study of possible operations, there are only three operations for transforming one system of linear equations into another. These three *elementary row operations* are

(1) interchanging any two rows in the matrix,

(2) multiplying any row by a nonzero scalar, and

(3) adding any arbitrary multiple of any row to any other row.

Armed with our elementary row operations, let us now solve the following set of linear equations:

$$x_1 - 3x_2 + 7x_3 = 2, \tag{11.4.5}$$

$$2x_1 + 4x_2 - 3x_3 = -1 \tag{11.4.6}$$

and

$$-x_1 + 13x_2 - 21x_3 = 2. \tag{11.4.7}$$

We begin by writing (11.4.5)–(11.4.7) in matrix notation:

$$\begin{pmatrix} 1 & -3 & 7 \\ 2 & 4 & -3 \\ -1 & 13 & -21 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}. \tag{11.4.8}$$

The matrix in (11.4.8) is called the *coefficient matrix* of the system.

We now introduce the concept of the *augmented matrix*: a matrix B composed of A plus the column vector $\mathbf{b}$ or

$$B = \left(\begin{array}{ccc|c} 1 & -3 & 7 & 2 \\ 2 & 4 & -3 & -1 \\ -1 & 13 & -21 & 2 \end{array}\right). \tag{11.4.9}$$

We can solve our original system by performing elementary row operations on the augmented matrix. Because the x_i's function essentially as placeholders, we can omit them until the end of the computation.

Returning to the problem, the first row may be used to eliminate the elements in the first column of the remaining rows. For this reason the first row is called the *pivotal* row and the element a_{11} is the *pivot.* By using the third elementary row operation twice (to eliminate the 2 and -1 in the first column), we finally have the equivalent system

$$B = \left(\begin{array}{ccc|c} 1 & -3 & 7 & 2 \\ 0 & 10 & -17 & -5 \\ 0 & 10 & -14 & 4 \end{array}\right). \qquad \textbf{(11.4.10)}$$

At this point we choose the second row as our new pivotal row and again apply the third row operation to eliminate the last element in the second column. This yields

$$B = \left(\begin{array}{ccc|c} 1 & -3 & 7 & 2 \\ 0 & 10 & -17 & -5 \\ 0 & 0 & 3 & 9 \end{array}\right). \qquad \textbf{(11.4.11)}$$

Thus, elementary row operations have transformed (11.4.5)–(11.4.7) into the triangular system:

$$x_1 - 3x_2 + 7x_3 = 2, \qquad \textbf{(11.4.12)}$$

$$10x_2 - 17x_3 = -5, \qquad \textbf{(11.4.13)}$$

$$3x_3 = 9, \qquad \textbf{(11.4.14)}$$

which is *equivalent* to the original system. The final solution is obtained by *back substitution*, solving from (11.4.14) back to (11.4.12). In the present case, $x_3 = 3$. Then, $10x_2 = 17(3) - 5$ or $x_2 = 4.6$. Finally, $x_1 = 3x_2 - 7x_3 + 2 = -5.2$.

In general, if an $n \times n$ linear system can be reduced to triangular form, then it will have a unique solution that we can obtain by performing back substitution. This reduction involves $n - 1$ steps. In the first step, a pivot element, and thus the pivotal row, is chosen from the nonzero entries in the first column of the matrix. We interchange rows (if necessary) so that the pivotal row is the first row. Multiples of the pivotal row are then subtracted from each of the remaining $n - 1$ rows so that there are 0's in the $(2, 1), ..., (n, 1)$ positions. In the second step, a pivot element is chosen from the nonzero entries in column 2, rows 2 through n, of the matrix. The row containing the pivot is then interchanged with the second row (if necessary) of the matrix and is used as the pivotal row. Multiples of the pivotal row are then subtracted from the remaining $n - 2$ rows, eliminating all entries below the diagonal in the second column. The same procedure is repeated for columns 3 through $n-1$. Note that in the second step, row 1 and column 1 remain unchanged, in the third step the first two rows and first two columns remain unchanged, and so on.

If elimination is carried out as described, we will arrive at an equivalent upper triangular system after $n-1$ steps. However, the procedure will fail if, at any step, all possible choices for a pivot element equal zero. Let us now examine such cases.

Consider now the system

$$x_1 + 2x_2 + x_3 = -1, \tag{11.4.15}$$

$$2x_1 + 4x_2 + 2x_3 = -2, \tag{11.4.16}$$

$$x_1 + 4x_2 + x_3 = 2. \tag{11.4.17}$$

Its augmented matrix is

$$B = \left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 2 & 4 & 2 & -2 \\ 1 & 4 & 2 & 2 \end{array}\right). \tag{11.4.18}$$

Choosing the first row as our pivotal row, we find that

$$B = \left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \end{array}\right) \tag{11.4.19}$$

or

$$B = \left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right). \tag{11.4.20}$$

The difficulty here is the presence of the zeros in the third row. Clearly any finite numbers will satisfy the equation $0x_1 + 0x_2 + 0x_3 = 0$ and we have an infinite number of solutions. Closer examination of the original system shows a underdetermined system; (11.4.15) and (11.4.16) differ by a factor of 2. An important aspect of this problem is the fact that the final augmented matrix is of the form of a staircase or *echelon form* rather than of triangular form.

Let us modify (11.4.15)–(11.4.17) to read

$$x_1 + 2x_2 + x_3 = -1, \tag{11.4.21}$$

$$2x_1 + 4x_2 + 2x_3 = 3, \tag{11.4.22}$$

$$x_1 + 4x_2 + x_3 = 2, \tag{11.4.23}$$

then the final augmented matrix is

$$B = \left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 5 \end{array}\right). \tag{11.4.24}$$

We again have a problem with the third row because $0x_1+0x_2+0x_3=5$, which is impossible. There is no solution in this case and we have an *overdetermined system*. Note, once again, that our augmented matrix has a row echelon form rather than a triangular form.

In summary, to include all possible situations in our procedure, we must rewrite the augmented matrix in row echelon form. *Row echelon form* consists of:

(1) The first nonzero entry in each row is 1.

(2) If row k does not consist entirely of zeros, the number of leading zero entries in row $k+1$ is greater than the number of leading zero entries in row k.

(3) If there are rows whose entries are all zero, they are below the rows having nonzero entries.

The number of nonzero rows in the row echelon form of a matrix is known as its *rank*. *Gaussian elimination* is the process of using elementary row operations to transform a linear system into one whose augmented matrix is in row echelon form.

• Example 11.4.3

Each of the following matrices is *not* of row echelon form because they violate one of the conditions for row echelon form:

$$\begin{pmatrix} 2 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{11.4.25}$$

• Example 11.4.4

The following matrices are in row echelon form:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 4 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{11.4.26}$$

• Example 11.4.5

Gaussian elimination may also be used to solve the general problem $AX = B$. One of the most common applications is in finding the inverse. For example, let us find the inverse of the matrix

$$A = \begin{pmatrix} 4 & -2 & 2 \\ -2 & -4 & 4 \\ -4 & 2 & 8 \end{pmatrix} \tag{11.4.27}$$

by Gaussian elimination.

Because the inverse is defined by $AA^{-1} = I$, our augmented matrix is

$$\left(\begin{array}{ccc|ccc} 4 & -2 & 2 & 1 & 0 & 0 \\ -2 & -4 & 4 & 0 & 1 & 0 \\ -4 & 2 & 8 & 0 & 0 & 1 \end{array}\right). \tag{11.4.28}$$

Then, by elementary row operations,

$$\left(\begin{array}{ccc|ccc} 4 & -2 & 2 & 1 & 0 & 0 \\ -2 & -4 & 4 & 0 & 1 & 0 \\ -4 & 2 & 8 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} -2 & -4 & 4 & 0 & 1 & 0 \\ 4 & -2 & 2 & 1 & 0 & 0 \\ -4 & 2 & 8 & 0 & 0 & 1 \end{array}\right) \tag{11.4.29}$$

$$= \left(\begin{array}{ccc|ccc} -2 & -4 & 4 & 0 & 1 & 0 \\ 4 & -2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 10 & 1 & 0 & 1 \end{array}\right) \tag{11.4.30}$$

$$= \left(\begin{array}{ccc|ccc} -2 & -4 & 4 & 0 & 1 & 0 \\ 0 & -10 & 10 & 1 & 2 & 0 \\ 0 & 0 & 10 & 1 & 0 & 1 \end{array}\right) \tag{11.4.31}$$

$$= \left(\begin{array}{ccc|ccc} -2 & -4 & 4 & 0 & 1 & 0 \\ 0 & -10 & 0 & 0 & 2 & -1 \\ 0 & 0 & 10 & 1 & 0 & 1 \end{array}\right) \tag{11.4.32}$$

$$= \left(\begin{array}{ccc|ccc} -2 & -4 & 0 & -2/5 & 1 & -2/5 \\ 0 & -10 & 0 & 0 & 2 & -1 \\ 0 & 0 & 10 & 1 & 0 & 1 \end{array}\right) \tag{11.4.33}$$

$$= \left(\begin{array}{ccc|ccc} -2 & 0 & 0 & -2/5 & 1/5 & 0 \\ 0 & -10 & 0 & 0 & 2 & -1 \\ 0 & 0 & 10 & 1 & 0 & 1 \end{array}\right) \tag{11.4.34}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & -1/10 & 0 \\ 0 & 1 & 0 & 0 & -1/5 & 1/10 \\ 0 & 0 & 1 & 1/10 & 0 & 1/10 \end{array}\right). \tag{11.4.35}$$

Thus, the right half of the augmented matrix yields the inverse and it equals

$$A^{-1} = \begin{pmatrix} 1/5 & -1/10 & 0 \\ 0 & -1/5 & 1/10 \\ 1/10 & 0 & 1/10 \end{pmatrix}. \tag{11.4.36}$$

Of course, we can always check our answer by multiplying A^{-1} by A.

Gaussian elimination may be used with overdetermined systems. *Overdetermined systems* are linear systems where there are more equations than unknowns ($m > n$). These systems are usually (but not always) inconsistent.

• **Example 11.4.6**

Consider the linear system

$$x_1 + x_2 = 1, \tag{11.4.37}$$

$$-x_1 + 2x_2 = -2, \tag{11.4.38}$$

$$x_1 - x_2 = 4. \tag{11.4.39}$$

After several row operations, the augmented matrix

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 2 & -2 \\ 1 & -1 & 4 \end{array}\right) \tag{11.4.40}$$

becomes

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -7 \end{array}\right). \tag{11.4.41}$$

From the last row of the augmented matrix (11.4.41) we see that the system is inconsistent. However, if we change the system to

$$x_1 + x_2 = 1, \tag{11.4.42}$$

$$-x_1 + 2x_2 = 5, \tag{11.4.43}$$

$$x_1 = -1, \tag{11.4.44}$$

the final form of the augmented matrix is

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right). \tag{11.4.45}$$

which has the unique solution $x_1 = -1$ and $x_2 = 2$.

Finally, by introducing the set:

$$x_1 + x_2 = 1, \tag{11.4.46}$$
$$2x_1 + 2x_2 = 2, \tag{11.4.47}$$
$$3x_1 + 3x_3 = 3, \tag{11.4.48}$$

the final form of the augmented matrix is

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right). \tag{11.4.49}$$

There are an infinite number of solutions: $x_1 = 1 - \alpha$ and $x_2 = \alpha$.

Gaussian elimination can also be employed with underdetermined systems. An *underdetermined linear system* is one where there are fewer equations than unknowns ($m < n$). These systems usually have an infinite number of solutions although they can be inconsistent.

- **Example 11.4.7**

Consider the underdetermined system:

$$2x_1 + 2x_2 + x_3 = -1, \tag{11.4.50}$$
$$4x_1 + 4x_2 + 2x_3 = 3. \tag{11.4.51}$$

Its augmented matrix may be transformed into the form:

$$\left(\begin{array}{ccc|c} 2 & 2 & 1 & -1 \\ 0 & 0 & 0 & 4 \end{array}\right). \tag{11.4.52}$$

Clearly this case corresponds to an inconsistent set of equations. On the other hand, if (11.4.51) is changed to

$$4x_1 + 4x_2 + 2x_3 = -2, \tag{11.4.53}$$

then the final form of the augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{11.4.54}$$

and we have an infinite number of solutions, namely $x_3 = \alpha$, $x_2 = \beta$, and $2x_1 = -1 - \alpha - 2\beta$.

Consider now one of most important classes of linear equations: the homogeneous equations $A\mathbf{x} = \mathbf{0}$. If $\det(A) \neq 0$, then by Cramer's rule

$x_1 = x_2 = x_3 = \cdots = x_n = 0$. Thus, the only possibility for a nontrivial solution is $\det(A) = 0$. In this case, A is singular, no inverse exists, and nontrivial solutions exist but they are not unique.

• **Example 11.4.8**

Consider the two homogeneous equations:

$$x_1 + x_2 = 0 \tag{11.4.55}$$

$$x_1 - x_2 = 0. \tag{11.4.56}$$

Note that $\det(A) = -2$. Solving this system yields $x_1 = x_2 = 0$.

However, if we change the system to

$$x_1 + x_2 = 0 \tag{11.4.57}$$

$$x_1 + x_2 = 0 \tag{11.4.58}$$

which has the $\det(A) = 0$ so that A is singular. Both equations yield $x_1 = -x_2 = \alpha$, any constant. Thus, there is an infinite number of solutions for this set of homogeneous equations.

We close this section by outlining the proof of the theorem which we introduced at the beginning.

Consider the system $A\mathbf{x} = \mathbf{b}$. By elementary row operations, the first equation in this system can be reduced to

$$x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n = \beta_1. \tag{11.4.59}$$

The second equation has the form

$$x_p + \alpha_{2,p+1}x_{p+1} + \cdots + \alpha_{2n}x_n = \beta_2, \tag{11.4.60}$$

where $p > 1$. The third equation has the form

$$x_q + \alpha_{3,q+1}x_{q+1} + \cdots + \alpha_{3n}x_n = \beta_3, \tag{11.4.61}$$

where $q > p$, and so on. To simplify the notation, we introduce z_i where we choose the first k values so that $z_1 = x_1$, $z_2 = x_p$, $z_3 = x_q$, ... Thus, the question of the existence of solutions depends upon the three integers: m, n, and k. The resulting set of equations have the form:

$$\begin{pmatrix} 1 & \gamma_{12} & \cdots & \gamma_{1,k} & \gamma_{1,k+1} & \cdots & \gamma_{1n} \\ 0 & 1 & \cdots & \gamma_{2,k} & \gamma_{2,k+1} & \cdots & \gamma_{2n} \\ & & & \vdots & & & \\ 0 & 0 & \cdots & 1 & \gamma_{k,k+1} & \cdots & \gamma_{kn} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ & & & \vdots & & & \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \\ \beta_{k+1} \\ \vdots \\ \beta_m \end{pmatrix}. \tag{11.4.62}$$

Note that $\beta_{k+1}, \ldots, \beta_m$ need not be all zero.

There are three possibilities:

(a) $k < m$ and at least one of the elements $\beta_{k+1}, \ldots, \beta_m$ is nonzero. Suppose that an element β_p is nonzero ($p > k$). Then the pth equation is

$$0z_1 + 0z_2 + \cdots + 0z_n = \beta_p \neq 0. \tag{11.4.63}$$

However, this is a contradiction and the equations are inconsistent.

(b) $k = n$ and either (i) $k < m$ and all of the elements $\beta_{k+1}, \ldots, \beta_m$ are zero, or (ii) $k = m$. Then the equations have a unique solution which can be obtained by back-substitution.

(c) $k < n$ and either (i) $k < m$ and all of the elements $\beta_{k+1}, \ldots, \beta_m$ are zero, or (ii) $k = m$. Then, arbitrary values can be assigned to the $n-k$ variables $z_{k+1}, \ldots, z_n$. The equations can be solved for $z_1, z_2, \ldots, z_k$ and there is an infinity of solutions.

For homogeneous equations $\mathbf{b} = \mathbf{0}$, all of the β_i are zero. In this case, we have only two cases:

(b′) $k = n$, then (11.4.62) has the solution $\mathbf{z} = \mathbf{0}$ which leads to the trivial solution for the original system $A\mathbf{x} = \mathbf{0}$.

(c′) $k < n$, the equations possess an infinity of solutions given by assigning arbitrary values to $z_{k+1}, \ldots, z_n$. □

Problems

Solve the following systems of linear equations by Gaussian elimination:

1. $2x_1 + x_2 = 4, \quad 5x_1 - 2x_2 = 1$
2. $x_1 + x_2 = 0, \quad 3x_1 - 4x_2 = 1$
3. $-x_1 + x_2 + 2x_3 = 0, \quad 3x_1 + 4x_2 + x_3 = 0, \quad -x_1 + x_2 + 2x_3 = 0$
4. $4x_1 + 6x_2 + x_3 = 2, \quad 2x_1 + x_2 - 4x_3 = 3, \quad 3x_1 - 2x_2 + 5x_3 = 8$
5. $3x_1 + x_2 - 2x_3 = -3, \quad x_1 - x_2 + 2x_3 = -1, \quad -4x_1 + 3x_2 - 6x_3 = 4$
6. $x_1 - 3x_2 + 7x_3 = 2, \quad 2x_1 + 4x_2 - 3x_3 = -1,$
   $-3x_1 + 7x_2 + 2x_3 = 3$
7. $x_1 - x_2 + 3x_3 = 5, \quad 2x_1 - 4x_2 + 7x_3 = 7,$
   $4x_1 - 9x_2 + 2x_3 = -15$
8. $x_1 + x_2 + x_3 + x_4 = -1, \quad 2x_1 - x_2 + 3x_3 = 1,$
   $2x_2 + 3x_4 = 15, \quad -x_1 + 2x_2 + x_4 = -2$

Find the inverse of each of the following matrices by Gaussian elimination:

9. $\begin{pmatrix} -3 & 5 \\ 2 & 1 \end{pmatrix}$
10. $\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$

11. $\begin{pmatrix} 19 & 2 & -9 \\ -4 & -1 & 2 \\ -2 & 0 & 1 \end{pmatrix}$ 12. $\begin{pmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 2 & 4 & 11 \end{pmatrix}$

13. Does $(A^2)^{-1} = (A^{-1})^2$? Justify your answer.

11.5 EIGENVALUES AND EIGENVECTORS

One of the classic problems of linear algebra[4] is finding all of the λ's which satisfy the $n \times n$ system

$$A\mathbf{x} = \lambda\mathbf{x}. \tag{11.5.1}$$

The nonzero quantity λ is the *eigenvalue* or *characteristic value* of A. The vector $\mathbf{x}$ is the *eigenvector* or *characteristic vector* belonging to λ. The set of the eigenvalues of A is called the *spectrum* of A. The largest of the absolute values of the eigenvalues of A is called the *spectral radius* of A.

To find λ and $\mathbf{x}$, we first rewrite (11.5.1) as a set of homogeneous equations:

$$(A - \lambda I)\mathbf{x} = \mathbf{0}. \tag{11.5.2}$$

From the theory of linear equations, (11.5.2) has trivial solutions unless its determinant equals zero. On the other hand, if

$$\det(A - \lambda I) = 0, \tag{11.5.3}$$

there are an infinity of solutions.

The expansion of the determinant (11.5.3) yields an nth-degree polynomial in λ, the *characteristic polynomial.* The roots of the characteristic polynomial are the eigenvalues of A. Because the characteristic polynomial has exactly n roots, A will have n eigenvalues, some of which may be repeated (with multiplicity $k \le n$) and some of which may be complex numbers. For each eigenvalue λ_i, there will be a corresponding eigenvector $\mathbf{x}_i$. This eigenvector is the solution of the homogeneous equations $(A - \lambda_i I)\mathbf{x}_i = \mathbf{0}$.

An important property of eigenvectors is their *linear independence* if there are n distinct eigenvalues. Vectors are linearly independent if the equation

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_n\mathbf{x}_n = \mathbf{0} \tag{11.5.4}$$

can be satisfied only by taking *all* of the α's equal to zero.

[4] The standard reference is Wilkinson, J. H., 1965: *The Algebraic Eigenvalue Problem,* Clarendon Press, Oxford.

To show that this is true in the case of n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, each eigenvalue λ_i having a corresponding eigenvector $\mathbf{x}_i$, we first write down the linear dependence condition

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_n\mathbf{x}_n = \mathbf{0}. \qquad \textbf{(11.5.5)}$$

Premultiplying (11.5.5) by A,

$$\alpha_1 A\mathbf{x}_1 + \alpha_2 A\mathbf{x}_2 + \cdots + \alpha_n A\mathbf{x}_n = \alpha_1\lambda_1\mathbf{x}_1 + \alpha_2\lambda_2\mathbf{x}_2 + \cdots + \alpha_n\lambda_n\mathbf{x}_n = \mathbf{0}. \qquad \textbf{(11.5.6)}$$

Premultiplying (11.5.5) by A^2,

$$\alpha_1 A^2\mathbf{x}_1 + \alpha_2 A^2\mathbf{x}_2 + \cdots + \alpha_n A^2\mathbf{x}_n = \alpha_1\lambda_1^2\mathbf{x}_1 + \alpha_2\lambda_2^2\mathbf{x}_2 + \cdots + \alpha_n\lambda_n^2\mathbf{x}_n = \mathbf{0}. \qquad \textbf{(11.5.7)}$$

In similar manner, we obtain the system of equations:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix} \begin{pmatrix} \alpha_1\mathbf{x}_1 \\ \alpha_2\mathbf{x}_2 \\ \alpha_3\mathbf{x}_3 \\ \vdots \\ \alpha_n\mathbf{x}_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \qquad \textbf{(11.5.8)}$$

Because

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix} = \begin{array}{c}(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)(\lambda_4-\lambda_3) \\ (\lambda_4-\lambda_2)\cdots(\lambda_n-\lambda_1) \neq 0,\end{array} \qquad \textbf{(11.5.9)}$$

since it is a Vandermonde determinant, $\alpha_1\mathbf{x}_1 = \alpha_2\mathbf{x}_2 = \alpha_3\mathbf{x}_3 = \cdots = \alpha_n\mathbf{x}_n = 0$. Because the eigenvectors are nonzero, $\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$ and the eigenvectors are linearly independent. □

This property of eigenvectors allows us to express any arbitrary vector $\mathbf{x}$ as a linear sum of the eigenvectors $\mathbf{x}_i$ or

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n. \qquad \textbf{(11.5.10)}$$

We will make good use of this property in Example 11.5.3.

• **Example 11.5.1**

Let us find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} -4 & 2 \\ -1 & -1 \end{pmatrix}. \qquad \textbf{(11.5.11)}$$

We begin by setting up the characteristic equation:

$$\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 2 \\ -1 & -1-\lambda \end{vmatrix} = 0. \tag{11.5.12}$$

Expanding the determinant,

$$(-4-\lambda)(-1-\lambda) + 2 = \lambda^2 + 5\lambda + 6 = (\lambda+3)(\lambda+2) = 0. \tag{11.5.13}$$

Thus, the eigenvalues of the matrix A are $\lambda_1 = -3$ and $\lambda_2 = -2$.

To find the corresponding eigenvectors, we must solve the linear system:

$$\begin{pmatrix} -4-\lambda & 2 \\ -1 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{11.5.14}$$

For example, for $\lambda_1 = -3$,

$$\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11.5.15}$$

or

$$x_1 = 2x_2. \tag{11.5.16}$$

Thus, any nonzero multiple of the vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector belonging to $\lambda_1 = -3$. Similarly, for $\lambda_2 = -2$, the eigenvector is any nonzero multiple of the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

• **Example 11.5.2**

Let us now find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} -4 & 5 & 5 \\ -5 & 6 & 5 \\ -5 & 5 & 6 \end{pmatrix}. \tag{11.5.17}$$

Setting up the characteristic equation:

$$\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 5 & 5 \\ -5 & 6-\lambda & 5 \\ -5 & 5 & 6-\lambda \end{vmatrix} = \begin{vmatrix} -4-\lambda & 5 & 5 \\ -5 & 6-\lambda & 5 \\ 0 & \lambda-1 & 1-\lambda \end{vmatrix} \tag{11.5.18}$$

$$= (\lambda-1) \begin{vmatrix} -4-\lambda & 5 & 5 \\ -5 & 6-\lambda & 5 \\ 0 & 1 & -1 \end{vmatrix} = (\lambda-1)^2 \begin{vmatrix} -1 & 1 & 0 \\ -5 & 6-\lambda & 5 \\ 0 & 1 & -1 \end{vmatrix} \tag{11.5.19}$$

$$= (\lambda-1)^2 \begin{vmatrix} -1 & 0 & 0 \\ -5 & 6-\lambda & 0 \\ 0 & 1 & -1 \end{vmatrix} = (\lambda-1)^2(6-\lambda) = 0. \tag{11.5.20}$$

Thus, the eigenvalues of the matrix A are $\lambda_{1,2} = 1$ (twice) and $\lambda_3 = 6$.

To find the corresponding eigenvectors, we must solve the linear system:

$$(-4 - \lambda)x_1 + 5x_2 + 5x_3 = 0, \tag{11.5.21}$$

$$-5x_1 + (6 - \lambda)x_2 + 5x_3 = 0 \tag{11.5.22}$$

and

$$-5x_1 + 5x_2 + (6 - \lambda)x_3 = 0. \tag{11.5.23}$$

For $\lambda_3 = 6$, (11.5.21)–(11.5.23) become

$$-10x_1 + 5x_2 + 5x_3 = 0, \tag{11.5.24}$$

$$-5x_1 + 5x_3 = 0 \tag{11.5.25}$$

and

$$-5x_1 + 5x_2 = 0. \tag{11.5.26}$$

Thus, $x_1 = x_2 = x_3$ and the eigenvector is any nonzero multiple of the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

The interesting aspect of this example involves finding the eigenvector for the eigenvalue $\lambda_{1,2} = 1$. If $\lambda_{1,2} = 1$, then (11.5.21)–(11.5.23) collapses into one equation

$$-x_1 + x_2 + x_3 = 0 \tag{11.5.27}$$

and we have *two* free parameters at our disposal. Let us take $x_2 = \alpha$ and $x_3 = \beta$. Then the eigenvector equals $\alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ for $\lambda_{1,2} = 1$.

In this example our 3×3 matrix has three *linearly independent* eigenvectors: $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ associated with $\lambda_1 = 1$, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ associated with $\lambda_2 = 1$, and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ associated with $\lambda_3 = 6$. However, with repeated eigenvalues this is not always true. For example,

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \tag{11.5.28}$$

has the repeated eigenvalues $\lambda_{1,2} = 1$. However, there is only a single eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for *both* λ_1 and λ_2.

• **Example 11.5.3**

When we discussed the stability of numerical schemes for the wave equation in Section 7.6, we examined the behavior of a prototypical Fourier harmonic to variation in the parameter $c\Delta t/\Delta x$. In this example we shall show another approach to determining the stability of a numerical scheme via matrices.

Consider the explicit scheme for the numerical integration of the wave equation (7.6.11). We can rewrite that single equation as the coupled difference equations:

$$u_m^{n+1} = 2(1-r^2)u_m^n + r^2(u_{m+1}^n + u_{m-1}^n) - v_m^n \tag{11.5.29}$$

and

$$v_m^{n+1} = u_m^n, \tag{11.5.30}$$

where $r = c\Delta t/\Delta x$. Let $u_{m+1}^n = e^{i\beta\Delta x}u_m^n$ and $u_{m-1}^n = e^{-i\beta\Delta x}u_m^n$, where β is real. Then (11.5.29)–(11.5.30) becomes

$$u_m^{n+1} = 2\left[1 - 2r^2\sin^2\left(\frac{\beta\Delta x}{2}\right)\right]u_m^n - v_m^n \tag{11.5.31}$$

and

$$v_m^{n+1} = u_m^n \tag{11.5.32}$$

or in the matrix form

$$\mathbf{u}_m^{n+1} = \begin{pmatrix} 2\left[1 - 2r^2\sin^2\left(\frac{\beta\Delta x}{2}\right)\right] & -1 \\ 1 & 0 \end{pmatrix}\mathbf{u}_m^n, \tag{11.5.33}$$

where $\mathbf{u}_m^n = \begin{pmatrix} u_m^n \\ v_m^n \end{pmatrix}$. The eigenvalues λ of this *amplification matrix* are given by

$$\lambda^2 - 2\left[1 - 2r^2\sin^2\left(\frac{\beta\Delta x}{2}\right)\right]\lambda + 1 = 0 \tag{11.5.34}$$

or

$$\lambda_{1,2} = 1 - 2r^2\sin^2\left(\frac{\beta\Delta x}{2}\right) \pm 2r\sin\left(\frac{\beta\Delta x}{2}\right)\sqrt{r^2\sin^2\left(\frac{\beta\Delta x}{2}\right) - 1}. \tag{11.5.35}$$

Because each successive time step consists of multiplying the solution from the previous time step by the amplification matrix, the solution will be stable only if $\mathbf{u}_m^n$ remains bounded. This will occur only if all of the eigenvalues have a magnitude less or equal to one because

$$\mathbf{u}_m^n = \sum_k c_k A^n \mathbf{x}_k = \sum_k c_k \lambda_k^n \mathbf{x}_k, \tag{11.5.36}$$

where A denotes the amplification matrix and $\mathbf{x}_k$ denotes the eigenvectors corresponding to the eigenvalues λ_k. Equation (11.5.36) follows from our ability to express any initial condition in terms of an eigenvector expansion:

$$\mathbf{u}_m^0 = \sum_k c_k \mathbf{x}_k. \tag{11.5.37}$$

In our particular example, two cases arise. If $r^2 \sin^2(\beta\Delta x/2) \leq 1$,

$$\lambda_{1,2} = 1 - 2r^2 \sin^2\left(\frac{\beta\Delta x}{2}\right) \pm 2ri \sin\left(\frac{\beta\Delta x}{2}\right)\sqrt{1 - r^2 \sin^2\left(\frac{\beta\Delta x}{2}\right)} \tag{11.5.38}$$

and $|\lambda_{1,2}| = 1$. On the other hand, if $r^2 \sin^2(\beta\Delta x/2) > 1$, $|\lambda_{1,2}| > 1$. Thus, we will have stability only if $c\Delta t/\Delta x \leq 1$.

Problems

Find the eigenvalues and corresponding eigenvectors for the following matrices:

1. $A = \begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix}$

2. $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$

3. $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$

4. $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

5. $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

6. $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{pmatrix}$

7. $A = \begin{pmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$

8. $A = \begin{pmatrix} -2 & 0 & 1 \\ 3 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

Project: Numerical Solution of the Sturm-Liouville Problem

You may have been struck by the similarity of the algebraic eigenvalue problem to the Sturm-Liouville problem. In both cases nontrivial solutions exist only for characteristic values of λ. The purpose of this project is to further deepen your insight into these similarities.

Consider the Sturm-Liouville problem:

$$y'' + \lambda y = 0, \qquad y(0) = y(\pi) = 0. \tag{11.5.39}$$

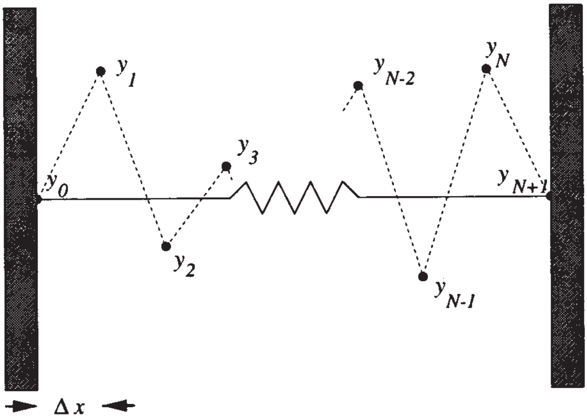

Figure 11.5.1: Schematic for finite-differencing a Sturm-Liouville problem into a set of difference equations.

We know that it has the nontrivial solutions $\lambda_m = m^2$, $y_m(x) = \sin(mx)$, where $m = 1, 2, 3, \ldots$

Step 1: Let us solve this problem numerically. Introducing centered finite differencing and the grid shown in Figure 11.5.1, show that

$$y'' \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta x}, \qquad n = 1, 2, \ldots, N, \tag{11.5.40}$$

where $\Delta x = \pi/(N+1)$. Show that the finite-differenced form of (11.5.39) is

$$-h^2 y_{n+1} + 2h^2 y_n - h^2 y_{n-1} = \lambda y_n \tag{11.5.41}$$

with $y_0 = y_{N+1} = 0$ and $h = 1/(\Delta x)$.

Step 2: Solve (11.5.41) as an algebraic eigenvalue problem using $N = 1, 2, \ldots$. Show that (11.5.41) can be written in the matrix form of

$$\begin{pmatrix} 2h^2 & -h^2 & 0 & \cdots & 0 & 0 & 0 \\ -h^2 & 2h^2 & -h^2 & \cdots & 0 & 0 & 0 \\ 0 & -h^2 & 2h^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -h^2 & 2h^2 & -h^2 \\ 0 & 0 & 0 & \cdots & 0 & -h^2 & 2h^2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-1} \\ y_N \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-1} \\ y_N \end{pmatrix}. \tag{11.5.42}$$

Note that the coefficient matrix is symmetric. Except for very small N, computing the values of λ using determinants is very difficult. Consequently you must use one of the numerical schemes that have been

Table 11.5.1: Eigenvalues computed from (11.5.42) as a numerical approximation of the Sturm-Liouville problem (11.5.39).

N	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
1	0.81057						
2	0.91189	2.73567					
3	0.94964	3.24228	5.53491				
4	0.96753	3.50056	6.63156	9.16459			
5	0.97736	3.64756	7.29513	10.94269	13.61289		
6	0.98333	3.73855	7.71996	12.13899	16.12040	18.87563	
7	0.98721	3.79857	8.00605	12.96911	17.93217	22.13966	24.95100
8	0.98989	3.84016	8.20702	13.56377	19.26430	24.62105	28.98791
20	0.99813	3.97023	8.84993	15.52822	23.85591	33.64694	44.68265
50	0.99972	3.99498	8.97438	15.91922	24.80297	35.59203	48.24538

developed for the efficient solution of the algebraic eigenvalue problem.[5] Packages for numerically solving the algebraic eigenvalue problem may already exist on your system or you may find code in a numerical methods book.

In Table 11.5.1 I have given the computed values of λ as a function of N using the IMSL routine EVLSF so that you may check your answers. How do your computed eigenvalues compare to the eigenvalues given by the Sturm-Liouville problem? What happens as you increase N? Which computed eigenvalues agree best with those given by the Sturm-Liouville problem? Which ones compare the worst?

Step 3: Let us examine the eigenfunctions now. First, reorder (if necessary) your eigenvectors so that each consecutive eigenvalue increases in magnitude. Starting with the smallest eigenvalue, construct an xy plot for each consecutive eigenvectors where $x_i = i\Delta x$, $i = 1, 2, \ldots, N$, and y_i are the corresponding element from the eigenvector. On the same plot, graph $y_m(x) = \sin(mx)$. Which eigenvectors and eigenfunctions agree the best? Which eigenvectors and eigenfunctions agree the worst? Why? Why are there N eigenvectors and an infinite number of eigenfunctions?

Step 4: The most important property of eigenfunctions is orthogonality. But what do we mean by orthogonality in the case of eigenvectors? Recall from three-dimensional vectors we had the scalar dot product:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3. \qquad \textbf{(11.5.43)}$$

[5] See Press, W. H., Flannery, B. F., Teukolsky, S. A., and Vetterling, W. T., 1986: *Numerical Recipes: The Art of Scientific Computing*, Cambridge University Press, Cambridge, chap. 11.

For n-dimensional vectors, this dot product is generalized to the inner product

$$\mathbf{x} \cdot \mathbf{y} = \sum_{k=1}^{n} x_k y_k. \tag{11.5.44}$$

Orthogonality implies that $\mathbf{x} \cdot \mathbf{y} = 0$ if $\mathbf{x} \neq \mathbf{y}$. Are your eigenvectors orthogonal? How might you use this property with eigenvectors?

11.6 SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

In this section we show how we may apply the classic algebraic eigenvalue problem to solve a system of ordinary differential equations.

Let us solve the following system:

$$x_1' = x_1 + 3x_2 \tag{11.6.1}$$

and

$$x_2' = 3x_1 + x_2, \tag{11.6.2}$$

where the primes denote the time derivative.

We begin by rewriting (11.6.1)–(11.6.2) in linear algebra notation:

$$\mathbf{x}' = A\mathbf{x}, \tag{11.6.3}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}. \tag{11.6.4}$$

Note that

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{x}'. \tag{11.6.5}$$

Assuming a solution of the form

$$\mathbf{x} = \mathbf{x}_0 e^{\lambda t}, \quad \text{where} \quad \mathbf{x}_0 = \begin{pmatrix} a \\ b \end{pmatrix} \tag{11.6.6}$$

is a constant vector, we substitute (11.6.6) into (11.6.3) and find that

$$\lambda e^{\lambda t}\mathbf{x}_0 = A e^{\lambda t}\mathbf{x}_0. \tag{11.6.7}$$

Because $e^{\lambda t}$ does not generally equal zero, we have that

$$(A - \lambda I)\mathbf{x}_0 = \mathbf{0}, \tag{11.6.8}$$

which we solved in the previous section. This set of homogeneous equations is the *classic eigenvalue problem*. In order for this set not to have trivial solutions,

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 0. \tag{11.6.9}$$

Expanding the determinant,

$$(1-\lambda)^2 - 9 = 0 \quad \text{or} \quad \lambda = -2, 4. \tag{11.6.10}$$

Thus, we have two real and distinct eigenvalues: $\lambda = -2$ and 4.

We must now find the corresponding $\mathbf{x}_0$ or *eigenvector* for each eigenvalue. From (11.6.8),

$$(1-\lambda)a + 3b = 0 \tag{11.6.11}$$

and

$$3a + (1-\lambda)b = 0. \tag{11.6.12}$$

If $\lambda = 4$, these equations are consistent and yield $a = b = c_1$. If $\lambda = -2$, we have that $a = -b = c_2$. Therefore, the general solution in matrix notation is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}. \tag{11.6.13}$$

To evaluate c_1 and c_2, we must have initial conditions. For example, if $x_1(0) = x_2(0) = 1$, then

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{11.6.14}$$

Solving for c_1 and c_2, $c_1 = 1$ and $c_2 = 0$ and the solution with this particular set of initial conditions is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}. \tag{11.6.15}$$

• Example 11.6.1

Let us solve the following set of linear ordinary differential equations:

$$x_1' = -x_2 + x_3, \tag{11.6.16}$$

$$x_2' = 4x_1 - x_2 - 4x_3 \tag{11.6.17}$$

and

$$x_3' = -3x_1 - x_2 + 4x_3; \tag{11.6.18}$$

or in matrix form,

$$\mathbf{x}' = \begin{pmatrix} 0 & -1 & 1 \\ 4 & -1 & -4 \\ -3 & -1 & 4 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{11.6.19}$$

Assuming the solution $\mathbf{x} = \mathbf{x}_0 e^{\lambda t}$,

$$\begin{pmatrix} 0 & -1 & 1 \\ 4 & -1 & -4 \\ -3 & -1 & 4 \end{pmatrix} \mathbf{x}_0 = \lambda \mathbf{x}_0 \tag{11.6.20}$$

or

$$\begin{pmatrix} -\lambda & -1 & 1 \\ 4 & -1-\lambda & -4 \\ -3 & -1 & 4-\lambda \end{pmatrix} \mathbf{x}_0 = \mathbf{0}. \tag{11.6.21}$$

For nontrivial solutions,

$$\begin{vmatrix} -\lambda & -1 & 1 \\ 4 & -1-\lambda & -4 \\ -3 & -1 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 4-4\lambda & -5-\lambda & -4 \\ -3+4\lambda-\lambda^2 & 3-\lambda & 4-\lambda \end{vmatrix} = 0 \tag{11.6.22}$$

and

$$(\lambda-1)(\lambda-3)(\lambda+1) = 0 \quad \text{or} \quad \lambda = -1, 1, 3. \tag{11.6.23}$$

To determine the eigenvectors, we rewrite (11.6.21) as

$$-\lambda a - b + c = 0, \tag{11.6.24}$$

$$4a - (1+\lambda)b - 4c = 0 \tag{11.6.25}$$

and

$$-3a - b + (4-\lambda)c = 0. \tag{11.6.26}$$

For example, if $\lambda = 1$,

$$-a - b + c = 0, \tag{11.6.27}$$

$$4a - 2b - 4c = 0 \tag{11.6.28}$$

and

$$-3a - b + 3c = 0; \tag{11.6.29}$$

or $a = c$ and $b = 0$. Thus, the eigenfunction for $\lambda = 1$ is $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Similarly, for $\lambda = -1$, $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and for $\lambda = 3$, $\mathbf{x}_0 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. Thus, the most general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} e^{3t}. \tag{11.6.30}$$

• **Example 11.6.2**

Let us solve the following set of linear ordinary differential equations:

$$x_1' = x_1 - 2x_2 \tag{11.6.31}$$

and

$$x_2' = 2x_1 - 3x_2; \tag{11.6.32}$$

or in matrix form,

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{11.6.33}$$

Assuming the solution $\mathbf{x} = \mathbf{x}_0 e^{\lambda t}$,

$$\begin{pmatrix} 1-\lambda & -2 \\ 2 & -3-\lambda \end{pmatrix} \mathbf{x}_0 = \mathbf{0}. \tag{11.6.34}$$

For nontrivial solutions,

$$\begin{vmatrix} 1-\lambda & -2 \\ 2 & -3-\lambda \end{vmatrix} = (\lambda+1)^2 = 0. \tag{11.6.35}$$

Thus, we have the solution

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}. \tag{11.6.36}$$

The interesting aspect of this example is the single solution that the traditional approach yields because we have repeated roots. To find the second solution, we try a solution of the form

$$\mathbf{x} = \begin{pmatrix} a+ct \\ b+dt \end{pmatrix} e^{-t}. \tag{11.6.37}$$

Equation (11.6.37) was guessed based upon our knowledge of solutions to differential equations when the characteristic polynomial has repeated roots. Substituting (11.6.37) into (11.6.33), we find that $c = d = 2c_2$ and $a - b = c_2$. Thus, we have one free parameter, which we will choose to be b, and set it equal to zero. This is permissible because (11.6.37) can be broken into two terms: $b \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$ and $c_2 \begin{pmatrix} 1+2t \\ 2t \end{pmatrix} e^{-t}$. The first term may be incorporated into the $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$ term. Thus, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + 2c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t}. \tag{11.6.38}$$

- **Example 11.6.3**

Let us solve the system of linear differential equations:

$$x_1' = 2x_1 - 3x_2 \tag{11.6.39}$$

and

$$x_2' = 3x_1 + 2x_2; \tag{11.6.40}$$

or in matrix form,

$$\mathbf{x}' = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{11.6.41}$$

Assuming the solution $\mathbf{x} = \mathbf{x}_0 e^{\lambda t}$,

$$\begin{pmatrix} 2-\lambda & -3 \\ 3 & 2-\lambda \end{pmatrix} \mathbf{x}_0 = \mathbf{0}. \tag{11.6.42}$$

For nontrivial solutions,

$$\begin{vmatrix} 2-\lambda & -3 \\ 3 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 + 9 = 0 \tag{11.6.43}$$

and $\lambda = 2 \pm 3i$. If $\mathbf{x}_0 = \begin{pmatrix} a \\ b \end{pmatrix}$, then $b = -ai$ if $\lambda = 2 + 3i$ and $b = ai$ if $\lambda = 2 - 3i$. Thus, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{2t+3it} + c_2 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{2t-3it}, \tag{11.6.44}$$

where c_1 and c_2 are arbitrary complex constants. Using Euler relationships, we can rewrite (11.6.44) as

$$\mathbf{x} = c_3 \begin{bmatrix} \cos(3t) \\ \sin(3t) \end{bmatrix} e^{2t} + c_4 \begin{bmatrix} \sin(3t) \\ -\cos(3t) \end{bmatrix} e^{2t}, \tag{11.6.45}$$

where $c_3 = c_1 + c_2$ and $c_4 = i(c_1 - c_2)$.

Problems

Find the general solution of the following sets of ordinary differential equations using matrix techniques:

1. $x_1' = x_1 + 2x_2$ $\qquad x_2' = 2x_1 + x_2.$
2. $x_1' = x_1 - 4x_2$ $\qquad x_2' = 3x_1 - 6x_2.$

3. $x_1' = x_1 + x_2$ $\quad x_2' = 4x_1 + x_2.$
4. $x_1' = x_1 + 5x_2$ $\quad x_2' = -2x_1 - 6x_2.$
5. $x_1' = -\frac{3}{2}x_1 - 2x_2$ $\quad x_2' = 2x_1 + \frac{5}{2}x_2.$
6. $x_1' = -3x_1 - 2x_2$ $\quad x_2' = 2x_1 + x_2.$
7. $x_1' = x_1 - x_2$ $\quad x_2' = x_1 + 3x_2.$
8. $x_1' = 3x_1 + 2x_2$ $\quad x_2' = -2x_1 - x_2.$
9. $x_1' = -2x_1 - 13x_2$ $\quad x_2' = x_1 + 4x_2.$
10. $x_1' = 3x_1 - 2x_2$ $\quad x_2' = 5x_1 - 3x_2.$
11. $x_1' = 4x_1 - 2x_2$ $\quad x_2' = 25x_1 - 10x_2.$
12. $x_1' = -3x_1 - 4x_2$ $\quad x_2' = 2x_1 + x_2.$
13. $x_1' = 3x_1 + 4x_2$ $\quad x_2' = -2x_1 - x_2.$
14. $x_1' + 5x_1 + x_2' + 3x_2 = 0$ $\quad 2x_1' + x_1 + x_2' + x_2 = 0.$
15. $x_1' - x_1 + x_2' - 2x_2 = 0$ $\quad x_1' - 5x_1 + 2x_2' - 7x_2 = 0.$
16. $x_1' = x_1 - 2x_2$ $\quad x_2' = 0$ $\quad x_3' = -5x_1 + 7x_3.$
17. $x_1' = 2x_1$ $\quad x_2' = x_1 + 2x_3$ $\quad x_3' = x_3.$
18. $x_1' = 3x_1 - 2x_3$ $\quad x_2' = -x_1 + 2x_2 + x_3$ $\quad x_3' = 4x_1 - 3x_3.$
19. $x_1' = 3x_1 - x_3$ $\quad x_2' = -2x_1 + 2x_2 + x_3$ $\quad x_3' = 8x_1 - 3x_3.$

Answers
To the Odd-Numbered Problems

Section 1.1

1. $1 + 2i$

3. $-2/5$

5. $2 + 2i\sqrt{3}$

7. $4e^{\pi i}$

9. $5\sqrt{2}e^{3\pi i/4}$

11. $2e^{2\pi i/3}$

Section 1.2

1.
$$\pm\sqrt{2}, \qquad \pm\sqrt{2}\left[\frac{1}{2} + \frac{\sqrt{3}i}{2}\right], \qquad \pm\sqrt{2}\left[-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right]$$

3.
$$i, \qquad -\frac{\sqrt{3}}{2} - \frac{i}{2}, \qquad z_2 = \frac{\sqrt{3}}{2} - \frac{i}{2}$$

5.
$$w_1 = \frac{1}{\sqrt{2}}\left[-\sqrt{\sqrt{a^2+b^2}+a} + i\sqrt{\sqrt{a^2+b^2}+a}\right], \qquad w_2 = -w_1.$$

7. $z_{1,2} = \pm(1+i); z_{3,4} = \pm 2(1-i)$

Section 1.3

1. $u = 2 - y,\ v = x$

3. $u = x^3 - 3xy^2,\ v = 3x^2y - y^3$

5. $f'(z) = 3z(1+z^2)^{1/2}$

7. $f'(z) = 2(1+4i)z - 3$

9. $f'(z) = -3i(iz-1)^{-4}$

11. $1/6$

13. $v(x,y) = 2xy + \text{constant}$

15. $v(x,y) = x\sin(x)e^{-y} + ye^{-y}\cos(x) + \text{constant}.$

Section 1.4

1. 0

3. $2i$

5. $14/15 - i/3$

Section 1.5

1. $(e^{-2} - e^{-4})/2$

3. $\pi/2$

Section 1.6

1. $\pi i/32$

3. $\pi i/2$

5. $-2\pi i$

7. $2\pi i$

9. -6π

Section 1.7

1.
$$\sum_{n=0}^{\infty}(n+1)z^n.$$

3.
$$f(z) = z^{10} - z^9 + \frac{z^8}{2} - \frac{z^7}{6} + \cdots - \frac{1}{11!z} + \cdots$$

We have an essential singularity and the residue equals $-1/11!$

5.
$$f(z) = \frac{1}{2!} + \frac{z^2}{4!} + \frac{z^4}{6!} + \cdots$$

We have a removable singularity where the value of the residue equals zero.

7.

$$f(z) = -\frac{2}{z} - 2 - \frac{7z}{6} - \frac{z^2}{2} - \cdots$$

We have a simple pole and the residue equals −2.

9.

$$f(z) = \frac{1}{2}\frac{1}{z-2} - \frac{1}{4} + \frac{z-2}{8} - \cdots$$

We have a simple pole and the residue equals 1/2.

Section 1.8

1. $-3\pi i/4$ 3. $-2\pi i$. 5. $2\pi i$ 7. $2\pi i$

Section 2.1

1.

$$f(t) = \frac{1}{2} - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\sin[(2m-1)t]}{2m-1}$$

3.

$$f(t) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2\pi}\cos(nt) + \frac{1-2(-1)^n}{n}\sin(nt)$$

5.

$$f(t) = \frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{\cos[(2m-1)\pi t]}{(2m-1)^2}$$

7.

$$f(t) = \frac{1}{\pi} + \frac{1}{2}\sin(t) - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\cos(2mt)}{4m^2-1}$$

9.

$$f(t) = -\frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m-1)^2}\sin[(2m-1)\pi t]$$

11.

$$f(t) = \frac{a}{2} - \frac{4a}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos\left[\frac{(2m-1)\pi t}{a}\right] - \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi t}{a}\right)$$

13.

$$f(t) = \frac{L}{2\pi}\sin\left(\frac{\pi t}{L}\right) - \frac{2L}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n n}{n^2-1}\sin\left(\frac{n\pi t}{L}\right)$$

15.

$$f(t) = \frac{4a\cosh(a\pi/2)}{\pi}\sum_{m=1}^{\infty}\frac{\cos[(2m-1)t]}{a^2+(2m-1)^2}.$$

Section 2.3

1.

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos[(2m-1)x]}{(2m-1)^2}$$

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin(nx)}{n}$$

3.

$$f(x) = \frac{1}{4} - \frac{2}{\pi^2}\sum_{m=1}^{\infty}\frac{\cos[2(2m-1)\pi x]}{(2m-1)^2}$$

$$f(x) = \frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}\sin[(2m-1)\pi x]}{(2m-1)^2}$$

5.

$$f(x) = \frac{3}{4} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos^2\left(\frac{n\pi}{4}\right)\cos\left(\frac{n\pi x}{2}\right)$$

$$f(x) = \frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^2}\sin\left[\frac{(2m-1)\pi x}{2}\right] - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{2}\right)$$

7.

$$f(x) = \frac{3}{4} + \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{2m-1}\cos\left[\frac{(2m-1)\pi x}{a}\right]$$

$$f(x) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1+\cos(n\pi/2)-2(-1)^n}{n}\sin\left(\frac{n\pi x}{a}\right)$$

9.

$$f(x) = \frac{3a}{8} + \frac{2a}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(n\pi/2) - 1}{n^2} \cos\left(\frac{n\pi x}{a}\right)$$

$$f(x) = \frac{a}{\pi} \sum_{n=1}^{\infty} \left[\frac{2}{n^2\pi} \sin\left(\frac{n\pi}{2}\right) - \frac{(-1)^n}{n}\right] \sin\left(\frac{n\pi x}{a}\right)$$

11.

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sin\left(\frac{m\pi}{2}\right) \cos\left(\frac{2m\pi x}{a}\right)$$

$$f(x) = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2m-1} \sin\left[\frac{(2m-1)\pi}{4}\right] \sin\left[\frac{(2m-1)\pi x}{a}\right]$$

13.

$$f(x) = \frac{e^{ak} - 1}{ak} + 2ka \sum_{n=1}^{\infty} \frac{(-1)^n e^{ka} - 1}{k^2a^2 + n^2\pi^2} \cos\left(\frac{n\pi x}{a}\right)$$

$$f(x) = -2\pi \sum_{n=1}^{\infty} \frac{n[(-1)^n e^{ka} - 1]}{k^2a^2 + n^2\pi^2} \sin\left(\frac{n\pi x}{a}\right)$$

Section 2.4

1.

$$f(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin[(2n-1)t]}{2n-1}$$

$$f(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos[(2n-1)t - \pi/2]}{2n-1}$$

3.

$$f(t) = 2 \sum_{n=1}^{\infty} \frac{1}{n} \cos\left[nt + (-1)^n \frac{\pi}{2}\right]$$

$$f(t) = 2 \sum_{n=1}^{\infty} \frac{1}{n} \sin\left\{nt + [1 + (-1)^n] \frac{\pi}{2}\right\}$$

Section 2.5

1.

$$f(t) = \frac{\pi}{2} - \frac{2}{\pi} \sum_{m=-\infty}^{\infty} \frac{e^{i(2m-1)t}}{(2m-1)^2}$$

3.

$$f(t) = 1 + \frac{i}{\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{n\pi it}}{n}.$$

5.

$$f(t) = \frac{1}{2} - \frac{i}{\pi} \sum_{m=-\infty}^{\infty} \frac{e^{2(2m-1)it}}{2m-1}.$$

Section 2.6

1.

$$y(t) = A\cosh(t) + B\sinh(t) - \frac{1}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin[(2n-1)t]}{(2n-1)+(2n-1)^3}$$

3.

$$y(t) = Ae^{2t} + Be^{t} + \frac{1}{4} + \frac{6}{\pi} \sum_{n=1}^{\infty} \frac{\cos[(2n-1)t]}{[2-(2n-1)^2]^2 + 9(2n-1)^2}$$
$$+ \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{[2-(2n-1)^2]\sin[(2n-1)t]}{(2n-1)\{[2-(2n-1)^2]^2 + 9(2n-1)^2\}}$$

5.

$$y_p(t) = \frac{\pi}{8} - \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{e^{i(2n-1)t}}{(2n-1)^2[4-(2n-1)^2]}$$

7.

$$q(t) = \sum_{n=-\infty}^{\infty} \frac{\omega^2 \varphi_n}{(in\omega_0)^2 + 2i\alpha n\omega_0 + \omega^2} e^{in\omega_0 t}$$

Section 2.7

1. $x(t) = \frac{3}{2} - \cos(\pi x/2) - \sin(\pi x/2) - \frac{1}{2}\cos(\pi x)$

Section 3.3

1. $\pi e^{-|\omega/a|}/|a|$

Section 3.4

1. $-t/(1+t^2)^2$

3. $f(t) = \frac{1}{2}e^{-t}H(t) + \frac{1}{2}e^{t}H(-t)$

5. $f(t) = e^{-t}H(t) - e^{-t/2}H(t) + \frac{1}{2}te^{-t/2}H(t)$

7.
$$f(t) = \frac{i}{2}\text{sgn}(t)e^{-a|t|}, \quad \text{where} \quad \text{sgn}(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0. \end{cases}$$

9.
$$f(t) = \frac{1}{4a}(1 - a|t|)e^{-a|t|}$$

11.
$$f(t) = \frac{(-1)^{n+1}}{(2n+1)!}t^{2n+1}e^{-at}H(t)$$

13.
$$f(t) = \begin{cases} e^{2t}, & t > 0 \\ e^{-t}, & t < 0. \end{cases}$$

Section 3.6

1.
$$y(t) = [(t-1)e^{-t} + e^{-2t}]H(t)$$

3.
$$y(t) = \begin{cases} \frac{1}{9}e^{-t}, & t > 0 \\ \frac{1}{9}e^{2t} - \frac{1}{3}te^{2t}, & t < 0. \end{cases}$$

Section 4.1

1. $F(s) = s/(s^2 - a^2)$

3. $F(s) = 1/s + 2/s^2 + 2/s^3$

5. $F(s) = \left[1 - e^{-2(s-1)}\right]/(s-1)$

7. $F(s) = 2/(s^2+1) - s/(s^2+4) + \cos(3)/s - 1/s^2$

9. $f(t) = e^{-3t}$

11. $f(t) = \frac{1}{3}\sin(3t)$

13. $f(t) = 2\sin(t) - \frac{15}{2}t^2 + 2e^{-t} - 6\cos(2t)$

15. $sF(s) - f(0) = as/(s^2 + a^2) - 0 = \mathcal{L}[f'(t)]$

17. $F(s) = 1/(2s) - sT^2/[2(s^2T^2 + \pi^2)]$

Section 4.2

1. $f(t) = (t-2)H(t-2) - (t-2)H(t-3)$

3. $y'' + 3y' + 2y = H(t-1)$

5. $y'' + 4y' + 4y = tH(t-2)$

7. $y'' - 3y' + 2y = e^{-t}H(t-2)$

9. $y'' + y = \sin(t)[1 - H(t-\pi)]$

Section 4.3

1. $F(s) = 2/(s^2 + 2s + 5)$

3. $F(s) = 1/(s-1)^2 + 3/(s^2 - 2s + 10) + (s-2)/(s^2 - 4s + 29)$

5. $F(s) = 2/(s+1)^3 + 2/(s^2 - 2s + 5) + (s+3)/(s^2 + 6s + 18)$

7. $F(s) = e^6 e^{-3s}/(s-2)$

9. $F(s) = 2e^{-s}/s^3 + 2e^{-s}/s^2 + 3e^{-s}/s + e^{-2s}/s$

11. $F(s) = (1 + e^{-s\pi})/(s^2 + 1)$

13. $F(s) = 4(s+3)/(s^2 + 6s + 13)^2$

15. $f(t) = \frac{1}{2}t^2e^{-2t} - \frac{1}{3}t^3e^{-2t}$

17. $f(t) = e^{-t}\cos(t) + 2e^{-t}\sin(t)$

19. $f(t) = e^{-2t} - 2te^{-2t} + \cos(t)e^{-t} + \sin(t)e^{-t}$

21. $f(t) = e^{t-3}H(t-3)$

23. $f(t) = e^{-(t-1)}[\cos(t-1) - \sin(t-1)]H(t-1)$

25. $f(t) = \cos[2(t-1)]H(t-1) + \frac{1}{6}(t-3)^3e^{2(t-3)}H(t-3)$

27. $f(t) = \{\cos[2(t-1)] + \frac{1}{2}\sin[2(t-1)]\}H(t-1) + \frac{1}{6}(t-3)^3H(t-3)$

29. $f(t) = t[H(t) - H(t-a)]$; $F(s) = 1/s^2 - e^{-as}/s^2 - ae^{-as}/s$

31. $F(s) = 1/s^2 - e^{-s}/s^2 - e^{-2s}/s$

33. $F(s) = e^{-s}/s^2 - e^{-2s}/s^2 - e^{-3s}/s$

35. $Y(s) = s/(s^2+4) + 3e^{-4s}/[s(s^2+4)]$

37. $Y(s) = e^{-(s-1)}/[(s-1)(s+1)(s+2)]$

39. $Y(s) = 5s/[(s-1)(s-2)] + e^{-s}/[s^3(s-1)(s-2)]$
$+2e^{-s}/[s^2(s-1)(s-2)] + e^{-s}/[s(s-1)(s-2)]$

41. $Y(s) = 1/[s^2(s+2)(s+1)] + ae^{-as}/[(s+1)^2(s+2)]$
$-e^{-as}/[s^2(s+1)(s+2)] - e^{-as}/[s(s+1)(s+2)]$

43. $\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s^2/(s^2+a^2) = 1 = f(0)$.

45. $\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} 3s/(s^2-2s+10) = 0 = f(0)$.

47. Yes

49. No

51. No

Section 4.4

1. $F(s) = \dfrac{1}{s^2+1}\coth\left(\dfrac{s\pi}{2}\right)$

3. $F(s) = \dfrac{1-(1+as)e^{-as}}{s^2(1-e^{-2as})}$

Section 4.5

1. $f(t) = e^{-t} - e^{-2t}$

3. $f(t) = \frac{5}{4}e^{-t} - \frac{6}{5}e^{-2t} - \frac{1}{20}e^{3t}$

5. $f(t) = e^{-2t}\cos\left(t + \frac{3\pi}{2}\right)$

7. $f(t) = 2.3584\cos(4t + 0.5586)$

9. $f(t) = \frac{1}{2} + \frac{\sqrt{2}}{2}\cos\left(2t + \frac{5\pi}{4}\right)$

Section 4.6

1.
$$\mathcal{L}(t) = \frac{1}{s^2} = \mathcal{L}(1)\mathcal{L}(1)$$

3.
$$\mathcal{L}\left(e^t - 1\right) = \frac{1}{s-1} - \frac{1}{s} = \frac{1}{s(s-1)} = \mathcal{L}(1)\mathcal{L}(e^t)$$

5.
$$\mathcal{L}[t - \sin(t)] = \frac{1}{s^2} - \frac{1}{s^2+1} = \frac{1}{s^2(s^2+1)} = \mathcal{L}(t)\mathcal{L}[\sin(t)]$$

7.

$$\mathcal{L}\left\{\frac{t^2}{a} - \frac{2}{a^3}[1-\cos(at)]\right\} = \frac{2}{s^3}\left(\frac{a}{s^2+a^2}\right) = \mathcal{L}(t^2)\mathcal{L}[\sin(at)]$$

9.

$$H(t-b) * H(t-a) = \int_a^t H(t-b-x)\,dx = -\int_{t-b-a}^{-b} H(\eta)\,d\eta,$$

if $t > a$ and $\eta = t - b - x$.

11.

$$f(t) = e^t - t - 1$$

13. Assuming that $a, b > 0$,

$$\int_0^t \delta(t-x-a)\delta(x-b)\,dx = \delta(t-b-a)$$

Section 4.7

1. $f(t) = 1 + 2t$

3. $f(t) = t + \frac{1}{2}t^2$

5. $f(t) = t^3 + \frac{1}{20}t^5$

7. $f(t) = t^2 - \frac{1}{3}t^4$

9. $f(t) = 5e^{2t} - 4e^t - 2te^t$

11. $f(t) = (1-t)^2 e^{-t}$

13. $f(t) = 4 + \frac{5}{2}t^2 + \frac{1}{24}t^4$

15. $x(t) = 2A\sqrt{t}/(\pi C) - Bt/(2C)$

17.

$$f(t) = \frac{\alpha}{\beta^2}\left(e^{\beta^2 t} - 1 + \frac{2\beta\sqrt{t}}{\sqrt{\pi}} - \frac{2e^{\beta^2 t}}{\sqrt{\pi}}\int_0^{\beta\sqrt{t}} e^{-u^2}\,du\right)$$

Section 4.8

1. $y(t) = \frac{5}{4}e^{2t} - \frac{1}{4} + \frac{1}{2}t$

3. $y(t) = e^{3t} - e^{2t}$

5. $y(t) = -\frac{3}{4}e^{-3t} + \frac{7}{4}e^{-t} + \frac{1}{2}te^{-t}$

7. $y(t) = \frac{3}{4}e^{-t} + \frac{1}{8}e^t - \frac{7}{8}e^{-3t}$

9. $y(t) = (t-1)H(t-1)$

11. $y(t) = e^{2t} - e^t + \left[\frac{1}{2} + \frac{1}{2}e^{2(t-1)} - e^{t-1}\right]H(t-1)$

13. $y(t) = \left[1 - e^{-2(t-2)} - 2(t-2)e^{-2(t-2)}\right] H(t-2)$

15. $y(t) = \left[\frac{1}{3}e^{2(t-2)} - \frac{1}{2}e^{t-2} + \frac{1}{6}e^{-(t-2)}\right] H(t-2)$

17. $y(t) = 1 - \cos(t) - [1 - \cos(t-T)]\, H(t-T)$

19.
$$\begin{aligned} y(t) &= e^{-t} - \tfrac{1}{4}e^{-2t} - \tfrac{3}{4} + \tfrac{1}{2}t \\ &- \left[e^{-(t-a)} - \tfrac{1}{4}e^{-2(t-a)} - \tfrac{3}{4} + \tfrac{1}{2}(t-a)\right] H(t-a) \\ &+ a\left[\tfrac{1}{2}e^{-2(t-a)} + (t-a)e^{-(t-a)} - \tfrac{1}{2}\right] H(t-a). \end{aligned}$$

21. $y(t) = te^{t} + 3(t-2)e^{t-2}H(t-2)$

23. $y(t) = 3\left[e^{-2(t-2)} - e^{-3(t-2)}\right] H(t-2)$
$\quad + 4\left[e^{-3(t-5)} - e^{-2(t-5)}\right] H(t-5)$

25. $x(t) = 2e^{t/2} - 2 - t; y(t) = e^{t/2} - 1 - t$

27. $x(t) = \frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t}; y(t) = e^{-t} - 1$

Section 4.9

1. $G(s) = 1/(s+k)$ — $g(t) = e^{-kt}$

$a(t) = \left(1 - e^{-kt}\right)/k$

3. $G(s) = 1/(s^2 + 4s + 3)$ — $g(t) = \frac{1}{2}\left(e^{-t} - e^{-3t}\right)$

$a(t) = \frac{1}{6}e^{-3t} - \frac{1}{2}e^{-t} + \frac{1}{3}$

5. $G(s) = 1/[(s-2)(s-1)]$ — $g(t) = e^{2t} - e^{t}$

$a(t) = \frac{1}{2} + \frac{1}{2}e^{2t} - e^{t}$

7. $G(s) = 1/(s^2 - 9)$ — $g(t) = \frac{1}{6}\left(e^{3t} - e^{-3t}\right)$

$a(t) = \frac{1}{18}\left(e^{3t} + e^{-3t} - 2\right)$

9. $G(s) = 1/[s(s-1)]$ — $g(t) = e^{t} - 1$

$a(t) = e^{t} - t - 1$

Section 4.10

1. $f(t) = (2-t)e^{-2t} - 2e^{-3t}$

3. $f(t) = \left(\frac{1}{4}t^2 - \frac{1}{4}t + \frac{1}{8}\right)e^{2t} - \frac{1}{8}$

5. $f(t) = \left[\frac{1}{2}(t-1) - \frac{1}{4} + \frac{1}{4}e^{-2(t-1)}\right]H(t-1)$

7.

$$f(t) = \frac{e^{-bt}}{\cosh(ab)} - 8ab\sum_{n=1}^{\infty}(-1)^n \frac{\sin[(2n-1)\pi t/(2a)]}{4a^2b^2 + (2n-1)^2\pi^2}$$
$$+ 4\sum_{n=1}^{\infty}(-1)^n \frac{(2n-1)\pi\cos[(2n-1)\pi t/(2a)]}{4a^2b^2 + (2n-1)^2\pi^2}.$$

Section 5.1

1. $F(z) = 2z/(2z-1)$ if $|z| > 1/2$.

3. $F(z) = (z^6 - 1)/(z^6 - z^5)$ if $|z| > 0$.

5. $F(z) = (a^2 + a - z)/[z(z-a)]$ if $|z| > a$.

Section 5.2

1. $F(z) = zTe^{aT}/(ze^{aT} - 1)^2$ 3. $F(z) = z(z+a)/(z-a)^3$

5. $F(z) = [z - \cos(1)]/\{z[z^2 - 2z\cos(1) + 1]\}$

7. $F(z) = z[z\sin(\theta) + \sin(\omega_0 T - \theta)]/[z^2 - 2z\cos(\omega_0 T) + 1]$

9. $F(z) = z/(z+1)$ 11. $f_n * g_n = n + 1$ 13. $f_n * g_n = 2^n/n!$

Section 5.3

1. $f_0 = 0.007143$, $f_1 = 0.08503$, $f_2 = 0.1626$, $f_3 = 0.2328$

3. $f_0 = 0.09836$, $f_1 = 0.3345$, $f_2 = 0.6099$, $f_3 = 0.7935$

5. $f_n = 8 - 8\left(\frac{1}{2}\right)^n - 6n\left(\frac{1}{2}\right)^n$ 7. $f_n = (1 - \alpha^{n+1})/(1-\alpha)$

9. $f_n = \left(\frac{1}{2}\right)^{n-10} H_{n-10} + \left(\frac{1}{2}\right)^{n-11} H_{n-11}$

11. $f_n = \frac{1}{9}(6n-4)(-1)^n + \frac{4}{9}\left(\frac{1}{2}\right)^n$ 13. $f_n = a^n/(n!)$

Section 5.4

1. $y_n = 1 + \frac{1}{6}n(n-1)(2n-1)$ 3. $y_n = \frac{1}{2}n(n-1)$

5. $y_n = \frac{1}{6}\left[5^n - (-1)^n\right]$ 7. $y_n = (2n-1)\left(\frac{1}{2}\right)^n + \left(-\frac{1}{2}\right)^n$

9. $y_n = 2^n - n - 1.$ 11. $x_n = 2 + (-1)^n; y_n = 1 + (-1)^n$

13. $x_n = 1 - 2(-6)^n; y_n = -7(-6)^n$

Section 5.5

1. marginally stable 3. unstable

Section 6.1

1. $\lambda_n = (2n-1)^2\pi^2/(4L^2)$ with $y_n(x) = \cos\left[(2n-1)\pi x/(2L)\right]$

3. $\lambda_0 = -1$, $y_0(x) = e^{-x}$ and $\lambda_n = n^2$, $y_n(x) = \sin(nx) - n\cos(nx)$

5. $\lambda_n = -n^4\pi^4/L^4$, $y_n(x) = \sin(n\pi x/L)$

7. $\lambda_n = k_n^2$, $y_n(x) = \sin(k_n x)$ with $k_n = -\tan(k_n)$

9. $\lambda_0 = -m_0^2$, $y_0(x) = \sinh(m_0 x) - m_0\cosh(m_0 x)$ with $\coth(m_0\pi) = m_0$ and $\lambda_n = k_n^2$, $y_n(x) = \sin(k_n x) - k_n\cos(k_n x)$ with $k_n = -\cot(k_n\pi)$

11.

(a) $\lambda_n = n^2\pi^2, \quad y_n(x) = \sin[n\pi\ln(x)]$

(b) $\lambda_n = (2n-1)^2\pi^2/4, \quad y_n(x) = \sin[(2n-1)\pi\ln(x)/2]$

(c) $\lambda_0 = 0, \quad y_0(x) = 1; \qquad \lambda_n = n^2\pi^2, \quad y_n(x) = \cos[n\pi\ln(x)]$

13. $\lambda_n = n^2 + 1$, $y_n(x) = x^{-1}\sin[n\ln(x)]$

Section 6.3

1.

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{L}\right)$$

3.

$$f(x) = \frac{8L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left[\frac{(2n-1)\pi x}{2L}\right]$$

Section 6.4

1.
$$f(x) = \tfrac{1}{4}P_0(x) + \tfrac{1}{2}P_1(x) + \tfrac{5}{16}P_2(x) + \cdots$$

3.
$$f(x) = \tfrac{1}{2}P_0(x) + \tfrac{5}{8}P_2(x) - \tfrac{3}{16}P_4(x) + \cdots$$

5.
$$f(x) = \tfrac{3}{2}P_1(x) - \tfrac{7}{8}P_3(x) + \tfrac{11}{16}P_5(x) + \cdots$$

Section 7.3

1.
$$u(x,t) = \frac{4L}{c\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\sin\left[\frac{(2m-1)\pi x}{L}\right]\sin\left[\frac{(2m-1)\pi ct}{L}\right]$$

3.
$$u(x,t) = \frac{9h}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{2n\pi}{3}\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

5.
$$\begin{aligned}u(x,t) &= \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi ct}{L}\right)\\ &\quad + \frac{4aL}{\pi^2 c}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left[\frac{(2n-1)\pi}{4}\right]\sin\left[\frac{(2n-1)\pi x}{L}\right]\\ &\qquad \times\sin\left[\frac{(2n-1)\pi ct}{L}\right]\end{aligned}$$

7.
$$u(x,t) = \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left[\frac{(2n-1)\pi x}{L}\right]\cos\left[\frac{(2n-1)\pi ct}{L}\right]$$

9.
$$\begin{aligned}u(x,t) = \frac{8L}{\pi^2}e^{-ht}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left[\frac{(2n-1)\pi x}{2L}\right]\Big\{\cos\left[t\sqrt{\lambda_n c^2 - h^2}\right]\\ + h\sin\left[t\sqrt{\lambda_n c^2 - h^2}\right]\Big/\sqrt{\lambda_n c^2 - h^2}\Big\},\end{aligned}$$

where $\lambda_n = (2n-1)^2\pi^2/4L^2$.

Section 7.4

1.

$$u(x,t) = \sin(2x)\cos(2ct) + \cos(x)\sin(ct)/c$$

3.

$$u(x,t) = \frac{1+x^2+c^2t^2}{(1+x^2+c^2t^2)^2+4x^2c^2t^2} + \frac{e^x\sinh(ct)}{c}$$

5.

$$u(x,t) = \cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi ct}{2}\right) + \frac{\sinh(ax)\sinh(act)}{ac}$$

Section 7.5

1.

$$u(x,t) = \frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{\sin[(2m-1)\pi x]\sin[(2m-1)\pi t]}{(2m-1)^2}$$

3.

$$u(x,t) = \sin(\pi x)\cos(\pi t) - \sin(\pi x)\sin(\pi t)/\pi$$

5.

$$u(x,t) = xt - te^{-x} + \sinh(t)e^{-x} + \left[1 - e^{-(t-x)} + t - x - \sinh(t-x)\right] H(t-x)$$

7.

$$u(x,t) = \frac{gx}{\omega^2} - \frac{2g\omega^2}{L}\sum_{n=1}^{\infty}\frac{\sin(\lambda_n x)\cos(\lambda_n t)}{\lambda_n^2(\omega^4+\omega^2/L+\lambda_n^2)\sin(\lambda_n L)},$$

where λ_n is the nth root of $\lambda = \omega^2\cot(\lambda L)$.

9.

$$u(x,t) = E - \frac{4E}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left[\frac{(2n-1)\pi x}{2\ell}\right]\cos\left[\frac{(2n-1)c\pi t}{2\ell}\right]$$

or

$$u(x,t) = E\sum_{n=0}^{\infty}(-1)^n H\left(t - \frac{x+2n\ell}{c}\right) + E\sum_{n=0}^{\infty}(-1)^n H\left\{t - \frac{[(2n+2)\ell - x]}{c}\right\}$$

11.

$$p(x,t) = p_0 - \frac{4\rho u_0 c}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\sin\left[\frac{(2n-1)\pi x}{2L}\right]\sin\left[\frac{(2n-1)c\pi t}{2L}\right]$$

13.

$$u(x,t) = \frac{gt^2}{2} - \frac{gL^2}{6c^2} - \frac{2gL^2}{c^2\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

Section 8.3

1.

$$u(x,t) = \frac{4A}{\pi}\sum_{m=1}^{\infty}\frac{\sin[(2m-1)x]}{2m-1}e^{-a^2(2m-1)^2t}$$

3.

$$u(x,t) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nx)e^{-a^2n^2t}$$

5.

$$u(x,t) = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^2}\sin[(2m-1)x]e^{-a^2(2m-1)^2t}$$

7.

$$u(x,t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos[(2m-1)x]}{(2m-1)^2}e^{-a^2(2m-1)^2t}$$

9.

$$u(x,t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos[(2n-1)x]}{(2n-1)^2}e^{-a^2(2n-1)^2t}$$

11.

$$u(x,t) = \frac{32}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3}\cos\left[\frac{(2n-1)x}{2}\right]e^{-a^2(2n-1)^2t/4}$$

13.

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)x/2]}{2n-1}e^{-a^2(2n-1)^2t/4}$$

15.

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{4}{2n-1} - \frac{8(-1)^{n+1}}{(2n-1)^2\pi}\right]\sin\left[\frac{(2n-1)x}{2}\right]e^{-a^2(2n-1)^2t/4}$$

17.

$$u(x,t) = \frac{T_0x}{\pi} + \frac{2T_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(nx)e^{-a^2n^2t}$$

19.

$$u(x,t) = h_1 + \frac{(h_2-h_1)x}{L} + \frac{2(h_2-h_1)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{a^2n^2\pi^2}{L^2}t\right)$$

21.

$$u(x,t) = h_0 - \frac{4h_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left[\frac{(2n-1)\pi x}{L}\right]\exp\left[-\frac{(2n-1)^2\pi^2a^2t}{L^2}\right]$$

23.

$$u(x,t) = \frac{1}{3} - t - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x)e^{-a^2n^2\pi^2t}$$

25.

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^4}\sin[(2n-1)x]\left[1 - e^{-(2n-1)^2t}\right]$$

27.

$$u(x,t) = \frac{A_0(L^2 - x^2)}{2\kappa} + \frac{A_0 L}{h}$$
$$- \frac{2L^2 A_0}{\kappa} \sum_{n=1}^{\infty} \frac{\sin(\beta_n)}{\beta_n^4[1 + \kappa \sin^2(\kappa)/hL]} \cos\left(\frac{\beta_n x}{L}\right) \exp\left(-\frac{a^2 \beta_n^2 t}{L^2}\right),$$

where β_n is the nth root of $\beta \tan(\beta) = \kappa/hL$.

29.

$$u(r,t) = \frac{2}{\pi r} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi r) e^{-a^2 n^2 \pi^2 t}$$

31.

$$u(r,t) = \frac{G}{4\rho\nu}(b^2 - r^2) - \frac{2Gb^2}{\rho\nu} \sum_{n=1}^{\infty} \frac{J_0(k_n r/b)}{k_n^3 J_1(k_n)} \exp\left(-\frac{\nu k_n^2}{b^2} t\right),$$

where k_n is the nth root of $J_0(k) = 0$.

Section 8.4

1.

$$u(x,t) = T_0\left(1 - e^{-a^2 t}\right)$$

3.

$$u(x,t) = x + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(n\pi x) e^{-n^2 \pi^2 t}$$

5.

$$u(x,t) = \frac{x(1-x)}{2} - \frac{4}{\pi^3} \sum_{m=1}^{\infty} \frac{\sin[(2m-1)\pi x]}{(2m-1)^3} e^{-(2m-1)^2 \pi^2 t}$$

7.

$$u(x,t) = x \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) - 2\sqrt{\frac{t}{\pi}} \exp\left(-\frac{x^2}{4t}\right)$$

9.

$$u(x,t) = \frac{u_0}{2} e^{-\delta x} \operatorname{erfc}\left(\frac{x}{2a\sqrt{t}} + \frac{a(1-\delta)\sqrt{t}}{2}\right)$$
$$+ \frac{u_0}{2} e^{-x} \operatorname{erfc}\left(\frac{x}{2a\sqrt{t}} - \frac{a(1-\delta)\sqrt{t}}{2}\right)$$

11.

$$u(x,t) = \frac{t(L-x)}{L} + \frac{Px(x-L)}{2a^2} - \frac{x(x-L)(x-2L)}{6a^2L}$$
$$- \frac{2PL^2}{a^2\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{a^2n^2\pi^2t}{L^2}\right)$$
$$+ \frac{2(P+1)L^2}{a^2\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{a^2n^2\pi^2t}{L^2}\right).$$

13.

$$u(r,t) = \frac{r^2}{2} + 3t - \frac{3}{10} - \frac{2}{r}\sum_{n=1}^{\infty}\frac{\sin(\lambda_n r)}{\lambda_n^2\sin(\lambda_n)}e^{-\lambda_n^2 t},$$

where $\tan(\lambda_n) = \lambda_n$.

15.

$$y(t) = \frac{4\mu A\omega^2}{mL}\sum_{n=1}^{\infty}\frac{\lambda_n e^{\lambda_n t}}{\lambda_n^4 - (\frac{2\mu}{mL})(1+\frac{2\mu L}{m\nu})\lambda_n^3 + 2\omega^2\lambda_n^2 + \frac{6\omega^2\mu}{mL}\lambda_n + \omega^4},$$

where λ_n is the nth root of $\lambda^2 + 2\mu\lambda^{3/2}\coth(L\sqrt{\lambda/\nu})/(m\sqrt{\nu}) + \omega^2 = 0$.

17.

$$u(x,t) = 1 - 2e^{Vx/2 - V^2t/4}$$
$$\times \sum_{n=1}^{\infty}\frac{\lambda_n\{(V/2)\sin[\lambda_n(1-x)] + \lambda_n\cos[\lambda_n(1-x)]\}e^{-\lambda_n^2 t}}{(\lambda_n^2 + V^2/4)[\lambda_n\sin(\lambda_n) - (1+V/2)\cos(\lambda_n)]},$$

where λ_n is the nth root of $\lambda\cot(\lambda) = -V/2$.

19.

$$u(r,t) = \frac{a^2 - r^2}{4} - 2a^2\sum_{n=1}^{\infty}\frac{J_0(k_n r/a)}{k_n^3 J_1(k_n)}e^{-k_n^2 t/a^2},$$

where k_n is the nth root of $J_0(k) = 0$.

Section 8.5

1.

$$u(x,t) = \tfrac{1}{2}\text{erf}\left(\frac{b-x}{\sqrt{4a^2t}}\right) + \tfrac{1}{2}\text{erf}\left(\frac{b+x}{\sqrt{4a^2t}}\right)$$

3.

$$u(x,t) = \tfrac{1}{2}T_0\mathrm{erf}\left(\frac{b-x}{\sqrt{4a^2t}}\right) + \tfrac{1}{2}T_0\mathrm{erf}\left(\frac{x}{\sqrt{4a^2t}}\right).$$

Section 8.6

1.

$$u(x,t) = \frac{4a^2\pi}{L^2}\sum_{m=1}^{\infty}(2m-1)\sin\left[\frac{(2m-1)\pi x}{L}\right]e^{-a^2(2m-1)^2\pi^2t/L^2}$$
$$\times \int_0^t f(\tau)e^{a^2(2m-1)^2\pi^2\tau/L^2}\,d\tau$$

3.

$$u(x,t) = \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4\nu t}}^{\infty} V\left(t - \frac{x^2}{4\nu\eta^2}\right)e^{-\eta^2}\,d\eta$$

Section 9.3

1.

$$u(x,y) = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\sinh[(2m-1)\pi(a-x)/b]\sin[(2m-1)\pi y/b]}{(2m-1)\sinh[(2m-1)\pi a/b]}.$$

3.

$$u(x,y) = -\frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{\sinh(n\pi y/a)\sin(n\pi x/a)}{n\ \sinh(n\pi b/a)}$$

5.

$$u(x,y) = \frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sinh[(2n-1)\pi y/2a]\cos[(2n-1)\pi x/2a]}{(2n-1)\sinh[(2n-1)\pi b/2a]}$$

7.

$$u(x,y) = 1$$

9.

$$u(x,y) = 1 - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cosh[(2m-1)\pi y/a]\sin[(2m-1)\pi x/a]}{(2m-1)\cosh[(2m-1)\pi b/a]}$$

11.
$$u(x,y) = 1$$

13.
$$u(x,y) = T_0 + \Delta T \cos(2\pi x/\lambda) e^{-2\pi y/\lambda}$$

15.
$$u(r,z) = 2a \sum_{n=1}^{\infty} \frac{\sinh(k_n z/a)\, J_0(k_n r/a)}{k_n^2 \cosh(k_n L/a) J_1(k_n)},$$

where k_n is the nth root of $J_0(k) = 0$.

17.
$$u(r,z) = \frac{2}{b^2 - a^2} \sum_{n=1}^{\infty} \frac{[bJ_1(k_n b) - aJ_1(k_n a)] J_0(k_n r) \cosh(k_n z)}{k_n \cosh(k_n d) J_0^2(k_n)},$$

where k_n is the nth root of $J_1(k) = 0$.

19.
$$u(r,z) = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n I_1(n\pi r) \sin(n\pi z)}{n\, I_1(n\pi a)}$$

21.
$$u(r,z) = 2B \sum_{n=1}^{\infty} \frac{\exp[z(1 - \sqrt{1 + 4k_n^2}\,)/2] J_0\,(k_n r)}{(k_n^2 + B^2) J_0(k_n)},$$

where k_n is the nth root of $k\, J_1(k) = B\, J_0(k)$.

23.
$$u(r,\theta) = 50 \sum_{m=1}^{\infty} [P_{2m-2}(0) - P_{2m}(0)] \left(\frac{r}{a}\right)^{2m-1} P_{2m-1}[\cos(\theta)]$$

25. T_0

Section 9.4

1.
$$u(x,y) = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{1-x}{y}\right) + \tan^{-1}\left(\frac{x}{y}\right)\right]$$

3.

$$u(x,y) = \frac{T_0}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{x}{y}\right)\right]$$

5.

$$\begin{aligned} u(x,y) = {} & \frac{T_0}{\pi}\left[\tan^{-1}\left(\frac{1-x}{y}\right) + \tan^{-1}\left(\frac{1+x}{y}\right)\right] \\ & + \frac{T_1 - T_0}{2\pi}\, y \ln\left[\frac{(x-1)^2 + y^2}{x^2 + y^2}\right] \\ & + \frac{T_1 - T_0}{\pi}\, x \left[\tan^{-1}\left(\frac{1-x}{y}\right) + \tan^{-1}\left(\frac{x}{y}\right)\right] \end{aligned}$$

Section 9.5

1.

$$\begin{aligned} u(x,y) = {} & \frac{64\,R}{\pi^4 T}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+1}(-1)^{m+1}}{(2n-1)(2m-1)} \\ & \times \frac{\cos[(2n-1)\pi x/2a]\cos[(2m-1)\pi y/b]}{(2n-1)(2m-1)[(2n-1)^2/a^2 + (2m-1)^2/b^2]} \end{aligned}$$

Section 9.6

1.

$$u(x,y) = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{2m-1}\exp\left[-\frac{(2m-1)\pi x}{a}\right]\sin\left[\frac{(2m-1)\pi y}{a}\right]$$

Section 10.1

1. $\mathbf{a}\times\mathbf{b} = -3\mathbf{i} + 19\mathbf{j} + 10\mathbf{k}$

3. $\mathbf{a}\times\mathbf{b} = \mathbf{i} - 8\mathbf{j} + 7\mathbf{k}$

5. $\mathbf{a}\times\mathbf{b} = -3\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$

7.

$$\begin{aligned} \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} + (\mathbf{b}\cdot\mathbf{a})\mathbf{c} \\ &\quad - (\mathbf{b}\cdot\mathbf{c})\mathbf{a} + (\mathbf{c}\cdot\mathbf{b})\mathbf{a} - (\mathbf{c}\cdot\mathbf{a})\mathbf{b} \\ &= 0 \end{aligned}$$

9.

$$\nabla f = y\cos(yz)\mathbf{i} + [x\cos(yz) - xyz\sin(yz)]\mathbf{j} - xy^2\sin(yz)\mathbf{k}$$

11.

$$\nabla f = 2xy^2(2z+1)^2\mathbf{i} + 2x^2y(2z+1)^2\mathbf{j} + 4x^2y^2(2z+1)\mathbf{k}$$

13. Plane parallel to the xy plane at height of $z = 3$, $\mathbf{n} = \mathbf{k}$

15. Paraboloid,

$$\mathbf{n} = -\frac{2x}{\sqrt{1+4x^2+4y^2}}\mathbf{i} - \frac{2y}{\sqrt{1+4x^2+4y^2}}\mathbf{j} + \frac{1}{\sqrt{1+4x^2+4y^2}}\mathbf{k}$$

17. A plane, $\mathbf{n} = \mathbf{j}/\sqrt{2} - \mathbf{k}/\sqrt{2}$

19. A parabola of infinite extent along the y-axis, $\mathbf{n} = -2x\mathbf{i}/\sqrt{1+4x^2} + \mathbf{k}/\sqrt{1+4x^2}$

21. $y = 2/(x+1)$; $\quad z = \exp[(y-1)/y]$

23. $y = x$; $\quad z^2 = y/(3y-2)$

Section 10.2

1.

$$\nabla \cdot \mathbf{F} = 2xz + z^2$$
$$\nabla \times \mathbf{F} = (2xy - 2yz)\mathbf{i} + (x^2 - y^2)\mathbf{j}$$
$$\nabla(\nabla \cdot \mathbf{F}) = 2z\mathbf{i} + (2x + 2z)\mathbf{k}$$

3.

$$\nabla \cdot \mathbf{F} = 2(x-y) - xe^{-xy} + xe^{2y}$$
$$\nabla \times \mathbf{F} = 2xze^{2y}\mathbf{i} - ze^{2y}\mathbf{j} + \left[2(x-y) - ye^{-xy}\right]\mathbf{k}$$
$$\nabla(\nabla \cdot \mathbf{F}) = \left(2 - e^{-xy} + xye^{-xy} + e^{2y}\right)\mathbf{i} + \left(x^2e^{-xy} + 2xe^{2y} - 2\right)\mathbf{j}$$

5.

$$\nabla \cdot \mathbf{F} = 0$$
$$\nabla \times \mathbf{F} = -x^2\mathbf{i} + (5y - 9x^2)\mathbf{j} + (2xz - 5z)\mathbf{k}$$
$$\nabla(\nabla \cdot \mathbf{F}) = \mathbf{0}$$

7.

$$\nabla \cdot \mathbf{F} = e^{-y} + z^2 - 3e^{-z}$$

$$\nabla \times \mathbf{F} = -2yz\mathbf{i} + xe^{-y}\mathbf{k}$$

$$\nabla(\nabla \cdot \mathbf{F}) = -e^{-y}\mathbf{j} + (2z + 3e^{-z})\mathbf{k}$$

9.

$$\nabla \cdot \mathbf{F} = yz + x^3ze^z + xye^z$$

$$\nabla \times \mathbf{F} = (xe^z - x^3ye^z - x^3yze^z)\mathbf{i} + (xy - ye^z)\mathbf{j} + (3x^2yze^z - xz)\mathbf{k}$$

$$\nabla(\nabla \cdot \mathbf{F}) = \left(3x^2ze^z + ye^z\right)\mathbf{i} + \left(z + xe^z\right)\mathbf{j} + \left(y + x^3e^z + x^3ze^z + xye^z\right)\mathbf{k}$$

11.

$$\nabla \cdot \mathbf{F} = y^2 + xz^2 - xy\sin(z)$$

$$\nabla \times \mathbf{F} = [x\cos(z) - 2xyz]\mathbf{i} - y\cos(z)\mathbf{j} + (yz^2 - 2xy)\mathbf{k}$$

$$\nabla(\nabla \cdot \mathbf{F}) = [z^2 - y\sin(z)]\mathbf{i} + [2y - x\sin(z)]\mathbf{j} + [2xz - xy\cos(z)]\mathbf{k}$$

13.

$$\nabla \cdot \mathbf{F} = y^2 + xz - xy\sin(z)$$

$$\nabla \times \mathbf{F} = [x\cos(z) - xy]\mathbf{i} - y\cos(z)\mathbf{j} + (yz - 2xy)\mathbf{k}$$

$$\nabla(\nabla \cdot \mathbf{F}) = [z - y\sin(z)]\mathbf{i} + [2y - x\sin(z)]\mathbf{j} + [x - xy\cos(z)]\mathbf{k}$$

Section 10.3

1. $16/7 + 2/(3\pi)$ 3. $e^2 + 2e^8/3 + e^{64}/2 - 13/6$ 5. -4π

7. 0 9. 2π

Section 10.4

1. $\varphi(x, y, z) = x^2y + y^2z + 4z + \text{constant}$

3. $\varphi(x, y, z) = xyz + \text{constant}$

5. $\varphi(x, y, z) = x^2\sin(y) + xe^{3z} + 4z + \text{constant}$

7. $\varphi(x, y, z) = xe^{2z} + y^3 + \text{constant}$

9. $\varphi(x, y, z) = xy + xz + \text{constant}$

Section 10.5

1. 1/2 3. 0 5. 27/2

7. 5 9. 0 11. 40/3

13. 86/3 15. 96π

Section 10.6

1. −5 3. 1 5. 0

7. 0 9. -16π 11. −2

Section 10.7

1. −10 3. 2 5. π 7. 45/2

Section 10.8

1. 3 3. −16 5. 4π 7. 5/12

Section 11.1

1.
$$A+B=\begin{pmatrix}4 & 5\\ 3 & 4\end{pmatrix}=B+A$$

3.
$$3A-2B=\begin{pmatrix}7 & 10\\ -1 & 2\end{pmatrix},\qquad 3(2A-B)=\begin{pmatrix}15 & 21\\ 0 & 6\end{pmatrix}$$

5.
$$(A+B)^T=\begin{pmatrix}4 & 3\\ 5 & 4\end{pmatrix},\qquad A^T+B^T=\begin{pmatrix}4 & 3\\ 5 & 4\end{pmatrix}$$

7.
$$AB=\begin{pmatrix}11 & 11\\ 5 & 5\end{pmatrix},\qquad A^TB=\begin{pmatrix}5 & 5\\ 8 & 8\end{pmatrix}$$
$$BA=\begin{pmatrix}4 & 6\\ 8 & 12\end{pmatrix},\qquad B^TA=\begin{pmatrix}5 & 8\\ 5 & 8\end{pmatrix}$$

9.
$$BB^T=\begin{pmatrix}2 & 4\\ 4 & 8\end{pmatrix},\qquad B^TB=\begin{pmatrix}5 & 5\\ 5 & 5\end{pmatrix}$$

11.

$$A^3 + 2A = \begin{pmatrix} 65 & 100 \\ 25 & 40 \end{pmatrix}$$

13. yes $\begin{pmatrix} 27 & 11 \\ 2 & 5 \end{pmatrix}$ 15. yes $\begin{pmatrix} 11 & 8 \\ 8 & 4 \\ 5 & 3 \end{pmatrix}$ 17. no

19.

$$5(2A) = \begin{pmatrix} 10 & 10 \\ 10 & 20 \\ 30 & 10 \end{pmatrix} = 10A$$

21.

$$(A + B) + C = \begin{pmatrix} 4 & 0 \\ 8 & 2 \end{pmatrix} = A + (B + C)$$

23.

$$A(B + C) = \begin{pmatrix} 9 & -1 \\ 11 & -2 \end{pmatrix} = AB + AC$$

25.

$$\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

27.

$$\begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

29.

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & -4 & -4 \\ 1 & 1 & 1 & 1 \\ 2 & -3 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -3 \\ 7 \end{pmatrix}$$

Section 11.2

1. 7 3. 1 5. −24 7. 3

Section 11.3

1. $x_1 = \frac{9}{5}$, $x_2 = \frac{3}{5}$ 3. $x_1 = 0$, $x_2 = 0$, $x_3 = -2$

Section 11.4

1. $x_2 = 2,\ x_1 = 1$

3. $x_3 = \alpha,\ x_2 = -\alpha,\ x_1 = \alpha$

5. $x_3 = \alpha,\ x_2 = 2\alpha,\ x_1 = -1$

7. $x_3 = 2.2,\ x_2 = 2.6,\ x_1 = 1$

9. $A^{-1} = \begin{pmatrix} -1/13 & 5/13 \\ 2/13 & 3/13 \end{pmatrix}$

11. $A^{-1} = \begin{pmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 2 & 4 & 11 \end{pmatrix}$

Section 11.5

1.
$$\lambda = 4, \quad \mathbf{x}_0 = \alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix}; \qquad \lambda = -3 \quad \mathbf{x}_0 = \beta \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

3.
$$\lambda = 1 \quad \mathbf{x}_0 = \alpha \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}; \qquad \lambda = 0, \quad \mathbf{x}_0 = \gamma \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

5.
$$\lambda = 1, \quad \mathbf{x}_0 = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}; \qquad \lambda = 2, \quad \mathbf{x}_0 = \gamma \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

7.
$$\lambda = 0, \quad \mathbf{x}_0 = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \ \lambda = 1, \quad \mathbf{x}_0 = \beta \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}; \ \lambda = 2, \quad \mathbf{x}_0 = \gamma \begin{pmatrix} 7 \\ 3 \\ 1 \end{pmatrix}$$

Section 11.6

1.
$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}.$$

3.
$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.$$

5.

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2} + c_2 \begin{pmatrix} t \\ -1/2 - t \end{pmatrix} e^{t/2}.$$

7.

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 + t \\ -t \end{pmatrix} e^{2t}.$$

9.

$$\mathbf{x} = c_3 \begin{pmatrix} -3\cos(2t) - 2\sin(2t) \\ \cos(2t) \end{pmatrix} e^{t} + c_4 \begin{pmatrix} 2\cos(2t) - 3\sin(2t) \\ \sin(2t) \end{pmatrix} e^{t}$$

11.

$$\mathbf{x} = c_3 \begin{pmatrix} 2\cos(t) \\ 7\cos(t) + \sin(t) \end{pmatrix} e^{-3t} + c_4 \begin{pmatrix} 2\sin(t) \\ 7\sin(t) - \cos(t) \end{pmatrix} e^{-3t}$$

13.

$$\mathbf{x} = c_3 \begin{pmatrix} -\cos(2t) + \sin(2t) \\ \cos(2t) \end{pmatrix} e^{t} + c_4 \begin{pmatrix} -\cos(2t) - \sin(2t) \\ \sin(2t) \end{pmatrix} e^{t}$$

15.

$$\mathbf{x} = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -3 \\ 2 \end{pmatrix} e^{-t}$$

17.

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} e^{2t}$$

19.

$$\mathbf{x} = c_1 \begin{pmatrix} 3 \\ -2 \\ 12 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t}$$

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