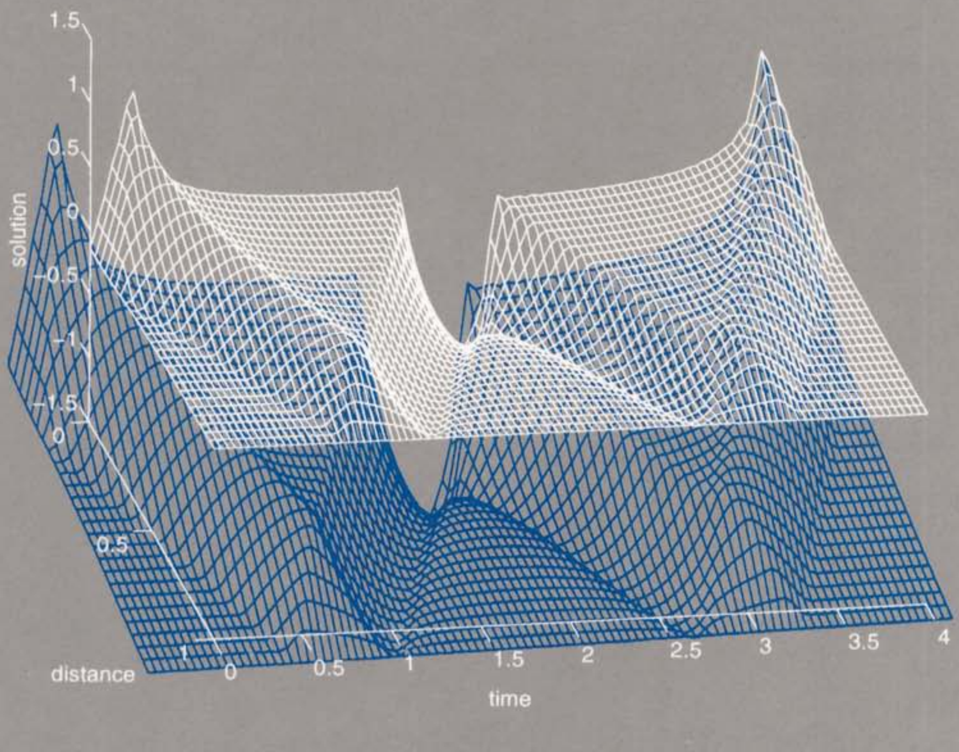
ADVANCED ENGINEERING MATHEMATICS



Dean G. Duffy

ADVANCED ENGINEERING MATHEMATICS

Dean G. Duffy

CRC Press
Boca Raton Boston New York Washington London

Library of Congress Cataloging-in-Publication Data

Duffy Dean G.
Advanced engineering mathematics / Dean G. Duffy.
p. cm.
Includes bibliographical references and index.
ISBN 0-8493-7854-0 (alk. paper)
1. Mathematics. I. Title.
QA37.2.D78 1997
510—dc21
for Library of Congress 97-33991
CIP

Direct all inquiries to CRC Press LLC, 2000 Corporate Blvd., N.W., Boca Raton, Florida 33431.

International Standard Book Number 0-8493-7854-0
Library of Congress Card Number 97-33991
Printed in the United States of America 1 2 3 4 5 6 7 8 9 0
Printed on acid-free paper

Introduction

This book grew out of a two-semester course given to sophomore and junior engineering majors at the U.S. Naval Academy. These students had just completed three semesters of traditional calculus and a fourth semester of ordinary differential equations. Consequently, it was assumed that they understood single and multivariable calculus, the calculus of single-variable, vector-valued functions, and how to solve a constant coefficient, ordinary differential equation.

The first five chapters were taught to system and electrical engineers because they needed transform methods to solve ordinary differential and difference equations. The last six chapters served mechanical, aeronautical, and other engineering majors. These students focused on the general topics of boundary-value problems, linear algebra, and vector calculus.

The book has been designed so that the instructor may inject his own personality into the course. For example, the instructor who enjoys the more theoretical aspects may dwell on them during his lecture with the confidence that the mechanics of how to solve the problems are completely treated in the text. Those who enjoy working problems may choose from a wealth of problems and topics. References are given to original sources and classic expositions so that the theoretically inclined may deepen their understanding of a given subject.

Overall this book consists of two parts. The first half involves advanced topics in single variable calculus, either with real or complex variables, while the second portion involves advanced topics in multivariable calculus. Unlike most engineering mathematics books, we begin with complex variables because they provide powerful techniques in understanding and computing Fourier, Laplace, and z-transforms.

Chapter 1 starts by reviewing complex numbers; in particular, we find all of the roots of a complex number, $z^{1/n}$, where n is an integer and z is a complex number. This naturally leads to complex algebra and complex functions. Finally, we define the derivative of a complex function.

The remaining portion of Chapter 1 is devoted to contour integration on the complex plane. First, we compute contour integrals by straightforward line integration. Focusing on closed contours, we introduce the Cauchy-Goursat theorem, Cauchy's integral theorem, and Cauchy's residue theorem to greatly facilitate the evaluation of these integrals. This analysis includes the classification of singularities. Although Chapter 1 is not necessary for most of this book, some sections or portions of some sections (2.5, 2.6, 3.1–3.6, 4.5, 4.10, 5.1, 5.3–5.5, 6.1, 6.5, 7.5–7.6, 8.4, 8.7, 9.4, 9.6, 11.6) require this material and must therefore be excluded when encountered. If the students have had elementary complex arithmetic (Section 1.1), the affected sections drop to 3.4, 3.6, 4.10, 5.3, 5.5, 7.5, 8.4, and 9.6.

Chapter 2 lays the foundation for transform methods and the solution of partial differential equations. We begin by deriving the classic Fourier series and working out some interesting problems. Next we investigate the properties of Fourier series, including Gibbs phenomena, and whether we can differentiate or integrate a Fourier series. Then we reexpress the classic Fourier series in alternative forms. Finally we use Fourier series to solve ordinary differential equations with periodic forcing. As a postscript we apply Fourier series to situations where there is a finite number of data values.

In Chapter 3 we introduce the Fourier transform. We compute some Fourier transforms and find their inverse by partial fractions and contour integration. Furthermore, we explore various properties of this transform, including convolution. Finally, we find the particular solution of an ordinary differential equation using Fourier transforms.

Chapter 4 presents Laplace transforms. This chapter includes finding a Laplace transform from its definition and using various theorems. We find the inverse by partial fractions, convolution, and contour integration. With these tools, the student can then solve an ordinary differential equation with initial conditions and a piece-wise continuous forcing. We also include systems of ordinary differential equations. Finally, we examine the importance of the transfer function, impulse response, and step response.

With the rise of digital technology and its associated difference equations, a version of the Laplace transform, the z-transform, was developed. In Chapter 5 we find a z-transform from its definition or by using various theorems. We also illustrate how to compute the inverse by long division, partial fractions, and contour integration. Finally, we use z-transforms to solve difference equations, especially with respect to the stability of the system.

Chapter 6 is a transitional chapter. We expand the concept of Fourier series so that it includes solutions to the Sturm-Liouville problem and show how any piece-wise continuous function can be reexpressed in terms of an expansion of these solutions. In particular, we focus on expansions that involve Bessel functions and Legendre polynomials.

Chapter 7, 8, and 9 deal with solutions to the wave, heat, and Laplace's equations, respectively. They serve as prototypes of much wider classes of partial differential equations. Of course, considerable attention is given to the technique of separation of variables. However, additional methods such as Laplace and Fourier transforms and integral representations are also included. Finally, we include a section on the numerical solution of each of these equations.

Chapter 10 is devoted to vector calculus. In this book we focus on the use of the del operator. This includes such topics as line integrals, surface integrals, the divergence theorem, and Stokes' theorem.

Finally, in Chapter 11 we present some topics from linear algebra. From this vast field of mathematics we study the solution of systems of linear equations because this subject is of greatest interest to engineers. Consequently, we shall cover such topics as matrices, determinants, and Cramer's rule. For the solution of systems of ordinary differential equations we discuss the classic eigenvalue problem.

This book contains a wealth of examples. Furthermore, in addition to the standard rote problems, I have sought to include many problems from the scientific and engineering literature. I have formulated many of the more complicated problems or computations as multistep projects. These problems may be given outside of class to deepen the students' understanding of a particular topic.

The answers to the odd problems are given in the back of the book while the worked solutions to all of the problems are available from the publisher. It is hoped that by including problems from the open literature some of the academic staleness that often pervades college texts will be removed.

Acknowledgments

I would like to thank the many midshipmen and cadets who have taken engineering mathematics from me. They have been willing or unwilling guinea pigs in testing out many of the ideas and problems in this book.

Special thanks goes to Dr. Mike Marcozzi for his many useful and often humorous suggestions for improving this book. The three-dimensional plots were done on MATLAB. Finally, I would like to express my appreciation to all those authors and publishers who allowed me the use of their material from the scientific and engineering literature.

Dedicated to the Brigade of Midshipmen
and the Corps of Cadets

Contents

Chapter 1
Complex Variables

The theory of complex variables was originally developed by mathematicians as an aid in understanding functions. Functions of a complex variable enjoy many powerful properties that their real counterparts do not. That is *not* why we will study them. For us they provide the keys for the complete mastery of transform methods and differential equations.

In this chapter all of our work points to one objective: integration on the complex plane by the method of residues. For this reason we will minimize discussions of limits and continuity which play such an important role in conventional complex variables in favor of the computational aspects. We begin by introducing some simple facts about complex variables. Then we progress to differential and integral calculus on the complex plane.

1.1 COMPLEX NUMBERS

A *complex number* is any number of the form $a + bi$, where a and b are real and $i = \sqrt{-1}$. We denote any member of a *set* of complex numbers by the *complex variable* $z = x + iy$. The real part of z, usually denoted by $\text{Re}(z)$, is x while the imaginary part of z, $\text{Im}(z)$, is y. The *complex conjugate*, $\bar{z}$ or z^*, of the complex number $a + bi$ is $a - bi$.

Complex numbers obey the fundamental rules of algebra. Thus, two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$. Just as real numbers have the fundamental operations of addition, subtraction, multiplication, and division, so too do complex numbers. These operations are defined:

Addition

$$(a + bi) + (c + di) = (a + c) + (b + d)i \tag{1.1.1}$$

Subtraction

$$(a + bi) - (c + di) = (a - c) + (b - d)i \tag{1.1.2}$$

Multiplication

$$(a + bi)(c + di) = ac + bci + adi + i^2bd = (ac - bd) + (ad + bc)i \tag{1.1.3}$$

Division

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di}\frac{c - di}{c - di} = \frac{ac - adi + bci - bdi^2}{c^2 + d^2} = \frac{ac + bd + (bc - ad)i}{c^2 + d^2}. \tag{1.1.4}$$

The *absolute value* or *modulus* of a complex number $a + bi$, written $|a + bi|$, equals $\sqrt{a^2 + b^2}$. Additional properties include:

$$|z_1 z_2 z_3 \cdots z_n| = |z_1||z_2||z_3|\cdots|z_n| \tag{1.1.5}$$

$$|z_1/z_2| = |z_1|/|z_2| \quad \text{if} \quad z_2 \neq 0 \tag{1.1.6}$$

$$|z_1 + z_2 + z_3 + \cdots + z_n| \leq |z_1| + |z_2| + |z_3| + \cdots + |z_n| \tag{1.1.7}$$

and

$$|z_1 + z_2| \geq |z_1| - |z_2|. \tag{1.1.8}$$

The use of inequalities with complex variables has meaning only when they involve absolute values.

It is often useful to plot the complex number $x + iy$ as a point (x, y) in the xy plane, now called the *complex plane*. Figure 1.1.1 illustrates this representation.

This geometrical interpretation of a complex number suggests an alternative method of expressing a complex number: the polar form. From the polar representation of x and y,

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta), \tag{1.1.9}$$

where $r = \sqrt{x^2 + y^2}$ is the *modulus*, *amplitude*, or *absolute value* of z and θ is the *argument* or *phase*, we have that

$$z = x + iy = r[\cos(\theta) + i\sin(\theta)]. \tag{1.1.10}$$

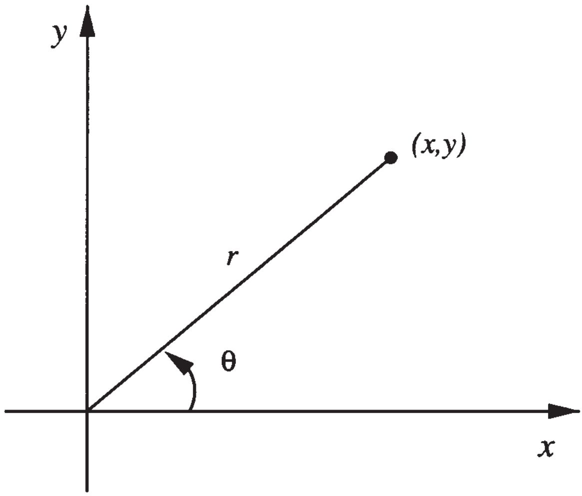


Figure 1.1.1: The complex plane.

However, from the Taylor expansion of the exponential in the real case,

$$e^{i\theta} = \sum_{k=0}^{\infty} \frac{(\theta i)^k}{k!}. \quad \textbf{(1.1.11)}$$

Expanding (1.1.11),

$$e^{i\theta} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \quad \textbf{(1.1.12)}$$

$$= \cos(\theta) + i\sin(\theta). \quad \textbf{(1.1.13)}$$

Equation (1.1.13) is *Euler's formula*. Consequently, we may express (1.1.10) as

$$z = re^{i\theta}, \quad \textbf{(1.1.14)}$$

which is the *polar form* of a complex number. Furthermore, because

$$z^n = r^n e^{in\theta} \quad \textbf{(1.1.15)}$$

by the law of exponents,

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]. \quad \textbf{(1.1.16)}$$

Equation (1.1.16) is *De Moivre's theorem*.

• Example 1.1.1

Let us simplify the following complex number:

$$\frac{3-2i}{-1+i} = \frac{3-2i}{-1+i} \times \frac{-1-i}{-1-i} = \frac{-3-3i+2i+2i^2}{1+1} = \frac{-5-i}{2} = -\frac{5}{2} - \frac{i}{2}. \tag{1.1.17}$$

• Example 1.1.2

Let us reexpress the complex number $-\sqrt{6} - i\sqrt{2}$ in polar form. From (1.1.9) $r = \sqrt{6+2}$ and $\theta = \tan^{-1}(b/a) = \tan^{-1}(1/\sqrt{3}) = \pi/6$ or $7\pi/6$. Because $-\sqrt{6} - i\sqrt{2}$ lies in the third quadrant of the complex plane, $\theta = 7\pi/6$ and

$$-\sqrt{6} - i\sqrt{2} = 2\sqrt{2}e^{7\pi i/6}. \tag{1.1.18}$$

Note that (1.1.18) is not a unique representation because $\pm 2n\pi$ may be added to $7\pi/6$ and we still have the same complex number since

$$e^{i(\theta \pm 2n\pi)} = \cos(\theta \pm 2n\pi) + i\sin(\theta \pm 2n\pi) = \cos(\theta) + i\sin(\theta) = e^{i\theta}. \tag{1.1.19}$$

For uniqueness we will often choose $n = 0$ and define this choice as the *principal branch.* Other branches correspond to different values of n.

• Example 1.1.3

Find the curve described by the equation $|z - z_0| = a$.
From the definition of the absolute value,

$$\sqrt{(x-x_0)^2 + (y-y_0)^2} = a \tag{1.1.20}$$

or

$$(x-x_0)^2 + (y-y_0)^2 = a^2. \tag{1.1.21}$$

Equation (1.1.21), and hence $|z - z_0| = a$, describes a circle of radius a with its center located at (x_0, y_0). Later on, we shall use equations such as this to describe curves in the complex plane.

• Example 1.1.4

As an example in manipulating complex numbers, let us show that

$$\left|\frac{a+bi}{b+ai}\right| = 1. \tag{1.1.22}$$

We begin by simplifying

$$\frac{a+bi}{b+ai} = \frac{a+bi}{b+ai} \times \frac{b-ai}{b-ai} = \frac{2ab}{a^2+b^2} + \frac{b^2-a^2}{a^2+b^2}i. \tag{1.1.23}$$

Therefore,

$$\left|\frac{a+bi}{b+ai}\right| = \sqrt{\frac{4a^2b^2}{(a^2+b^2)^2} + \frac{b^4-2a^2b^2+a^4}{(a^2+b^2)^2}} = \sqrt{\frac{a^4+2a^2b^2+b^4}{(a^2+b^2)^2}} = 1. \tag{1.1.24}$$

Problems

Simplify the following complex numbers. Represent the solution in the Cartesian form $a + bi$:

1. $\dfrac{5i}{2+i}$

2. $\dfrac{5+5i}{3-4i} + \dfrac{20}{4+3i}$

3. $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i}$

4. $(1-i)^4$

5. $i(1-i\sqrt{3})(\sqrt{3}+i)$

Represent the following complex numbers in polar form:

6. $-i$

7. -4

8. $2+2\sqrt{3}\,i$

9. $-5+5i$

10. $2-2i$

11. $-1+\sqrt{3}\,i$

12. By the law of exponents, $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$. Use Euler's formula to obtain expressions for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$ in terms of sines and cosines of α and β.

13. Using the property that $\sum_{n=0}^{N} q^n = (1-q^{N+1})/(1-q)$ and the geometric series $\sum_{n=0}^{N} e^{int}$, obtain the following sums of trigonometric functions:

$$\sum_{n=0}^{N} \cos(nt) = \cos\left(\frac{Nt}{2}\right) \frac{\sin[(N+1)t/2]}{\sin(t/2)}$$

and

$$\sum_{n=1}^{N} \sin(nt) = \sin\left(\frac{Nt}{2}\right) \frac{\sin[(N+1)t/2]}{\sin(t/2)}.$$

These results are often called *Lagrange's trigonometric identities.*

14. (a) Using the property that $\sum_{n=0}^{\infty} q^n = 1/(1-q)$, if $|q| < 1$, and the geometric series $\sum_{n=0}^{\infty} \epsilon^n e^{int}$, $|\epsilon| < 1$, show that

$$\sum_{n=0}^{\infty} \epsilon^n \cos(nt) = \frac{1 - \epsilon \cos(t)}{1 + \epsilon^2 - 2\epsilon \cos(t)}$$

and

$$\sum_{n=1}^{\infty} \epsilon^n \sin(nt) = \frac{\epsilon \sin(t)}{1 + \epsilon^2 - 2\epsilon \cos(t)}.$$

(b) Let $\epsilon = e^{-a}$, where $a > 0$. Show that

$$2\sum_{n=1}^{\infty} e^{-na} \sin(nt) = \frac{\sin(t)}{\cosh(a) - \cos(t)}.$$

1.2 FINDING ROOTS

The concept of finding roots of a number, which is rather straightforward in the case of real numbers, becomes more difficult in the case of complex numbers. By finding the *roots* of a complex number, we wish to find all the solutions w of the equation $w^n = z$, where n is a positive integer for a given z.

We begin by writing z in the polar form:

$$z = re^{i\varphi} \tag{1.2.1}$$

while we write

$$w = Re^{i\Phi} \tag{1.2.2}$$

for the unknown. Consequently,

$$w^n = R^n e^{in\Phi} = re^{i\varphi} = z. \tag{1.2.3}$$

We satisfy (1.2.3) if

$$R^n = r \quad \text{and} \quad n\Phi = \varphi + 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots, \tag{1.2.4}$$

because the addition of any multiple of 2π to the argument is also a solution. Thus, $R = r^{1/n}$, where R is the uniquely determined real positive root, and

$$\Phi_k = \frac{\varphi}{n} + \frac{2\pi k}{n}, \qquad k = 0, \pm 1, \pm 2, \ldots \tag{1.2.5}$$

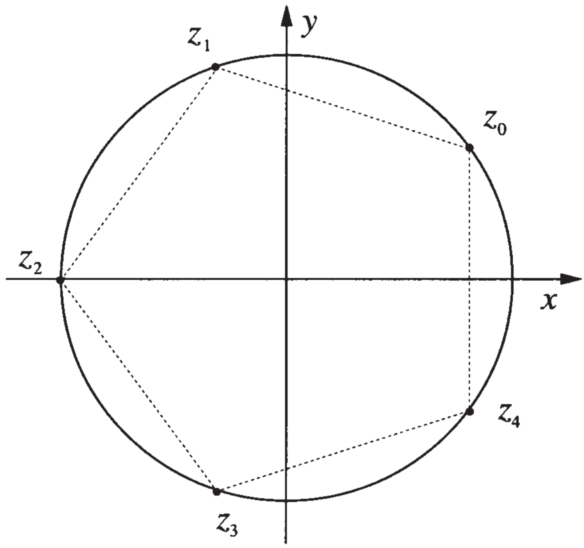


Figure 1.2.1: The zeros of $z^5 = -32$.

Because $w_k = w_{k\pm n}$, it is sufficient to take $k = 0, 1, 2, \ldots, n-1$. Therefore, there are exactly n solutions:

$$w_k = Re^{\Phi_k i} = r^{1/n} \exp\left[i\left(\frac{\varphi}{n} + \frac{2\pi k}{n}\right)\right] \qquad \textbf{(1.2.6)}$$

with $k = 0, 1, 2, \ldots, n-1$. They are the n roots of z. Geometrically we can locate these w_k's on a circle, centered at the point (0,0), with radius R and separated from each other by $2\pi/n$ radians. These roots also form the vertices of a regular polygon of n sides inscribed inside of a circle of radius R. (See Example 1.2.1.)

In summary, the method for finding the n roots of a complex number z_0 is as follows. First, write z_0 in its polar form: $z_0 = re^{i\varphi}$. Then multiply the polar form by $e^{2i\pi k}$. Using the law of exponents, take the $1/n$ power of both sides of the equation. Finally, using Euler's formula, evaluate the roots for $k = 0, 1, \ldots, n-1$.

• **Example 1.2.1**

Let us find all of the values of z for which $z^5 = -32$ and locate these values on the complex plane.

Because

$$-32 = 32e^{\pi i} = 2^5 e^{\pi i}, \qquad \textbf{(1.2.7)}$$

$$z_k = 2\exp\left(\frac{\pi i}{5} + \frac{2\pi i k}{5}\right), \qquad k = 0, 1, 2, 3, 4, \qquad \textbf{(1.2.8)}$$

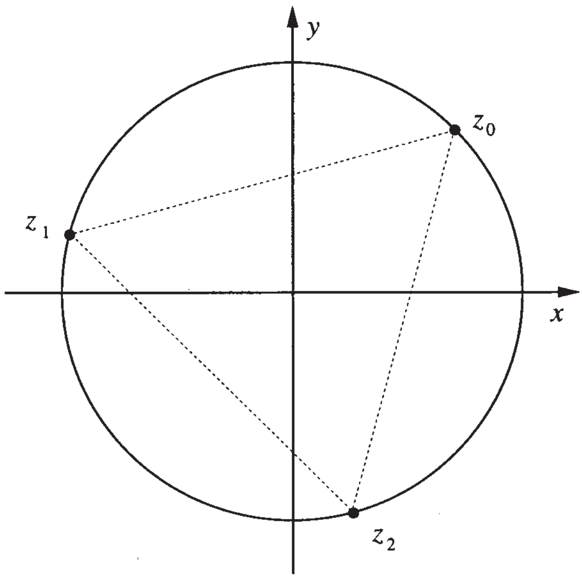


Figure 1.2.2: The zeros of $z^3 = -1 + i$.

or

$$z_0 = 2\exp\left(\frac{\pi i}{5}\right) = 2\left[\cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right)\right], \tag{1.2.9}$$

$$z_1 = 2\exp\left(\frac{3\pi i}{5}\right) = 2\left[\cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right)\right], \tag{1.2.10}$$

$$z_2 = 2\exp(\pi i) = -2, \tag{1.2.11}$$

$$z_3 = 2\exp\left(\frac{7\pi i}{5}\right) = 2\left[\cos\left(\frac{7\pi}{5}\right) + i\sin\left(\frac{7\pi}{5}\right)\right] \tag{1.2.12}$$

and

$$z_4 = 2\exp\left(\frac{9\pi i}{5}\right) = 2\left[\cos\left(\frac{9\pi}{5}\right) + i\sin\left(\frac{9\pi}{5}\right)\right]. \tag{1.2.13}$$

Figure 1.2.1 shows the location of these roots in the complex plane.

• Example 1.2.2

Let us find the cube roots of $-1 + i$ and locate them graphically.

Because $-1+i = \sqrt{2}\exp(3\pi i/4)$,

$$z_k = 2^{1/6}\exp\left(\frac{\pi i}{4} + \frac{2i\pi k}{3}\right), \qquad k = 0, 1, 2, \tag{1.2.14}$$

or

$$z_0 = 2^{1/6}\exp\left(\frac{\pi i}{4}\right) = 2^{1/6}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right], \tag{1.2.15}$$

$$z_1 = 2^{1/6}\exp\left(\frac{11\pi i}{12}\right) = 2^{1/6}\left[\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right] \tag{1.2.16}$$

and

$$z_2 = 2^{1/6}\exp\left(\frac{19\pi i}{12}\right) = 2^{1/6}\left[\cos\left(\frac{19\pi}{12}\right) + i\sin\left(\frac{19\pi}{12}\right)\right]. \tag{1.2.17}$$

Figure 1.2.2 gives the location of these zeros on the complex plane.

Problems

Extract all of the possible roots of the following complex numbers:

1. $8^{1/6}$
2. $(-1)^{1/3}$
3. $(-i)^{1/3}$
4. $(-27i)^{1/6}$

5. Find algebraic expressions for the square roots of $a - bi$, where $a > 0$ and $b > 0$.

6. Find all of the roots for the algebraic equation $z^4 - 3iz^2 - 2 = 0$.

7. Find all of the roots for the algebraic equation $z^4 + 6iz^2 + 16 = 0$.

1.3 THE DERIVATIVE IN THE COMPLEX PLANE: THE CAUCHY-RIEMANN EQUATIONS

In the previous two sections, we have done complex arithmetic. We are now ready to introduce the concept of function as it applies to complex variables.

We have already introduced the complex variable $z = x + iy$, where x and y are variable. We now define another complex variable $w = u + iv$ so that for each value of z there corresponds a value of $w = f(z)$. From all of the possible complex functions that we might invent, we will focus on those functions where for each z there is one, and only one, value of w. These functions are *single-valued*. They differ from functions such

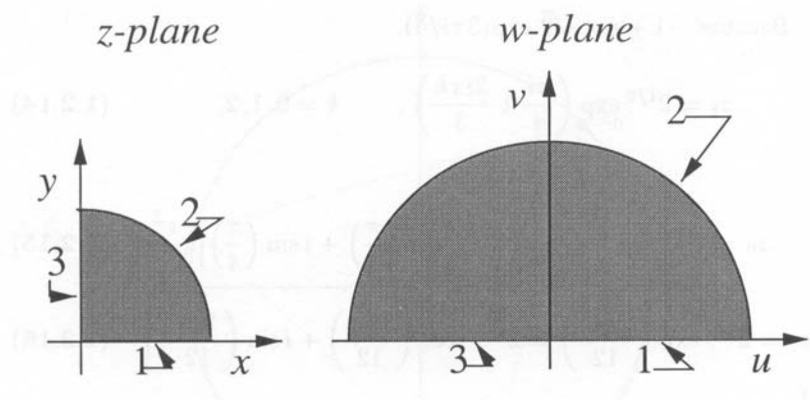


Figure 1.3.1: The complex function $w = z^2$.

as the square root, logarithm, and inverse sine and cosine, where there are multiple answers for each z. These *multivalued functions* do arise in various problems. However, they are beyond the scope of this book and we shall always assume that we are dealing with single-valued functions.

A popular method for representing a complex function involves drawing some closed domain in the z-plane and then showing the corresponding domain in the w-plane. This procedure is called *mapping* and the z-plane illustrates the *domain* of the function while the w-plane illustrates its *image* or *range*. Figure 1.3.1 shows the z-plane and w-plane for $w = z^2$; a pie-shaped wedge in the z-plane maps into a semicircle on the w-plane.

• **Example 1.3.1**

Given the complex function $w = e^{-z^2}$, let us find the corresponding $u(x, y)$ and $v(x, y)$.

From Euler's formula,

$$w = e^{-z^2} = e^{-(x+iy)^2} = e^{y^2-x^2}e^{-2ixy} = e^{y^2-x^2}[\cos(2xy) - i\sin(2xy)]. \qquad \mathbf{(1.3.1)}$$

Therefore, by inspection,

$$u(x, y) = e^{y^2-x^2}\cos(2xy) \qquad \text{and} \qquad v(x, y) = -e^{y^2-x^2}\sin(2xy). \qquad \mathbf{(1.3.2)}$$

Note that there is no i in the expression for $v(x, y)$. The function $w = f(z)$ is single-valued because for each distinct value of z, there is an unique value of $u(x, y)$ and $v(x, y)$.

• **Example 1.3.2**

As counterpoint, let us show that $w = \sqrt{z}$ is a multivalued function.

We begin by writing $z = re^{i\theta+2\pi ik}$, where $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$. Then,

$$w_k = \sqrt{r}e^{i\theta/2+\pi ik}, \qquad k = 0, 1, \tag{1.3.3}$$

or

$$w_0 = \sqrt{r}\left[\cos(\theta/2) + i\sin(\theta/2)\right] \qquad \text{and} \qquad w_1 = -w_0. \tag{1.3.4}$$

Therefore,

$$u_0(x,y) = \sqrt{r}\cos(\theta/2), \qquad v_0(x,y) = \sqrt{r}\sin(\theta/2) \tag{1.3.5}$$

and

$$u_1(x,y) = -\sqrt{r}\cos(\theta/2), \qquad v_1(x,y) = -\sqrt{r}\sin(\theta/2). \tag{1.3.6}$$

Each solution w_0 or w_1 is a *branch* of the multivalued function $\sqrt{z}$. We can make $\sqrt{z}$ single-valued by restricting ourselves to a single branch, say w_0. In that case, the $\text{Re}(w) > 0$ if we restrict $-\pi < \theta < \pi$. Although this is not the only choice that we could have made, it is a popular one. For example, most digital computers use this definition in their complex square root function. The point here is our ability to make a multivalued function single-valued by defining a particular branch.

Although the requirement that a complex function be single-valued is important, it is still too general and would cover all functions of two real variables. To have a useful theory, we must introduce additional constraints. Because an important property associated with most functions is the ability to take their derivative, let us examine the derivative in the complex plane.

Following the definition of a derivative for a single real variable, the derivative of a complex function $w = f(z)$ is defined as

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}. \tag{1.3.7}$$

A function of a complex variable that has a derivative at every point within a region of the complex plane is said to be *analytic* (or *regular* or *holomorphic*) over that region. If the function is analytic everywhere in the complex plane, it is *entire*.

Because the derivative is defined as a limit and limits are well behaved with respect to elementary algebraic operations, the following operations carry over from elementary calculus:

$$\frac{d}{dz}\left[cf(z)\right] = cf'(z), \qquad c \text{ a constant} \tag{1.3.8}$$

$$\frac{d}{dz}\left[f(z) \pm g(z)\right] = f'(z) \pm g'(z) \tag{1.3.9}$$

$$\frac{d}{dz}\left[f(z)g(z)\right] = f'(z)g(z) + f(z)g'(z) \tag{1.3.10}$$

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)f'(z) - g'(z)f(z)}{g^2(z)} \tag{1.3.11}$$

$$\frac{d}{dz}\left\{f[g(z)]\right\} = f'[g(z)]g'(z), \qquad \text{the chain rule.} \tag{1.3.12}$$

Another important property that carries over from real variables is l'Hôspital rule: Let $f(z)$ and $g(z)$ be analytic at z_0, where $f(z)$ has a zero[1] of order m and $g(z)$ has a zero of order n. Then, if $m > n$,

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = 0; \tag{1.3.13}$$

if $m = n$,

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f^{(m)}(z_0)}{g^{(m)}(z_0)} \tag{1.3.14}$$

and if $m < n$,

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \infty. \tag{1.3.15}$$

• **Example 1.3.3**

Let us evaluate $\lim_{z \to i}(z^{10} + 1)/(z^6 + 1)$. From l'Hôspital rule,

$$\lim_{z \to i} \frac{z^{10} + 1}{z^6 + 1} = \lim_{z \to i} \frac{10z^9}{6z^5} = \frac{5}{3} \lim_{z \to i} z^4 = \frac{5}{3}. \tag{1.3.16}$$

So far we have introduced the derivative and some of its properties. But how do we actually know whether a function is analytic or how do we compute its derivative? At this point we must develop some relationships involving the known quantities $u(x, y)$ and $v(x, y)$.

We begin by returning to the definition of the derivative. Because $\Delta z = \Delta x + i\Delta y$, there is an infinite number of different ways of approaching the limit $\Delta z \to 0$. Uniqueness of that limit requires that (1.3.7) must be independent of the manner in which Δz approaches zero. A simple

---

[1] An analytic function $f(z)$ has a zero of order m at z_0 if and only if $f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$ and $f^{(m)}(z_0) \neq 0$.

Figure 1.3.2: Although educated as an engineer, Augustin-Louis Cauchy (1789–1857) would become a mathematician's mathematician, publishing 789 papers and 7 books in the fields of pure and applied mathematics. His greatest writings established the discipline of mathematical analysis as he refined the notions of limit, continuity, function, and convergence. It was this work on analysis that led him to develop complex function theory via the concept of residues. (Portrait courtesy of the Archives de l'Académie des sciences, Paris.)

example is to take Δz in the x-direction so that $\Delta z = \Delta x$; another is to take Δz in the y-direction so that $\Delta z = i\Delta y$. These examples yield

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta u + i\Delta v}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{1.3.17}$$

and

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta y \to 0} \frac{\Delta u + i\Delta v}{i\Delta y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \tag{1.3.18}$$

Figure 1.3.3: Despite his short life, (Georg Friedrich) Bernhard Riemann's (1826–1866) mathematical work contained many imaginative and profound concepts. It was in his doctoral thesis on complex function theory (1851) that he introduced the Cauchy-Riemann differential equations. Riemann's later work dealt with the definition of the integral and the foundations of geometry and non-Euclidean (elliptic) geometry. (Portrait courtesy of Photo AKG, London.)

In both cases we are approaching zero from the positive side. For the limit to be unique and independent of path, (1.3.17) must equal (1.3.18), or

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.3.19}$$

These equations which u and v must both satisfy are the *Cauchy-Riemann* equations. They are necessary but not sufficient to ensure that a function is differentiable. The following example will illustrate this.

• **Example 1.3.4**

Consider the complex function

$$w = \begin{cases} z^5/|z|^4, & z \neq 0 \\ 0, & z = 0. \end{cases} \tag{1.3.20}$$

The derivative at $z = 0$ is given by

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{(\Delta z)^5/|\Delta z|^4 - 0}{\Delta z} = \lim_{\Delta z \to 0} \frac{(\Delta z)^4}{|\Delta z|^4}, \tag{1.3.21}$$

provided that this limit exists. However, this limit does not exist because, in general, the numerator depends upon the path used to approach zero. For example, if $\Delta z = re^{\pi i/4}$ with $r \to 0$, $dw/dz = -1$. On the other hand, if $\Delta z = re^{\pi i/2}$ with $r \to 0$, $dw/dz = 1$.

Are the Cauchy-Riemann equations satisfied in this case? To check this, we first compute

$$u_x(0,0) = \lim_{\Delta x \to 0} \left(\frac{\Delta x}{|\Delta x|}\right)^4 = 1, \tag{1.3.22}$$

$$v_y(0,0) = \lim_{\Delta y \to 0} \left(\frac{i\Delta y}{|\Delta y|}\right)^4 = 1, \tag{1.3.23}$$

$$u_y(0,0) = \lim_{\Delta y \to 0} \operatorname{Re}\left[\frac{(i\Delta y)^5}{\Delta y|\Delta y|^4}\right] = 0 \tag{1.3.24}$$

and

$$v_x(0,0) = \lim_{\Delta x \to 0} \operatorname{Im}\left[\left(\frac{\Delta x}{|\Delta x|}\right)^4\right] = 0. \tag{1.3.25}$$

Hence, the Cauchy-Riemann equations are satisfied at the origin. Thus, even though the derivative is not uniquely defined, (1.3.21) happens to have the same value for paths taken along the coordinate axes so that the Cauchy-Riemann equations are satisfied.

In summary, if a function is differentiable at a point, the Cauchy-Riemann equations hold. Similarly, if the Cauchy-Riemann equations are not satisfied at a point, then the function is not differentiable at that point. This is one of the important uses of the Cauchy-Riemann equations: the location of nonanalytic points. Isolated nonanalytic points of an otherwise analytic function are called *isolated singularities*. Functions that contain isolated singularities are called *meromorphic*.

The Cauchy-Riemann condition can be modified so that it is a sufficient condition for the derivative to exist. Let us require that u_x, u_y, v_x, and v_y be continuous in some region surrounding a point z_0 and satisfy the Cauchy-Riemann conditions there. Then

$$f(z) - f(z_0) = [u(z) - u(z_0)] + i[v(z) - v(z_0)] \tag{1.3.26}$$

$$\begin{aligned} &= [u_x(z_0)(x - x_0) + u_y(z_0)(y - y_0) \\ &\qquad + \epsilon_1(x - x_0) + \epsilon_2(y - y_0)] \\ &+ i[v_x(z_0)(x - x_0) + v_y(z_0)(y - y_0) \\ &\qquad + \epsilon_3(x - x_0) + \epsilon_4(y - y_0)] \end{aligned} \tag{1.3.27}$$

$$\begin{aligned} &= [u_x(z_0) + iv_x(z_0)](z - z_0) \\ &+ (\epsilon_1 + i\epsilon_3)(x - x_0) + (\epsilon_2 + i\epsilon_4)(y - y_0), \end{aligned} \tag{1.3.28}$$

where we have used the Cauchy-Riemann equations and $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \to 0$ as $\Delta x, \Delta y \to 0$. Hence,

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z) - f(z_0)}{\Delta z} = u_x(z_0) + iv_x(z_0), \tag{1.3.29}$$

because $|\Delta x| \leq |\Delta z|$ and $|\Delta y| \leq |\Delta z|$. Using (1.3.29) and the Cauchy-Riemann equations, we can obtain the derivative from any of the following formulas:

$$\boxed{\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}} \tag{1.3.30}$$

and

$$\boxed{\frac{dw}{dz} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.} \tag{1.3.31}$$

Furthermore, $f'(z_0)$ is continuous because the partial derivatives are.

• **Example 1.3.5**

Let us show that $\sin(z)$ is an entire function.

$$w = \sin(z) \tag{1.3.32}$$

$$u + iv = \sin(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy) \tag{1.3.33}$$

$$= \sin(x)\cosh(y) + i\cos(x)\sinh(y), \tag{1.3.34}$$

because

$$\cos(iy) = \tfrac{1}{2}\left[e^{i(iy)} + e^{-i(iy)}\right] = \tfrac{1}{2}\left[e^{y} + e^{-y}\right] = \cosh(y) \tag{1.3.35}$$

and

$$\sin(iy) = \tfrac{1}{2i}\left[e^{i(iy)} - e^{-i(iy)}\right] = -\tfrac{1}{2i}\left[e^{y} - e^{-y}\right] = i\sinh(y) \tag{1.3.36}$$

so that

$$u(x,y) = \sin(x)\cosh(y) \quad \text{and} \quad v(x,y) = \cos(x)\sinh(y). \tag{1.3.37}$$

Differentiating both $u(x,y)$ and $v(x,y)$ with respect to x and y, we have that

$$\frac{\partial u}{\partial x} = \cos(x)\cosh(y) \qquad \frac{\partial u}{\partial y} = \sin(x)\sinh(y) \tag{1.3.38}$$

$$\frac{\partial v}{\partial x} = -\sin(x)\sinh(y) \qquad \frac{\partial v}{\partial y} = \cos(x)\cosh(y) \tag{1.3.39}$$

and $u(x,y)$ and $v(x,y)$ satisfy the Cauchy-Riemann equations for all values of x and y. Furthermore, u_x, u_y, v_x, and v_y are continuous for all x and y. Therefore, the function $w = \sin(z)$ is an entire function.

• **Example 1.3.6**

Consider the function $w = 1/z$. Then

$$w = u + iv = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}. \tag{1.3.40}$$

Therefore,

$$u(x,y) = \frac{x}{x^2+y^2} \quad \text{and} \quad v(x,y) = -\frac{y}{x^2+y^2}. \tag{1.3.41}$$

Now

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}, \tag{1.3.42}$$

$$\frac{\partial v}{\partial y} = -\frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial u}{\partial x}, \tag{1.3.43}$$

$$\frac{\partial v}{\partial x} = -\frac{0-2xy}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2} \tag{1.3.44}$$

and

$$\frac{\partial u}{\partial y} = \frac{0-2xy}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2} = -\frac{\partial v}{\partial x}. \tag{1.3.45}$$

The function is analytic at all points except the origin because the function itself ceases to exist when both x and y are zero and the modulus of w becomes infinite.

• **Example 1.3.7**

Let us find the derivative of $\sin(z)$.
Using (1.3.30) and (1.3.34),

$$\frac{d}{dz}\Big[\sin(z)\Big] = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{1.3.46}$$

$$= \cos(x)\cosh(y) - i\sin(x)\sinh(y) \tag{1.3.47}$$

$$= \cos(x+iy) = \cos(z). \tag{1.3.48}$$

Similarly,

$$\frac{d}{dz}\left(\frac{1}{z}\right) = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{2ixy}{(x^2+y^2)^2} \tag{1.3.49}$$

$$= -\frac{1}{(x+iy)^2} = -\frac{1}{z^2}. \tag{1.3.50}$$

The results in the above examples are identical to those for z real. As we showed earlier, the fundamental rules of elementary calculus apply to complex differentiation. Consequently, it is usually simpler to apply those rules to find the derivative rather than breaking $f(z)$ down into its real and imaginary parts, applying either (1.3.30) or (1.3.31), and then putting everything back together.

An additional property of analytic functions follows by cross differentiating the Cauchy-Riemann equations or

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2} \quad \text{or} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1.3.51}$$

and

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial y^2} \quad \text{or} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{1.3.52}$$

Any function that has continuous partial derivatives of second order and satisfies Laplace's equation (1.3.51) or (1.3.52) is called a *harmonic function*. Because both $u(x,y)$ and $v(x,y)$ satisfy Laplace's equation if $f(z) = u + iv$ is analytic, $u(x,y)$ and $v(x,y)$ are called *conjugate harmonic functions.*

- **Example 1.3.8**

Given that $u(x,y) = e^{-x}[x\sin(y) - y\cos(y)]$, let us show that u is harmonic and find a conjugate harmonic function $v(x,y)$ such that $f(z) = u + iv$ is analytic.

Because

$$\frac{\partial^2 u}{\partial x^2} = -2e^{-x}\sin(y) + xe^{-x}\sin(y) - ye^{-x}\cos(y) \tag{1.3.53}$$

and

$$\frac{\partial^2 u}{\partial y^2} = -xe^{-x}\sin(y) + 2e^{-x}\sin(y) + ye^{-x}\cos(y), \tag{1.3.54}$$

it follows that $u_{xx} + u_{yy} = 0$. Therefore, $u(x,y)$ is harmonic. From the Cauchy-Riemann equations,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x}\sin(y) - xe^{-x}\sin(y) + ye^{-x}\cos(y) \tag{1.3.55}$$

and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^{-x}\cos(y) - xe^{-x}\cos(y) - ye^{-x}\sin(y). \tag{1.3.56}$$

Integrating (1.3.55) with respect to y,

$$v(x,y) = ye^{-x}\sin(y) + xe^{-x}\cos(y) + g(x). \tag{1.3.57}$$

Using (1.3.56),

$$\begin{aligned} v_x = -ye^{-x}\sin(y) - xe^{-x}\cos(y) + e^{-x}\cos(y) + g'(x) \\ = e^{-x}\cos(y) - xe^{-x}\cos(y) - ye^{-x}\sin(x). \end{aligned} \tag{1.3.58}$$

Therefore, $g'(x) = 0$ or $g(x) =$ constant. Consequently,

$$v(x,y) = e^{-x}[y\sin(y) + x\cos(y)] + \text{constant}. \tag{1.3.59}$$

Hence, for our real harmonic function $u(x,y)$, there are infinitely many harmonic conjugates $v(x,y)$ which differ from each other by an additive constant.

Problems

Show that the following functions are entire:

1. $f(z) = iz + 2$
2. $f(z) = e^{-z}$
3. $f(z) = z^3$
4. $f(z) = \cosh(z)$

Find the derivative of the following functions:

5. $f(z) = (1 + z^2)^{3/2}$
6. $f(z) = (z + 2z^{1/2})^{1/3}$
7. $f(z) = (1 + 4i)z^2 - 3z - 2$
8. $f(z) = (2z - i)/(z + 2i)$
9. $f(z) = (iz - 1)^{-3}$

Evaluate the following limits:

10. $\lim_{z\to i} \frac{z^2 - 2iz - 1}{z^4 + 2z^2 + 1}$
11. $\lim_{z\to 0} \frac{z - \sin(z)}{z^3}$

12. Show that the function $f(z) = z^*$ is nowhere differentiable.

For each of the following $u(x, y)$, show that it is harmonic and then find a corresponding $v(x, y)$ such that $f(z) = u + iv$ is analytic.

13.
$$u(x, y) = x^2 - y^2$$

14.
$$u(x, y) = x^4 - 6x^2y^2 + y^4 + x$$

15.
$$u(x, y) = x\cos(x)e^{-y} - y\sin(x)e^{-y}$$

16.
$$u(x, y) = (x^2 - y^2)\cos(y)e^x - 2xy\sin(y)e^x$$

1.4 LINE INTEGRALS

So far, we discussed complex numbers, complex functions, and complex differentiation. We are now ready for integration.

Just as we have integrals involving real variables, we can define an integral that involves complex variables. Because the z-plane is two-dimensional there is clearly greater freedom in what we mean by a complex integral. For example, we might ask whether the integral of some function between points A and B depends upon the curve along which we integrate. (In general it does.) Consequently, an important ingredient in any complex integration is the *contour* that we follow during the integration.

The result of a line integral is a complex number or expression. Unlike its counterpart in real variables, there is no physical interpretation for this quantity, such as area under a curve. Generally, integration in the complex plane is an intermediate process with a physically realizable quantity occurring only after we take its real or imaginary part. For example, in potential fluid flow, the lift and drag are found by taking the real and imaginary part of a complex integral, respectively.

How do we compute $\int_C f(z)\,dz$? Let us deal with the definition; we will illustrate the actual method by examples.

A popular method for evaluating complex line integrals consists of breaking everything up into real and imaginary parts. This reduces the integral to line integrals of real-valued functions which we know how to handle. Thus, we write $f(z) = u(x,y) + iv(x,y)$ as usual, and because $z = x + iy$, formally $dz = dx + i\,dy$. Therefore,

$$\int_C f(z)\,dz = \int_C [u(x,y) + iv(x,y)][dx + i\,dy] \tag{1.4.1}$$

$$= \int_C u(x,y)\,dx - v(x,y)\,dy + i\int_C v(x,y)\,dx + u(x,y)\,dy. \tag{1.4.2}$$

The exact method used to evaluate (1.4.2) depends upon the exact path specified.

From the definition of the line integral, we have the following self-evident properties:

$$\int_C f(z)\,dz = -\int_{C'} f(z)\,dz, \tag{1.4.3}$$

where C' is the contour C taken in the opposite direction of C and

$$\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz. \tag{1.4.4}$$

• **Example 1.4.1**

Let us evaluate $\int_C z^*\,dz$ from $z = 0$ to $z = 4+2i$ along two different contours. The first consists of the parametric equation $z = t^2 + it$. The second consists of two "dog legs": the first leg runs along the imaginary

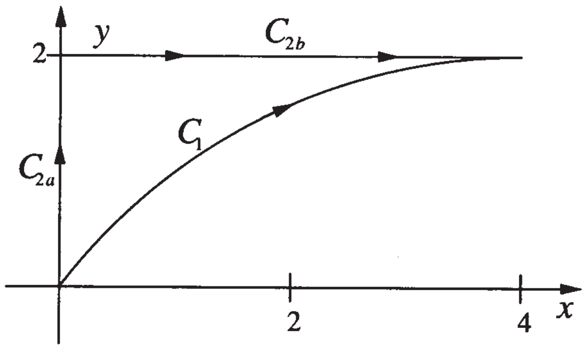


Figure 1.4.1: Contour used in Example 1.4.1.

axis from $z = 0$ to $z = 2i$ and then along a line parallel to the x-axis from $z = 2i$ to $z = 4 + 2i$. See Figure 1.4.1.

For the first case, the points $z = 0$ and $z = 4+2i$ on C_1 correspond to $t = 0$ and $t = 2$, respectively. Then the line integral equals

$$\int_{C_1} z^* \, dz = \int_0^2 (t^2 + it)^* \, d(t^2 + it) = \int_0^2 (2t^3 - it^2 + t) \, dt = 10 - \tfrac{8i}{3}. \tag{1.4.5}$$

The line integral for the second contour C_2 equals

$$\int_{C_2} z^* dz = \int_{C_{2a}} z^* dz + \int_{C_{2b}} z^* dz, \tag{1.4.6}$$

where C_{2a} denotes the integration from $z = 0$ to $z = 2i$ while C_{2b} denotes the integration from $z = 2i$ to $z = 4 + 2i$. For the first integral,

$$\int_{C_{2a}} z^* dz = \int_{C_{2a}} (x - iy)(dx + i\,dy) = \int_0^2 y\,dy = 2, \tag{1.4.7}$$

because $x = 0$ and $dx = 0$ along C_{2a}. On the other hand, along C_{2b}, $y = 2$ and $dy = 0$ so that

$$\int_{C_{2b}} z^* dz = \int_{C_{2b}} (x - iy)(dx + i\,dy) = \int_0^4 x\,dx + i\int_0^4 -2\,dx = 8 - 8i. \tag{1.4.8}$$

Thus the value of entire C_2 contour integral equals the sum of the two parts or $10 - 8i$.

The point here is that integration along two different paths has given us different results even though we integrated from $z = 0$ to $z = 4 + 2i$ both times. This results foreshadows a general result that is extremely important. Because the integrand contains nonanalytic

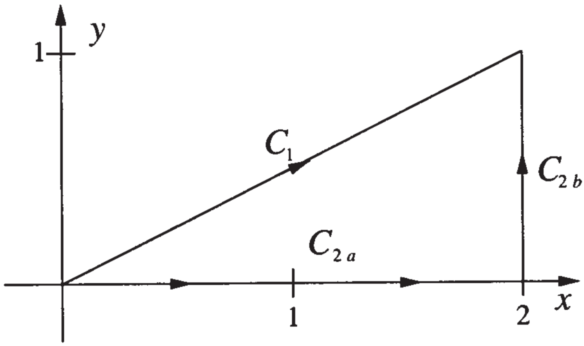


Figure 1.4.2: Contour used in Example 1.4.2.

points along and inside the region enclosed by our two curves, as shown by the Cauchy-Riemann equations, the results depend upon the path taken. Since complex integrations often involve integrands that have nonanalytic points, many line integrations depend upon the contour taken.

• Example 1.4.2

Let us integrate the *entire* function $f(z) = z^2$ along the two paths from $z = 0$ to $z = 2 + i$ shown in Figure 1.4.2. For the first integration, $x = 2y$ while along the second path we have two straight paths: $z = 0$ to $z = 2$ and $z = 2$ to $z = 2 + i$.

For the first contour integration,

$$\int_{C_1} z^2 dz = \int_0^1 (2y + iy)^2 (2\,dy + i\,dy) \tag{1.4.9}$$

$$= \int_0^1 (3y^2 + 4y^2 i)(2\,dy + i\,dy) \tag{1.4.10}$$

$$= \int_0^1 6y^2\,dy + 8y^2 i\,dy + 3y^2 i\,dy - 4y^2\,dy \tag{1.4.11}$$

$$= \int_0^1 2y^2\,dy + 11y^2 i\,dy \tag{1.4.12}$$

$$= \tfrac{2}{3}y^3\big|_0^1 + \tfrac{11}{3}iy^3\big|_0^1 = \tfrac{2}{3} + \tfrac{11i}{3}. \tag{1.4.13}$$

For our second integration,

$$\int_{C_2} z^2\,dz = \int_{C_{2a}} z^2\,dz + \int_{C_{2b}} z^2\,dz. \tag{1.4.14}$$

Along C_{2a} we find that $y = dy = 0$ so that

$$\int_{C_{2a}} z^2\,dz = \int_0^2 x^2\,dx = \tfrac{1}{3}x^3\big|_0^2 = \tfrac{8}{3} \tag{1.4.15}$$

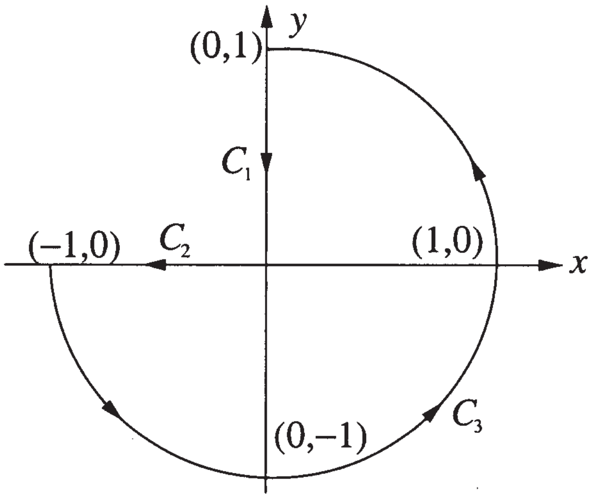


Figure 1.4.3: Contour used in Example 1.4.3.

and

$$\int_{C_{2b}} z^2\, dz = \int_0^1 (2+iy)^2 i\, dy = i\left(4y + 2iy^2 - \frac{y^3}{3}\right)\bigg|_0^1 = 4i - 2 - \tfrac{i}{3}, \tag{1.4.16}$$

because $x = 2$ and $dx = 0$. Consequently,

$$\int_{C_2} z^2\, dz = \frac{2}{3} + \frac{11i}{3}. \tag{1.4.17}$$

In this problem we obtained the same results from two different contours of integration. Exploring other contours, we would find that the results are always the same; the integration is path-independent. But what makes these results path-independent while the integration in Example 1.4.1 was not? Perhaps it is the fact that the integrand is analytic everywhere on the complex plane and there are no nonanalytic points. We will explore this later.

Finally, an important class of line integrals involves *closed contours.* We denote this special subclass of line integrals by placing a circle on the integral sign: $\oint$. Consider now the following examples:

• **Example 1.4.3**

Let us integrate $f(z) = z$ around the closed contour shown in Figure 1.4.3.

From Figure 1.4.3,

$$\oint_C z\,dz = \int_{C_1} z\,dz + \int_{C_2} z\,dz + \int_{C_3} z\,dz. \tag{1.4.18}$$

Now

$$\int_{C_1} z\,dz = \int_1^0 iy\,(i\,dy) = -\int_1^0 y\,dy = -\left.\frac{y^2}{2}\right|_1^0 = \frac{1}{2}, \tag{1.4.19}$$

$$\int_{C_2} z\,dz = \int_0^{-1} x\,dx = \left.\frac{x^2}{2}\right|_0^{-1} = \frac{1}{2} \tag{1.4.20}$$

and

$$\int_{C_3} z\,dz = \int_{-\pi}^{\pi/2} e^{\theta i} i e^{\theta i}\,d\theta = \left.\frac{e^{2\theta i}}{2}\right|_{-\pi}^{\pi/2} = -1, \tag{1.4.21}$$

where we have used $z = e^{\theta i}$ around the portion of the unit circle. Therefore, the closed line integral equals zero.

- **Example 1.4.4**

Let us integrate $f(z) = 1/(z - a)$ around any circle centered on $z = a$. The Cauchy-Riemann equations show that $f(z)$ is a meromorphic function. It is analytic everywhere except at the isolated singularity $z = a$.

If we introduce polar coordinates by letting $z - a = re^{\theta i}$ and $dz = ire^{\theta i}\,d\theta$,

$$\oint_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{\theta i}}{re^{\theta i}}\,d\theta = i\int_0^{2\pi} d\theta = 2\pi i. \tag{1.4.22}$$

Note that the integrand becomes undefined at $z = a$. Furthermore, the answer is independent of the size of the circle. Our example suggests that when we have a closed contour integration it is the behavior of the function within the contour rather than the exact shape of the closed contour that is of importance. We will return to this point in later sections.

Problems

1. Evaluate $\oint_C (z^*)^2\,dz$ around the circle $|z| = 1$ taken in the counterclockwise direction.

2. Evaluate $\oint_C |z|^2\,dz$ around the square with vertices at (0,0), (1,0), (1,1), and (0,1) taken in the counterclockwise direction.

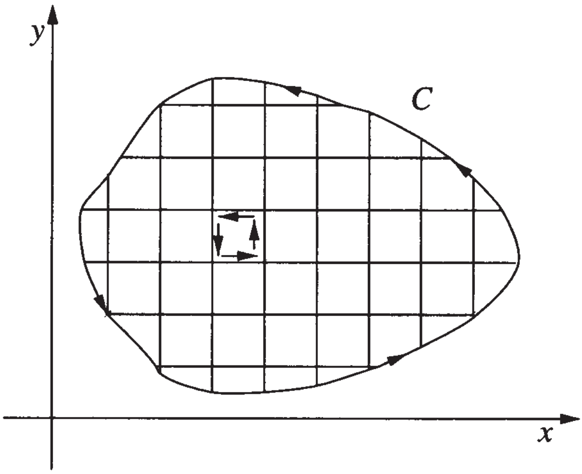


Figure 1.5.1: Diagram used in proving the Cauchy-Goursat theorem.

3. Evaluate $\int_C |z|\, dz$ along the right half of the circle $|z| = 1$ from $z = -i$ to $z = i$.

4. Evaluate $\int_C e^z\, dz$ along the line $y = x$ from $(-1, -1)$ to $(1, 1)$.

5. Evaluate $\int_C (z^*)^2\, dz$ along the line $y = x^2$ from $(0, 0)$ to $(1, 1)$.

6. Evaluate $\int_C z^{-1/2}\, dz$, where C is (a) the upper semicircle $|z| = 1$ and (b) the lower semicircle $|z| = 1$. If $z = re^{\theta i}$, restrict $-\pi < \theta < \pi$. Take both contours in the counterclockwise direction.

1.5 THE CAUCHY-GOURSAT THEOREM

In the previous section we showed how to evaluate line integrations by brute-force reduction to real-valued integrals. In general, this direct approach is quite difficult and we would like to apply some of the deeper properties of complex analysis to work smarter. In the remaining portions of this chapter we will introduce several theorems that will do just that.

If we scan over the examples worked in the previous section, we see considerable differences when the function was analytic inside and on the contour and when it was not. We may formalize this anecdotal evidence into the following theorem:

Cauchy-Goursat theorem[2]: *Let $f(z)$ be analytic in a domain D and*

[2] See Goursat, E., 1900: Sur la définition générale des fonctions analytiques, d'après Cauchy. *Trans. Am. Math. Soc.*, **1**, 14–16.

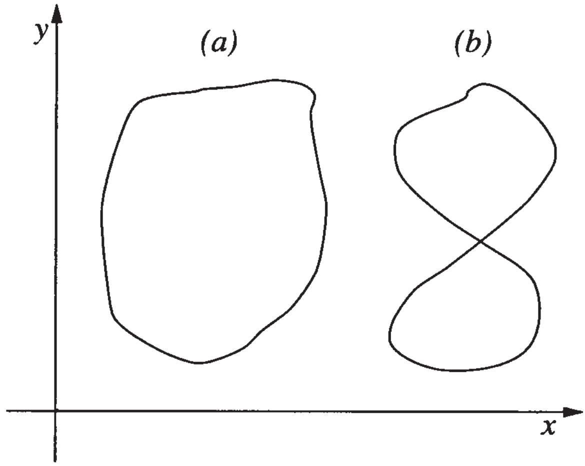


Figure 1.5.2: Examples of a (a) simply closed curve and (b) not simply closed curve.

let C be a simple Jordan curve[3] *inside D so that $f(z)$ is analytic on and inside of C. Then $\oint_C f(z)\,dz = 0$.*

Proof: Let C denote the contour around which we will integrate $w = f(z)$. We divide the region within C into a series of infinitesimal rectangles. See Figure 1.5.1. The integration around each rectangle equals the product of the average value of w on each side and its length,

$$\begin{aligned}&\left[w + \frac{\partial w}{\partial x}\frac{dx}{2}\right] dx + \left[w + \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial (iy)}\frac{d(iy)}{2}\right] d(iy)\\ &\quad + \left[w + \frac{\partial w}{\partial x}\frac{dx}{2} + \frac{\partial w}{\partial (iy)}d(iy)\right](-dx) + \left[w + \frac{\partial w}{\partial (iy)}\frac{d(iy)}{2}\right] d(-iy)\\ &\quad = \left(\frac{\partial w}{\partial x} - \frac{\partial w}{i\partial y}\right)(i\,dx\,dy) \end{aligned} \tag{1.5.1}$$

Substituting $w = u + iv$ into (1.5.1),

$$\frac{\partial w}{\partial x} - \frac{\partial w}{i\,\partial y} = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right). \tag{1.5.2}$$

Because the function is analytic, the right side of (1.5.1) and (1.5.2) equals zero. Thus, the integration around each of these rectangles also equals zero.

---

[3] A Jordan curve is a simply closed curve. It looks like a closed loop that does not cross itself. See Figure 1.5.2.

We note next that in integrating around adjoining rectangles we transverse each side in opposite directions, the net result being equivalent to integrating around the outer curve C. We therefore arrive at the result $\oint_C f(z)\,dz = 0$, where $f(z)$ is analytic within and on the closed contour. □

The Cauchy-Goursat theorem has several useful implications. Suppose we have a domain where $f(z)$ is analytic. Within this domain let us evaluate a line integral from point A to B along two different contours C_1 and C_2. Then, the integral around the closed contour formed by integrating along C_1 and then back along C_2, only in the opposite direction, is

$$\oint_C f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0 \tag{1.5.3}$$

or

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz. \tag{1.5.4}$$

Because C_1 and C_2 are completely arbitrary, we have the result that if, in a domain, $f(z)$ is analytic, the integral between any two points within the domain is *path independent.*

One obvious advantage of path independence is the ability to choose the contour so that the computations are made easier. This obvious choice immediately leads to

The principle of deformation of contours: *The value of a line integral of an analytic function around any simple closed contour remains unchanged if we deform the contour in such a manner that we do not pass over a nonanalytic point.*

• Example 1.5.1

Let us integrate $f(z) = z^{-1}$ around the closed contour C in the counterclockwise direction. This contour consists of a square, centered on the origin, with vertices at $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$.

The direct integration of $\oint_C z^{-1}dz$ around the original contour is very cumbersome. However, because the integrand is analytic everywhere except at the origin, we may deform the origin contour into a circle of radius r, centered on the origin. Then, $z = re^{\theta i}$ and $dz = rie^{\theta i}d\theta$ so that

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{rie^{\theta i}}{re^{\theta i}}\,d\theta = i\int_0^{2\pi} d\theta = 2\pi i. \tag{1.5.5}$$

The point here is that no matter how bizarre the contour is, as long as it encircles the origin and is a simply closed contour, we can deform it into a circle and we will get the same answer for the contour integral. This suggests that it is not the shape of the closed contour that makes the difference but whether we enclose any singularities [points where $f(z)$ becomes undefined] that matters. We shall return to this idea many times in the next few sections.

Finally, suppose that we have a function $f(z)$ such that $f(z)$ is analytic in some domain. Furthermore, let us introduce the analytic function $F(z)$ such that $f(z) = F'(z)$. We would like to evaluate $\int_a^b f(z)\,dz$ in terms of $F(z)$.

We begin by noting that we can represent F, f as $F(z) = U + iV$ and $f(z) = u + iv$. From (1.3.30) we have that $u = U_x$ and $v = V_x$. Therefore,

$$\int_a^b f(z)\,dz = \int_a^b (u + iv)(dx + i\,dy) \tag{1.5.6}$$

$$= \int_a^b U_x\,dx - V_x\,dy + i\int_a^b V_x\,dx + U_x\,dy \tag{1.5.7}$$

$$= \int_a^b U_x\,dx + U_y\,dy + i\int_a^b V_x\,dx + V_y\,dy \tag{1.5.8}$$

$$= \int_a^b dU + i\int_a^b dV = F(b) - F(a) \tag{1.5.9}$$

or

$$\int_a^b f(z)\,dz = F(b) - F(a). \tag{1.5.10}$$

Equation (1.5.10) is the complex variable form of the fundamental theorem of calculus. Thus, if we can find the antiderivative of a function $f(z)$ that is analytic within a specific region, we can evaluate the integral by evaluating the antiderivative at the endpoints for any curves within that region.

• **Example 1.5.2**

Let us evaluate $\int_0^{\pi i} z\,\sin(z^2)\,dz$.

The integrand $f(z) = z\,\sin(z^2)$ is an entire function and has the antiderivative $-\frac{1}{2}\cos(z^2)$. Therefore,

$$\int_0^{\pi i} z\,\sin(z^2)\,dz = -\tfrac{1}{2}\cos(z^2)\Big|_0^{\pi i} \tag{1.5.11}$$

$$= \tfrac{1}{2}[\cos(0) - \cos(-\pi^2)] \tag{1.5.12}$$

$$= \tfrac{1}{2}[1 - \cos(\pi^2)]. \tag{1.5.13}$$

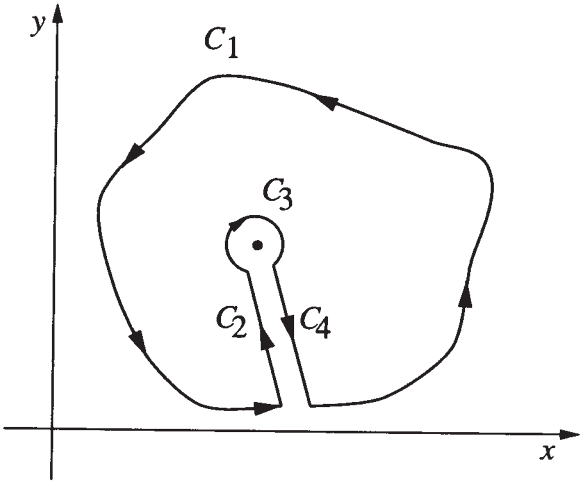


Figure 1.6.1: Diagram used to prove Cauchy's integral formula.

Problems

For the following integrals, show that they are path independent and determine the value of the integral:

1. $\int_{1-\pi i}^{2+3\pi i} e^{-2z}\,dz$
2. $\int_{0}^{2\pi} [e^z - \cos(z)]\,dz$
3. $\int_{0}^{\pi} \sin^2(z)\,dz$
4. $\int_{-i}^{2i} (z+1)\,dz$

1.6 CAUCHY'S INTEGRAL FORMULA

In the previous section, our examples suggested that the presence of a singularity within a contour really determines the value of a closed contour integral. Continuing with this idea, let us consider a class of closed contour integrals that explicitly contain a single singularity within the contour, namely $\oint_C g(z)\,dz$, where $g(z) = f(z)/(z - z_0)$ and $f(z)$ is analytic within and on the contour C. We have closed the contour in the *positive sense* where the enclosed area lies to your left as you move along the contour.

We begin by examining a closed contour integral where the closed contour consists of the C_1, C_2, C_3, and C_4 as shown in Figure 1.6.1. The gap or cut between C_2 and C_4 is very small. Because $g(z)$ is analytic within and on the closed integral, we have that

$$\int_{C_1} \frac{f(z)}{z-z_0}\,dz + \int_{C_2} \frac{f(z)}{z-z_0}\,dz + \int_{C_3} \frac{f(z)}{z-z_0}\,dz + \int_{C_4} \frac{f(z)}{z-z_0}\,dz = 0. \tag{1.6.1}$$

It can be shown that the contribution to the integral from the path C_2 going into the singularity will cancel the contribution from the path C_4 going away from the singularity as the gap between them vanishes. Because $f(z)$ is analytic at z_0, we can approximate its value on C_3 by $f(z) = f(z_0) + \delta(z)$, where δ is a small quantity. Substituting into (1.6.1),

$$\oint_{C_1} \frac{f(z)}{z - z_0}\, dz = -f(z_0) \int_{C_3} \frac{1}{z - z_0}\, dz - \int_{C_3} \frac{\delta(z)}{z - z_0}\, dz. \qquad \textbf{(1.6.2)}$$

Consequently, as the gap between C_2 and C_4 vanishes, the contour C_1 becomes the closed contour C so that (1.6.2) may be written

$$\oint_{C} \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0) + i \int_0^{2\pi} \delta\, d\theta, \qquad \textbf{(1.6.3)}$$

where we have set $z - z_0 = \epsilon e^{\theta i}$ and $dz = i\epsilon e^{\theta i} d\theta$.

Let M denote the value of the integral on the right side of (1.6.3) and Δ equal the greatest value of the modulus of δ along the circle. Then

$$|M| < \int_0^{2\pi} |\delta| d\theta \le \int_0^{2\pi} \Delta d\theta = 2\pi\Delta. \qquad \textbf{(1.6.4)}$$

As the radius of the circle diminishes to zero, Δ also diminishes to zero. Therefore, $|M|$, which is positive, becomes less than any finite quantity, however small, and M itself equals zero. Thus, we have that

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz. \qquad \textbf{(1.6.5)}$$

This equation is *Cauchy's integral formula*. By taking n derivatives of (1.6.5), we can extend Cauchy's integral formula[4] to

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \qquad \textbf{(1.6.6)}$$

[4] See Carrier, G. F., Krook, M., and Pearson, C. E., 1966: *Functions of a Complex Variable: Theory and Technique*, McGraw-Hill, New York, pp. 39–40 for the proof.

for $n = 1, 2, 3, \ldots$. For computing integrals, it is convenient to rewrite (1.6.6) as

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0). \tag{1.6.7}$$

• **Example 1.6.1**

Let us find the value of the integral

$$\oint_C \frac{\cos(\pi z)}{(z-1)(z-2)}\,dz, \tag{1.6.8}$$

where C is the circle $|z| = 5$. Using partial fractions,

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} \tag{1.6.9}$$

and

$$\oint_C \frac{\cos(\pi z)}{(z-1)(z-2)}\,dz = \oint_C \frac{\cos(\pi z)}{z-2}\,dz - \oint_C \frac{\cos(\pi z)}{z-1}\,dz. \tag{1.6.10}$$

By Cauchy's integral formula with $z_0 = 2$ and $z_0 = 1$,

$$\oint_C \frac{\cos(\pi z)}{z-2}\,dz = 2\pi i\ \cos(2\pi) = 2\pi i \tag{1.6.11}$$

and

$$\oint_C \frac{\cos(\pi z)}{z-1}\,dz = 2\pi i \cos(\pi) = -2\pi i, \tag{1.6.12}$$

because $z_0 = 1$ and $z_0 = 2$ lie inside C and $\cos(\pi z)$ is analytic there. Thus the required integral has the value

$$\oint_C \frac{\cos(\pi z)}{(z-1)(z-2)}\,dz = 4\pi i. \tag{1.6.13}$$

• **Example 1.6.2**

Let us use Cauchy's integral formula to evaluate

$$I = \oint_{|z|=2} \frac{e^z}{(z-1)^2(z-3)}\,dz. \tag{1.6.14}$$

We need to convert (1.6.14) into the form (1.6.7). To do this, we rewrite (1.6.14) as

$$\oint_{|z|=2} \frac{e^z}{(z-1)^2(z-3)}\,dz = \oint_{|z|=2} \frac{e^z/(z-3)}{(z-1)^2}\,dz. \tag{1.6.15}$$

Therefore, $f(z) = e^z/(z-3)$, $n = 1$ and $z_0 = 1$. The function $f(z)$ is analytic within the closed contour because the point $z_0 = 3$ lies outside of the contour. Applying Cauchy's integral formula,

$$\oint_{|z|=2} \frac{e^z}{(z-1)^2(z-3)}\,dz = \frac{2\pi i}{1!}\frac{d}{dz}\left(\frac{e^z}{z-3}\right)\Bigg|_{z=1} \tag{1.6.16}$$

$$= 2\pi i\left[\frac{e^z}{z-3} - \frac{e^z}{(z-3)^2}\right]\Bigg|_{z=1} \tag{1.6.17}$$

$$= -\frac{3\pi i e}{2}. \tag{1.6.18}$$

Problems

Use Cauchy's integral formula to evaluate the following integrals. Assume all of the contours are in the positive sense.

1. $\displaystyle\oint_{|z|=1} \frac{\sin^6(z)}{z-\pi/6}\,dz$

2. $\displaystyle\oint_{|z|=1} \frac{\sin^6(z)}{(z-\pi/6)^3}\,dz$

3. $\displaystyle\oint_{|z|=1} \frac{1}{z(z^2+4)}\,dz$

4. $\displaystyle\oint_{|z|=1} \frac{\tan(z)}{z}\,dz$

5. $\displaystyle\oint_{|z-1|=1/2} \frac{1}{(z-1)(z-2)}\,dz$

6. $\displaystyle\oint_{|z|=5} \frac{\exp(z^2)}{z^3}\,dz$

7. $\displaystyle\oint_{|z-1|=1} \frac{z^2+1}{z^2-1}\,dz$

8. $\displaystyle\oint_{|z|=2} \frac{z^2}{(z-1)^4}\,dz$

9. $\displaystyle\oint_{|z|=2} \frac{z^3}{(z+i)^3}\,dz$

10. $\displaystyle\oint_{|z|=1} \frac{\cos(z)}{z^{2n+1}}\,dz$

1.7 TAYLOR AND LAURENT EXPANSIONS AND SINGULARITIES

In the previous section we showed what a crucial role singularities play in complex integration. Before we can find the most general way of computing a closed complex integral, our understanding of singularities must deepen. For this, we employ power series.

One reason why power series are so important is their ability to provide locally a general representation of a function even when its arguments are complex. For example, when we were introduced to trigonometric functions in high school, it was in the context of a right triangle and a real angle. However, when the argument becomes complex this geometrical description disappears and power series provide a formalism for defining the trigonometric functions, regardless of the nature of the argument.

Let us begin our analysis by considering the complex function $f(z)$ which is analytic everywhere on the boundary and the interior of a circle whose center is at $z = z_0$. Then, if z denotes any point within the circle, we have from Cauchy's integral formula that

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z_0}\left[\frac{1}{1-(z-z_0)/(\zeta - z_0)}\right]d\zeta, \tag{1.7.1}$$

where C denotes the closed contour. Expanding the bracketed term as a geometric series, we find that

$$f(z) = \frac{1}{2\pi i}\left[\oint_C \frac{f(\zeta)}{\zeta - z_0}\,d\zeta + (z-z_0)\oint_C \frac{f(\zeta)}{(\zeta - z_0)^2}\,d\zeta + \cdots \right.$$
$$\left. + (z-z_0)^n \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,d\zeta + \cdots\right]. \tag{1.7.2}$$

Applying Cauchy's integral formula to each integral in (1.7.2), we finally obtain

$$f(z) = f(z_0) + \frac{(z-z_0)}{1!}f'(z_0) + \cdots + \frac{(z-z_0)^n}{n!}f^{(n)}(z_0) + \cdots \tag{1.7.3}$$

or the familiar formula for a Taylor expansion. Consequently, *we can expand any analytic function into a Taylor series.* Interestingly, the radius of convergence[5] of this series may be shown to be the distance between z_0 and the nearest nonanalytic point of $f(z)$.

• Example 1.7.1

Let us find the expansion of $f(z) = \sin(z)$ about the point $z_0 = 0$.

Because $f(z)$ is an entire function, we can construct a Taylor expansion anywhere on the complex plane. For $z_0 = 0$,

$$f(z) = f(0) + \tfrac{1}{1!}f'(0)z + \tfrac{1}{2!}f''(0)z^2 + \tfrac{1}{3!}f'''(0)z^3 + \cdots \tag{1.7.4}$$

Because $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$ and so forth,

$$f(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \tag{1.7.5}$$

Because $\sin(z)$ is an entire function, the radius of convergence is $|z-0| < \infty$, i.e., all z.

[5] A positive number h such that the series diverges for $|z - z_0| > h$ but converges absolutely for $|z - z_0| < h$.

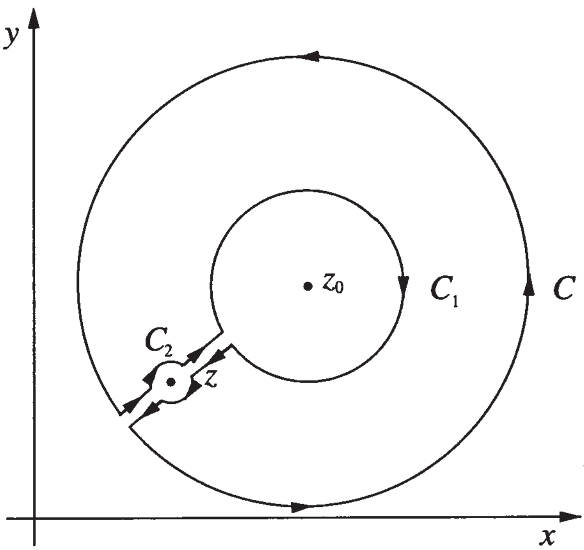


Figure 1.7.1: Contour used in deriving the Laurent expansion.

• **Example 1.7.2**

Let us find the expansion of $f(z) = 1/(1-z)$ about the point $z_0 = 0$. From the formula for a Taylor expansion,

$$f(z) = f(0) + \tfrac{1}{1!}f'(0)z + \tfrac{1}{2!}f''(0)z^2 + \tfrac{1}{3!}f'''(0)z^3 + \cdots \qquad (1.7.6)$$

Because $f^{(n)}(0) = n!$, we find that

$$f(z) = 1 + z + z^2 + z^3 + z^4 + \cdots = \frac{1}{1-z}. \qquad (1.7.7)$$

Equation (1.7.7) is the familiar result for a geometric series. Because the only nonanalytic point is at $z = 1$, the radius of convergence is $|z - 0| < 1$, the unit circle centered at $z = 0$.

Consider now the situation where we draw two concentric circles about some arbitrary point z_0; we denote the outer circle by C while we denote the inner circle by C_1. See Figure 1.7.1. Let us assume that $f(z)$ is analytic inside the annulus between the two circles. Outside of this area, the function may or may not be analytic. Within the annulus we pick a point z and construct a small circle around it, denoting the circle by C_2. As the gap or *cut* in the annulus becomes infinitesimally small, the line integrals that connect the circle C_2 to C_1 and C sum to zero, leaving

$$\oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta = \oint_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \oint_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{1.7.8}$$

Because $f(\zeta)$ is analytic everywhere within C_2,

$$2\pi i f(z) = \oint_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{1.7.9}$$

Using the relationship:

$$\oint_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta = -\oint_{C_1} \frac{f(\zeta)}{z - \zeta}\, d\zeta, \tag{1.7.10}$$

(1.7.8) becomes

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{z - \zeta}\, d\zeta. \tag{1.7.11}$$

Now,

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - z + z_0} = \frac{1}{\zeta - z_0}\frac{1}{1 - (z - z_0)/(\zeta - z_0)} \tag{1.7.12}$$

$$= \frac{1}{\zeta - z_0}\left[1 + \left(\frac{z - z_0}{\zeta - z_0}\right) + \left(\frac{z - z_0}{\zeta - z_0}\right)^2 + \cdots + \left(\frac{z - z_0}{\zeta - z_0}\right)^n + \cdots\right], \tag{1.7.13}$$

where $|z - z_0|/|\zeta - z_0| < 1$ and

$$\frac{1}{z - \zeta} = \frac{1}{z - z_0 - \zeta + z_0} = \frac{1}{z - z_0}\frac{1}{1 - (\zeta - z_0)/(z - z_0)} \tag{1.7.14}$$

$$= \frac{1}{z - z_0}\left[1 + \left(\frac{\zeta - z_0}{z - z_0}\right) + \left(\frac{\zeta - z_0}{z - z_0}\right)^2 + \cdots + \left(\frac{\zeta - z_0}{z - z_0}\right)^n + \cdots\right], \tag{1.7.15}$$

where $|\zeta - z_0|/|z - z_0| < 1$. Upon substituting these expressions into (1.7.11),

$$\begin{aligned} f(z) = &\left[\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z_0}\, d\zeta + \frac{z - z_0}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z_0)^2}\, d\zeta + \cdots \right.\\ &\left. + \frac{(z - z_0)^n}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta + \cdots\right]\\ &+ \left[\frac{1}{z - z_0}\frac{1}{2\pi i}\oint_{C_1} f(\zeta)\, d\zeta + \frac{1}{(z - z_0)^2}\frac{1}{2\pi i}\oint_{C_1} f(\zeta)(\zeta - z_0)\, d\zeta + \cdots \right.\\ &\left. + \frac{1}{(z - z_0)^n}\frac{1}{2\pi i}\oint_{C_1} f(\zeta)(\zeta - z_0)^{n-1}\, d\zeta + \cdots\right] \end{aligned} \tag{1.7.16}$$

or

$$f(z) = \frac{a_1}{z - z_0} + \frac{a_2}{(z - z_0)^2} + \cdots + \frac{a_n}{(z - z_0)^n} + \cdots$$
$$+ b_0 + b_1(z - z_0) + \cdots + b_n(z - z_0)^n + \cdots \quad (1.7.17)$$

Equation (1.7.17) is a *Laurent expansion*.[6] If $f(z)$ is analytic at z_0, then $a_1 = a_2 = \cdots = a_n = \cdots = 0$ and the Laurent expansion reduces to a Taylor expansion. If z_0 is a singularity of $f(z)$, then the Laurent expansion will include both positive and *negative* powers. The coefficient of the $(z - z_0)^{-1}$ term, a_1, is the *residue*, for reasons that will appear in the next section.

Unlike the Taylor series, there is no straightforward method for obtaining a Laurent series. For the remaining portions of this section we will illustrate their construction. These techniques include replacing a function by its appropriate power series, the use of geometric series to expand the denominator, and the use of algebraic tricks to assist in applying the first two method.

• **Example 1.7.3**

Laurent expansions provide a formalism for the classification of singularities of a function. *Isolated singularities* fall into three types; they are

• *Essential Singularity*: Consider the function $f(z) = \cos(1/z)$. Using the expansion for cosine,

$$\cos\left(\frac{1}{z}\right) = 1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} - \frac{1}{6!z^6} + \cdots \quad (1.7.18)$$

for $0 < |z| < \infty$. Note that this series never truncates in the inverse powers of z. Essential singularities have Laurent expansions which have an infinite number of inverse powers of $z - z_0$. The value of the residue for this essential singularity at $z = 0$ is zero.

• *Removable Singularity*: Consider the function $f(z) = \sin(z)/z$. This function has a singularity at $z = 0$. Upon applying the expansion for sine,

$$\frac{\sin(z)}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \ldots\right) \quad (1.7.19)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \frac{z^8}{9!} - \ldots \quad (1.7.20)$$

[6] See Laurent, M., 1843: Extension du théorème de M. Cauchy relatif à la convergence du développement d'une fonction suivant les puissances ascendantes de la variable x. *C. R. l'Acad. Sci.*, **17**, 938–942.

for all z, if the division is permissible. We have made $f(z)$ analytic by defining it by (1.7.20) and, in the process, removed the singularity. The residue for a removable singularity always equals zero.

- *Pole of order n*: Consider the function

$$f(z) = \frac{1}{(z-1)^3(z+1)}. \tag{1.7.21}$$

This function has two singularities: one at $z = 1$ and the other at $z = -1$. We shall only consider the case $z = 1$. After a little algebra,

$$f(z) = \frac{1}{(z-1)^3}\frac{1}{2+(z-1)} \tag{1.7.22}$$

$$= \frac{1}{2}\frac{1}{(z-1)^3}\frac{1}{1+(z-1)/2} \tag{1.7.23}$$

$$= \frac{1}{2}\frac{1}{(z-1)^3}\left[1 - \frac{z-1}{2} + \frac{(z-1)^2}{4} - \frac{(z-1)^3}{8} + \cdots\right] \tag{1.7.24}$$

$$= \frac{1}{2(z-1)^3} - \frac{1}{4(z-1)^2} + \frac{1}{8(z-1)} - \frac{1}{16} + \cdots \tag{1.7.25}$$

for $0 < |z-1| < 2$. Because the largest inverse (negative) power is three, the singularity at $z = 1$ is a third-order pole; the value of the residue is 1/8. Generally, we refer to a first-order pole as a *simple* pole.

- **Example 1.7.4**

Let us find the Laurent expansion for

$$f(z) = \frac{z}{(z-1)(z-3)} \tag{1.7.26}$$

about the point $z = 1$.

We begin by rewriting $f(z)$ as

$$f(z) = \frac{1+(z-1)}{(z-1)[-2+(z-1)]} \tag{1.7.27}$$

$$= -\frac{1}{2}\frac{1+(z-1)}{(z-1)[1-\frac{1}{2}(z-1)]} \tag{1.7.28}$$

$$= -\frac{1}{2}\frac{1+(z-1)}{(z-1)}[1 + \tfrac{1}{2}(z-1) + \tfrac{1}{4}(z-1)^2 + \cdots] \tag{1.7.29}$$

$$= -\frac{1}{2}\frac{1}{z-1} - \frac{3}{4} - \frac{3}{8}(z-1) - \frac{3}{16}(z-1)^2 - \cdots \tag{1.7.30}$$

provided $0 < |z - 1| < 2$. Therefore we have a simple pole at $z = 1$ and the value of the residue is $-1/2$. A similar procedure would yield the Laurent expansion about $z = 3$.

For complicated complex functions, it is very difficult to determine the nature of the singularities by finding the complete Laurent expansion and we must try another method. We shall call it "a poor man's Laurent expansion". The idea behind this method is the fact that we generally need only the first few terms of the Laurent expansion to discover its nature. Consequently, we compute these terms through the application of power series where we retain only the leading terms. Consider the following example.

• **Example 1.7.5**

Let us discover the nature of the singularity at $z = 0$ of the function

$$f(z) = \frac{e^{tz}}{z\sinh(az)}, \tag{1.7.31}$$

where a and t are real.

We begin by replacing the exponential and hyperbolic sine by their Taylor expansion about $z = 0$. Then

$$f(z) = \frac{1 + tz + t^2z^2/2 + \cdots}{z(az - a^3z^3/6 + \cdots)}. \tag{1.7.32}$$

Factoring out az in the denominator,

$$f(z) = \frac{1 + tz + t^2z^2/2 + \cdots}{az^2(1 - a^2z^2/6 + \cdots)}. \tag{1.7.33}$$

Within the parentheses all of the terms except the leading one are small. Therefore, by long division, we formally have that

$$f(z) = \frac{1}{az^2}(1 + tz + t^2z^2/2 + \cdots)(1 + a^2z^2/6 + \cdots) \tag{1.7.34}$$

$$= \frac{1}{az^2}(1 + tz + t^2z^2/2 + a^2z^2/6 + \cdots) \tag{1.7.35}$$

$$= \frac{1}{az^2} + \frac{t}{az} + \frac{3t^2 + a^2}{6a} + \cdots \tag{1.7.36}$$

Thus, we have a second-order pole at $z = 0$ and the residue equals t/a.

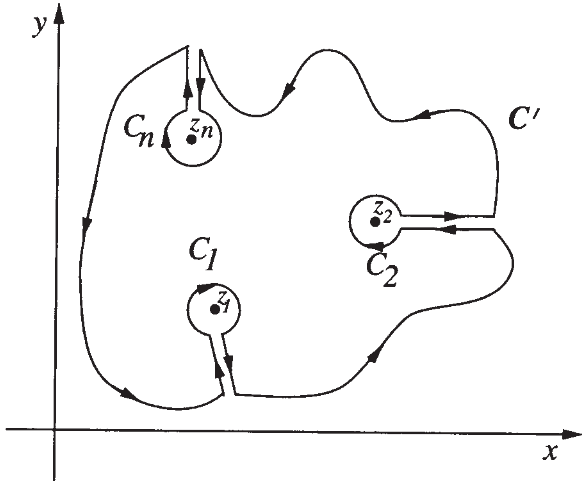


Figure 1.8.1: Contour used in deriving the residue theorem.

Problems

1. Find the Taylor expansion of $f(z) = (1-z)^{-2}$ about the point $z = 0$.

2. Find the Taylor expansion of $f(z) = (z-1)e^z$ about the point $z = 1$. [Hint: Don't find the expansion by taking derivatives.]

By constructing a Laurent expansion, describe the type of singularity and give the residue at z_0 for each of the following functions:

3. $f(z) = z^{10}e^{-1/z}; \quad z_0 = 0$

4. $f(z) = z^{-3}\sin^2(z); \quad z_0 = 0$

5. $f(z) = \dfrac{\cosh(z)-1}{z^2}; \quad z_0 = 0$

6. $f(z) = \dfrac{z}{(z+2)^2}; \quad z_0 = -2$

7. $f(z) = \dfrac{e^z+1}{e^{-z}-1}; \quad z_0 = 0$

8. $f(z) = \dfrac{e^{iz}}{z^2+b^2}; \quad z_0 = bi$

9. $f(z) = \dfrac{1}{z(z-2)}; \quad z_0 = 2$

10. $f(z) = \dfrac{\exp(z^2)}{z^4}; \quad z_0 = 0$

1.8 THEORY OF RESIDUES

Having shown that around any singularity we may construct a Laurent expansion, we now use this result in the integration of closed complex integrals. Consider a closed contour in which the function $f(z)$ has a number of isolated singularities. As we did in the case of Cauchy's integral formula, we introduce a new contour C' which excludes all of the singularities because they are isolated. See Figure 1.8.1. Therefore,

$$\oint_C f(z)\,dz - \oint_{C_1} f(z)\,dz - \cdots - \oint_{C_n} f(z)\,dz = \oint_{C'} f(z)\,dz = 0. \quad \mathbf{(1.8.1)}$$

Consider now the mth integral, where $1 \leq m \leq n$. Constructing a Laurent expansion for the function $f(z)$ at the isolated singularity $z = z_m$, this integral equals

$$\oint_{C_m} f(z)\,dz = \sum_{k=1}^{\infty} a_k \oint_{C_m} \frac{1}{(z - z_m)^k}\,dz + \sum_{k=0}^{\infty} b_k \oint_{C_m} (z - z_m)^k\,dz. \quad \mathbf{(1.8.2)}$$

Because $(z - z_m)^k$ is an entire function if $k \geq 0$, the integrals equal zero for each term in the second summation. We use Cauchy's integral formula to evaluate the remaining terms. The analytic function in the numerator is 1. Because $d^{k-1}(1)/dz^{k-1} = 0$ if $k > 1$, all of the terms vanish except for $k = 1$. In that case, the integral equals $2\pi i a_1$, where a_1 is the value of the residue for that particular singularity. Applying this approach to each of the singularities, we obtain

Cauchy's residue theorem[7]: *If $f(z)$ is analytic inside and on a closed contour C (taken in the positive sense) except at points $z_1, z_2, \ldots, z_n$ where $f(z)$ has singularities, then*

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} \text{Res}[f(z); z_j], \quad \mathbf{(1.8.3)}$$

where $\text{Res}[f(z); z_j]$ denotes the residue of the jth isolated singularity of $f(z)$ located at $z = z_j$.

• **Example 1.8.1**

Let us compute $\oint_{|z|=2} z^2/(z+1)\,dz$ by the residue theorem, assuming that we take the contour in the positive sense.

Because the contour is a circle of radius 2, centered on the origin, the singularity at $z = -1$ lies within the contour. If the singularity were

[7] See Mitrinović, D. S. and Kečkić, J. D., 1984: *The Cauchy Method of Residues: Theory and Applications*, D. Reidel Publishing, Boston. Section 10.3 gives the historical development of the residue theorem.

not inside the contour, then the integrand would have been analytic inside and on the contour C. In this case, the answer would then be zero by the Cauchy-Goursat theorem.

Returning to the original problem, we construct the Laurent expansion for the integrand around the point $z = 1$ by noting that

$$\frac{z^2}{z+1} = \frac{[(z+1)-1]^2}{z+1} = \frac{1}{z+1} - 2 + (z+1). \tag{1.8.4}$$

The singularity at $z = -1$ is a simple pole and by inspection the value of the residue equals 1. Therefore,

$$\oint_{|z|=2} \frac{z^2}{z+1}\,dz = 2\pi i. \tag{1.8.5}$$

As it presently stands, it would appear that we must always construct a Laurent expansion for each singularity if we wish to use the residue theorem. This becomes increasingly difficult as the structure of the integrand becomes more complicated. In the following paragraphs we will show several techniques that avoid this problem in practice.

We begin by noting that many functions that we will encounter consist of the ratio of two *polynomials*, i.e., rational functions: $f(z) = g(z)/h(z)$. Generally, we can write $h(z)$ as $(z-z_1)^{m_1}(z-z_2)^{m_2}\cdots$. Here we have assumed that we have divided out any common factors between $g(z)$ and $h(z)$ so that $g(z)$ does not vanish at $z_1, z_2, \ldots$. Clearly $z_1, z_2, \ldots,$ are singularities of $f(z)$. Further analysis shows that the nature of the singularities are a pole of order m_1 at $z = z_1$, a pole of order m_2 at $z = z_2$, and so forth.

Having found the nature and location of the singularity, we compute the residue as follows. Suppose we have a pole of order n. Then we know that its Laurent expansion is

$$f(z) = \frac{a_n}{(z-z_0)^n} + \frac{a_{n-1}}{(z-z_0)^{n-1}} + \cdots + b_0 + b_1(z-z_0) + \cdots \tag{1.8.6}$$

Multiplying both sides of (1.8.6) by $(z-z_0)^n$,

$$\begin{aligned} F(z) &= (z-z_0)^n f(z) \\ &= a_n + a_{n-1}(z-z_0) + \cdots + b_0(z-z_0)^n + b_1(z-z_0)^{n+1} + \cdots \end{aligned} \tag{1.8.7}$$

Because $F(z)$ is analytic at $z = z_0$, it has the Taylor expansion

$$F(z) = F(z_0) + F'(z_0)(z-z_0) + \cdots + \frac{F^{(n-1)}(z_0)}{(n-1)!}(z-z_0)^{n-1} + \cdots \tag{1.8.8}$$

Matching powers of $z - z_0$ in (1.8.7) and (1.8.8), the residue equals

$$\text{Res}[f(z); z_0] = a_1 = \frac{F^{(n-1)}(z_0)}{(n-1)!}. \tag{1.8.9}$$

Substituting in $F(z) = (z - z_0)^n f(z)$, we can compute the residue of a pole of order n by

$$\boxed{\text{Res}[f(z); z_j] = \frac{1}{(n-1)!} \lim_{z \to z_j} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_j)^n f(z) \right].} \tag{1.8.10}$$

For a simple pole (1.8.10) simplifies to

$$\boxed{\text{Res}[f(z); z_j] = \lim_{z \to z_j} (z - z_j) f(z).} \tag{1.8.11}$$

Quite often, $f(z) = p(z)/q(z)$. From l'Hôspital's rule, it follows that

$$\boxed{\text{Res}[f(z); z_j] = \frac{p(z_j)}{q'(z_j)}.} \tag{1.8.12}$$

Remember that these formulas work only for finite-order poles. For an essential singularity we must compute the residue from its Laurent expansion; however, essential singularities are very rare in applications.

• **Example 1.8.2**

Let us evaluate

$$\oint_C \frac{e^{iz}}{z^2 + a^2}\, dz, \tag{1.8.13}$$

where C is any contour that includes both $z = \pm ai$ and is in the positive sense.

From Cauchy's residue theorem,

$$\oint_C \frac{e^{iz}}{z^2 + a^2}\, dz = 2\pi i \left[\text{Res}\left(\frac{e^{iz}}{z^2 + a^2}; ai \right) + \text{Res}\left(\frac{e^{iz}}{z^2 + a^2}; -ai \right) \right]. \tag{1.8.14}$$

The singularities at $z = \pm ai$ are simple poles. The corresponding residues are

$$\text{Res}\left(\frac{e^{iz}}{z^2+a^2}; ai\right) = \lim_{z\to ai}(z-ai)\frac{e^{iz}}{(z-ai)(z+ai)} = \frac{e^{-a}}{2ia} \tag{1.8.15}$$

and

$$\text{Res}\left(\frac{e^{iz}}{z^2+a^2}; -ai\right) = \lim_{z\to -ai}(z+ai)\frac{e^{iz}}{(z-ai)(z+ai)} = -\frac{e^{a}}{2ia}. \tag{1.8.16}$$

Consequently,

$$\oint_C \frac{e^{iz}}{z^2+a^2}\,dz = -\frac{2\pi}{2a}\left(e^a - e^{-a}\right) = -\frac{2\pi}{a}\sinh(a). \tag{1.8.17}$$

• **Example 1.8.3**

Let us evaluate

$$\frac{1}{2\pi i}\oint_C \frac{e^{tz}}{z^2(z^2+2z+2)}\,dz, \tag{1.8.18}$$

where C includes all of the singularities and is in the positive sense.

The integrand has a second-order pole at $z = 0$ and two simple poles at $z = -1 \pm i$ which are the roots of $z^2 + 2z + 2 = 0$. Therefore, the residue at $z = 0$ is

$$\text{Res}\left[\frac{e^{tz}}{z^2(z^2+2z+2)}; 0\right] = \lim_{z\to 0}\frac{1}{1!}\frac{d}{dz}\left\{(z-0)^2\left[\frac{e^{tz}}{z^2(z^2+2z+2)}\right]\right\} \tag{1.8.19}$$

$$= \lim_{z\to 0}\left[\frac{te^{tz}}{z^2+2z+2} - \frac{(2z+2)e^{tz}}{(z^2+2z+2)^2}\right] = \frac{t-1}{2}. \tag{1.8.20}$$

The residue at $z = -1 + i$ is

$$\text{Res}\left[\frac{e^{tz}}{z^2(z^2+2z+2)}; -1+i\right] = \lim_{z\to -1+i}[z-(-1+i)]\frac{e^{tz}}{z^2(z^2+2z+2)} \tag{1.8.21}$$

$$= \left(\lim_{z\to -1+i}\frac{e^{tz}}{z^2}\right)\left(\lim_{z\to -1+i}\frac{z+1-i}{z^2+2z+2}\right) \tag{1.8.22}$$

$$= \frac{\exp[(-1+i)t]}{2i(-1+i)^2} = \frac{\exp[(-1+i)t]}{4}. \tag{1.8.23}$$

Similarly, the residue at $z = -1 - i$ is

$$\operatorname{Res}\left[\frac{e^{tz}}{z^2(z^2+2z+2)}; -1-i\right] = \lim_{z\to-1-i}[z-(-1-i)]\frac{e^{tz}}{z^2(z^2+2z+2)} \tag{1.8.24}$$

$$= \left(\lim_{z\to-1-i}\frac{e^{tz}}{z^2}\right)\left(\lim_{z\to-1-i}\frac{z+1+i}{z^2+2z+2}\right) \tag{1.8.25}$$

$$= \frac{\exp[(-1-i)t]}{2i(-1-i)^2} = \frac{\exp[(-1-i)t]}{4}. \tag{1.8.26}$$

Then by the residue theorem,

$$\frac{1}{2\pi i}\oint_C \frac{e^{tz}}{z^2(z^2+2z+2)}\,dz = \operatorname{Res}\left[\frac{e^{tz}}{z^2(z^2+2z+2)}; 0\right] + \operatorname{Res}\left[\frac{e^{tz}}{z^2(z^2+2z+2)}; -1+i\right] + \operatorname{Res}\left[\frac{e^{tz}}{z^2(z^2+2z+2)}; -1-i\right] \tag{1.8.27}$$

$$= \frac{t-1}{2} + \frac{\exp[(-1+i)t]}{4} + \frac{\exp[(-1-i)t]}{4} \tag{1.8.28}$$

$$= \tfrac{1}{2}\left[t - 1 + e^{-t}\cos(t)\right]. \tag{1.8.29}$$

Problems

Assuming that all of the following closed contours are in the positive sense, use the residue theorem to evaluate the following integrals:

1. $\displaystyle\oint_{|z|=1}\frac{z+1}{z^4-2z^3}\,dz$

2. $\displaystyle\oint_{|z|=1}\frac{(z+4)^3}{z^4+5z^3+6z^2}\,dz$

3. $\displaystyle\oint_{|z|=1}\frac{1}{1-e^z}\,dz$

4. $\displaystyle\oint_{|z|=2}\frac{z^2-4}{(z-1)^4}\,dz$

5. $\displaystyle\oint_{|z|=2}\frac{z^3}{z^4-1}\,dz$

6. $\displaystyle\oint_{|z|=1} z^n e^{2/z}\,dz, \quad n > 0$

7. $\displaystyle\oint_{|z|=1} e^{1/z}\cos(1/z)\,dz$

8. $\displaystyle\oint_{|z|=2}\frac{2+4\cos(\pi z)}{z(z-1)^2}\,dz$

1.9 EVALUATION OF REAL DEFINITE INTEGRALS

One of the important applications of the theory of residues consists in the evaluation of certain types of real definite integrals. Similar techniques apply when the integrand contains a sine or cosine. See Section 3.4.

• Example 1.9.1

Let us evaluate the integral

$$\int_0^\infty \frac{dx}{x^2+1} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{x^2+1}. \tag{1.9.1}$$

This integration occurs along the real axis. In terms of complex variables we can rewrite (1.9.1) as

$$\int_0^\infty \frac{dx}{x^2+1} = \frac{1}{2}\int_{C_1} \frac{dz}{z^2+1}, \tag{1.9.2}$$

where the contour C_1 is the line $\text{Im}(z) = 0$. However, the use of the residue theorem requires an integration along a closed contour. Let us choose the one pictured in Figure 1.9.1. Then

$$\oint_C \frac{dz}{z^2+1} = \int_{C_1} \frac{dz}{z^2+1} + \int_{C_2} \frac{dz}{z^2+1}, \tag{1.9.3}$$

where C denotes the complete closed contour and C_2 denotes the integration path along a semicircle at infinity. Clearly we want the second integral on the right side of (1.9.3) to vanish; otherwise, our choice of the contour C_2 is poor. Because $z = Re^{\theta i}$ and $dz = iRe^{\theta i}d\theta$,

$$\left|\int_{C_2} \frac{dz}{z^2+1}\right| = \left|\int_0^\pi \frac{iR\exp(\theta i)\,d\theta}{1+R^2\exp(2\theta i)}\right| \le \int_0^\pi \frac{R\,d\theta}{R^2-1}, \tag{1.9.4}$$

which tends to zero as $R \to \infty$. On the other hand, the residue theorem gives

$$\oint_C \frac{dz}{z^2+1} = 2\pi i \,\text{Res}\left(\frac{1}{z^2+1}; i\right) = 2\pi i \lim_{z\to i} \frac{z-i}{z^2+1} = 2\pi i \times \frac{1}{2i} = \pi. \tag{1.9.5}$$

Therefore,

$$\int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}. \tag{1.9.6}$$

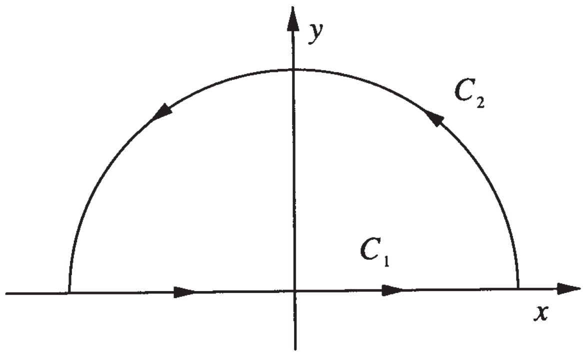


Figure 1.9.1: Contour used in evaluating the integral (1.9.1).

Note that we only evaluated the residue in the upper half-plane because it is the only one inside the contour.

This example illustrates the basic concepts of evaluating definite integrals by the residue theorem. We introduce a closed contour that includes the real axis and an additional contour. We must then evaluate the integral along this additional contour as well as the closed contour integral. If we have properly chosen our closed contour, this additional integral will vanish. For certain classes of general integrals, we shall now show that this additional contour is a circular arc at infinity.

Theorem: *If, on a circular arc C_R with a radius R and center at the origin, $zf(z) \to 0$ uniformly with $|z| \in C_R$ and as $R \to \infty$, then*

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0. \tag{1.9.7}$$

This follows from the fact that if $|zf(z)| \leq M_R$, then $|f(z)| \leq M_R/R$. Because the length of C_R is αR, where α is the subtended angle,

$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{M_R}{R}\,\alpha R = \alpha M_R \to 0, \tag{1.9.8}$$

because $M_R \to 0$ as $R \to \infty$. □

• **Example 1.9.2**

A simple illustration of this theorem is the integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+x+1} = \int_{C_1} \frac{dz}{z^2+z+1}. \tag{1.9.9}$$

A quick check shows that $z/(z^2+z+1)$ tends to zero uniformly as $R \to \infty$. Therefore, if we use the contour pictured in Figure 1.9.1,

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+x+1} = \oint_C \frac{dz}{z^2+z+1} = 2\pi i \operatorname{Res}\left(\frac{1}{z^2+z+1}; -\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right) \tag{1.9.10}$$

$$= 2\pi i \lim_{z \to -\frac{1}{2}+\frac{\sqrt{3}}{2}i} \left(\frac{1}{2z+1}\right) = \frac{2\pi}{\sqrt{3}}. \tag{1.9.11}$$

- **Example 1.9.3**

Let us evaluate

$$\int_0^{\infty} \frac{dx}{x^6+1}. \tag{1.9.12}$$

In place of an infinite semicircle in the upper half-plane, consider the following integral

$$\oint_C \frac{dz}{z^6+1}, \tag{1.9.13}$$

where we show the closed contour in Figure 1.9.2. We chose this contour for two reasons. First, we only have to evaluate one residue rather than the three enclosed in a traditional upper half-plane contour. Second, the contour integral along C_3 simplifies to a particularly simple and useful form.

Because the only enclosed singularity lies at $z = e^{\pi i/6}$,

$$\oint_C \frac{dz}{z^6+1} = 2\pi i \operatorname{Res}\left(\frac{1}{z^6+1}; e^{\pi i/6}\right) = 2\pi i \lim_{z \to e^{\pi i/6}} \frac{z-e^{\pi i/6}}{z^6+1} \tag{1.9.14}$$

$$= 2\pi i \lim_{z \to e^{\pi i/6}} \frac{1}{6z^5} = -\frac{\pi i}{3} e^{\pi i/6}. \tag{1.9.15}$$

Let us now evaluate (1.9.12) along each of the legs of the contour:

$$\int_{C_1} \frac{dz}{z^6+1} = \int_0^{\infty} \frac{dx}{x^6+1}, \tag{1.9.16}$$

$$\int_{C_2} \frac{dz}{z^6+1} = 0, \tag{1.9.17}$$

because of (1.9.7) and

$$\int_{C_3} \frac{dz}{z^6+1} = \int_{\infty}^{0} \frac{e^{\pi i/3}\, dr}{r^6+1} = -e^{\pi i/3} \int_0^{\infty} \frac{dx}{x^6+1}, \tag{1.9.18}$$

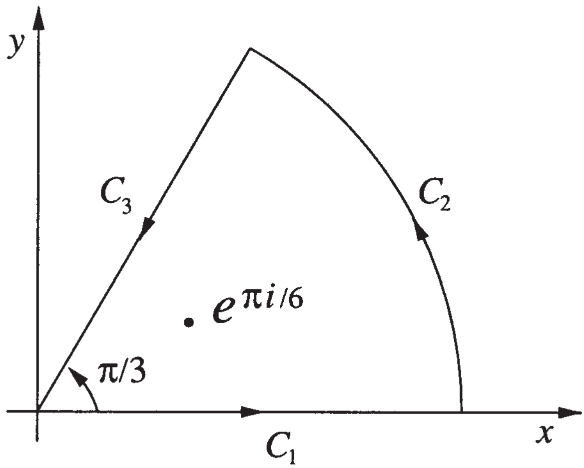


Figure 1.9.2: Contour used in evaluating the integral (1.9.13).

since $z = re^{\pi i/3}$.

Substituting into (1.9.15),

$$\left(1 - e^{\pi i/3}\right) \int_0^\infty \frac{dx}{x^6+1} = -\frac{\pi i}{3} e^{\pi i/6} \tag{1.9.19}$$

or

$$\int_0^\infty \frac{dx}{x^6+1} = \frac{\pi i}{6} \frac{2ie^{\pi i/6}}{e^{\pi i/6}\left(e^{\pi i/6} - e^{-\pi i/6}\right)} = \frac{\pi}{6\sin(\pi/6)} = \frac{\pi}{3}. \tag{1.9.20}$$

Problems

Use the residue theorem to verify the following integral:

1.

$$\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi\sqrt{2}}{4}$$

2.

$$\int_{-\infty}^\infty \frac{dx}{(x^2+4x+5)^2} = \frac{\pi}{2}$$

3.

$$\int_{-\infty}^\infty \frac{x\,dx}{(x^2+1)(x^2+2x+2)} = -\frac{\pi}{5}$$

4.

$$\int_0^\infty \frac{x^2}{x^6+1}\,dx = \frac{\pi}{6}$$

5.
$$\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$$

6.
$$\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)^2} = \frac{5\pi}{288}$$

7.
$$\int_0^\infty \frac{t^2}{(t^2+1)[t^2(a/h+1)+(a/h-1)]}\,dt = \frac{\pi}{4}\left[1-\sqrt{\frac{1-h/a}{1+h/a}}\right],$$

where $h/a < 1$.

8. During an electromagnetic calculation, Strutt[8] needed to prove that

$$\pi\frac{\sinh(\sigma x)}{\cosh(\sigma\pi)} = 2\sigma\sum_{n=0}^{\infty}\frac{\cos\left[\left(n+\frac{1}{2}\right)(x-\pi)\right]}{\sigma^2+\left(n+\frac{1}{2}\right)^2}, \qquad |x| \le \pi.$$

Verify his proof.

Step 1: Using the residue theorem, show that

$$\frac{1}{2\pi i}\oint_{C_N} \pi\frac{\sinh(xz)}{\cosh(\pi z)}\frac{dz}{z-\sigma} = \pi\frac{\sinh(\sigma x)}{\cosh(\sigma\pi)} - \sum_{n=-N-1}^{N}\frac{(-1)^n\sin\left[\left(n+\frac{1}{2}\right)x\right]}{\sigma - i\left(n+\frac{1}{2}\right)},$$

where C_N is a circular contour that includes the poles $z=\sigma$ and $z_n = \pm i\left(n+\frac{1}{2}\right)$, $n = 0, 1, 2, \ldots, N$.

Step 2: Show that in the limit of $N \to \infty$, the contour integral vanishes. [Hint: Examine the behavior of $z\sinh(xz)/[(z-\sigma)\cosh(\pi z)]$ as $|z| \to \infty$. Use (1.9.7) where C_R is the circular contour.]

Step 3: Break the infinite series in Step 1 into two parts and simplify.

In the next chapter we shall show how we can obtain the same series by direct integration.

[8] Strutt, M. J. O., 1934: Berechnung des hochfrequenten Feldes einer Kreiszylinderspule in einer konzentrischen leitenden Schirmhülle mit ebenen Deckeln. *Hochfrequenztechn. Elecktroak.*, **43**, 121–123.

Chapter 2
Fourier Series

Fourier series arose during the eighteenth century as a formal solution to the classic wave equation. Later on, it was used to describe physical processes in which events recur in a regular pattern. For example, a musical note usually consists of a simple note, called the fundamental, and a series of auxiliary vibrations, called overtones. Fourier's theorem provides the mathematical language which allows us to precisely describe this complex structure.

2.1 FOURIER SERIES

One of the crowning glories[1] of nineteenth century mathematics

[1] "Fourier's Theorem ... is not only one of the most beautiful results of modern analysis, but may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics. To mention only sonorous vibrations, the propagation of electric signals along a telegraph wire, and the conduction of heat by the earth's crust, as subjects in their generality intractable without it, is to give but a feeble idea of its importance." (Quote taken from Thomson, W. and Tait, P. G., 1879: *Treatise on Natural Philosophy, Part I,* Cambridge University Press, Cambridge, Section 75.)

was the discovery that the infinite series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \tag{2.1.1}$$

can represent a function $f(t)$ under certain general conditions. This series, called a *Fourier series*, converges to the value of the function $f(t)$ at every point in the interval $[-L, L]$ with the possible exceptions of the points at any discontinuities and the endpoints of the interval. Because each term has a period of $2L$, the sum of the series also has the same period. The *fundamental* of the periodic function $f(t)$ is the $n = 1$ term while the *harmonics* are the remaining terms whose frequencies are integer multiples of the fundamental.

We must now find some easy method for computing the a_n's and b_n's for a given function $f(t)$. As a first attempt, we integrate (2.1.1) term by term[2] from $-L$ to L. On the right side, all of the integrals multiplied by a_n and b_n vanish because the average of $\cos(n\pi t/L)$ and $\sin(n\pi t/L)$ is zero. Therefore, we are left with

$$\boxed{a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\, dt.} \tag{2.1.2}$$

Consequently a_0 is twice the mean value of $f(t)$ over one period.

We next multiply each side of (2.1.1) by $\cos(m\pi t/L)$, where m is a fixed integer. Integrating from $-L$ to L,

$$\begin{aligned} \int_{-L}^{L} f(t)\cos\left(\frac{m\pi t}{L}\right) dt &= \frac{a_0}{2}\int_{-L}^{L} \cos\left(\frac{m\pi t}{L}\right) dt \\ &+ \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos\left(\frac{n\pi t}{L}\right) \cos\left(\frac{m\pi t}{L}\right) dt \\ &+ \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin\left(\frac{n\pi t}{L}\right) \cos\left(\frac{m\pi t}{L}\right) dt. \end{aligned} \tag{2.1.3}$$

The a_0 and b_n terms vanish by direct integration. Finally all of the a_n

---

[2] We assume that the integration of the series can be carried out term by term. This is sometimes difficult to justify but we do it anyway.

integrals vanish when $n \neq m$. Consequently, (2.1.3) simplifies to

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right) dt, \tag{2.1.4}$$

because $\int_{-L}^{L} \cos^2(n\pi t/L)\, dt = L$. Finally, by multiplying both sides of (2.1.1) by $\sin(m\pi t/L)$ (m is again a fixed integer) and integrating from $-L$ to L,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right) dt. \tag{2.1.5}$$

Although (2.1.2), (2.1.4), and (2.1.5) give us a_0, a_n, and b_n for periodic functions over the interval $[-L, L]$, in certain situations it is convenient to use the interval $[\tau, \tau + 2L]$, where τ is any real number. In that case, (2.1.1) still gives the Fourier series of $f(t)$ and

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{\tau}^{\tau+2L} f(t)\, dt, \\ a_n &= \frac{1}{L}\int_{\tau}^{\tau+2L} f(t)\cos\left(\frac{n\pi t}{L}\right) dt, \\ b_n &= \frac{1}{L}\int_{\tau}^{\tau+2L} f(t)\sin\left(\frac{n\pi t}{L}\right) dt. \end{aligned} \tag{2.1.6}$$

These results follow when we recall that the function $f(t)$ is a periodic function that extends from minus infinity to plus infinity. The results must remain unchanged, therefore, when we shift from the interval $[-L, L]$ to the new interval $[\tau, \tau + 2L]$.

We now ask the question: what types of functions have Fourier series? Secondly, if a function is discontinuous at a point, what value will the Fourier series give? Dirichlet[3,4] answered these questions in the first half of the nineteenth century. He showed that if any arbitrary function is finite over one period and has a finite number of maxima and minima, then the Fourier series converges. If $f(t)$ is discontinuous at the point t and has two different values at $f(t^-)$ and $f(t^+)$, where t^+ and t^- are points infinitesimally to the right and left of t, the Fourier series converges to the mean value of $[f(t^+) + f(t^-)]/2$. Because *Dirichlet's conditions* are very mild, it is very rare that a convergent Fourier series does not exist for a function that appears in an engineering or scientific problem.

• Example 2.1.1

Let us find the Fourier series for the function

$$f(t) = \begin{cases} 0, & -\pi < t \le 0 \\ t, & 0 \le t < \pi. \end{cases} \tag{2.1.7}$$

We compute the Fourier coefficients a_n and b_n using (2.1.6) by letting $L = \pi$ and $\tau = -\pi$. We then find that

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_0^{\pi} t\,dt = \frac{\pi}{2}, \tag{2.1.8}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} t\cos(nt)\,dt = \frac{1}{\pi}\left[\frac{t\sin(nt)}{n} + \frac{\cos(nt)}{n^2}\right]\Bigg|_0^{\pi} \tag{2.1.9}$$

$$= \frac{\cos(n\pi) - 1}{n^2\pi} = \frac{(-1)^n - 1}{n^2\pi} \tag{2.1.10}$$

because $\cos(n\pi) = (-1)^n$ and

$$b_n = \frac{1}{\pi}\int_0^{\pi} t\sin(nt)\,dt = \frac{1}{\pi}\left[\frac{-t\cos(nt)}{n} + \frac{\sin(nt)}{n^2}\right]\Bigg|_0^{\pi} \tag{2.1.11}$$

$$= -\frac{\cos(n\pi)}{n} = \frac{(-1)^{n+1}}{n} \tag{2.1.12}$$

---

[3] Dirichlet, P. G. L., 1829: Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données. *J. reine angew. Math.*, **4**, 157–169.

[4] Dirichlet, P. G. L., 1837: Sur l'usage des intégrales définies dans la sommation des séries finies ou infinies. *J. reine angew. Math.*, **17**, 57–67.

Figure 2.1.1: A product of the French Revolution, (Jean Baptiste) Joseph Fourier (1768–1830) held positions within the Napoleonic Empire during his early career. After Napoleon's fall from power, Fourier devoted his talents exclusively to science. Although he won the Institut de France prize in 1811 for his work on heat diffusion, criticism of its mathematical rigor and generality led him to publish the classic book *Théorie analytique de la chaleur* in 1823. Within this book he introduced the world to the series that bears his name. (Portrait courtesy of the Archives de l'Académie des sciences, Paris.)

for $n = 1, 2, 3, \ldots$. Thus, the Fourier series for $f(t)$ is

$$f(t) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi} \cos(nt) + \frac{(-1)^{n+1}}{n} \sin(nt) \qquad \textbf{(2.1.13)}$$

$$= \frac{\pi}{4} - \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\cos[(2m-1)t]}{(2m-1)^2} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nt). \qquad \textbf{(2.1.14)}$$

Figure 2.1.2: Second to Gauss, Peter Gustav Lejeune Dirichlet (1805–1859) was Germany's leading mathematician during the first half of the nineteenth century. Initially drawn to number theory, his later studies in analysis and applied mathematics led him to consider the convergence of Fourier series. These studies eventually produced the modern concept of a function as a correspondence that associates with each real x in an interval some unique value denoted by $f(x)$. (Taken from the frontispiece of Dirichlet, P. G. L., 1889: *Werke.* Druck und Verlag von Georg Reimer, Berlin, 644 pp.)

We note that at the points $t = \pm(2n-1)\pi$, where $n = 1, 2, 3, \ldots$, the function jumps from zero to π. To what value does the Fourier series converge at these points? From Dirichlet's conditions, the series converges to the average of the values of the function just to the right and left of the point of discontinuity, i.e., $(\pi + 0)/2 = \pi/2$. At the remaining points the series converges to $f(t)$.

In Figure 2.1.3 we show how well (2.1.13) approximates the function by graphing various partial sums of (2.1.13) as we include more and more

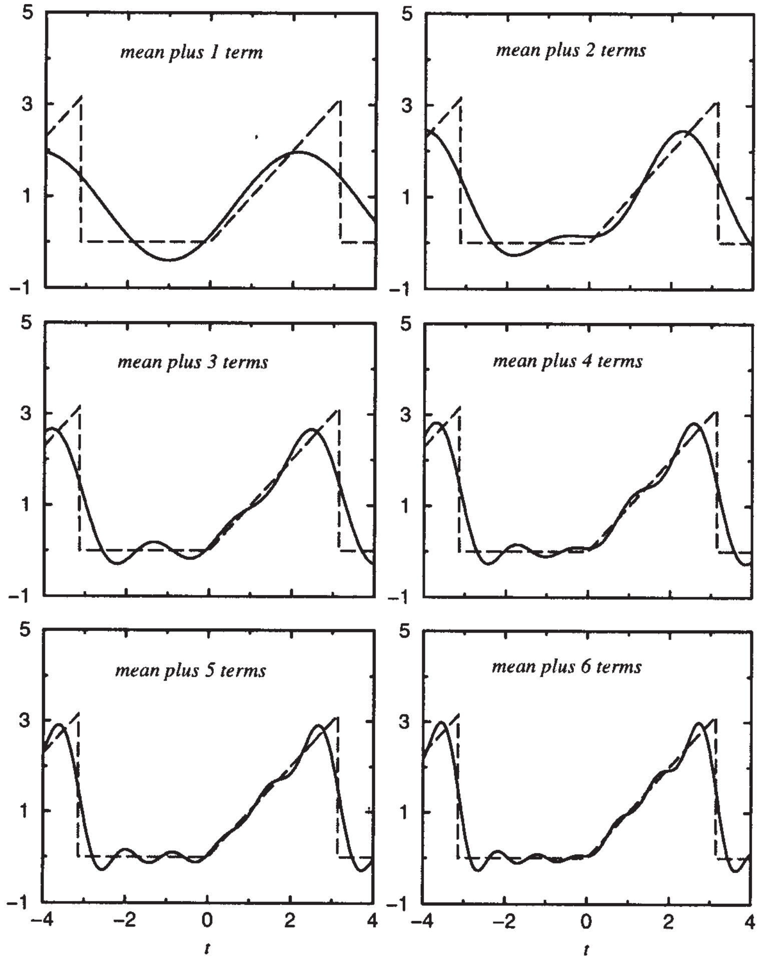


Figure 2.1.3: Partial sum of the Fourier series for (2.1.7).

terms (harmonics). As the figure shows, successive corrections are made to the mean value of the series, $\pi/2$. As each harmonic is added, the Fourier series fits the function better in the sense of least squares:

$$\int_{\tau}^{\tau+2L} [f(x) - f_N(x)]^2 \, dx = \text{ minimum}, \qquad \textbf{(2.1.15)}$$

where $f_N(x)$ is the truncated Fourier series of N terms.

• **Example 2.1.2**

Let us calculate the Fourier series of the function $f(t) = |t|$ which is defined over the range $-\pi \le t \le \pi$.

From the definition of the Fourier coefficients,

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -t\,dt + \int_0^{\pi} t\,dt\right] = \frac{\pi}{2} + \frac{\pi}{2} = \pi, \tag{2.1.16}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -t\cos(nt)\,dt + \int_0^{\pi} t\cos(nt)\,dt\right] \tag{2.1.17}$$

$$= -\left.\frac{nt\sin(nt)+\cos(nt)}{n^2\pi}\right|_{-\pi}^{0} + \left.\frac{nt\sin(nt)+\cos(nt)}{n^2\pi}\right|_{0}^{\pi} \tag{2.1.18}$$

$$= \frac{2}{n^2\pi}[(-1)^n - 1] \tag{2.1.19}$$

and

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -t\sin(nt)\,dt + \int_0^{\pi} t\sin(nt)\,dt\right] \tag{2.1.20}$$

$$= \left.\frac{nt\cos(nt)-\sin(nt)}{n^2\pi}\right|_{-\pi}^{0} - \left.\frac{nt\cos(nt)-\sin(nt)}{n^2\pi}\right|_{0}^{\pi} = 0 \tag{2.1.21}$$

for $n = 1, 2, 3, \ldots$. Therefore,

$$|t| = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n - 1]}{n^2}\cos(nt) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos[(2m-1)t]}{(2m-1)^2} \tag{2.1.22}$$

for $-\pi \le t \le \pi$.

In Figure 2.1.4 we show how well (2.1.22) approximates the function by graphing various partial sums of (2.1.22). As the figure shows, the Fourier series does very well even when we use very few terms. The reason for this rapid convergence is the nature of the function: it does not possess any jump discontinuities.

• **Example 2.1.3**

Sometimes the function $f(t)$ is an even or odd function.[5] Can we use this property to simplify our work? The answer is yes.

Let $f(t)$ be an even function. Then

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt = \frac{2}{L}\int_0^{L} f(t)\,dt \tag{2.1.23}$$

[5] An even function $f_e(t)$ has the property that $f_e(-t) = f_e(t)$; an odd function $f_o(t)$ has the property that $f_o(-t) = -f_o(t)$.

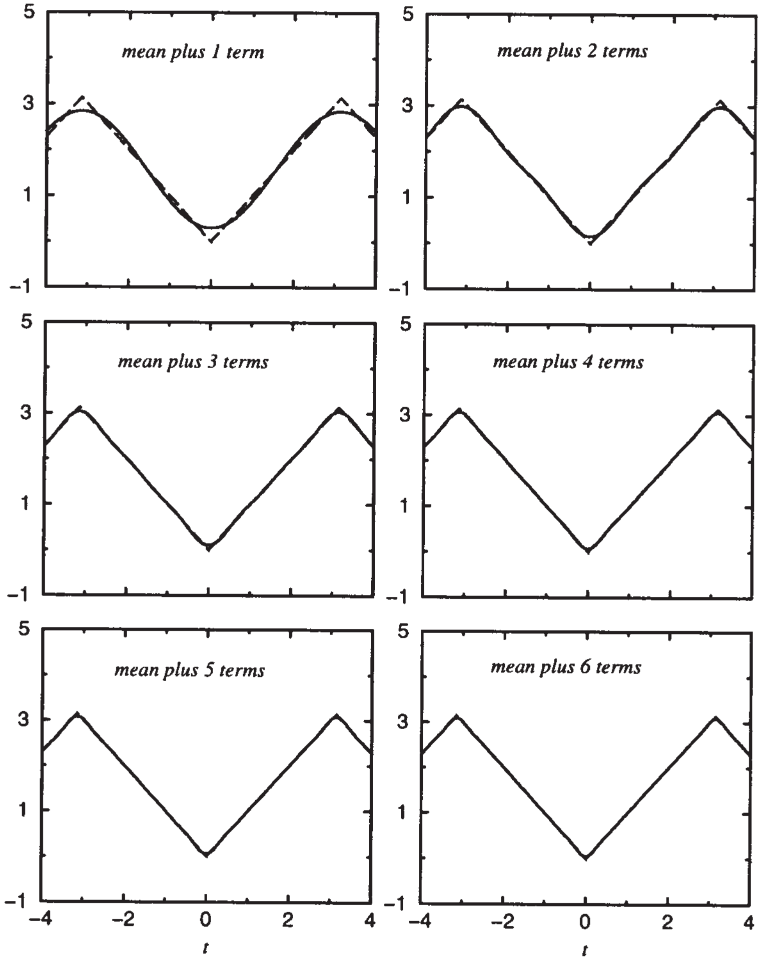


Figure 2.1.4: Partial sum of the Fourier series for $f(t) = |t|$.

and

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)\,dt = \frac{2}{L}\int_{0}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)\,dt \quad \textbf{(2.1.24)}$$

whereas

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)\,dt = 0. \quad \textbf{(2.1.25)}$$

Here we have used the properties that $\int_{-L}^{L} f_e(x)\,dx = 2\int_0^L f_e(x)\,dx$ and $\int_{-L}^{L} f_o(x)\,dx = 0$. Thus, if we have an even function, we merely compute a_0 and a_n via (2.1.23)–(2.1.24) and $b_n = 0$. Because the corresponding series contains only cosine terms, it is often called a *Fourier cosine series.*

Similarly, if $f(t)$ is odd, then

$$a_0 = a_n = 0 \qquad \text{and} \qquad b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)\,dt. \tag{2.1.26}$$

Thus, if we have an odd function, we merely compute b_n via (2.1.26) and $a_0 = a_n = 0$. Because corresponding series contains only sine terms, it is often called a *Fourier sine series.*

• Example 2.1.4

In the case when $f(x)$ consists of a constant and/or trigonometric functions, it is much easier to find the corresponding Fourier series by inspection rather than by using (2.1.6). For example, let us find the Fourier series for $f(x) = \sin^2(x)$ defined over the range $-\pi \le x \le \pi$.

We begin by rewriting $f(x) = \sin^2(x)$ as $f(x) = \frac{1}{2}[1 - \cos(2x)]$. Next, we note that any function defined over the range $-\pi < x < \pi$ has the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx) \tag{2.1.27}$$

$$= \frac{a_0}{2} + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \cdots \tag{2.1.28}$$

On the other hand,

$$f(x) = \tfrac{1}{2} - \tfrac{1}{2}\cos(2x) \tag{2.1.29}$$

$$= \tfrac{1}{2} + 0\cos(x) + 0\sin(x) - \tfrac{1}{2}\cos(2x) + 0\sin(2x) + \cdots \tag{2.1.30}$$

Consequently, by inspection, we can immediately write that

$$a_0 = 1, a_1 = b_1 = 0, a_2 = -\tfrac{1}{2}, b_2 = 0, a_n = b_n = 0, n \ge 3. \tag{2.1.31}$$

Thus, instead of the usual expansion involving an infinite number of sine and cosine terms, our Fourier series contains only two terms and is simply

$$f(x) = \tfrac{1}{2} - \tfrac{1}{2}\cos(2x), \qquad -\pi \le x \le \pi. \tag{2.1.32}$$

• Example 2.1.5: Quieting snow tires

An application of Fourier series to a problem in industry occurred several years ago, when drivers found that snow tires produced a loud whine[6] on dry pavement. Tire sounds are produced primarily by the dynamic interaction of the tread elements with the road surface.[7] As each tread element passes through the contact patch, it contributes a pulse of acoustic energy to the total sound field radiated by the tire.

For evenly spaced treads we envision that the release of acoustic energy resembles the top of Figure 2.1.5. If we perform a Fourier analysis of this distribution, we find that

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi/2-\epsilon}^{-\pi/2+\epsilon} 1\,dt + \int_{\pi/2-\epsilon}^{\pi/2+\epsilon} 1\,dt\right] = \frac{4\epsilon}{\pi}, \tag{2.1.33}$$

where ϵ is half of the width of the tread and

$$a_n = \frac{1}{\pi}\left[\int_{-\pi/2-\epsilon}^{-\pi/2+\epsilon} \cos(nt)\,dt + \int_{\pi/2-\epsilon}^{\pi/2+\epsilon} \cos(nt)\,dt\right] \tag{2.1.34}$$

$$= \frac{1}{n\pi}\left[\sin(nt)\Big|_{-\pi/2-\epsilon}^{-\pi/2+\epsilon} + \sin(nt)\Big|_{\pi/2-\epsilon}^{\pi/2+\epsilon}\right] \tag{2.1.35}$$

$$= \frac{1}{n\pi}\left[\sin\left(-\frac{n\pi}{2}+n\epsilon\right) - \sin\left(-\frac{n\pi}{2}-n\epsilon\right)\right.$$

$$\left. + \sin\left(\frac{n\pi}{2}+n\epsilon\right) - \sin\left(\frac{n\pi}{2}-n\epsilon\right)\right] \tag{2.1.36}$$

$$= \frac{1}{n\pi}\left[2\cos\left(-\frac{n\pi}{2}\right) + 2\cos\left(\frac{n\pi}{2}\right)\right]\sin(n\epsilon) \tag{2.1.37}$$

$$= \frac{4}{n\pi}\cos\left(\frac{n\pi}{2}\right)\sin(n\epsilon). \tag{2.1.38}$$

Because $f(t)$ is an even function, $b_n = 0$.

The question now arises of how to best illustrate our Fourier coefficients. In Section 2.4 we will show that any harmonic can be represented as a single wave $A_n \cos(n\pi t/L + \varphi_n)$ or $A_n \sin(n\pi t/L + \psi_n)$, where the amplitude $A_n = \sqrt{a_n^2 + b_n^2}$. At the bottom of Figure 2.1.5, we have plotted this amplitude, usually called the *amplitude* or *frequency spectrum* $\frac{1}{2}\sqrt{a_n^2 + b_n^2}$, as a function of n for an arbitrarily chosen $\epsilon = \pi/12$. Although the value of ϵ will affect the exact shape of the spectrum, the qualitative arguments that we will present remain unchanged. We have added the factor $\frac{1}{2}$ so that our definition of the frequency spectrum is consistent with that for a complex Fourier series stated after (2.5.15). The amplitude spectrum in Figure 2.1.5 shows that the spectrum for periodically placed tire treads has its largest amplitude at small

---

[6] Information based on Varterasian, J. H., 1969: Math quiets rotating machines. *SAE J.*, **77(10)**, 53.

[7] Willett, P. R., 1975: Tire tread pattern sound generation. *Tire Sci. Tech.*, **3**, 252–266.

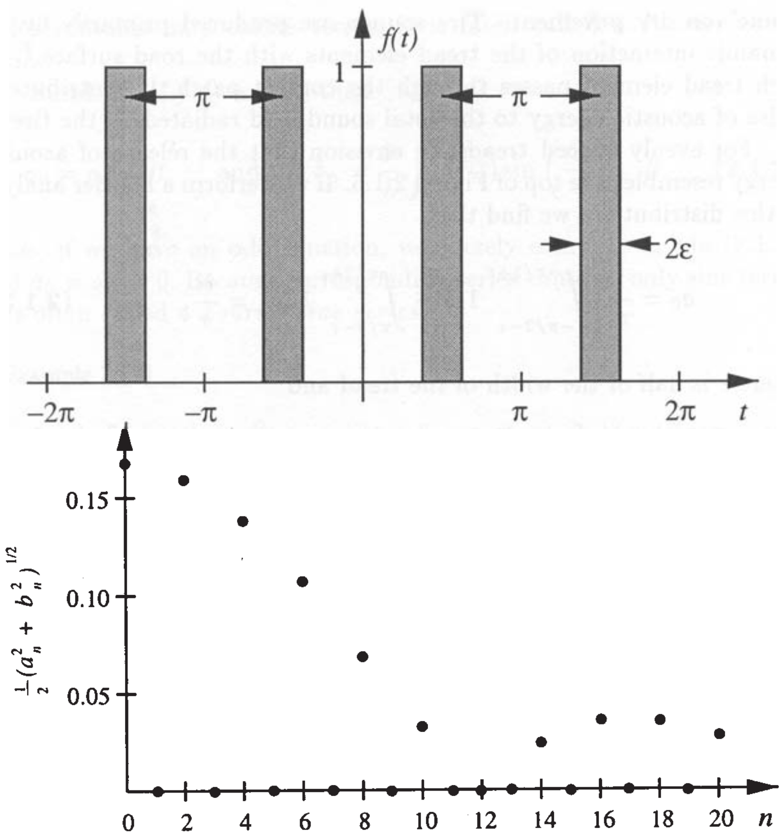


Figure 2.1.5: Temporal spacing (over two periods) and frequency spectrum of a uniformly spaced snow tire.

n. This produces one loud tone plus strong harmonic overtones because the $n = 1$ term (the fundamental) and its overtones are the dominant terms in the Fourier series representation.

Clearly this loud, monotone whine is undesirable. How might we avoid it? Just as soldiers marching in step produce a loud uniform sound, we suspect that our uniform tread pattern is the problem. Therefore, let us now vary the interval between the treads so that the distance between any tread and its nearest neighbor is not equal. Figure 2.1.6 illustrates a simple example. Again we perform a Fourier analysis and obtain that

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi/2-\epsilon}^{-\pi/2+\epsilon} 1\,dt + \int_{\pi/4-\epsilon}^{\pi/4+\epsilon} 1\,dt\right] = \frac{4\epsilon}{\pi}, \qquad \textbf{(2.1.39)}$$

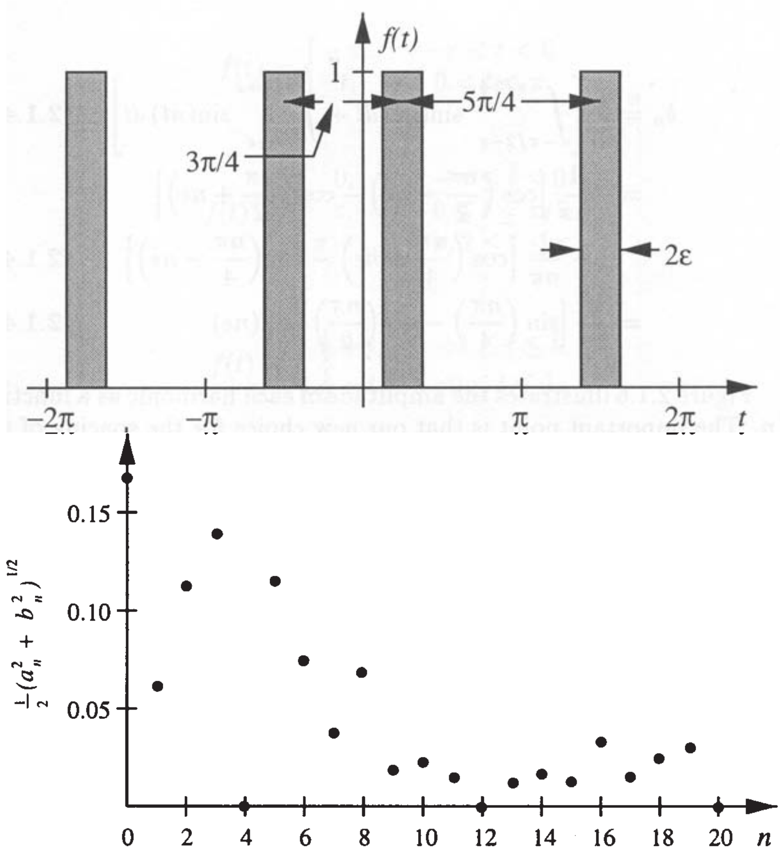


Figure 2.1.6: Temporal spacing and frequency spectrum of a nonuniformly spaced snow tire.

$$a_n = \frac{1}{\pi}\left[\int_{-\pi/2-\epsilon}^{-\pi/2+\epsilon} \cos(nt)\,dt + \int_{\pi/4-\epsilon}^{\pi/4+\epsilon} \cos(nt)\,dt\right] \tag{2.1.40}$$

$$= \frac{1}{n\pi}\sin(nt)\Big|_{-\pi/2-\epsilon}^{-\pi/2+\epsilon} + \frac{1}{n\pi}\sin(nt)\Big|_{\pi/4-\epsilon}^{\pi/4+\epsilon} \tag{2.1.41}$$

$$= -\frac{1}{n\pi}\left[\sin\left(\frac{n\pi}{2} - n\epsilon\right) - \sin\left(\frac{n\pi}{2} + n\epsilon\right)\right] + \frac{1}{n\pi}\left[\sin\left(\frac{n\pi}{4} + n\epsilon\right) - \sin\left(\frac{n\pi}{4} - n\epsilon\right)\right] \tag{2.1.42}$$

$$= \frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) + \cos\left(\frac{n\pi}{4}\right)\right]\sin(n\epsilon) \tag{2.1.43}$$

and

$$b_n = \frac{1}{\pi}\left[\int_{-\pi/2-\epsilon}^{-\pi/2+\epsilon} \sin(nt)\,dt + \int_{\pi/4-\epsilon}^{\pi/4+\epsilon} \sin(nt)\,dt\right] \tag{2.1.44}$$

$$= -\frac{1}{n\pi}\left[\cos\left(\frac{n\pi}{2} - n\epsilon\right) - \cos\left(\frac{n\pi}{2} + n\epsilon\right)\right]$$

$$-\frac{1}{n\pi}\left[\cos\left(\frac{n\pi}{4} + n\epsilon\right) - \cos\left(\frac{n\pi}{4} - n\epsilon\right)\right] \tag{2.1.45}$$

$$= \frac{2}{n\pi}\left[\sin\left(\frac{n\pi}{4}\right) - \sin\left(\frac{n\pi}{2}\right)\right]\sin(n\epsilon). \tag{2.1.46}$$

Figure 2.1.6 illustrates the amplitude of each harmonic as a function of n. The important point is that our new choice for the spacing of the treads has reduced or eliminated some of the harmonics compared to the case of equally spaced treads. On the negative side we have excited some of the harmonics that were previously absent. However, the net effect is advantageous because the treads produce less noise at more frequencies rather than a lot of noise at a few select frequencies.

If we were to extend this technique so that the treads occurred at completely random positions, then the treads would produce very little noise at many frequencies and the total noise would be comparable to that generated by other sources within the car. To find the distribution of treads with the whitest noise[8] is a process of trial and error. Assuming a distribution, we can perform a Fourier analysis to obtain the frequency spectrum. If annoying peaks are present in the spectrum, we can then adjust the elements in the tread distribution that may contribute to the peak and analyze the revised distribution. You are finished when no peaks appear.

Problems

Find the Fourier series for the following functions. Plot various partial sums and compare them against the exact function.

1.

$$f(t) = \begin{cases} 1, & -\pi < t < 0 \\ 0, & 0 < t < \pi \end{cases}$$

2.

$$f(t) = \begin{cases} t, & -\pi < t < 0 \\ 0, & 0 < t < \pi \end{cases}$$

[8] White noise is sound that is analogous to white light in that it is uniformly distributed throughout the complete audible sound spectrum.

3.

$$f(t) = \begin{cases} -\pi, & -\pi < t < 0 \\ t, & 0 < t < \pi \end{cases}$$

4.

$$f(t) = \begin{cases} 0, & -\pi \le t \le 0 \\ t, & 0 \le t \le \pi/2 \\ \pi - t, & \pi/2 \le t \le \pi \end{cases}$$

5.

$$f(t) = \begin{cases} \frac{1}{2} + t, & -1 \le t \le 0 \\ \frac{1}{2} - t, & 0 \le t \le 1 \end{cases}$$

6.

$$f(t) = e^{at}, \qquad -L < t < L$$

7.

$$f(t) = \begin{cases} 0, & -\pi \le t \le 0 \\ \sin(t), & 0 \le t \le \pi \end{cases}$$

8.

$$f(t) = t + t^2, \qquad -L < t < L$$

9.

$$f(t) = \begin{cases} t, & -\frac{1}{2} \le t \le \frac{1}{2} \\ 1 - t, & \frac{1}{2} \le t \le \frac{3}{2} \end{cases}$$

10.

$$f(t) = \begin{cases} 0, & -\pi \le t \le -\pi/2 \\ \sin(2t), & -\pi/2 \le t \le \pi/2 \\ 0, & \pi/2 \le t \le \pi \end{cases}$$

11.

$$f(t) = \begin{cases} 0, & -a < t < 0 \\ 2t, & 0 < t < a \end{cases}$$

12.

$$f(t) = \frac{\pi - t}{2}, \qquad 0 < t < 2$$

13.

$$f(t) = t \cos\left(\frac{\pi t}{L}\right), \qquad -L < t < L$$

14.

$$f(t) = \begin{cases} 0, & -\pi < t \le 0 \\ t^2, & 0 \le t < \pi \end{cases}$$

15.

$$f(t) = \sinh\left[a\left(\frac{\pi}{2} - |t|\right)\right], \qquad -\pi \le t \le \pi$$

2.2 PROPERTIES OF FOURIER SERIES

In the previous section we introduced the Fourier series and showed how to compute one given the function $f(t)$. In this section we examine some particular properties of these series.

Differentiation of a Fourier series

In certain instances we only have the Fourier series representation of a function $f(t)$. Can we find the derivative or the integral of $f(t)$ merely by differentiating or integrating the Fourier series term by term? Is this permitted? Let us consider the case of differentiation first.

Consider a function $f(t)$ of period $2L$ which has the derivative $f'(t)$. Let us assume that we can expand $f'(t)$ as a Fourier series. This implies that $f'(t)$ is continuous except for a finite number of discontinuities and $f(t)$ is continuous over an interval that starts at $t = \tau$ and ends at $t = \tau + 2L$. Then

$$f'(t) = \frac{a_0'}{2} + \sum_{n=1}^{\infty} a_n' \cos\left(\frac{n\pi t}{L}\right) + b_n' \sin\left(\frac{n\pi t}{L}\right), \tag{2.2.1}$$

where we have denoted the Fourier coefficients of $f'(t)$ with a prime. Computing the Fourier coefficients,

$$a_0' = \frac{1}{L}\int_{\tau}^{\tau+2L} f'(t)\, dt = \frac{1}{L}\left[f(\tau + 2L) - f(\tau)\right] = 0, \tag{2.2.2}$$

if $f(\tau + 2L) = f(\tau)$. Similarly, by integrating by parts,

$$a_n' = \frac{1}{L}\int_{\tau}^{\tau+2L} f'(t)\cos\left(\frac{n\pi t}{L}\right) dt \tag{2.2.3}$$

$$= \frac{1}{L}\left[f(t)\cos\left(\frac{n\pi t}{L}\right)\right]\Bigg|_{\tau}^{\tau+2L} + \frac{n\pi}{L^2}\int_{\tau}^{\tau+2L} f(t)\sin\left(\frac{n\pi t}{L}\right) dt \tag{2.2.4}$$

$$= \frac{n\pi b_n}{L} \tag{2.2.5}$$

and

$$b_n' = \frac{1}{L}\int_\tau^{\tau+2L} f'(t)\sin\left(\frac{n\pi t}{L}\right)\,dt \tag{2.2.6}$$

$$= \frac{1}{L}\left[f(t)\sin\left(\frac{n\pi t}{L}\right)\right]\Bigg|_\tau^{\tau+2L} - \frac{n\pi}{L^2}\int_\tau^{\tau+2L} f(t)\cos\left(\frac{n\pi t}{L}\right)\,dt \tag{2.2.7}$$

$$= -\frac{n\pi a_n}{L}. \tag{2.2.8}$$

Consequently, if we have a function $f(t)$ whose derivative $f'(t)$ is continuous except for a finite number of discontinuities and $f(\tau) = f(\tau + 2L)$, then

$$f'(t) = \sum_{n=1}^{\infty} \frac{n\pi}{L}\left[b_n \cos\left(\frac{n\pi t}{L}\right) - a_n \sin\left(\frac{n\pi t}{L}\right)\right]. \tag{2.2.9}$$

That is, the derivative of $f(t)$ is given by a term-by-term differentiation of the Fourier series of $f(t)$.

• **Example 2.2.1**

The Fourier series for the function

$$f(t) = \begin{cases} 0, & -\pi \le t \le 0 \\ t, & 0 \le t \le \pi/2 \\ \pi - t, & \pi/2 \le t \le \pi \end{cases} \tag{2.2.10}$$

is

$$f(t) = \frac{\pi}{8} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos[2(2n-1)t]}{(2n-1)^2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin[(2n-1)t]. \tag{2.1.11}$$

Because $f(t)$ is continuous over the entire interval $(-\pi, \pi)$ and $f(-\pi) = f(\pi) = 0$, we can find $f'(t)$ by taking the derivative of (2.2.11) term by term:

$$f'(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin[2(2n-1)t]}{2n-1} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos[(2n-1)t]. \tag{2.2.12}$$

This is the Fourier series that we would have obtained by computing the Fourier series for

$$f'(t) = \begin{cases} 0, & -\pi < t < 0 \\ 1, & 0 < t < \pi/2 \\ -1, & \pi/2 < t < \pi. \end{cases} \tag{2.2.13}$$

Integration of a Fourier series

To determine whether we can find the integral of $f(t)$ by term-by-term integration of its Fourier series, consider a form of the antiderivative of $f(t)$:

$$F(t) = \int_0^t \left[f(\tau) - \frac{a_0}{2} \right] d\tau. \tag{2.2.14}$$

Now

$$F(t+2L) = \int_0^t \left[f(\tau) - \frac{a_0}{2} \right] d\tau + \int_t^{t+2L} \left[f(\tau) - \frac{a_0}{2} \right] d\tau \tag{2.2.15}$$

$$= F(t) + \int_{-L}^{L} \left[f(\tau) - \frac{a_0}{2} \right] d\tau \tag{2.2.16}$$

$$= F(t) + \int_{-L}^{L} f(\tau)\, d\tau - La_0 = F(t), \tag{2.2.17}$$

so that $F(t)$ has a period of $2L$. Consequently we may expand $F(t)$ as the Fourier series

$$F(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi t}{L}\right) + B_n \sin\left(\frac{n\pi t}{L}\right). \tag{2.2.18}$$

For A_n,

$$A_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right) dt \tag{2.2.19}$$

$$= \frac{1}{L} \left[F(t) \frac{\sin(n\pi t/L)}{n\pi/L} \right] \Bigg|_{-L}^{L} - \frac{1}{n\pi} \int_{-L}^{L} \left[f(t) - \frac{a_0}{2} \right] \sin\left(\frac{n\pi t}{L}\right) dt \tag{2.2.20}$$

$$= -\frac{b_n}{n\pi/L}. \tag{2.2.21}$$

Similarly,

$$B_n = \frac{a_n}{n\pi/L}. \tag{2.2.22}$$

Therefore,

$$\int_0^t f(\tau)\, d\tau = \frac{a_0 t}{2} + \frac{A_0}{2} + \sum_{n=1}^{\infty} \frac{a_n \sin(n\pi t/L) - b_n \cos(n\pi t/L)}{n\pi/L}. \tag{2.2.23}$$

This is identical to a term-by-term integration of the Fourier series for $f(t)$. Thus, we can always find the integral of $f(t)$ by a term-by-term integration of its Fourier series.

• **Example 2.2.2**

The Fourier series for $f(t) = t$ for $-\pi < t < \pi$ is

$$f(t) = -2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nt). \tag{2.2.24}$$

To find the Fourier series for $f(t) = t^2$, we integrate (2.2.24) term by term and find that

$$\frac{\tau^2}{2}\bigg|_0^t = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nt) - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}. \tag{2.2.25}$$

But $\sum_{n=1}^{\infty}(-1)^n/n^2 = -\pi^2/12$. Substituting and multiplying by 2, we obtain the final result that

$$t^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nt). \tag{2.2.26}$$

Parseval's equality

One of the fundamental quantities in engineering is power. The *power content* of a periodic signal $f(t)$ of period $2L$ is $\int_\tau^{\tau+2L} f^2(t)\,dt/L$. This mathematical definition mirrors the power dissipation I^2R that occurs in a resistor of resistance R where I is the root mean square (RMS) of the current. We would like to compute this power content as simply as possible given the coefficients of its Fourier series.

Assume that $f(t)$ has the Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right). \tag{2.2.27}$$

Then,

$$\begin{aligned}\frac{1}{L}\int_\tau^{\tau+2L} f^2(t)\,dt = {} & \frac{a_0}{2L}\int_\tau^{\tau+2L} f(t)\,dt \\ & + \sum_{n=1}^{\infty} \frac{a_n}{L}\int_\tau^{\tau+2L} f(t)\cos\left(\frac{n\pi t}{L}\right) dt \\ & + \sum_{n=1}^{\infty} \frac{b_n}{L}\int_\tau^{\tau+2L} f(t)\sin\left(\frac{n\pi t}{L}\right) dt \qquad (2.2.28) \\ = {} & \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2). \qquad (2.2.29)\end{aligned}$$

Equation (2.2.29) is *Parseval's equality.*[9] It allows us to sum squares of Fourier coefficients (which we have already computed) rather than performing the integration $\int_{\tau}^{\tau+2L} f^2(t)\,dt$ analytically or numerically.

• **Example 2.2.3**

The Fourier series for $f(t) = t^2$ over the interval $[-\pi, \pi]$ is

$$t^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nt). \tag{2.2.30}$$

Then, by Parseval's equality,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} t^4\,dt = \left.\frac{2t^5}{5\pi}\right|_0^{\pi} = \frac{4\pi^4}{18} + 16\sum_{n=1}^{\infty}\frac{1}{n^4} \tag{2.2.31}$$

$$\left(\frac{2}{5} - \frac{4}{18}\right)\pi^4 = 16\sum_{n=1}^{\infty}\frac{1}{n^4} \tag{2.2.32}$$

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty}\frac{1}{n^4}. \tag{2.2.33}$$

Gibbs phenomena

In the actual application of Fourier series, we cannot sum an infinite number of terms but must be content with N terms. If we denote this partial sum of the Fourier series by $S_N(t)$, we have from the definition of the Fourier series:

$$S_N(t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{N} a_n \cos(nt) + b_n \sin(nt) \tag{2.2.34}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

[9] Parseval, M.-A., 1805: Mémoire sur les séries et sur l'intégration complète d'une équation aux différences partielles linéaires du second ordre, à coefficients constants. *Mémoires présentés a l'Institut des sciences, lettres et arts, par divers savans, et lus dans ses assemblées: Sciences mathématiques et Physiques*, **1**, 638–648.

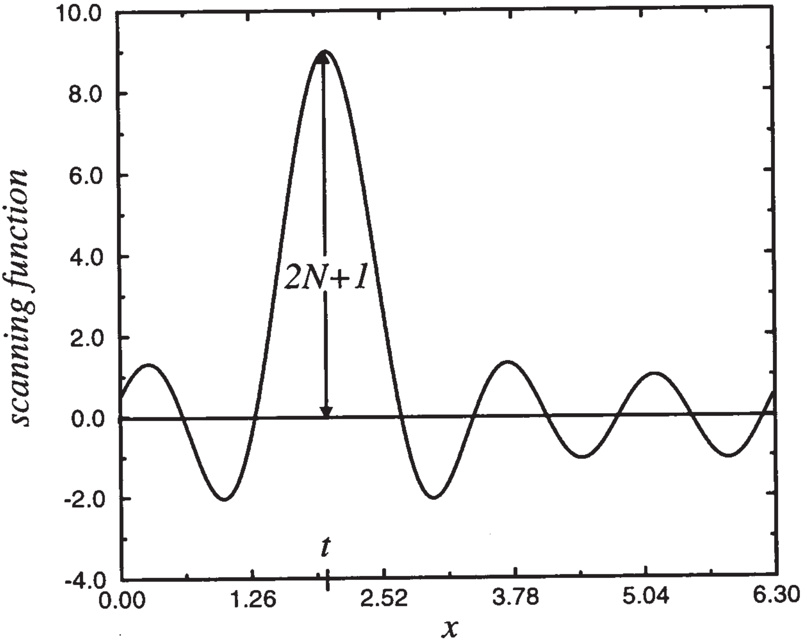


Figure 2.2.1: The scanning function over $0 \le x \le 2\pi$ for $N = 5$.

$$+\frac{1}{\pi}\int_0^{2\pi} f(x)\left[\sum_{n=1}^{N}\cos(nt)\cos(nx)+\sin(nt)\sin(nx)\right] dx \tag{2.2.35}$$

$$S_N(t) = \frac{1}{\pi}\int_0^{2\pi} f(x)\left\{\frac{1}{2}+\sum_{n=1}^{N}\cos[n(t-x)]\right\} dx \tag{2.2.36}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(x)\frac{\sin[(N+\frac{1}{2})(x-t)]}{\sin[\frac{1}{2}(x-t)]}\, dx. \tag{2.2.37}$$

The quantity $\sin[(N+\frac{1}{2})(x-t)]/\sin[\frac{1}{2}(x-t)]$ is called a *scanning function*. Over the range $0 \le x \le 2\pi$ it has a very large peak at $x = t$ where the amplitude equals $2N+1$. See Figure 2.2.1. On either side of this peak there are oscillations which decrease rapidly with distance from the peak. Consequently, as $N \to \infty$, the scanning function becomes essentially a long narrow slit corresponding to the area under the large peak at $x = t$. If we neglect for the moment the small area under the minor ripples adjacent to this slit, then the integral (2.2.37) essentially equals $f(t)$ times the area of the slit divided by 2π. If $1/2\pi$ times the area of the slit equals unity, then the value of $S_N(t) \approx f(t)$ to a good approximation for large N.

For a relatively small value of N, the scanning function deviates considerably from its ideal form, and the partial sum $S_N(t)$ only crudely approximates the given function $f(t)$. As the partial sum includes more terms and N becomes relatively large, the form of the scanning function improves and so does the degree of approximation between $S_N(t)$ and $f(t)$. The improvement in the scanning function is due to the large hump becoming taller and narrower. At the same time, the adjacent ripples become larger in number and hence also become narrower in the same proportion as the large hump becomes narrower.

The reason why $S_N(t)$ and $f(t)$ will never become identical, even in the limit of $N \to \infty$, is the presence of the positive and negative side lobes near the large peak. Because

$$\frac{\sin[(N+\frac{1}{2})(x-t)]}{\sin[\frac{1}{2}(x-t)]} = 1 + 2\sum_{n=1}^{N} \cos[n(t-x)], \tag{2.2.38}$$

an integration of the scanning function over the interval 0 to 2π shows that the total area under the scanning function equals 2π. However, from Figure 2.2.1 the net area contributed by the ripples is numerically negative so that the area under the large peak must exceed the value of 2π if the area of the entire function equals 2π. Although the exact value depends upon N, it is important to note that this excess does not become zero as $N \to \infty$.

Thus, the presence of these negative side lobes explains the departure of our scanning function from the idealized slit of area 2π. To illustrate this departure, consider the function:

$$f(t) = \begin{cases} 1, & 0 < t < \pi \\ -1, & \pi < t < 2\pi. \end{cases} \tag{2.2.39}$$

Then,

$$S_N(t) = \frac{1}{2\pi}\int_0^{\pi} \frac{\sin[(N+\frac{1}{2})(x-t)]}{\sin[\frac{1}{2}(x-t)]}\,dx - \frac{1}{2\pi}\int_{\pi}^{2\pi} \frac{\sin[(N+\frac{1}{2})(x-t)]}{\sin[\frac{1}{2}(x-t)]}\,dx \tag{2.2.40}$$

$$= \frac{1}{2\pi}\int_0^{\pi} \left\{ \frac{\sin[(N+\frac{1}{2})(x-t)]}{\sin[\frac{1}{2}(x-t)]}\,dx + \frac{\sin[(N+\frac{1}{2})(x+t)]}{\sin[\frac{1}{2}(x+t)]}\,dx \right\} \tag{2.2.41}$$

$$= \frac{1}{2\pi}\int_{-t}^{\pi-t} \frac{\sin[(N+\frac{1}{2})\theta]}{\sin(\frac{1}{2}\theta)}\,d\theta - \frac{1}{2\pi}\int_{t}^{\pi+t} \frac{\sin[(N+\frac{1}{2})\theta]}{\sin(\frac{1}{2}\theta)}\,d\theta. \tag{2.2.42}$$

The first integral in (2.2.42) gives the contribution to $S_N(t)$ from the jump discontinuity at $t = 0$ while the second integral gives the contribution from $t = \pi$. In Figure 2.2.2 we have plotted the numerical

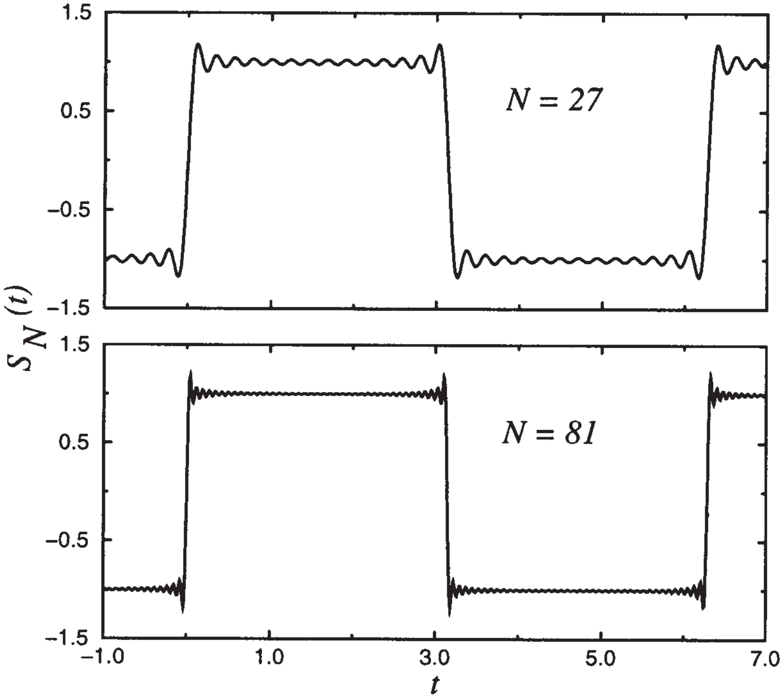


Figure 2.2.2: The finite Fourier series representation $S_N(t)$ for the function (2.2.39) for the range $-1 \leq t \leq 7$ for $N = 27$ and $N = 81$.

integration of (2.2.42) for $N = 27$ and $N = 81$. Residual discrepancies remain even for very large values of N. Indeed, even as N increases this figure changes only in that the ripples in the vicinity of the discontinuity of $f(t)$ show a proportionally increased rate of oscillation as a function of t while their relative magnitude remains the same. As $N \to \infty$ these ripples compress into a single vertical line at the point of discontinuity. True, these oscillations occupy smaller and smaller spaces but they still remain. Thus, we can never approximate a function in the vicinity of a discontinuity by a finite Fourier series without suffering from this over- and undershooting of the series. This peculiarity of Fourier series is called the *Gibbs phenomena*.[10] Gibbs phenomena can only be eliminated by removing the discontinuity.[11]

---

[10] Gibbs, J. W., 1898: Fourier's series. *Nature*, **59**, 200; Gibbs, J. W., 1899: Fourier's series. *Nature*, **59**, 606. For the historical development, see Hewitt, E. and Hewitt, R. E., 1979: The Gibbs-Wilbraham phenomenon: An episode in Fourier analysis. *Arch. Hist. Exact Sci.*, **21**, 129–160.

[11] For a particularly clever method for improving the convergence of

Problems

Additional Fourier series representation can be generated by differentiating or integrating known Fourier series. Work out the following two examples.

1. Given

$$\frac{\pi^2 - 2\pi x}{8} = \sum_{n=0}^{\infty} \frac{\cos[(2n+1)x]}{(2n+1)^2}, \qquad 0 \le x \le \pi,$$

obtain

$$\frac{\pi^2 x - \pi x^2}{8} = \sum_{n=0}^{\infty} \frac{\sin[(2n+1)x]}{(2n+1)^3}, \qquad 0 \le x \le \pi,$$

by term-by-term integration. Could we go the other way, i.e., take the derivative of the second equation to obtain the first? Explain.

2. Given

$$\frac{\pi^2 - 3x^2}{12} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos(nx)}{n^2}, \qquad -\pi \le x \le \pi,$$

obtain

$$\frac{\pi^2 x - x^3}{12} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(nx)}{n^3}, \qquad -\pi \le x \le \pi,$$

by term-by-term integration. Could we go the other way, i.e., take the derivative of the second equation to obtain the first? Explain.

3. (a) Show that the Fourier series for the odd function:

$$f(t) = \begin{cases} 2t + t^2, & -2 < t < 0 \\ 2t - t^2, & 0 < t < 2 \end{cases}$$

is

$$f(t) = \frac{32}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin\left[\frac{(2n-1)\pi t}{2}\right].$$

---

a trigonometric series, see Kantorovich, L. V. and Krylov, V. I., 1964: *Approximate Methods of Higher Analysis*. Interscience, New York, pp. 77–88.

(b) Use Parseval's equality to show that

$$\frac{\pi^6}{960} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}.$$

This series converges very rapidly to $\pi^6/960$ and provides a convenient method for computing π^6.

2.3 HALF-RANGE EXPANSIONS

In certain applications, we will find that we need a Fourier series representation for a function $f(x)$ that applies over the interval $(0, L)$ rather than $(-L, L)$. Because we are completely free to define the function over the interval $(-L, 0)$, it is simplest to have a series that consists only of sines or cosines. In this section we shall show how we can obtain these so-called *half-range expansions.*

Recall in Example 2.1.3 how we saw that if $f(x)$ is an even function $[f_o(x) = 0]$, then $b_n = 0$ for all n. Similarly, if $f(x)$ is an odd function $[f_e(x) = 0]$, then $a_0 = a_n = 0$ for all n. We now use these results to find a Fourier half-range expansion by extending the function defined over the interval $(0, L)$ as either an even or odd function into the interval $(-L, 0)$. If we extend $f(x)$ as an even function, we will get a half-range cosine series; if we extend $f(x)$ as an odd function, we obtain a half-range sine series.

It is important to remember that half-range expansions are a special case of the general Fourier series. For any $f(x)$ we can construct either a Fourier sine or cosine series over the interval $(-L, L)$. Both of these series will give the correct answer over the interval of $(0, L)$. Which one we choose to use depends upon whether we wish to deal with a cosine or sine series.

• Example 2.3.1

Let us find the half-range sine expansion of

$$f(x) = 1, \qquad 0 < x < \pi. \tag{2.3.1}$$

We begin by defining the periodic odd function

$$\tilde{f}(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases} \tag{2.3.2}$$

with $\tilde{f}(x + 2\pi) = \tilde{f}(x)$. Because $\tilde{f}(x)$ is odd, $a_0 = a_n = 0$ and

$$b_n = \frac{2}{\pi} \int_0^{\pi} 1 \sin(nx)\, dx = -\frac{2}{n\pi} \cos(nx)\Big|_0^{\pi} \tag{2.3.3}$$

$$= -\frac{2}{n\pi} [\cos(n\pi) - 1] = -\frac{2}{n\pi} [(-1)^n - 1]. \tag{2.3.4}$$

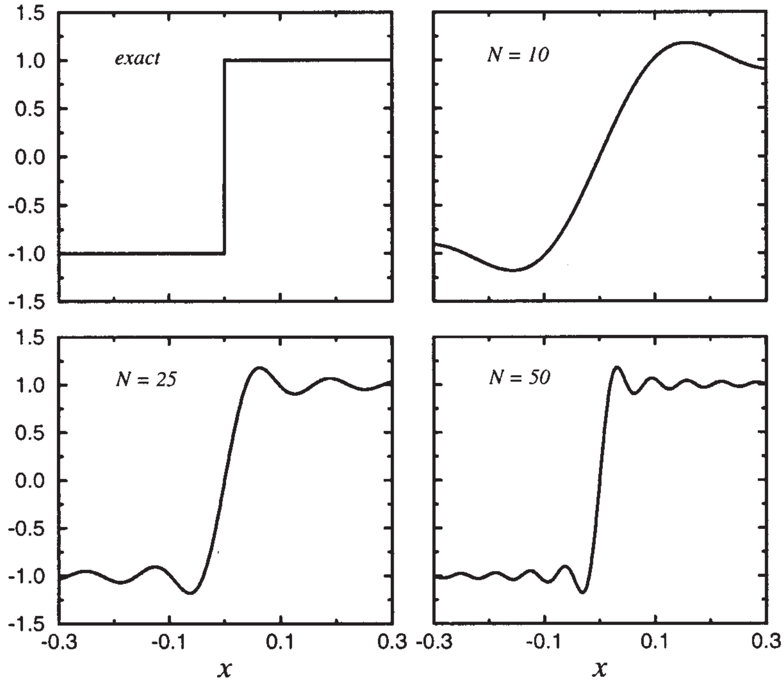


Figure 2.3.1: Partial sum of N terms in the Fourier half-range sine representation of a square wave.

The Fourier half-range sine series expansion of $f(x)$ is therefore

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\sin(nx) = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\sin[(2m-1)x]}{2m-1}. \qquad (2.3.5)$$

As counterpoint, let us find the half-range cosine expansion of $f(x) = 1$, $0 < x < \pi$. Now, we have that $b_n = 0$,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} 1\, dx = 2 \qquad (2.3.6)$$

and

$$a_n = \frac{2}{\pi}\int_0^{\pi}\cos(nx)\, dx = \frac{2}{n\pi}\sin(nx)\Big|_0^{\pi} = 0. \qquad (2.3.7)$$

Thus, the Fourier half-range cosine expansion equals the single term:

$$f(x) = 1, \qquad 0 < x < \pi. \qquad (2.3.8)$$

This is perfectly reasonable. To form a half-range cosine expansion we extend $f(x)$ as an even function into the interval $(-\pi, 0)$. In this case, we would obtain $\tilde{f}(x) = 1$ for $-\pi < x < \pi$. Finally, we note that the Fourier series of a constant is simply that constant.

In practice it is impossible to sum (2.3.5) exactly and we actually sum only the first N terms. Figure 2.3.1 illustrates $f(x)$ when the Fourier series (2.3.5) contains N terms. As seen from the figure, the truncated series tries to achieve the infinite slope at $x = 0$, but in the attempt, it *overshoots* the discontinuity by a certain amount (in this particular case, by 17.9%). This is another example of the Gibbs phenomena. Increasing the number of terms does not remove this peculiarity; it merely shifts it nearer to the discontinuity.

• **Example 2.3.2: Inertial supercharging of an engine**

An important aspect of designing any gasoline engine involves the motion of the fuel, air, and exhaust gas mixture through the engine. Ordinarily an engineer would consider the motion as steady flow; but in the case of a four-stroke, single-cylinder gasoline engine, the closing of the intake valve interrupts the steady flow of the gasoline-air mixture for nearly three quarters of the engine cycle. This periodic interruption sets up standing waves in the intake pipe – waves which can build up an appreciable pressure amplitude just outside the input value.

When one of the harmonics of the engine frequency equals one of the resonance frequencies of the intake pipe, then the pressure fluctuations at the valve will be large. If the intake valve closes during that portion of the cycle when the pressure is less than average, then the waves will reduce the power output. However, if the intake valve closes when the pressure is greater than atmospheric, then the waves will have a supercharging effect and will produce an increase of power. This effect is called *inertia supercharging*.

While studying this problem, Morse et al.[12] found it necessary to express the velocity of the air-gas mixture in the valve, given by

$$f(t) = \begin{cases} 0, & -\pi < \omega t < -\pi/4 \\ \pi\cos(2\omega t)/2, & -\pi/4 < \omega t < \pi/4 \\ 0, & \pi/4 < \omega t < \pi \end{cases} \tag{2.3.9}$$

in terms of a Fourier expansion. The advantage of working with the Fourier series rather than the function itself lies in the ability to write the velocity as a periodic forcing function that highlights the various harmonics that might be resonant with the structure comprising the fuel line.

[12] Morse, P. M., Boden, R. H., and Schecter, H., 1938: Acoustic vibrations and internal combustion engine performance. I. Standing waves in the intake pipe system. *J. Appl. Phys.*, **9**, 16–23.

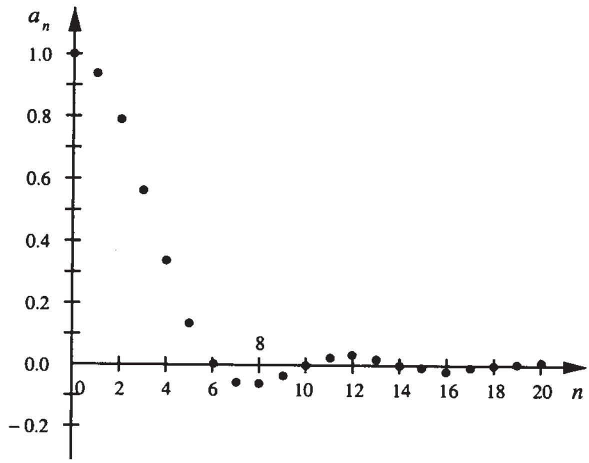


Figure 2.3.2: The spectral coefficients of the Fourier cosine series of the function (2.3.9).

Clearly $f(t)$ is an even function and its Fourier representation will be a cosine series. In this problem $\tau = -\pi/\omega$ and $L = \pi/\omega$. Therefore,

$$a_0 = \frac{2\omega}{\pi}\int_{-\pi/4\omega}^{\pi/4\omega} \frac{\pi}{2}\cos(2\omega t)\,dt = \tfrac{1}{2}\sin(2\omega t)\Big|_{-\pi/4\omega}^{\pi/4\omega} = 1 \tag{2.3.10}$$

and

$$a_n = \frac{2\omega}{\pi}\int_{-\pi/4\omega}^{\pi/4\omega} \frac{\pi}{2}\cos(2\omega t)\cos\left(\frac{n\pi t}{\pi/\omega}\right)dt \tag{2.3.11}$$

$$= \frac{\omega}{2}\int_{-\pi/4\omega}^{\pi/4\omega} \{\cos[(n+2)\omega t] + \cos[(n-2)\omega t]\}\,dt \tag{2.3.12}$$

$$= \begin{cases} \left.\frac{\sin[(n+2)\omega t]}{2(n+2)} + \frac{\sin[(n-2)\omega t]}{2(n-2)}\right|_{-\pi/4\omega}^{\pi/4\omega}, & n \neq 2 \\ \left.\frac{\omega t}{2} + \frac{\sin(4\omega t)}{4}\right|_{-\pi/4\omega}^{\pi/4\omega}, & n = 2 \end{cases} \tag{2.3.13}$$

$$= \begin{cases} -\frac{4}{n^2-4}\cos\left(\frac{n\pi}{4}\right), & n \neq 2 \\ \frac{\pi}{4}, & n = 2. \end{cases} \tag{2.3.14}$$

Because these Fourier coefficients become small rapidly (see Figure 2.3.2), Morse et al. showed that there are only about three resonances where the acoustic properties of the intake pipe can enhance engine performance. These peaks occur when $q = 30c/NL = 3, 4,$ or 5, where c is the velocity of sound in the air-gas mixture, L is the effective length

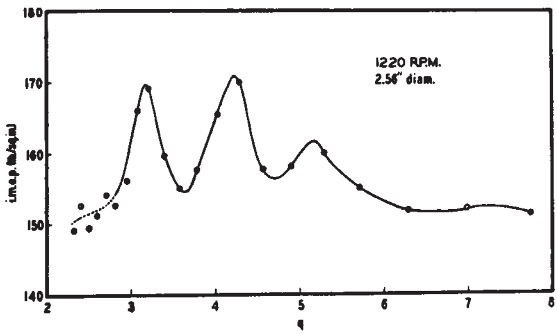


Figure 2.3.3: Experimental verification of the resonance of the $n = 3, 4$, and 5 harmonics of the Fourier representation (2.3.14) of the flow of an air-gas mixture with the intake pipe system. The parameter q is defined in the text. (From Morse, P., Boden, R. H., and Schecter, H., 1938: Acoustic vibrations and internal combustion engine performance. *J. Appl. Phys.*, **9**, 17 with permission.)

of the intake pipe, and N is the engine speed in rpm. See Figure 2.3.3. Subsequent experiments[13] verified these results.

Such analyses are valuable to automotive engineers. Engineers are always seeking ways to optimize a system with little or no additional cost. Our analysis shows that by tuning the length of the intake pipe so that it falls on one of the resonance peaks, we could obtain higher performance from the engine with little or no extra work. Of course, the problem is that no car always performs at some optimal condition.

Problems

Find the Fourier cosine and sine series for the following functions:

1.
$$f(t) = t, \qquad 0 < t < \pi$$

2.
$$f(t) = \pi - t, \qquad 0 < t < \pi$$

3.
$$f(t) = \begin{cases} t, & 0 \le t \le \frac{1}{2} \\ 1 - t, & \frac{1}{2} \le t \le 1 \end{cases}$$

[13] Boden, R. H. and Schecter, H., 1944: Dynamics of the inlet system of a four-stroke engine. *NACA Tech. Note 935.*

4.

$$f(t) = \pi^2 - t^2, \qquad 0 < t < \pi$$

5.

$$f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 1, & 1 \le t < 2 \end{cases}$$

6.

$$f(t) = \begin{cases} 0, & 0 < t < \frac{a}{3} \\ t - \frac{a}{3}, & \frac{a}{3} < t < \frac{2a}{3} \\ \frac{a}{3}, & \frac{2a}{3} < t < a \end{cases}$$

7.

$$f(t) = \begin{cases} \frac{1}{2}, & 0 < t < \frac{a}{2} \\ 1, & \frac{a}{2} < t < a \end{cases}$$

8.

$$f(t) = \begin{cases} \frac{2t}{a}, & 0 < t < \frac{a}{2} \\ \frac{3a-2t}{2a}, & \frac{a}{2} < t < a \end{cases}$$

9.

$$f(t) = \begin{cases} t, & 0 < t < \frac{a}{2} \\ \frac{a}{2}, & \frac{a}{2} < t < a \end{cases}$$

10.

$$f(t) = \frac{a-t}{a}, \qquad 0 < t < a$$

11.

$$f(t) = \begin{cases} 0, & 0 < t < \frac{a}{4} \\ 1, & \frac{a}{4} < t < \frac{3a}{4} \\ 0, & \frac{3a}{4} < t < a \end{cases}$$

12.

$$f(t) = t(a-t), \qquad 0 < t < a$$

13.

$$f(t) = e^{kt}, \qquad 0 < t < a$$

14.

$$f(t) = \begin{cases} 0, & 0 < t < \frac{a}{2} \\ 1, & \frac{a}{2} < t < a \end{cases}$$

15. The function

$$f(t) = 1 - (1+a)\frac{t}{\pi} + (a-1)\frac{t^2}{\pi^2} + (a+1)\frac{t^3}{\pi^3} - a\frac{t^4}{\pi^4}, \quad 0 < t < \pi$$

is a curve fit to the observed pressure trace of an explosion wave in the atmosphere. Because the observed transmission of atmospheric waves depends on the five-fourths power of the frequency, Reed[14] had to re-express this curve fit as a Fourier sine series before he could use the transmission law. He found that

$$f(t) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1 - \frac{3(a-1)}{2\pi^2 n^2}\right]\sin(2nt)$$
$$+ \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{2}{2n-1}\left[1 + \frac{2(a-1)}{\pi^2(2n-1)^2} - \frac{48a}{\pi^4(2n-1)^4}\right]\sin[(2n-1)t].$$

Confirm his result.

2.4 FOURIER SERIES WITH PHASE ANGLES

Sometimes it is desirable to rewrite a general Fourier series as a purely cosine or purely sine series with a phase angle. Engineers often like to speak of some quantity leading or lagging another quantity. Re-expressing a Fourier series in terms of amplitude and phase provides a convenient method for determining these phase relationships.

Suppose, for example, that we have a function $f(t)$ of period $2L$, given in the interval $[-L, L]$, whose Fourier series expansion is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right). \qquad \mathbf{(2.4.1)}$$

We wish to replace (2.4.1) by the series:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi t}{L} + \varphi_n\right). \qquad \mathbf{(2.4.2)}$$

[14] From Reed, J. W., 1977: Atmospheric attenuation of explosion waves. *J. Acoust. Soc. Am.*, **61**, 39–47 with permission.

To do this we note that

$$B_n \sin\left(\frac{n\pi t}{L} + \varphi_n\right) = a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \tag{2.4.3}$$

$$= B_n \sin\left(\frac{n\pi t}{L}\right)\cos(\varphi_n) + B_n \sin(\varphi_n)\cos\left(\frac{n\pi t}{L}\right). \tag{2.4.4}$$

We equate coefficients of $\sin(n\pi t/L)$ and $\cos(n\pi t/L)$ on both sides and obtain

$$a_n = B_n \sin(\varphi_n) \qquad \text{and} \qquad b_n = B_n \cos(\varphi_n). \tag{2.4.5}$$

Hence, upon squaring and adding,

$$B_n = \sqrt{a_n^2 + b_n^2}, \tag{2.4.6}$$

while taking the ratio gives

$$\varphi_n = \tan^{-1}(a_n/b_n). \tag{2.4.7}$$

Similarly we could rewrite (2.4.1) as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi t}{L} + \varphi_n\right), \tag{2.4.8}$$

where

$$A_n = \sqrt{a_n^2 + b_n^2} \qquad \text{and} \qquad \varphi_n = \tan^{-1}(-b_n/a_n) \tag{2.4.9}$$

and

$$a_n = A_n \cos(\varphi_n) \qquad \text{and} \qquad b_n = -A_n \sin(\varphi_n). \tag{2.4.10}$$

In both cases, we must be careful in computing φ_n because there are two possible values of φ_n which satisfy (2.4.7) or (2.4.9). These φ_n's must give the correct a_n and b_n using either (2.4.5) or (2.4.10).

• Example 2.4.1

The Fourier series for $f(t) = e^t$ over the interval $-L < t < L$ is

$$f(t) = \frac{\sinh(aL)}{aL} + 2\sinh(aL)\sum_{n=1}^{\infty} \frac{aL(-1)^n}{a^2L^2 + n^2\pi^2} \cos\left(\frac{n\pi t}{L}\right)$$
$$- 2\sinh(aL)\sum_{n=1}^{\infty} \frac{n\pi(-1)^n}{a^2L^2 + n^2\pi^2} \sin\left(\frac{n\pi t}{L}\right). \tag{2.4.11}$$

Let us rewrite (2.4.11) as a Fourier series with a phase angle. Regardless of whether we want the new series to contain $\cos(n\pi t/L + \varphi_n)$ or $\sin(n\pi t/L + \varphi_n)$, the amplitude A_n or B_n is the same in both series:

$$A_n = B_n = \sqrt{a_n^2 + b_n^2} = \frac{2\sinh(aL)}{\sqrt{a^2L^2 + n^2\pi^2}}. \tag{2.4.12}$$

If we want our Fourier series to read

$$f(t) = \frac{\sinh(aL)}{aL} + 2\sinh(aL)\sum_{n=1}^{\infty} \frac{\cos(n\pi t/L + \varphi_n)}{\sqrt{a^2L^2 + n^2\pi^2}}, \tag{2.4.13}$$

then

$$\varphi_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right) = \tan^{-1}\left(\frac{n\pi}{aL}\right), \tag{2.4.14}$$

where φ_n lies in the first quadrant if n is even and in the third quadrant if n is odd. This ensures that the sign from the $(-1)^n$ is correct.

On the other hand, if we prefer

$$f(t) = \frac{\sinh(aL)}{aL} + 2\sinh(aL)\sum_{n=1}^{\infty} \frac{\sin(n\pi t/L + \varphi_n)}{\sqrt{a^2L^2 + n^2\pi^2}}, \tag{2.4.15}$$

then

$$\varphi_n = \tan^{-1}\left(\frac{a_n}{b_n}\right) = -\tan^{-1}\left(\frac{aL}{n\pi}\right), \tag{2.4.16}$$

where φ_n lies in the fourth quadrant if n is odd and in the second quadrant if n is even.

Problems

Write the following Fourier series in both the cosine and sine phase angle form:

1.

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin[(2n-1)\pi t]}{2n-1}$$

2.

$$f(t) = \frac{3}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}\cos\left[\frac{(2n-1)\pi t}{2}\right]$$

3.

$$f(t) = -2\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin(nt)$$

4.

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos[(2n-1)t]}{(2n-1)^2}$$

2.5 COMPLEX FOURIER SERIES

So far in our discussion, we have expressed Fourier series in terms of sines and cosines. We are now ready to reexpress a Fourier series as a series of complex exponentials. There are two reasons for this. First, in certain engineering and scientific applications of Fourier series, the expansion of a function in terms of complex exponentials results in coefficients of considerable simplicity and clarity. Secondly, these complex Fourier series point the way to the development of the Fourier transform in the next chapter.

We begin by introducing the variable

$$\omega_n = \frac{n\pi}{L}, \qquad (2.5.1)$$

where $n = 0, \pm 1, \pm 2, \ldots$. Using Euler's formula we can replace the sine and cosine in the Fourier series by exponentials and find that

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{i\omega_n t} + e^{-i\omega_n t}\right) + \frac{b_n}{2i}\left(e^{i\omega_n t} - e^{-i\omega_n t}\right) \qquad (2.5.2)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} - \frac{b_n i}{2}\right) e^{i\omega_n t} + \left(\frac{a_n}{2} + \frac{b_n i}{2}\right) e^{-i\omega_n t}. \qquad (2.5.3)$$

If we define c_n as

$$c_n = \tfrac{1}{2}(a_n - ib_n), \qquad (2.5.4)$$

then

$$c_n = \tfrac{1}{2}(a_n - ib_n) = \frac{1}{2L}\int_\tau^{\tau+2L} f(t)[\cos(\omega_n t) - i\sin(\omega_n t)]\,dt \qquad (2.5.5)$$

$$= \frac{1}{2L}\int_\tau^{\tau+2L} f(t)e^{-i\omega_n t}dt. \qquad (2.5.6)$$

Similarly, the complex conjugate of c_n, c_n^*, equals

$$c_n^* = \tfrac{1}{2}(a_n + ib_n) = \frac{1}{2L}\int_\tau^{\tau+2L} f(t)e^{i\omega_n t}dt. \qquad (2.5.7)$$

To simplify (2.5.3) we note that

$$\omega_{-n} = \frac{(-n)\pi}{L} = -\frac{n\pi}{L} = -\omega_n, \qquad (2.5.8)$$

which yields the result that

$$c_{-n} = \frac{1}{2L}\int_{\tau}^{\tau+2L} f(t)e^{-i\omega_{-n}t}dt = \frac{1}{2L}\int_{\tau}^{\tau+2L} f(t)e^{i\omega_n t}dt = c_n^* \quad \textbf{(2.5.9)}$$

so that we can write (2.5.3) as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n e^{i\omega_n t} + c_n^* e^{-i\omega_n t} = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n e^{i\omega_n t} + c_{-n} e^{-i\omega_n t}. \quad \textbf{(2.5.10)}$$

Letting $n = -m$ in the second summation on the right side of (2.5.10),

$$\sum_{n=1}^{\infty} c_{-n} e^{-i\omega_n t} = \sum_{m=-1}^{-\infty} c_m e^{-i\omega_{-m} t} = \sum_{m=-\infty}^{-1} c_m e^{i\omega_m t} = \sum_{n=-\infty}^{-1} c_n e^{i\omega_n t}, \quad \textbf{(2.5.11)}$$

where we have introduced $m = n$ into the last summation in (2.5.11). Therefore,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n e^{i\omega_n t} + \sum_{n=-\infty}^{-1} c_n e^{i\omega_n t}. \quad \textbf{(2.5.12)}$$

On the other hand,

$$\frac{a_0}{2} = \frac{1}{2L}\int_{\tau}^{\tau+2L} f(t)\,dt = c_0 = c_0 e^{i\omega_0 t}, \quad \textbf{(2.5.13)}$$

because $\omega_0 = 0\pi/L = 0$. Thus, our final result is

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t},} \quad \textbf{(2.5.14)}$$

where

$$\boxed{c_n = \frac{1}{2L}\int_{\tau}^{\tau+2L} f(t)e^{-i\omega_n t}\,dt} \quad \textbf{(2.5.15)}$$

and $n = 0, \pm 1, \pm 2, \ldots$. Note that even though c_n is generally complex, the summation (2.5.14) always gives a *real-valued* function $f(t)$.

Just as we can represent the function $f(t)$ graphically by a plot of t against $f(t)$, we can plot c_n as a function of n, commonly called the frequency *spectrum*. Because c_n is generally complex, it is necessary to make two plots. Typically the plotted quantities are the amplitude spectra $|c_n|$ and the phase spectra φ_n, where φ_n is the phase of c_n. However, we could just as well plot the real and imaginary parts of c_n. Because n is an integer, these plots consist merely of a series of vertical lines representing the ordinates of the quantity $|c_n|$ or φ_n for each n. For this reason we refer to these plots as the *line spectra*.

Because $2c_n = a_n - ib_n$, the c_n's for an even function will be purely real; the c_n's for an odd function are purely imaginary. It is important to note that we lose the advantage of even and odd functions in the sense that we cannot just integrate over the interval 0 to L and then double the result. In the present case we have a line integral of a complex function along the real axis.

• **Example 2.5.1**

Let us find the complex Fourier series for

$$f(t) = \begin{cases} 1, & 0 < t < \pi \\ -1, & -\pi < t < 0, \end{cases} \tag{2.5.16}$$

which has the periodicity $f(t + 2\pi) = f(t)$.

With $L = \pi$ and $\tau = -\pi$, $\omega_n = n\pi/L = n$. Therefore,

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{0} (-1)e^{-int}\,dt + \frac{1}{2\pi}\int_{0}^{\pi} (1)e^{-int}\,dt \tag{2.5.17}$$

$$= \frac{1}{2n\pi i}e^{-int}\bigg|_{-\pi}^{0} - \frac{1}{2n\pi i}e^{-int}\bigg|_{0}^{\pi} \tag{2.5.18}$$

$$= -\frac{i}{2n\pi}\left(1 - e^{n\pi i}\right) + \frac{i}{2n\pi}\left(e^{-n\pi i} - 1\right), \tag{2.5.19}$$

if $n \neq 0$. Because $e^{n\pi i} = \cos(n\pi) + i\sin(n\pi) = (-1)^n$ and $e^{-n\pi i} = \cos(-n\pi) + i\sin(-n\pi) = (-1)^n$, then

$$c_n = -\frac{i}{n\pi}[1 - (-1)^n] = \begin{cases} 0, & n \text{ even} \\ -\frac{2i}{n\pi}, & n \text{ odd} \end{cases} \tag{2.5.20}$$

with

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}. \tag{2.5.21}$$

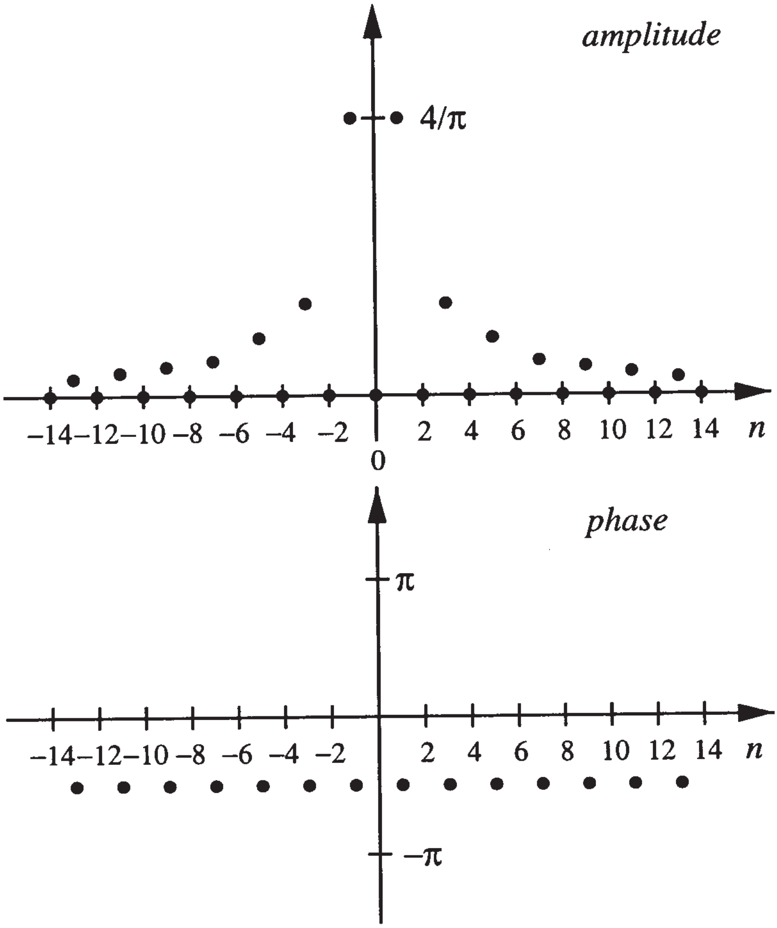


Figure 2.5.1: Amplitude and phase spectra for the function (2.5.16).

In this particular problem we must treat the case $n = 0$ specially because (2.5.18) is undefined for $n = 0$. In that case,

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{0}(-1)\,dt + \frac{1}{2\pi}\int_{0}^{\pi}(1)\,dt = \frac{1}{2\pi}(-t)\Big|_{-\pi}^{0} + \frac{1}{2\pi}(t)\Big|_{0}^{\pi} = 0. \tag{2.5.22}$$

Because $c_0 = 0$, we can write the expansion:

$$f(t) = -\frac{2i}{\pi}\sum_{m=-\infty}^{\infty}\frac{e^{(2m-1)it}}{2m-1}, \tag{2.5.23}$$

because we can write all odd integers as $2m - 1$, where $m = 0, \pm1, \pm2,$

$\pm 3, \ldots$. In Figure 2.5.1 we present the amplitude and phase spectra for the function (2.5.16).

Problems

Find the complex Fourier series for the following functions:

1. $f(t) = |t|, \quad -\pi \le t \le \pi$
2. $f(t) = e^t, \quad 0 < t < 2$
3. $f(t) = t, \quad 0 < t < 2$
4. $f(t) = t^2, \quad -\pi \le t \le \pi$
5. $f(t) = \begin{cases} 0, & -\pi/2 < t < 0 \\ 1, & 0 < t < \pi/2 \end{cases}$
6. $f(t) = t, \quad -1 < t < 1$

2.6 THE USE OF FOURIER SERIES IN THE SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

An important application of Fourier series is the solution of ordinary differential equations. Structural engineers especially use this technique because the occupants of buildings and bridges often subject these structures to forcings that are periodic in nature.[15]

• **Example 2.6.1**

Let us find the general solution to the ordinary differential equation

$$y'' + 9y = f(t), \tag{2.6.1}$$

where the forcing is

$$f(t) = |t|, \qquad -\pi \le t \le \pi, \qquad f(t + 2\pi) = f(t). \tag{2.6.2}$$

This equation represents an oscillator forced by a driver whose displacement is the saw-tooth function.

We begin by replacing the function $f(t)$ by its Fourier series representation because the forcing function is periodic. The advantage of expressing $f(t)$ as a Fourier series is its validity for any time t. The alternative would have been to construct a solution over each interval $n\pi < t < (n + 1)\pi$ and then piece together the final solution assuming that the solution and its first derivative is continuous at each junction

[15] Timoshenko, S. P., 1943: Theory of suspension bridges. Part II. *J. Franklin Inst.*, **235**, 327–349; Inglis, C. E., 1934: *A Mathematical Treatise on Vibrations in Railway Bridges*, Cambridge University Press, Cambridge.

$t = n\pi$. Because the function is an even function, all of the sine terms vanish and the Fourier series is

$$|t| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos[(2n-1)t]}{(2n-1)^2}. \tag{2.6.3}$$

Next, we note that the general solution consists of the complementary solution, which equals

$$y_H(t) = A\cos(3t) + B\sin(3t), \tag{2.6.4}$$

and the particular solution $y_p(t)$ which satisfies the differential equation

$$y_p'' + 9y_p = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos[(2n-1)t]}{(2n-1)^2}. \tag{2.6.5}$$

To determine this particular solution, we write (2.6.5) as

$$y_p'' + 9y_p = \frac{\pi}{2} - \frac{4}{\pi}\cos(t) - \frac{4}{9\pi}\cos(3t) - \frac{4}{25\pi}\cos(5t) - \cdots \tag{2.6.6}$$

By the method of undetermined coefficients, we would have guessed the particular solution:

$$y_p(t) = \frac{a_0}{2} + a_1\cos(t) + b_1\sin(t) + a_2\cos(3t) + b_2\sin(3t) + \cdots \tag{2.6.7}$$

or

$$y_p(t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos[(2n-1)t] + b_n\sin[(2n-1)t]. \tag{2.6.8}$$

Because

$$y_p''(t) = \sum_{n=1}^{\infty} -(2n-1)^2\{a_n\cos[(2n-1)t] + b_n\sin[(2n-1)t]\}, \tag{2.6.9}$$

$$\begin{aligned}\sum_{n=1}^{\infty} -(2n-1)^2\{a_n\cos[(2n-1)t] + b_n\sin[(2n-1)t]\}& \\ + \tfrac{9}{2}a_0 + 9\sum_{n=1}^{\infty} a_n\cos[(2n-1)t] + b_n\sin[(2n-1)t]& \\ = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos[(2n-1)t]}{(2n-1)^2}&\end{aligned} \tag{2.6.10}$$

or

$$\frac{9}{2}a_0 - \frac{\pi}{2} + \sum_{n=1}^{\infty} \left\{ [9 - (2n-1)^2]a_n + \frac{4}{\pi(2n-1)^2} \right\} \cos[(2n-1)t]$$
$$+ \sum_{n=1}^{\infty} [9 - (2n-1)^2] b_n \sin[(2n-1)t] = 0. \qquad \textbf{(2.6.11)}$$

Because (2.6.11) must hold true for any time, each harmonic must vanish separately:

$$a_0 = \frac{\pi}{9}, \qquad a_n = -\frac{4}{\pi(2n-1)^2[9-(2n-1)^2]} \qquad \textbf{(2.6.12)}$$

and $b_n = 0$. All of the a_n's are finite except for $n = 2$, where a_2 becomes undefined. The coefficient a_2 is undefined because the harmonic $\cos(3t)$ in the forcing function is resonating with the natural mode of the system.

Let us review our analysis to date. We found that each harmonic in the forcing function yields a corresponding harmonic in the particular solution (2.6.8). The only difficulty arises with the harmonic $n = 2$. Although our particular solution is not correct because it contains $\cos(3t)$, we suspect that if we remove that term then the remaining harmonic solutions are correct. The problem is linear, and difficulties with one harmonic term should not affect other harmonics. But how shall we deal with the $\cos(3t)$ term in the forcing function? Let us denote the particular solution for that harmonic by $Y(t)$ and modify our particular solution as follows:

$$y_p(t) = \tfrac{1}{2}a_0 + a_1\cos(t) + Y(t) + a_3\cos(5t) + \cdots \qquad \textbf{(2.6.13)}$$

Substituting this solution into the differential equation and simplifying, everything cancels except

$$Y'' + 9Y = -\frac{4}{9\pi}\cos(3t). \qquad \textbf{(2.6.14)}$$

The solution of this equation by the method of undetermined coefficients is

$$Y(t) = -\frac{2}{27\pi}t\sin(3t). \qquad \textbf{(2.6.15)}$$

This term, called a *secular term*, is the most important one in the solution. While the other terms merely represent simple oscillatory motion, the term $t\sin(3t)$ grows linearly with time and eventually becomes the dominant term in the series. Consequently, the general solution equals the complementary plus the particular solution:

$$y(t) = A\cos(3t) + B\sin(3t)$$
$$+ \frac{\pi}{18} - \frac{2}{27\pi}t\sin(3t) - \frac{4}{\pi}\sum_{\substack{n=1 \\ n\neq 2}}^{\infty} \frac{\cos[(2n-1)t]}{(2n-1)^2[9-(2n-1)^2]}. \qquad \textbf{(2.6.16)}$$

• **Example 2.6.2**

Let us redo the previous problem only using complex Fourier series. That is, let us find the general solution to the ordinary differential equation

$$y'' + 9y = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=-\infty}^{\infty} \frac{e^{i(2n-1)t}}{(2n-1)^2}. \qquad (2.6.17)$$

From the method of undetermined coefficients we guess the particular solution for (2.6.17) to be

$$y_p(t) = c_0 + \sum_{n=-\infty}^{\infty} c_n e^{i(2n-1)t}. \qquad (2.6.18)$$

Then

$$y_p''(t) = \sum_{n=-\infty}^{\infty} -(2n-1)^2 c_n e^{i(2n-1)t}. \qquad (2.6.19)$$

Substituting (2.6.18) and (2.6.19) into (2.6.17),

$$9c_0 + \sum_{n=-\infty}^{\infty} [9-(2n-1)^2] c_n e^{i(2n-1)t} = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=-\infty}^{\infty} \frac{e^{i(2n-1)t}}{(2n-1)^2}. \qquad (2.6.20)$$

Because (2.6.20) must be true for any t,

$$c_0 = \frac{\pi}{18} \quad \text{and} \quad c_n = \frac{2}{\pi(2n-1)^2[(2n-1)^2-9]}. \qquad (2.6.21)$$

Therefore,

$$y_p(t) = \frac{\pi}{18} + \frac{2}{\pi}\sum_{n=-\infty}^{\infty} \frac{e^{i(2n-1)t}}{(2n-1)^2[(2n-1)^2-9]} e^{i(2n-1)t}. \qquad (2.6.22)$$

However, there is a problem when $n = -1$ and $n = 2$. Therefore, we modify (2.6.22) to read

$$y_p(t) = \frac{\pi}{18} + c_2 t e^{3it} + c_{-1} t e^{-3it} + \frac{2}{\pi}\sum_{\substack{n=-\infty \\ n\neq -1,2}}^{\infty} \frac{e^{i(2n-1)t}}{(2n-1)^2[(2n-1)^2-9]} e^{i(2n-1)t}. \qquad (2.6.23)$$

Substituting (2.6.23) into (2.6.17) and simplifying,

$$c_2 = -\frac{1}{27\pi i} \quad \text{and} \quad c_{-1} = -\frac{1}{27\pi i}. \qquad (2.6.24)$$

The general solution is then

$$y_p(t) = Ae^{3it} + Be^{-3it} + \frac{\pi}{18} - \frac{te^{3it}}{27\pi i} + \frac{te^{-3it}}{27\pi i} + \frac{2}{\pi}\sum_{\substack{n=-\infty \\ n\neq -1,2}}^{\infty} \frac{e^{i(2n-1)t}}{(2n-1)^2[(2n-1)^2-9]}. \tag{2.6.25}$$

The first two terms on the right side of (2.6.25) represent the complementary solution. Although (2.6.25) is equivalent to (2.6.16), we have all of the advantages of dealing with exponentials rather than sines and cosines. These advantages include ease of differentiation and integration, and writing the series in terms of amplitude and phase.

• **Example 2.6.3: Temperature within a spinning satellite**

In the design of artificial satellites, it is important to determine the temperature distribution on the spacecraft's surface. An interesting special case is the temperature fluctuation in the skin due to the spinning of the vehicle. If the craft is thin-walled so that there is no radial dependence, Hrycak[16] showed that he could approximate the nondimensional temperature field at the equator of the rotating satellite by

$$\frac{d^2T}{d\eta^2} + b\frac{dT}{d\eta} - c\left(T - \frac{3}{4}\right) = -\frac{\pi c}{4}\frac{F(\eta) + \beta/4}{1 + \pi\beta/4}, \tag{2.6.26}$$

where

$$b = 4\pi^2 r^2 f/a, \qquad c = \frac{16\pi S}{\gamma T_\infty}\left(1 + \frac{\pi\beta}{4}\right), \tag{2.6.27}$$

$$F(\eta) = \begin{cases} \cos(2\pi\eta), & 0 < \eta < \frac{1}{4} \\ 0, & \frac{1}{4} < \eta < \frac{3}{4} \\ \cos(2\pi\eta), & \frac{3}{4} < \eta < 1, \end{cases} \tag{2.6.28}$$

$$T_\infty = \left(\frac{S}{\pi\sigma\epsilon}\right)^{1/4}\left(\frac{1 + \pi\beta/4}{1 + \beta}\right)^{1/4}, \tag{2.6.29}$$

a is the thermal diffusivity of the shell, f is the rate of spin, r is the radius of the spacecraft, S is the net direct solar heating, β is the ratio of the emissivity of the interior shell to the emissivity of the exterior surface, ϵ is the overall emissivity of the exterior surface, γ is the satellite's skin conductance, and σ is the Stefan-Boltzmann constant. The independent variable η is the longitude along the equator with the effect of rotation

[16] Hrycak, P., 1963: Temperature distribution in a spinning spherical space vehicle. *AIAA J.*, **1**, 96–99.

subtracted out ($2\pi\eta = \varphi - 2\pi ft$). The reference temperature T_∞ equals the temperature that the spacecraft would have if it spun with infinite angular speed so that the solar heating would be uniform around the craft. We have nondimensionalized the temperature with respect to T_∞.

We begin our analysis by introducing the new variables

$$y = T - \frac{3}{4} - \frac{\pi\beta}{16 + 4\pi\beta}, \quad \nu_0 = \frac{2\pi^2 r^2 f}{a\rho_0}, \quad A_0 = -\frac{\pi\rho^2}{4 + \pi\beta} \tag{2.6.30}$$

and $\rho_0^2 = c$ so that

$$\frac{d^2y}{d\eta^2} + 2\rho_0\nu_0\frac{dy}{d\eta} - \rho_0^2 y = A_0 F(\eta). \tag{2.6.31}$$

Next, we expand $F(\eta)$ as a Fourier series because it is a periodic function of period 1. Because it is an even function,

$$f(\eta) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(2n\pi\eta), \tag{2.6.32}$$

where

$$a_0 = \frac{1}{1/2}\int_0^{1/4} \cos(2\pi x)\,dx + \frac{1}{1/2}\int_{3/4}^{1} \cos(2\pi x)\,dx = \frac{2}{\pi}, \tag{2.6.33}$$

$$a_1 = \frac{1}{1/2}\int_0^{1/4} \cos^2(2\pi x)\,dx + \frac{1}{1/2}\int_{3/4}^{1} \cos^2(2\pi x)\,dx = \frac{1}{2} \tag{2.6.34}$$

and

$$a_n = \frac{1}{1/2}\int_0^{1/4} \cos(2\pi x)\cos(2n\pi x)\,dx + \frac{1}{1/2}\int_{3/4}^{1} \cos(2\pi x)\cos(2n\pi x)\,dx \tag{2.6.35}$$

$$= -\frac{2(-1)^n}{\pi(n^2-1)}\cos\left(\frac{n\pi}{2}\right), \tag{2.6.36}$$

if $n \geq 2$. Therefore,

$$f(\eta) = \frac{1}{\pi} + \frac{1}{2}\cos(2\pi\eta) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos(4n\pi\eta). \tag{2.6.37}$$

From the method of undetermined coefficients, the particular solution is

$$y_p(\eta) = \tfrac{1}{2}a_0 + a_1\cos(2\pi\eta) + b_1\sin(2\pi\eta) + \sum_{n=1}^{\infty} a_{2n}\cos(4n\pi\eta) + b_{2n}\sin(4n\pi\eta), \tag{2.6.38}$$

which yields

$$y_p'(\eta) = -2\pi a_1 \sin(2\pi\eta) + 2\pi b_1 \cos(2\pi\eta) + \sum_{n=1}^{\infty}[-4n\pi a_{2n} \sin(4n\pi\eta) + 4n\pi b_{2n} \cos(4n\pi\eta)] \quad \textbf{(2.6.39)}$$

and

$$y_p''(\eta) = -4\pi^2 a_1 \cos(2\pi\eta) - 4\pi^2 b_1 \sin(2\pi\eta) + \sum_{n=1}^{\infty}[-16n^2\pi^2 a_{2n} \cos(4n\pi\eta) - 16n^2\pi^2 b_{2n} \sin(4n\pi\eta)]. \quad \textbf{(2.6.40)}$$

Substituting into (2.6.31),

$$-\frac{1}{2}\rho_0^2 a_0 - \frac{A_0}{\pi} + \left(-4\pi^2 a_1 + 4\pi\rho_0\nu_0 b_1 - \rho_0^2 a_1 - \frac{A_0}{2}\right)\cos(2\pi\eta) + \left(-4\pi^2 b_1 - 4\pi\rho_0\nu_0 a_1 - \rho_0^2 b_1\right)\sin(2\pi\eta) + \sum_{n=1}^{\infty}\left[-16n^2\pi^2 a_{2n} + 8n\pi\rho_0\nu_0 b_{2n} - \rho_0^2 a_{2n} + \frac{2A_0(-1)^n}{\pi(4n^2-1)}\right]\cos(4n\pi\eta) + \sum_{n=1}^{\infty}\left(-16n^2\pi^2 b_{2n} - 8n\pi\rho_0\nu_0 a_{2n} - \rho_0^2 b_{2n}\right)\sin(4n\pi\eta) = 0. \quad \textbf{(2.6.41)}$$

In order to satisfy (2.6.41) for any η, we set

$$a_0 = -\frac{2A_0}{\pi\rho_0^2}, \quad \textbf{(2.6.42)}$$

$$-(4\pi^2 + \rho_0^2)a_1 + 4\pi\rho_0\nu_0 b_1 = \frac{A_0}{2}, \quad \textbf{(2.6.43)}$$

$$4\pi\rho_0\nu_0 a_1 + (4\pi^2 + \rho_0^2)b_1 = 0, \quad \textbf{(2.6.44)}$$

$$(16n^2\pi^2 + \rho_0^2)a_{2n} - 8n\pi\rho_0\nu_0 b_{2n} = \frac{2A_0(-1)^n}{\pi(4n^2-1)} \quad \textbf{(2.6.45)}$$

and

$$8n\pi\rho_0\nu_0 a_{2n} + (16n^2\pi^2 + \rho_0^2)b_{2n} = 0 \quad \textbf{(2.6.46)}$$

or

$$[16\pi^2\rho_0^2\nu_0^2 + (4\pi^2 + \rho_0^2)^2]a_1 = -\frac{(4\pi^2 + \rho_0^2)A_0}{2}, \quad \textbf{(2.6.47)}$$

$$[16\pi^2\rho_0^2\nu_0^2 + (4\pi^2 + \rho_0^2)^2]b_1 = 2\pi\rho_0\nu_0 A_0, \quad \textbf{(2.6.48)}$$

$$[64n^2\pi^2\rho_0^2\nu_0^2 + (16n^2\pi^2 + \rho_0^2)^2]a_{2n} = \frac{2A_0(-1)^n(16n^2\pi^2 + \rho_0^2)}{\pi(4n^2-1)} \quad \textbf{(2.6.49)}$$

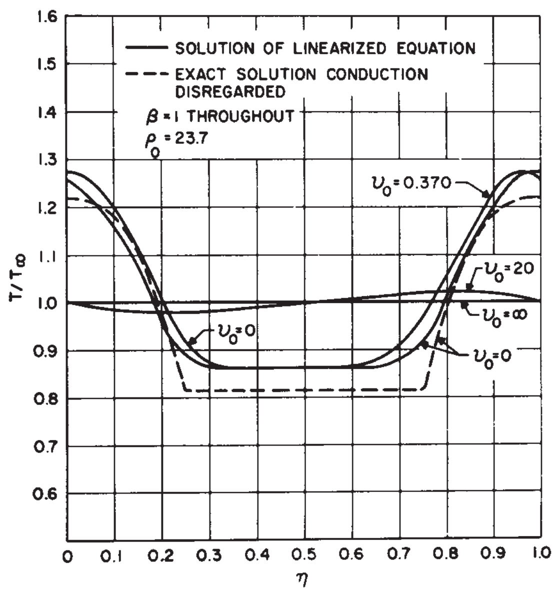


Figure 2.6.1: Temperature distribution along the equator of a spinning spherical satellite. (From Hrycak, P., 1963: Temperature distribution in a spinning spherical space vehicle. *AIAA J.*, **1**, 97. ©1963 AIAA, reprinted with permission.)

and

$$[64n^2\pi^2\rho_0^2\nu_0^2 + (16n^2\pi^2 + \rho_0^2)^2]b_{2n} = -\frac{16(-1)^n\rho_0\nu_0 nA_0}{4n^2-1}. \quad \textbf{(2.6.50)}$$

Substituting for a_0, a_1, b_1, a_{2n}, and b_{2n}, the particular solution is

$$\begin{aligned} y(\eta) = &-\frac{A_0}{\pi\rho_0^2} - \frac{(4\pi^2+\rho_0^2)A_0\cos(2\pi\eta)}{2[(4\pi^2+\rho_0^2)^2 + 16\pi^2\rho_0^2\nu_0^2]} + \frac{2\pi\rho_0\nu_0 A_0\sin(2\pi\eta)}{(4\pi^2+\rho_0^2)^2+16\pi^2\rho_0^2\nu_0^2} \\ &+ \frac{2A_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n(16n^2\pi^2+\rho_0^2)\cos(2n\pi\eta)}{(4n^2-1)[64n^2\pi^2\rho_0^2\nu_0^2+(16n^2\pi^2+\rho_0^2)^2]} \\ &- 16\rho_0\nu_0A_0\sum_{n=1}^{\infty}\frac{(-1)^n n\sin(2n\pi\eta)}{(4n^2-1)[64n^2\pi^2\rho_0^2\nu_0^2+(16n^2\pi^2+\rho_0^2)^2]}. \end{aligned} \quad \textbf{(2.6.51)}$$

In Figure 2.6.1 we reproduce a figure from Hrycak's paper showing the variation of the nondimensional temperature as a function of η for the spinning rate ν_0. The other parameters are typical of a satellite with aluminum skin and fully covered with glass-protected solar cells.

As a check on the solution, we show the temperature field (the dashed line) of a nonrotating satellite where we neglect the effects of conduction and only radiation occurs. The difference between the $\nu_0 = 0$ solid and dashed lines arises primarily due to the *linearization* of the nonlinear radiation boundary condition during the derivation of the governing equations.

Problems

Solve the following ordinary differential equations by Fourier series if the forcing is by the periodic function

$$f(t) = \begin{cases} 1, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

and $f(t) = f(t + 2\pi)$:

1. $y'' - y = f(t)$, 2. $y'' + y = f(t)$, 3. $y'' - 3y' + 2y = f(t)$.

Solve the following ordinary differential equations by *complex* Fourier series if the forcing is by the periodic function

$$f(t) = |t|, \qquad -\pi < t < \pi,$$

and $f(t) = f(t + 2\pi)$:

4. $y'' - y = f(t)$, 5. $y'' + 4y = f(t)$.

6. An object radiating into its nocturnal surrounding has a temperature $y(t)$ governed by the equation[17]:

$$\frac{dy}{dt} + ay = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega t) + B_n \sin(n\omega t),$$

where the constant a is the heat loss coefficient and the Fourier series describes the temporal variation of the atmospheric air temperature and the effective sky temperature. If $y(0) = T_0$, find $y(t)$.

7. The equation that governs the charge q on the capacitor of an LRC electrical circuit is

$$q'' + 2\alpha q' + \omega^2 q = \omega^2 E,$$

[17] Reprinted from *Solar Energy*, **28**, Sodha, M. S., Transient radiative cooling, 541, ©1982, with the kind permission from Elsevier Science Ltd, The Boulevard, Langford Lane, Kidlington, OX5 1GB, UK.

where $\alpha = R/2L$, $\omega^2 = 1/LC$, R denotes resistance, C denotes capacitance, L denotes the inductance, and E is the electromotive force driving the circuit. If E is given by

$$E = \sum_{n=-\infty}^{\infty} \varphi_n e^{in\omega_0 t},$$

find $q(t)$.

2.7 FINITE FOURIER SERIES

In many applications we must construct a Fourier series from values given by data or a graph. Unlike the situation for an analytic formula where we have an infinite number of data points and, consequently, an infinite number of terms in the Fourier series, the Fourier series contains a finite number of sine and cosines. This number is controlled by the number of data points; there must be at least two points (one for the crest, the other for the trough) to resolve the highest harmonic.

Assuming that these series are useful, the next question is how do we find the Fourier coefficients? We could compute them by numerically integrating (2.1.6). However, the results would suffer from the truncation errors that afflict all numerical schemes. On the other hand, we can avoid this problem if we again employ the orthogonality properties of sines and cosines, now in their discrete form. Just as in the case of conventional Fourier series, we can use these properties to derive formulas for computing the Fourier coefficients. These results will be *exact* except for roundoff errors.

We begin our analysis by deriving some preliminary results. Let us define $x_m = mP/(2N)$. Then, if k is an integer,

$$\sum_{m=0}^{2N-1} \exp\left(\frac{2\pi i k x_m}{P}\right) = \sum_{m=0}^{2N-1} \exp\left(\frac{km\pi i}{N}\right) = \sum_{m=0}^{2N-1} r^m \quad \textbf{(2.7.1)}$$

$$= \begin{cases} \frac{1-r^{2N}}{1-r} = 0, & r \neq 1 \\ 2N, & r = 1 \end{cases} \quad \textbf{(2.7.2)}$$

because $r^{2N} = \exp(2\pi k i) = 1$ if $r \neq 1$. If $r = 1$, then the sum consists of $2N$ terms, each of which equals one. The condition $r = 1$ corresponds to $k = 0, \pm 2N, \pm 4N, \ldots$. Taking the real and imaginary part of (2.7.2),

$$\sum_{m=0}^{2N-1} \cos\left(\frac{2\pi k x_m}{P}\right) = \begin{cases} 0, & k \neq 0, \pm 2N, \pm 4N, \ldots \\ 2N, & k = 0, \pm 2N, \pm 4N, \ldots \end{cases} \quad \textbf{(2.7.3)}$$

and

$$\sum_{m=0}^{2N-1} \sin\left(\frac{2\pi k x_m}{P}\right) = 0 \tag{2.7.4}$$

for all k.

Consider now the following sum:

$$\sum_{m=0}^{2N-1} \cos\left(\frac{2\pi k x_m}{P}\right) \cos\left(\frac{2\pi j x_m}{P}\right)$$

$$= \frac{1}{2}\sum_{m=0}^{2N-1} \left\{\cos\left[\frac{2\pi(k+j)x_m}{P}\right] + \cos\left[\frac{2\pi(k-j)x_m}{P}\right]\right\} \tag{2.7.5}$$

$$= \begin{cases} 0, & |k-j| \text{ and } |k+m| \neq 0, 2N, 4N, \ldots \\ N, & |k-j| \text{ or } |k+m| \neq 0, 2N, 4N, \ldots \\ 2N, & |k-j| \text{ and } |k+m| = 0, 2N, 4N, \ldots \end{cases} \tag{2.7.6}$$

Let us simplify the right side of (2.7.6) by restricting ourselves to $k+j$ lying between 0 to $2N$. This is permissible because of the periodic nature of (2.7.5). If $k+j=0$, $k=j=0$; if $k+j=2N$, $k=j=N$. In either case, $k-j=0$ and the right side of (2.7.6) equals $2N$. Consider now the case $k \neq j$. Then $k+j \neq 0$ or $2N$ and $k-j \neq 0$ or $2N$. The right side of (2.7.6) must equal 0. Finally, if $k=j \neq 0$ or N, then $k+j \neq 0$ or $2N$ but $k-j=0$ and the right side of (2.7.6) equals N. In summary,

$$\sum_{m=0}^{2N-1} \cos\left(\frac{2\pi k x_m}{P}\right) \cos\left(\frac{2\pi j x_m}{P}\right) = \begin{cases} 0, & k \neq j \\ N, & k = j \neq 0, N \\ 2N, & k = j = 0, N. \end{cases} \tag{2.7.7}$$

In a similar manner,

$$\sum_{m=0}^{2N-1} \cos\left(\frac{2\pi k x_m}{P}\right) \sin\left(\frac{2\pi j x_m}{P}\right) = 0 \tag{2.7.8}$$

for all k and j and

$$\sum_{m=0}^{2N-1} \sin\left(\frac{2\pi k x_m}{P}\right) \sin\left(\frac{2\pi j x_m}{P}\right) = \begin{cases} 0, & k \neq j \\ N, & k = j \neq 0, N \\ 0, & k = j = 0, N. \end{cases} \tag{2.7.9}$$

Armed with (2.7.7)–(2.7.9) we are ready to find the coefficients A_n and B_n of the finite Fourier series,

$$f(x) = \frac{A_0}{2} + \sum_{k=1}^{N-1}\left[A_k \cos\left(\frac{2\pi k x}{P}\right) + B_k \sin\left(\frac{2\pi k x}{P}\right)\right]$$
$$+ \frac{A_N}{2}\cos\left(\frac{2\pi N x}{P}\right), \tag{2.7.10}$$

where we have $2N$ data points and we now define P as the period of the function.

To find A_k we proceed as before and multiply (2.7.10) by $\cos(2\pi jx/P)$ (j may take on values from 0 to N) and sum from 0 to $2N-1$. At the point $x = x_m$,

$$\begin{aligned}\sum_{m=0}^{2N-1} f(x_m)\cos\left(\frac{2\pi j}{P}x_m\right) = {} & \frac{A_0}{2}\sum_{m=0}^{2N-1}\cos\left(\frac{2\pi j}{P}x_m\right) \\ & + \sum_{k=1}^{N-1} A_k \sum_{m=0}^{2N-1}\cos\left(\frac{2\pi k}{P}x_m\right)\cos\left(\frac{2\pi j}{P}x_m\right) \\ & + \sum_{k=1}^{N-1} B_k \sum_{m=0}^{2N-1}\sin\left(\frac{2\pi k}{P}x_m\right)\cos\left(\frac{2\pi j}{P}x_m\right) \\ & + \frac{A_N}{2}\sum_{m=0}^{2N-1}\cos\left(\frac{2\pi N}{P}x_m\right)\cos\left(\frac{2\pi j}{P}x_m\right).\end{aligned} \quad \textbf{(2.7.11)}$$

If $j \neq 0$ or N, then the first summation on the right side vanishes by (2.7.3), the third by (2.7.9), and the fourth by (2.7.7). The second summation does *not* vanish if $k = j$ and equals N. Similar considerations lead to the formulas for the calculation of A_k and B_k:

$$A_k = \frac{1}{N}\sum_{m=0}^{2N-1} f(x_m)\cos\left(\frac{2\pi k}{P}x_m\right), \qquad k = 0, 1, 2, \ldots, N \quad \textbf{(2.7.12)}$$

and

$$B_k = \frac{1}{N}\sum_{m=0}^{2N-1} f(x_m)\sin\left(\frac{2\pi k}{P}x_m\right), \qquad k = 1, 2, \ldots, N-1. \quad \textbf{(2.7.13)}$$

If there are $2N+1$ data points and $f(x_0) = f(x_{2N})$, then (2.7.12)–(2.7.13) is still valid and we need only consider the first $2N$ points. If $f(x_0) \neq f(x_{2N})$, we can still use our formulas if we require that the endpoints have the value of $[f(x_0)+f(x_{2N})]/2$. In this case the formulas for the coefficients A_k and B_k are

$$A_k = \frac{1}{N}\left[\frac{f(x_0)+f(x_{2N})}{2} + \sum_{m=1}^{2N-1} f(x_m)\cos\left(\frac{2\pi k}{P}x_m\right)\right], \quad \textbf{(2.7.14)}$$

where $k = 0, 1, 2, \ldots, N$ and

$$B_k = \frac{1}{N}\sum_{m=1}^{2N-1} f(x_m)\sin\left(\frac{2\pi k}{P}x_m\right), \quad \textbf{(2.7.15)}$$

Table 2.7.1: The Depth of Water in the Harbor at Buffalo, NY (Minus the Low-Water Datum of 568.8 ft) on the 15th Day of Each Month During 1977.

mo	n	depth	mo	n	depth	mo	n	depth
Jan	1	1.61	May	5	3.16	Sep	9	2.42
Feb	2	1.57	Jun	6	2.95	Oct	10	2.95
Mar	3	2.01	Jul	7	3.10	Nov	11	2.74
Apr	4	2.68	Aug	8	2.90	Dec	12	2.63

where $k = 1, 2, \ldots, N-1$.

It is important to note that $2N$ data points yield $2N$ Fourier coefficients A_k and B_k. Consequently our sampling frequency will always limit the amount of information, whether in the form of data points or Fourier coefficients. It might be argued that from the Fourier series representation of $f(t)$ we could find the value of $f(t)$ for any given t, which is more than we can do with the data alone. This is not true. Although we can calculate a value for $f(t)$ at any t using the finite Fourier series, we simply do not know whether those values are correct or not. They are simply those given by a finite Fourier series which fit the given data points. Despite this, the Fourier analysis of finite data sets yields valuable physical insights into the processes governing many physical systems.

• **Example 2.7.1: Water depth at Buffalo, NY**

Each entry[18] in Table 2.7.1 gives the observed depth of water at Buffalo, NY (minus the low-water datum of 568.6 ft) on the 15th of the corresponding month during 1977. Assuming that the water level is a periodic function of 1 year, and that we took the observations at equal intervals, we want to construct a finite Fourier series from these data. This corresponds to computing the Fourier coefficients $A_0, A_1, \ldots, A_6, B_1, \ldots, B_5$, which give the mean level and harmonic fluctuations of the depth of water, the harmonics having the periods 12 months, 6 months, 4 months, and so forth.

In this problem, P equals 12 months, $N = P/2 = 6$ and $x_m = mP/(2N) = m(12 \text{ mo})/12 = m$ mo. That is, there should be a data

[18] National Ocean Survey, 1977: Great Lakes Water Level, 1977, Daily and Monthly Average Water Surface Elevations, National Oceanic and Atmospheric Administration, Rockville, MD.

point for each month. From (2.7.12) and (2.7.13),

$$A_k = \frac{1}{6}\sum_{m=0}^{11} f(x_m)\cos\left(\frac{mk\pi}{6}\right), \quad k = 0, 1, 2, 3, 4, 5, 6 \qquad \textbf{(2.7.16)}$$

and

$$B_k = \frac{1}{6}\sum_{m=0}^{11} f(x_m)\sin\left(\frac{mk\pi}{6}\right), \quad k = 1, 2, 3, 4, 5. \qquad \textbf{(2.7.17)}$$

Substituting the data into (2.7.16)–(2.7.17) yields

A_0	= twice the mean level				= +5.120 ft
A_1	= harmonic component with a period of	12	mo		= −0.566 ft
B_1	= harmonic component with a period of	12	mo		= −0.128 ft
A_2	= harmonic component with a period of	6	mo		= −0.177 ft
B_2	= harmonic component with a period of	6	mo		= −0.372 ft
A_3	= harmonic component with a period of	4	mo		= −0.110 ft
B_3	= harmonic component with a period of	4	mo		= −0.123 ft
A_4	= harmonic component with a period of	3	mo		= +0.025 ft
B_4	= harmonic component with a period of	3	mo		= +0.052 ft
A_5	= harmonic component with a period of	2.4	mo		= −0.079 ft
B_5	= harmonic component with a period of	2.4	mo		= −0.131 ft
A_6	= harmonic component with a period of	2	mo		= −0.107 ft

Figure 2.7.1 is a plot of our results using (2.7.10). Note that when we include all of the harmonic terms, the finite Fourier series fits the data points exactly. The values given by the series at points between the data points may be right or they may not. To illustrate this, we also plotted the values for the first of each month. Sometimes the values given by the Fourier series and these intermediate data points are quite different.

Let us now examine our results in terms of various physical processes. In the long run the depth of water in the harbor at Buffalo, NY depends upon the three-way balance between precipitation, evaporation, and inflow-outflow of any rivers. Because the inflow and outflow of the rivers depends strongly upon precipitation, and evaporation is of secondary importance, the water level should correlate with the precipitation rate. It is well known that more precipitation falls during the warmer months rather than the colder months. The large amplitude of the Fourier coefficient A_1 and B_1, corresponding to the annual cycle ($k = 1$), reflects this.

Another important term in the harmonic analysis corresponds to the semiannual cycle ($k = 2$). During the winter months around Lake

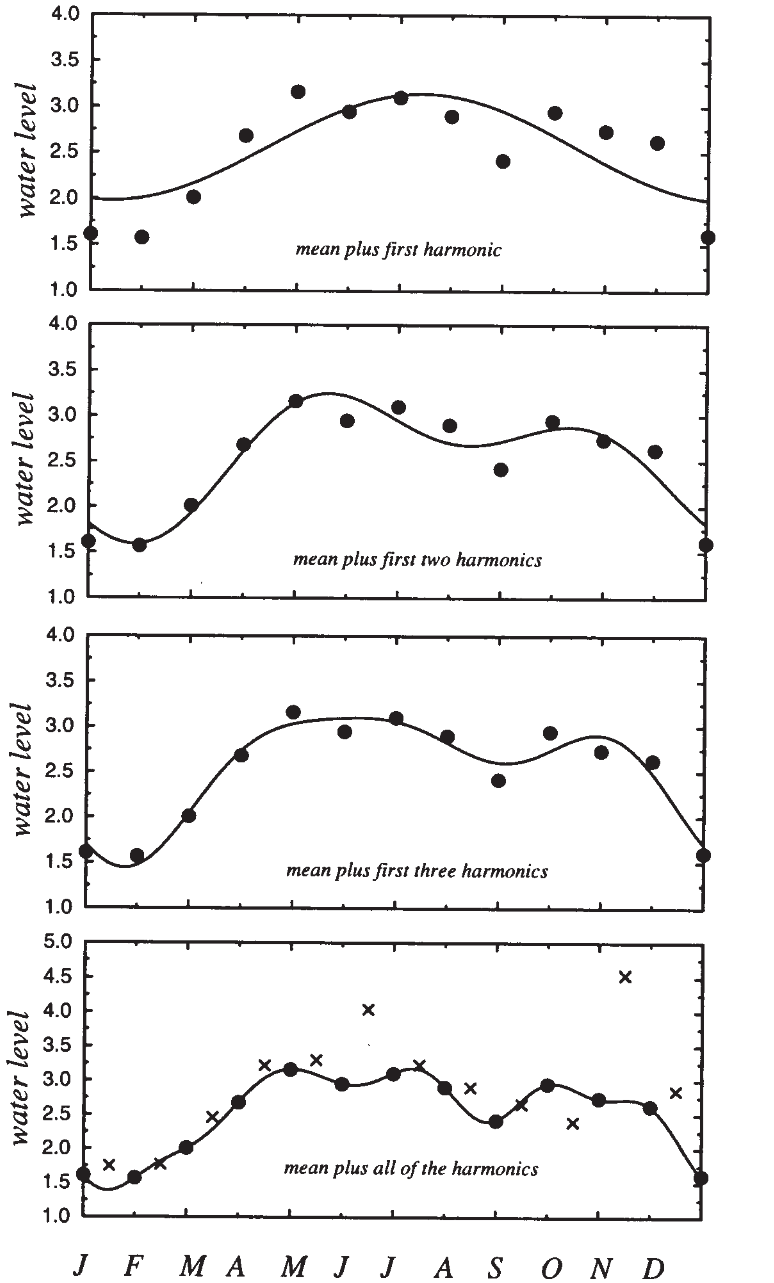


Figure 2.7.1: Partial sums of the finite Fourier series for the depth of water in the harbor of Buffalo, NY during 1977. Circles indicate observations on the 15^{th} of the month; crosses are observations on the first.

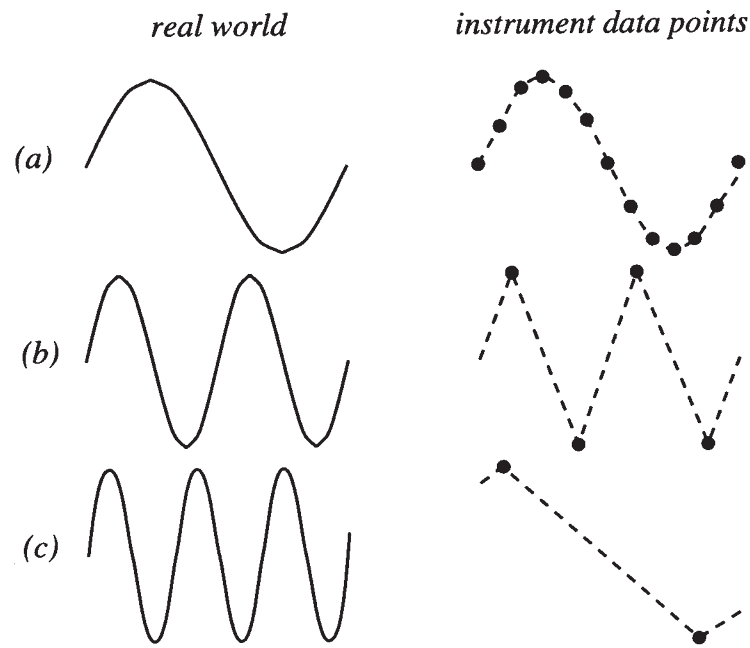


Figure 2.7.2: The effect of sampling in the representation of periodic functions.

Ontario, precipitation falls as snow. Therefore, the inflow from rivers is greatly reduced. When spring comes, the snow and ice melt and a jump in the water level occurs. Because the second harmonic gives periodic variations associated with seasonal variations, this harmonic is absolutely necessary if we want to get the correct answer while the higher harmonics do not represent any specific physical process.

• **Example 2.7.2: Aliasing**

In the previous example, we could only resolve phenomena with a period of 2 months or greater although we had data for each of the 12 months. This is an example of *Nyquist's sampling criteria*[19]: At least two samples are required to resolve the highest frequency in a periodically sampled record.

Figure 2.7.2 will help explain this phenomenon. In case (a) we have quite a few data points over one cycle. Consequently our picture, constructed from data, is fairly good. In case (b), we have only taken samples at the ridges and troughs of the wave. Although our picture

---

[19] Nyquist, H., 1928: Certain topics in telegraph transmission theory. *AIEE Trans.*, **47**, 617–644.

of the real phenomenon is poor, at least we know that there is a wave. From this picture we see that even if we are lucky enough to take our observations at the ridges and troughs of a wave, we need at least two data points per cycle (one for the ridge, the other for the trough) to resolve the highest-frequency wave.

In case (c) we have made a big mistake. We have taken a wave of frequency N Hz and misrepresented it as a wave of frequency $N/2$ Hz. This mispresentation of a high-frequency wave by a lower-frequency wave is called *aliasing*. It arises because we are sampling a continuous signal at equal intervals. By comparing cases (b) and (c), we see that there is a cutoff between aliased and nonaliased frequencies. This frequency is called the *Nyquist* or *folding* frequency. It corresponds to the highest frequency resolved by our finite Fourier analysis.

Because most periodic functions require an infinite number of harmonics for their representation, aliasing of signals is a common problem. Thus the question is not "can I avoid aliasing?" but "can I live with it?" Quite often, we can construct our experiments to say yes. An example where aliasing is unavoidable occurs in a Western at the movies when we see the rapidly rotating spokes of the stagecoach's wheel. A movie is a sampling of continuous motion where we present the data as a succession of pictures. Consequently, a film aliases the high rate of revolution of the stagecoach's wheel in such a manner so that it appears to be stationary or rotating very slowly.

• **Example 2.7.3: Spectrum of the Chesapeake Bay**

For our final example we will perform a Fourier analysis of hourly sea-level measurements taken at the mouth of the Chesapeake Bay during the 2000 days from 9 April 1985 to 29 June 1990. Figure 2.7.3 shows 200 days of this record, starting from 1 July 1985. As this figure shows, the measurements contain a wide range of oscillations. In particular, note the large peak near day 90 which corresponds to the passage of hurricane Gloria during the early hours of 27 September 1985.

Utilizing the entire 2000 days, we have plotted the amplitude of the Fourier coefficients as a function of period in Figure 2.7.4. We see a general rise of the amplitude as the period increases. Especially noteworthy are the sharp peaks near periods of 12 and 24 hours. The largest peak is at 12.417 hours and corresponds to the semidiurnal tide. Thus, our Fourier analysis has shown that the dominant oscillations at the mouth of the Chesapeake Bay are the tides. A similar situation occurs in Baltimore harbor. Furthermore, with this spectral information we could predict high and low tides very accurately.

Although the tides are of great interest to many, they are a nuisance to others because they mask other physical processes that might be occurring. For that reason we would like to remove them from the tidal

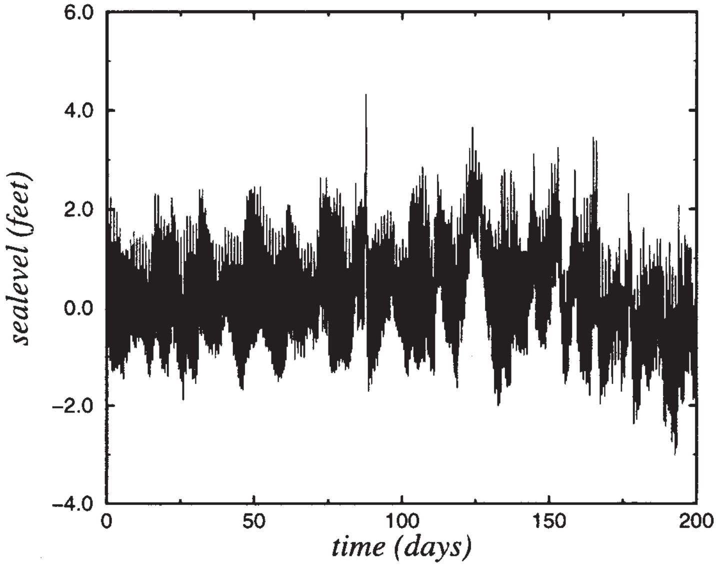


Figure 2.7.3: The sea elevation at the mouth of the Chesapeake Bay from its average depth as a function of time after 1 July 1985.

gauge history and see what is left. One way would be to zero out the Fourier coefficients corresponding to the tidal components and then plot the resulting Fourier series. Another method is to replace each hourly report with an average of hourly reports that occurred 24 hours ahead and behind of a particular report. We will construct this average in such a manner that waves with periods of the tides sum to zero.[20] Such a *filter* is a popular method for eliminating unwanted waves from a record. Filters play an important role in the analysis of data. We have plotted the filtered sea level data in Figure 2.7.5. Note that summertime (0–50 days) produces little variation in the sea level compared to wintertime (100–150 days) when intense coastal storms occur.

Problems

Find the finite Fourier series for the following pieces of data:

1. $x(0) = 0$, $x(1) = 1$, $x(2) = 2$, $x(3) = 3$ and $N = 2$.

2. $x(0) = 1$, $x(1) = 1$, $x(2) = -1$, $x(3) = -1$ and $N = 2$.

---

[20] See Godin, G., 1972: *The Analysis of Tides*, University of Toronto Press, Toronto, Section 2.1.

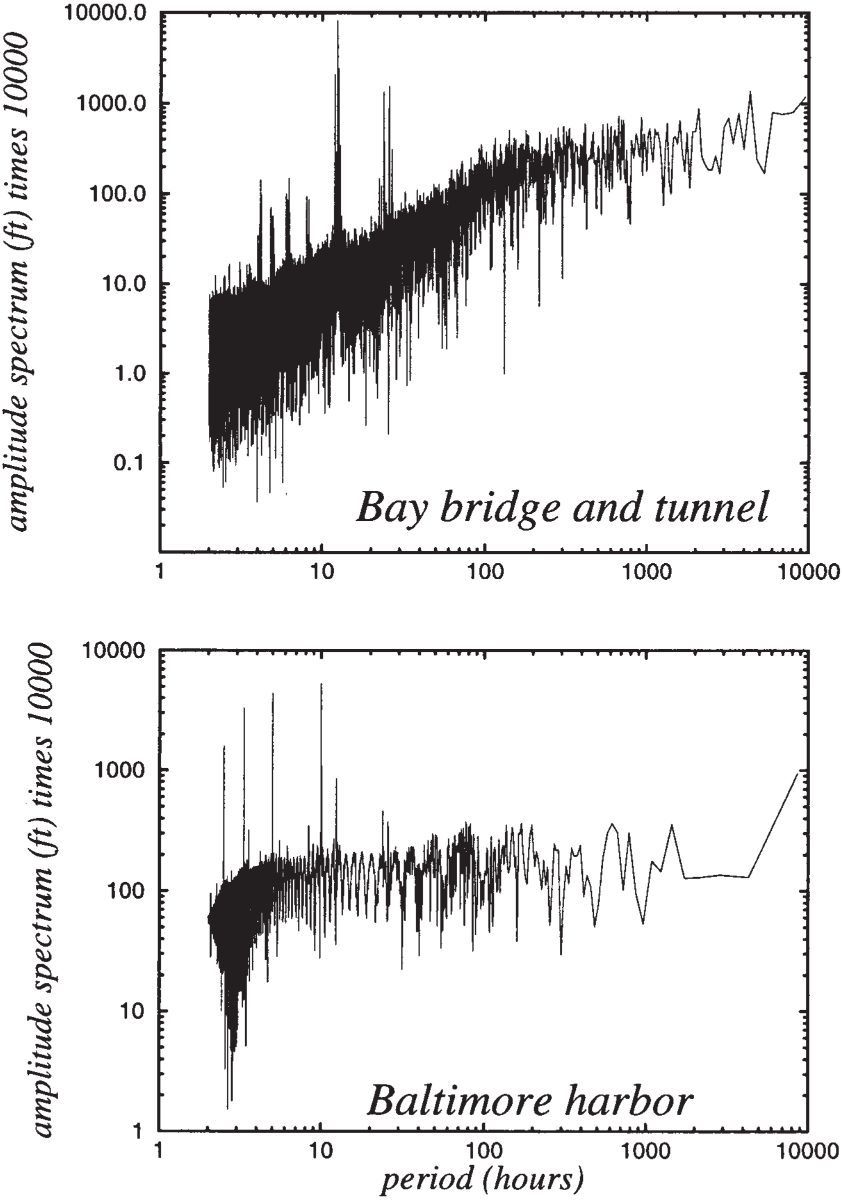


Figure 2.7.4: The amplitude of the Fourier coefficients for the sea elevation at the Chesapeake Bay bridge and tunnel (top) and Baltimore harbor (bottom) as a function of period.

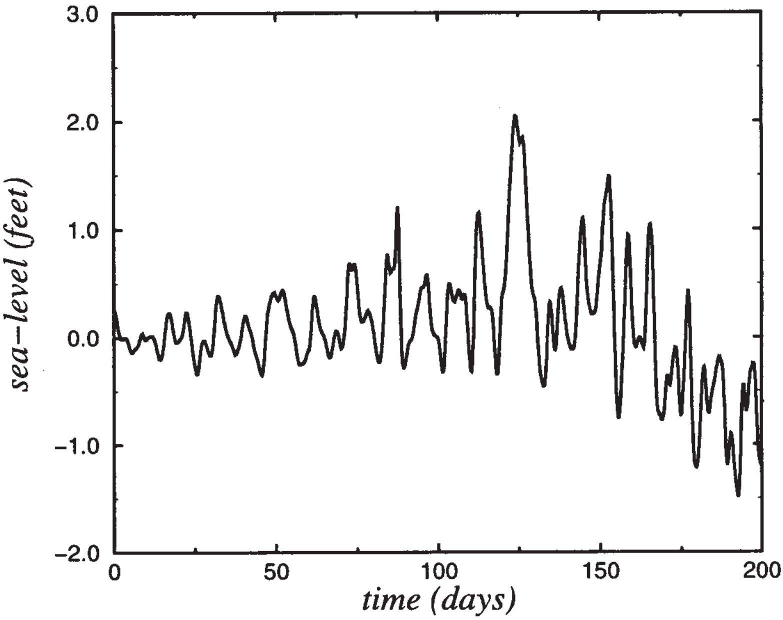


Figure 2.7.5: Same as Figure 2.7.3 but with the tides removed.

Project: Spectrum of the Earth's Orography

Table 2.7.3 gives the orographic height of the earth's surface used in an atmospheric general circulation model (GCM) at a resolution of 2.5° longitude along the latitude belts of 28°S, 36°N, and 66°N. In this project you will find the spectrum of this orographic field along the various latitude belts.

Step 1: Write code to read in the data and find A_n and B_n. Although you could code (2.7.12)–(2.7.13), no one does Fourier analysis that way any more. They use a fast Fourier transform (FFT) that is available as a system's routine on their computer or use one that is given in various computer books.[21] Many of these routines deal with finite Fourier series in its complex form. The only way that you can be confident of your results is to first create a data set with a known Fourier series, for example:

$$f(x) = 5 + \cos\left(\frac{2\pi x}{2N}\right) + 3\sin\left(\frac{2\pi x}{2N}\right) + 6\cos\left(\frac{6\pi x}{2N}\right),$$

[21] For example, Press, W. H., Flannery, B. P., Teukolsky, S. A., and Vetterling, W. T., 1986: *Numerical Recipes: The Art of Scientific Computing*, Cambridge University Press, New York, chap. 12.

Table 2.7.2: The Fourier Coefficients Generated by the IMSL Subroutine FFTRF with $N = 8$ for the Test Case Given in Step 1 of the Project.

x	$f(x)$	Fourier coefficient	Value of Fourier coefficient
0.00000	12.00000	$2NA_0$	80.00001
1.00000	9.36803	NA_1	8.00001
2.00000	3.58579	$-NB_1$	−24.00000
3.00000	2.61104	NA_2	0.00000
4.00000	8.00000	$-NB_2$	0.00001
5.00000	12.93223	NA_3	48.00000
6.00000	10.65685	$-NB_3$	0.00001
7.00000	2.92807	NA_4	−0.00001
8.00000	−2.00000	$-NB_4$	0.00000
9.00000	0.63197	NA_5	0.00000
10.00000	6.41421	$-NB_5$	0.00000
11.00000	7.38895	NA_6	0.00000
12.00000	2.00000	$-NB_6$	0.00000
13.00000	−2.93223	NA_7	−0.00001
14.00000	−0.65685	$-NB_7$	0.00000
15.00000	7.07193	$2NA_8$	0.00001

and then find the Fourier coefficients given by the subroutine. In Table 2.7.3 we show the results from using the IMSL routine FFTRF. From these results, you see that the Fourier coefficients given by the subroutine are multiplied by N and the B_ns are of opposite sign.

Step 2: Construct several spectra by using every data point, every other data point, etc. How do the magnitudes of the Fourier coefficient change? You might like to read about *leakage* from a book on harmonic analysis.[22]

Step 3: Compare and contrast the spectra from the various latitude belts. How do the magnitudes of the Fourier coefficients decrease with n? Why are there these differences?

Step 4: You may have noted that some of the heights are negative, even in the middle of the ocean! Take the original data (for any latitude belt) and zero out all of the negative heights. Find the spectra for this

[22] For example, Bloomfield, P., 1976: *Fourier Analysis of Time Series: An Introduction*, John Wiley & Sons, New York.

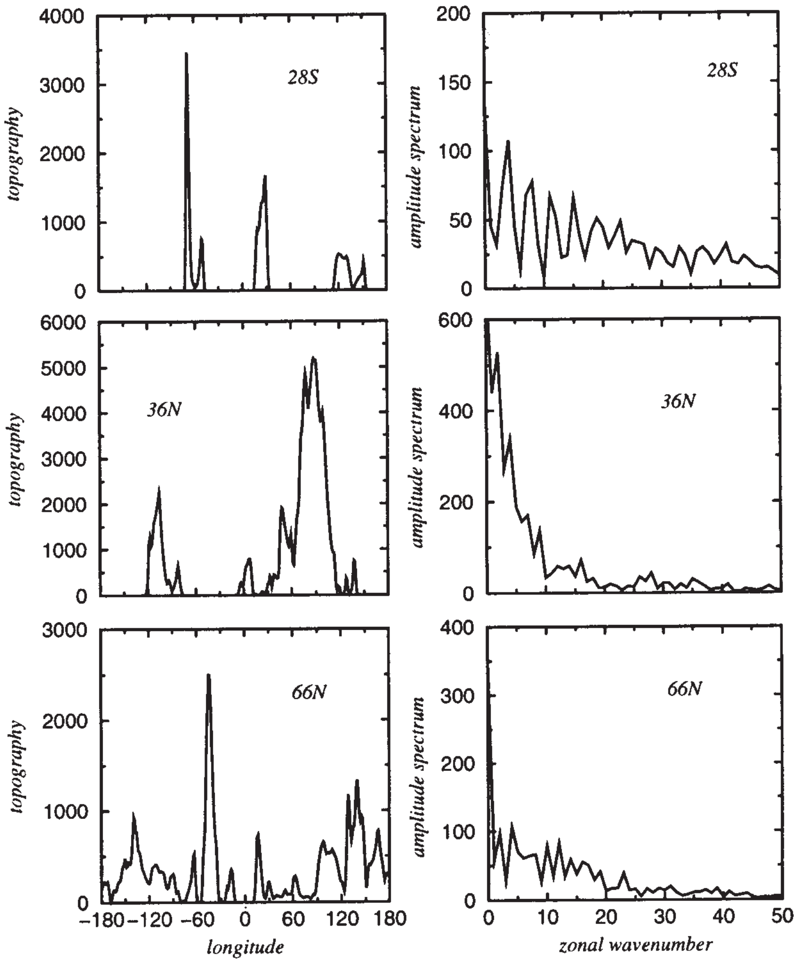


Figure 2.7.6: The orography of the earth and its spectrum in meters along three latitude belts.

new data set. How has the spectra changed? Is there a reason why the negative heights were introduced?

Table 2.7.3: Orographic Heights (in m) Along Three Latitude Belts.

Longitude	28°S	36°N	66°N	Longitude	28°S	36°N	66°N
−180.0	4.	3.	2532.	−82.5	36.	4047.	737.
−177.5	1.	−2.	1665.	−80.0	−64.	3938.	185.
−175.0	1.	2.	1432.	−77.5	138.	1669.	71.
−172.5	1.	−3.	1213.	−75.0	−363.	236.	160.
−170.0	1.	1.	501.	−72.5	4692.	31.	823.
−167.5	1.	−3.	367.	−70.0	19317.	−8.	1830.
−165.0	1.	1.	963.	−67.5	21681.	0.	3000.
−162.5	0.	0.	1814.	−65.0	9222.	−2.	3668.
−160.0	−1.	6.	2562.	−62.5	1949.	−2.	2147.
−157.5	0.	1.	3150.	−60.0	774.	0.	391.
−155.0	0.	3.	4008.	−57.5	955.	5.	−77.
−152.5	1.	−2.	4980.	−55.0	2268.	6.	601.
−150.0	−1.	4.	6011.	−52.5	4636.	−1.	3266.
−147.5	6.	−1.	6273.	−50.0	4621.	2.	9128.
−145.0	14.	3.	5928.	−47.5	1300.	−4.	17808.
−142.5	6.	−1.	6509.	−45.0	−91.	1.	22960.
−140.0	−2.	6.	7865.	−42.5	57.	−1.	20559.
−137.5	0.	3.	7752.	−40.0	−25.	4.	14296.
−135.0	−2.	5.	6817.	−37.5	13.	−1.	9783.
−132.5	1.	−2.	6272.	−35.0	−10.	6.	5969.
−130.0	−2.	0.	5582.	−32.5	8.	2.	1972.
−127.5	0.	5.	4412.	−30.0	−4.	22.	640.
−125.0	−2.	423.	3206.	−27.5	6.	33.	379.
−122.5	1.	3688.	2653.	−25.0	−2.	39.	286.
−120.0	−3.	10919.	2702.	−22.5	3.	2.	981.
−117.5	2.	16148.	3062.	−20.0	−3.	11.	1971.
−115.0	−3.	17624.	3344.	−17.5	1.	−6.	2576.
−112.5	7.	18132.	3444.	−15.0	−1.	19.	1692.
−110.0	12.	19511.	3262.	−12.5	0.	−18.	357.
−107.5	9.	22619.	3001.	−10.0	−1.	490.	−21.
−105.0	−5.	20273.	2931.	−7.5	0.	2164.	−5.
−102.5	3.	12914.	2633.	−5.0	1.	4728.	−10.
−100.0	−5.	7434.	1933.	−2.5	0.	5347.	0.
−97.5	6.	4311.	1473.	0.0	4.	2667.	−6.
−95.0	−8.	2933.	1689.	2.5	−5.	1213.	−1.
−92.5	8.	2404.	2318.	5.0	7.	1612.	−31.
−90.0	−12.	1721.	2285.	7.5	−13.	1744.	−58.
−87.5	18.	1681.	1561.	10.0	28.	1153.	381.
−85.0	−23.	2666.	1199.	12.5	107.	838.	2472.

Table 2.7.3, contd.: Orographic Heights (in m) Along Three Latitude Belts.

Longitude	28°S	36°N	66°N	Longitude	28°S	36°N	66°N
15.0	2208.	1313.	5263.	97.5	0.	35538.	6222.
17.5	6566.	862.	5646.	100.0	−2.	31985.	5523.
20.0	9091.	1509.	3672.	102.5	0.	23246.	4823.
22.5	10690.	2483.	1628.	105.0	−4.	17363.	4689.
25.0	12715.	1697.	889.	107.5	2.	14315.	4698.
27.5	14583.	3377.	1366.	110.0	−17.	12639.	4674.
30.0	11351.	7682.	1857.	112.5	302.	10543.	4435.
32.5	3370.	9663.	1534.	115.0	1874.	4967.	3646.
35.0	15.	10197.	993.	117.5	4005.	1119.	2655.
37.5	49.	10792.	863.	120.0	4989.	696.	2065.
40.0	−31.	11322.	756.	122.5	4887.	475.	1583.
42.5	20.	13321.	620.	125.0	4445.	1631.	3072.
45.0	−17.	15414.	626.	127.5	4362.	2933.	7290.
47.5	−19.	12873.	836.	130.0	4368.	1329.	8541.
50.0	−18.	6114.	1029.	132.5	3485.	88.	7078.
52.5	6.	2962.	946.	135.0	1921.	598.	7322.
55.0	−2.	4913.	828.	137.5	670.	1983.	9445.
57.5	3.	6600.	1247.	140.0	666.	2511.	10692.
60.0	−3.	4885.	2091.	142.5	1275.	866.	9280.
62.5	2.	3380.	2276.	145.0	1865.	13.	8372.
65.0	−1.	5842.	1870.	147.5	2452.	11.	6624.
67.5	2.	12106.	1215.	150.0	3160.	−4.	3617.
70.0	0.	23032.	680.	152.5	2676.	−1.	2717.
72.5	2.	35376.	531.	155.0	697.	0.	3474.
75.0	−1.	36415.	539.	157.5	−67.	−3.	4337.
77.5	1.	26544.	579.	160.0	25.	3.	4824.
80.0	0.	19363.	554.	162.5	−12.	−1.	5525.
82.5	1.	17915.	632.	165.0	10.	4.	6323.
85.0	−2.	22260.	791.	167.5	−5.	−2.	5899.
87.5	−1.	30442.	1455.	170.0	0.	1.	4330.
90.0	−3.	33601.	3194.	172.5	0.	−4.	3338.
92.5	−1.	30873.	4878.	175.0	4.	3.	3408.
95.0	0.	31865.	5903.	177.5	3.	−1.	3407.

Chapter 3

The Fourier Transform

In the previous chapter we showed how we could expand a periodic function in terms of an infinite sum of sines and cosines. However, most functions encountered in engineering are aperiodic. As we shall see, the extension of Fourier series to these functions leads to the Fourier transform.

3.1 FOURIER TRANSFORMS

The Fourier transform is the natural extension of Fourier series to a function $f(t)$ of infinite period. To show this, consider a periodic function $f(t)$ of period $2T$ that satisfies the so-called Dirichlet's conditions.[1] If the integral $\int_a^b |f(t)|\,dt$ exists, this function has the complex Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t/T}, \tag{3.1.1}$$

[1] A function $f(t)$ satisfies Dirichlet's conditions in the interval (a, b) if (1) it is bounded in (a, b), and (2) it has at most a finite number of discontinuities and a finite number of maxima and minima in the interval (a, b).

where

$$c_n = \frac{1}{2T}\int_{-T}^{T} f(t)e^{-in\pi t/T}\,dt. \qquad (3.1.2)$$

Equation (3.1.1) applies only if $f(t)$ is continuous at t; if $f(t)$ suffers from a jump discontinuity at t, then the left side of (3.1.1) equals $\frac{1}{2}[f(t^+) + f(t^-)]$, where $f(t^+) = \lim_{x\to t^+} f(x)$ and $f(t^-) = \lim_{x\to t^-} f(x)$. Substituting (3.1.2) into (3.1.1),

$$f(t) = \frac{1}{2T}\sum_{n=-\infty}^{\infty} e^{in\pi t/T}\int_{-T}^{T} f(x)e^{-in\pi x/T}\,dx. \qquad (3.1.3)$$

Let us now introduce the notation $\omega_n = n\pi/T$ so that $\Delta\omega_n = \omega_{n+1} - \omega_n = \pi/T$. Then,

$$f(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} F(\omega_n)e^{i\omega_n t}\Delta\omega_n, \qquad (3.1.4)$$

where

$$F(\omega_n) = \int_{-T}^{T} f(x)e^{-i\omega_n x}\,dx. \qquad (3.1.5)$$

As $T \to \infty$, ω_n approaches a continuous variable ω and $\Delta\omega_n$ may be interpreted as the infinitesimal $d\omega$. Therefore, ignoring any possible difficulties.[2]

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega} \qquad (3.1.6)$$

and

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt.} \qquad (3.1.7)$$

[2] For a rigorous derivation, see Titchmarsh, E. C., 1948: *Introduction to the Theory of Fourier Integrals*, Clarendon Press, Oxford, chap. 1.

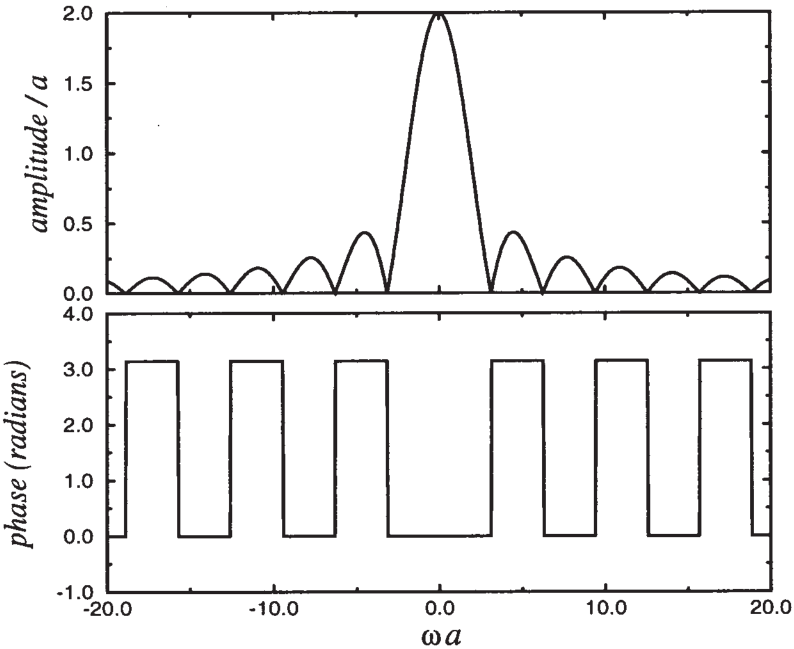


Figure 3.1.1: Graph of the Fourier transform for (3.1.9).

Equation (3.1.7) is the *Fourier transform* of $f(t)$ while (3.1.6) is the *inverse Fourier transform* which converts a Fourier transform back to $f(t)$. Alternatively, we may combine (3.1.6)–(3.1.7) to yield the equivalent real form:

$$f(t) = \frac{1}{\pi}\int_0^\infty \left\{\int_{-\infty}^\infty f(x)\cos[\omega(t-x)]\,dx\right\} d\omega. \qquad \textbf{(3.1.8)}$$

Hamming[3] has suggested the following analog in understanding the Fourier transform. Let us imagine that $f(t)$ is a light beam. Then the Fourier transform, like a glass prism, breaks up the function into its component frequencies ω, each of intensity $F(\omega)$. In optics, the various frequencies are called colors; by analogy the Fourier transform gives us the color spectrum of a function. On the other hand, the inverse Fourier transform blends a function's spectrum to give back the original function.

Most signals encountered in practice have Fourier transforms because they are absolutely integrable since they are bounded and of finite duration. However, there are some notable exceptions. Examples include the trigonometric functions sine and cosine.

---

[3] Hamming, R. W., 1977: *Digital Filters*, Prentice-Hall, Englewood Cliffs, NJ, p. 136.

• **Example 3.1.1**

Let us find the Fourier transform for

$$f(t) = \begin{cases} 1, & |t| < a \\ 0, & |t| > a. \end{cases} \tag{3.1.9}$$

From the definition of the Fourier transform,

$$F(\omega) = \int_{-\infty}^{-a} 0\, e^{-i\omega t}\,dt + \int_{-a}^{a} 1\, e^{-i\omega t}\,dt + \int_{a}^{\infty} 0\, e^{-i\omega t}\,dt \tag{3.1.10}$$

$$= \frac{e^{\omega a i} - e^{-\omega a i}}{\omega i} = \frac{2\sin(\omega a)}{\omega} = 2a\,\mathrm{sinc}(\omega a), \tag{3.1.11}$$

where $\mathrm{sinc}(x) = \sin(x)/x$ is the *sinc function.*

Although this particular example does not show it, the Fourier transform is, in general, a complex function. The most common method of displaying it is to plot its amplitude and phase on two separate graphs for all values of ω. See Figure 3.1.1. Of these two quantities, the amplitude is by far the more popular one and is given the special name of *frequency spectrum.*

From the definition of the inverse Fourier transform,

$$f(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\omega a)}{\omega} e^{i\omega t}\,d\omega = \begin{cases} 1, & |t| < a \\ 0, & |t| > a. \end{cases} \tag{3.1.12}$$

An important question is what value does $f(t)$ converge to in the limit as $t \to a$ and $t \to -a$? Because Fourier transforms are an extension of Fourier series, the behavior at a jump is the same as that for a Fourier series. For that reason, $f(a) = \frac{1}{2}[f(a^+) + f(a^-)] = \frac{1}{2}$ and $f(-a) = \frac{1}{2}[f(-a^+) + f(-a^-)] = \frac{1}{2}$.

• **Example 3.1.2: Dirac delta function**

Of the many functions that have a Fourier transform, a particularly important one is the *(Dirac) delta function.*[4] For example, in Section 3.6 we will use it to solve differential equations. We *define* it as the inverse of the Fourier transform $F(\omega) = 1$. Therefore,

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\,d\omega. \tag{3.1.13}$$

---

[4] Dirac, P. A. M., 1947: *The Principles of Quantum Mechanics,* Clarendon Press, Oxford, Section 15.

Table 3.1.1: The Fourier Transforms of Some Commonly Encountered Functions. The Heaviside Step Function $H(t)$ Is Defined by (3.2.16).

	$\mathbf{f(t)},\ \lvert\mathbf{t}\rvert < \infty$	$\mathbf{F}(\omega)$
1.	$e^{-at}H(t), \quad a > 0$	$\dfrac{1}{a+\omega i}$
2.	$e^{at}H(-t), \quad a > 0$	$\dfrac{1}{a-\omega i}$
3.	$te^{-at}H(t), \quad a > 0$	$\dfrac{1}{(a+\omega i)^2}$
4.	$te^{at}H(-t), \quad a > 0$	$\dfrac{-1}{(a-\omega i)^2}$
5.	$t^n e^{-at}H(t),\ \mathrm{Re}(a) > 0,\ n = 1, 2, \ldots$	$\dfrac{n!}{(a+\omega i)^{n+1}}$
6.	$e^{-a\lvert t\rvert}, \quad a > 0$	$\dfrac{2a}{\omega^2 + a^2}$
7.	$te^{-a\lvert t\rvert}, \quad a > 0$	$\dfrac{-4a\omega i}{(\omega^2+a^2)^2}$
8.	$\dfrac{1}{1+a^2t^2}$	$\dfrac{\pi}{\lvert a\rvert} e^{-\lvert\omega/a\rvert}$
9.	$\dfrac{\cos(at)}{1+t^2}$	$\frac{\pi}{2}\left(e^{-\lvert\omega-a\rvert} + e^{-\lvert\omega+a\rvert}\right)$
10.	$\dfrac{\sin(at)}{1+t^2}$	$\frac{\pi}{2i}\left(e^{-\lvert\omega-a\rvert} - e^{-\lvert\omega+a\rvert}\right)$
11.	$\begin{cases} 1, & \lvert t\rvert < a \\ 0, & \lvert t\rvert > a \end{cases}$	$\dfrac{2\sin(\omega a)}{\omega}$
12.	$\dfrac{\sin(at)}{at}$	$\begin{cases} \pi/a, & \lvert\omega\rvert < a \\ 0, & \lvert\omega\rvert > a \end{cases}$

To give some insight into the nature of the delta function, consider another band-limited transform:

$$F_\Omega(\omega) = \begin{cases} 1, & \lvert\omega\rvert < \Omega \\ 0, & \lvert\omega\rvert > \Omega, \end{cases} \tag{3.1.14}$$

where Ω is real and positive. Then,

$$f_\Omega(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} e^{i\omega t}\,d\omega = \frac{\Omega}{\pi}\frac{\sin(\Omega t)}{\Omega t}. \tag{3.1.15}$$

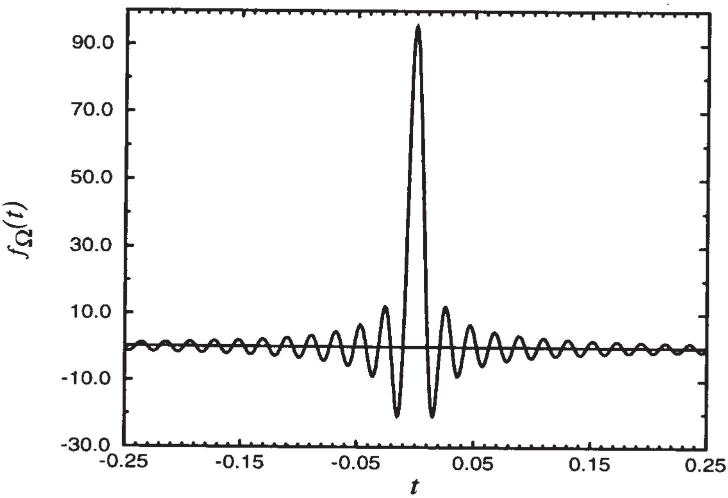


Figure 3.1.2: Graph of the function given in (3.1.15) for $\Omega = 300$.

Figure 3.1.2 illustrates $f_\Omega(t)$ for a large value of Ω. We observe that as $\Omega \to \infty$, $f_\Omega(t)$ becomes very large near $t = 0$ as well as very narrow. On the other hand, $f_\Omega(t)$ rapidly approaches zero as $|t|$ increases. Therefore, we may consider the delta function as the limit:

$$\delta(t) = \lim_{\Omega \to \infty} \frac{\sin(\Omega t)}{\pi t} \tag{3.1.16}$$

or

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0. \end{cases} \tag{3.1.17}$$

Because the Fourier transform of the delta function equals one,

$$\int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} \, dt = 1. \tag{3.1.18}$$

Since (3.1.18) must hold for any ω, we take $\omega = 0$ and find that

$$\int_{-\infty}^{\infty} \delta(t) \, dt = 1. \tag{3.1.19}$$

Thus, the area under the delta function equals unity. Taking (3.1.17) into account, we can also write (3.1.19) as

$$\int_{-a}^{b} \delta(t) \, dt = 1, \qquad a, b > 0. \tag{3.1.20}$$

Finally,

$$\int_{a}^{b} f(t) \delta(t - t_0) \, dt = f(t_0), \tag{3.1.21}$$

if $a < t_0 < b$. This follows from the law of the mean of integrals.

We may also use several other functions with equal validity to represent the delta function. These include the limiting case of the following rectangular or triangular distributions:

$$\delta(t) = \lim_{\epsilon \to 0} \begin{cases} \frac{1}{\epsilon}, & |t| < \frac{\epsilon}{2} \\ 0, & |t| > \frac{\epsilon}{2} \end{cases} \tag{3.1.22}$$

or

$$\delta(t) = \lim_{\epsilon \to 0} \begin{cases} \frac{1}{\epsilon}\left(1 - \frac{|t|}{\epsilon}\right), & |t| < \epsilon \\ 0, & |t| > \epsilon \end{cases} \tag{3.1.23}$$

and the Gaussian function:

$$\delta(t) = \lim_{\epsilon \to 0} \frac{\exp(-\pi t^2/\epsilon)}{\sqrt{\epsilon}}. \tag{3.1.24}$$

Note that the delta function is an even function.

Problems

1. Show that the Fourier transform of

$$f(t) = e^{-a|t|}, \qquad a > 0,$$

is

$$F(\omega) = \frac{2a}{\omega^2 + a^2}.$$

Now plot the amplitude and phase spectra for this transform.

2. Show that the Fourier transform of

$$f(t) = te^{-a|t|}, \qquad a > 0,$$

is

$$F(\omega) = -\frac{4a\omega i}{(\omega^2 + a^2)^2}.$$

Now plot the amplitude and phase spectra for this transform.

3. Show that the Fourier transform of

$$f(t) = \begin{cases} e^{2t}, & t < 0 \\ e^{-t}, & t > 0 \end{cases}$$

is

$$F(\omega) = \frac{3}{(2 - i\omega)(1 + i\omega)}.$$

Now plot the amplitude and phase spectra for this transform.

4. Show that the Fourier transform of

$$f(t) = \begin{cases} e^{-(1+i)t}, & t > 0 \\ -e^{(1-i)t}, & t < 0 \end{cases}$$

is

$$F(\omega) = \frac{-2i(\omega+1)}{(\omega+1)^2+1}.$$

Now plot the amplitude and phase spectra for this transform.

5. Show that the Fourier transform of

$$f(t) = \begin{cases} \cos(at), & |t| < 1 \\ 0, & |t| > 1 \end{cases}$$

is

$$F(\omega) = \frac{\sin(\omega - a)}{\omega - a} + \frac{\sin(\omega + a)}{\omega + a}.$$

Now plot the amplitude and phase spectra for this transform.

6. Show that the Fourier transform of

$$f(t) = \begin{cases} \sin(t), & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

is

$$F(\omega) = -\frac{1}{2}\left[\frac{1-\cos(\omega-1)}{\omega-1} + \frac{\cos(\omega+1)-1}{\omega+1}\right] - \frac{i}{2}\left[\frac{\sin(\omega-1)}{\omega-1} - \frac{\sin(\omega+1)}{\omega+1}\right].$$

Now plot the amplitude and phase spectra for this transform.

7. Show that the Fourier transform of

$$f(t) = \begin{cases} 1 - t/\tau, & 0 \le t < 2\tau \\ 0, & \text{otherwise} \end{cases}$$

is

$$F(\omega) = \frac{2e^{-i\omega t}}{i\omega}\left[\frac{\sin(\omega\tau)}{\omega\tau} - \cos(\omega\tau)\right]$$

Now plot the amplitude and phase spectra for this transform.

8. The integral representation[5] of the modified Bessel function $K_\nu(\,)$ is

$$K_\nu\left(a|\omega|\right) = \frac{\Gamma\left(\nu+\frac{1}{2}\right)(2a)^\nu}{|\omega|^\nu \Gamma\left(\frac{1}{2}\right)} \int_0^\infty \frac{\cos(\omega t)}{(t^2+a^2)^{\nu+1/2}}\,dt,$$

where $\Gamma(\,)$ is the gamma function, $\nu \geq 0$ and $a > 0$. Use this relationship to show that

$$\mathcal{F}\left[\frac{1}{(t^2+a^2)^{\nu+1/2}}\right] = \frac{2|\omega|^\nu \Gamma\left(\frac{1}{2}\right) K_\nu\left(a|\omega|\right)}{\Gamma\left(\nu+\frac{1}{2}\right)(2a)^\nu}.$$

9. Show that the Fourier transform of a constant K is $2\pi\delta(\omega)K$.

3.2 FOURIER TRANSFORMS CONTAINING THE DELTA FUNCTION

In the previous section we stressed the fact that such simple functions as cosine and sine are not absolutely integrable. Does this mean that these functions do not possess a Fourier transform? In this section we shall show that certain functions can still have a Fourier transform even though we cannot compute them directly.

The reason why we can find the Fourier transform of certain functions that are not absolutely integrable lies with the introduction of the delta function because

$$\int_{-\infty}^{\infty} \delta(\omega-\omega_0)e^{it\omega}\,d\omega = e^{i\omega_0 t} \qquad \textbf{(3.2.1)}$$

for all t. Thus, the inverse of the Fourier transform $\delta(\omega-\omega_0)$ is the complex exponential $e^{i\omega_0 t}/2\pi$ or

$$\mathcal{F}\left(e^{i\omega_0 t}\right) = 2\pi\delta(\omega-\omega_0). \qquad \textbf{(3.2.2)}$$

This yields immediately the result that

$$\mathcal{F}(1) = 2\pi\delta(\omega), \qquad \textbf{(3.2.3)}$$

if we set $\omega_0 = 0$. Thus, the Fourier transform of 1 is an impulse at $\omega = 0$ with weight 2π. Because the Fourier transform equals zero for all $\omega \neq 0$, $f(t) = 1$ does not contain a nonzero frequency and is consequently a DC signal.

---

[5] Watson, G. N., 1966: *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, p. 185.

Another set of transforms arises from Euler's formula because we have that

$$\mathcal{F}[\sin(\omega_0 t)] = \tfrac{1}{2i}\left[\mathcal{F}\left(e^{i\omega_0 t}\right) - \mathcal{F}\left(e^{-i\omega_0 t}\right)\right] \tag{3.2.4}$$

$$= \tfrac{\pi}{i}\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right] \tag{3.2.5}$$

$$= -\pi i\delta(\omega - \omega_0) + \pi i\delta(\omega + \omega_0) \tag{3.2.6}$$

and

$$\mathcal{F}[\cos(\omega_0 t)] = \tfrac{1}{2}\left[\mathcal{F}\left(e^{i\omega_0 t}\right) + \mathcal{F}\left(e^{-i\omega_0 t}\right)\right] \tag{3.2.7}$$

$$= \pi\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]. \tag{3.2.8}$$

Note that although the amplitude spectra of $\sin(\omega_0 t)$ and $\cos(\omega_0 t)$ are the same, their phase spectra are different.

Let us consider the Fourier transform of any arbitrary periodic function. Recall that any such function $f(t)$ with period $2L$ can be rewritten as the complex Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}, \tag{3.2.9}$$

where $\omega_0 = \pi/L$. The Fourier transform of $f(t)$ is

$$F(\omega) = \mathcal{F}[f(t)] = \sum_{n=-\infty}^{\infty} 2\pi c_n \delta(\omega - n\omega_0). \tag{3.2.10}$$

Therefore, the Fourier transform of any arbitrary periodic function is a sequence of impulses with weight $2\pi c_n$ located at $\omega = n\omega_0$ with $n = 0, \pm1, \pm2, \ldots$. Thus, the Fourier series and transform of a periodic function are closely related.

• **Example 3.2.1: Fourier transform of the sign function**

Consider the sign function

$$\operatorname{sgn}(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0. \end{cases} \tag{3.2.11}$$

The function is not absolutely integrable. However, let us approximate it by $e^{-\epsilon|t|}\operatorname{sgn}(t)$, where ϵ is a small positive number. This new function is absolutely integrable and we have that

$$\mathcal{F}[\operatorname{sgn}(t)] = \lim_{\epsilon\to0}\left[-\int_{-\infty}^{0} e^{\epsilon t} e^{-i\omega t}\,dt + \int_{0}^{\infty} e^{-\epsilon t} e^{-i\omega t}\,dt\right] \tag{3.2.12}$$

$$= \lim_{\epsilon\to0}\left(\frac{-1}{\epsilon - i\omega} + \frac{1}{\epsilon + i\omega}\right). \tag{3.2.13}$$

If $\omega \neq 0$, (3.2.13) equals $2/i\omega$. If $\omega = 0$, (3.2.13) equals 0 because

$$\lim_{\epsilon \to 0} \left(\frac{-1}{\epsilon} + \frac{1}{\epsilon} \right) = 0. \tag{3.2.14}$$

Thus, we conclude that

$$\mathcal{F}[\text{sgn}(t)] = \begin{cases} 2/i\omega, & \omega \neq 0 \\ 0, & \omega = 0. \end{cases} \tag{3.2.15}$$

• **Example 3.2.2: Fourier transform of the step function**

An important function in transform methods is the *(Heaviside) step function*:

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0. \end{cases} \tag{3.2.16}$$

In terms of the sign function it can be written

$$H(t) = \tfrac{1}{2} + \tfrac{1}{2}\text{sgn}(t). \tag{3.2.17}$$

Because the Fourier transforms of 1 and $\text{sgn}(t)$ are $2\pi\delta(\omega)$ and $2/i\omega$, respectively, we have that

$$\mathcal{F}[H(t)] = \pi\delta(\omega) + \frac{1}{i\omega}. \tag{3.2.18}$$

These transforms are used in engineering but the presence of the delta function requires extra care to ensure their proper use.

Problems

1. Verify that

$$\mathcal{F}[\sin(\omega_0 t)H(t)] = \frac{\omega_0}{\omega_0^2 - \omega^2} + \frac{\pi i}{2}[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)].$$

2. Verify that

$$\mathcal{F}[\cos(\omega_0 t)H(t)] = \frac{i\omega}{\omega_0^2 - \omega^2} + \frac{\pi}{2}[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)].$$

3.3 PROPERTIES OF FOURIER TRANSFORMS

In principle we can compute any Fourier transform from the definition. However, it is far more efficient to derive some simple relationships that relate transforms to each other. This is the purpose of this section.

Linearity

If $f(t)$ and $g(t)$ are functions with Fourier transforms $F(\omega)$ and $G(\omega)$, respectively, then

$$\mathcal{F}[c_1 f(t) + c_2 g(t)] = c_1 F(\omega) + c_2 G(\omega), \tag{3.3.1}$$

where c_1 and c_2 are (real or complex) constants.

This result follows from the integral definition:

$$\mathcal{F}[c_1 f(t) + c_2 g(t)] = \int_{-\infty}^{\infty} [c_1 f(t) + c_2 g(t)] e^{-i\omega t}\, dt \tag{3.3.2}$$

$$= c_1 \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt + c_2 \int_{-\infty}^{\infty} g(t) e^{-i\omega t}\, dt \tag{3.3.3}$$

$$= c_1 F(\omega) + c_2 G(\omega). \tag{3.3.4}$$

Time shifting

If $f(t)$ is a function with a Fourier transform $F(\omega)$, then $\mathcal{F}[f(t - \tau)] = e^{-i\omega\tau} F(\omega)$.

This follows from the definition of the Fourier transform:

$$\mathcal{F}[f(t-\tau)] = \int_{-\infty}^{\infty} f(t-\tau) e^{-i\omega t} dt = \int_{-\infty}^{\infty} f(x) e^{-i\omega(x+\tau)} dx \tag{3.3.5}$$

$$= e^{-i\omega\tau} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx = e^{-i\omega\tau} F(\omega). \tag{3.3.6}$$

• **Example 3.3.1**

The Fourier transform of $f(t) = \cos(at)H(t)$ is $F(\omega) = i\omega/(a^2 - \omega^2) + \pi[\delta(\omega + a) + \delta(\omega - a)]/2$. Therefore,

$$\mathcal{F}\{\cos[a(t-k)]H(t-k)\} = e^{-ik\omega} \mathcal{F}[\cos(at)H(t)] \tag{3.3.7}$$

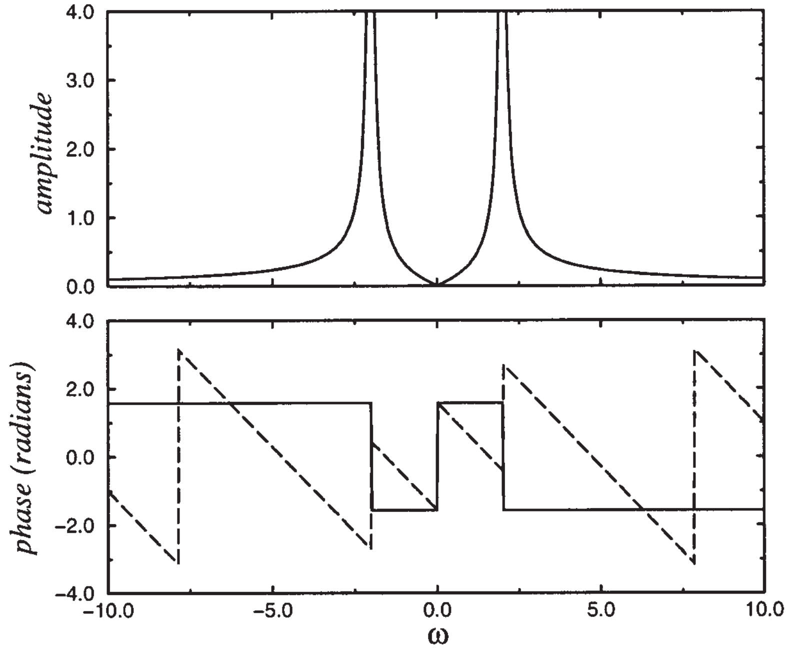


Figure 3.3.1: The amplitude and phase spectra of the Fourier transform for $\cos(2t)H(t)$ (solid line) and $\cos[2(t-1)]H(t-1)$ (dashed line). The amplitude becomes infinite at $\omega = \pm 2$.

or

$$\mathcal{F}\{\cos[a(t-k)]H(t-k)\} = \frac{i\omega e^{-ik\omega}}{a^2-\omega^2} + \frac{\pi}{2}e^{-ik\omega}[\delta(\omega+a)+\delta(\omega-a)]. \tag{3.3.8}$$

In Figure 3.3.1 we present the amplitude and phase spectra for $\cos(2t)$ $H(t)$ (the solid line) while the dashed line gives these spectra for $\cos[2(t-1)]H(t-1)$. This figure shows that the amplitude spectra are identical (why?) while the phase spectra are considerably different.

Scaling factor

Let $f(t)$ be a function with a Fourier transform $F(\omega)$ and k be a real, nonzero constant. Then $\mathcal{F}[f(kt)] = F(\omega/k)/|k|$.

From the definition of the Fourier transform:

$$\mathcal{F}[f(kt)] = \int_{-\infty}^{\infty} f(kt)e^{-i\omega t}\,dt = \frac{1}{|k|}\int_{-\infty}^{\infty} f(x)e^{-i(\omega/k)x}\,dx = \frac{1}{|k|}F\left(\frac{\omega}{k}\right). \tag{3.3.9}$$

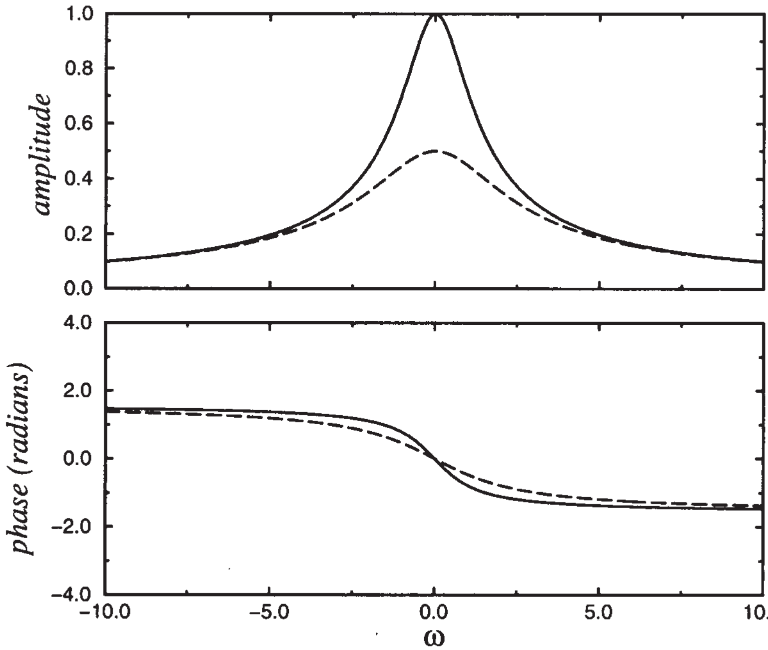


Figure 3.3.2: The amplitude and phase spectra of the Fourier transform for $e^{-t}H(t)$ (solid line) and $e^{-2t}H(t)$ (dashed line).

• Example 3.3.2

The Fourier transform of $f(t) = e^{-t}H(t)$ is $F(\omega) = 1/(1+\omega i)$. Therefore, the Fourier transform for $f(at) = e^{-at}H(t)$, $a > 0$, is

$$\mathcal{F}[f(at)] = \left(\frac{1}{a}\right)\left(\frac{1}{1+i\omega/a}\right) = \frac{1}{a+\omega i}. \tag{3.3.10}$$

In Figure 3.3.2 we present the amplitude and phase spectra for $e^{-t}H(t)$ (solid line) while the dashed line gives these spectra for $e^{-2t}H(t)$. This figure shows that the amplitude spectra has decreased by a factor of two for $e^{-2t}H(t)$ compared to $e^{-t}H(t)$ while the differences in the phase are smaller.

Symmetry

If the function $f(t)$ has the Fourier transform $F(\omega)$, then $\mathcal{F}[F(t)] = 2\pi f(-\omega)$.

From the definition of the inverse Fourier transform,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x)e^{ixt}\,dx. \tag{3.3.11}$$

Then

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(x)e^{-i\omega x}\,dx = \int_{-\infty}^{\infty} F(t)e^{-i\omega t}\,dt = \mathcal{F}[F(t)]. \tag{3.3.12}$$

• **Example 3.3.3**

The Fourier transform of $1/(1+t^2)$ is $\pi e^{-|\omega|}$. Therefore,

$$\mathcal{F}\left(\pi e^{-|t|}\right) = \frac{2\pi}{1+\omega^2} \tag{3.3.13}$$

or

$$\mathcal{F}\left(e^{-|t|}\right) = \frac{2}{1+\omega^2}. \tag{3.3.14}$$

Derivatives of functions

Let $f^{(k)}(t), k = 0, 1, 2, \ldots, n-1$, be continuous and $f^{(n)}(t)$ be piecewise continuous. Let $|f^{(k)}(t)| \leq Ke^{-bt}, b > 0, 0 \leq t < \infty; |f^{(k)}(t)| \leq Me^{at}, a > 0, -\infty < t \leq 0, k = 0, 1, .., n$. Then, $\mathcal{F}[f^{(n)}(t)] = (i\omega)^n F(\omega)$.

We begin by noting that if the transform $\mathcal{F}[f'(t)]$ exists, then

$$\mathcal{F}[f'(t)] = \int_{-\infty}^{\infty} f'(t)e^{-i\omega t}\,dt \tag{3.3.15}$$

$$= \int_{-\infty}^{\infty} f'(t)e^{\omega_i t}[\cos(\omega_r t) - i\sin(\omega_r t)]\,dt \tag{3.3.16}$$

$$= (-\omega_i + i\omega_r)\int_{-\infty}^{\infty} f(t)e^{\omega_i t}[\cos(\omega_r t) - i\sin(\omega_r t)]\,dt \tag{3.3.17}$$

$$= i\omega\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt = i\omega F(\omega). \tag{3.3.18}$$

Finally,

$$\mathcal{F}[f^{(n)}(t)] = i\omega\mathcal{F}[f^{(n-1)}(t)] = (i\omega)^2\mathcal{F}[f^{(n-2)}(t)] = \cdots = (i\omega)^n F(\omega). \tag{3.3.19}$$

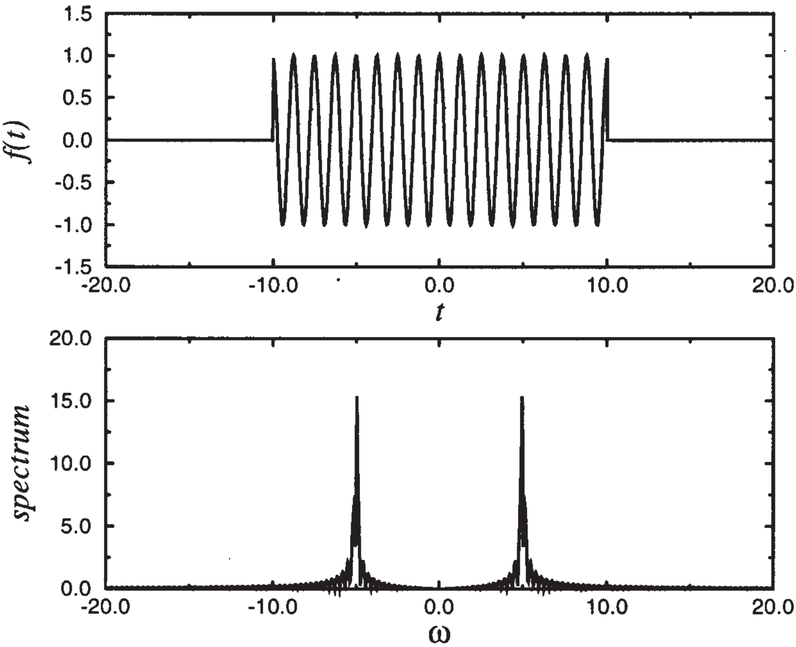


Figure 3.3.3: The (amplitude) spectrum of a rectangular pulse (3.1.9) with a half width $a = 10$ that has been modulated with $\cos(5t)$.

• **Example 3.3.4**

The Fourier transform of $f(t) = 1/(1+t^2)$ is $F(\omega) = \pi e^{-|\omega|}$. Therefore,

$$\mathcal{F}\left[-\frac{2t}{(1+t^2)^2}\right] = i\omega\pi e^{-|\omega|} \tag{3.3.20}$$

or

$$\mathcal{F}\left[\frac{t}{(1+t^2)^2}\right] = -\frac{i\omega\pi}{2}e^{-|\omega|}. \tag{3.3.21}$$

Modulation

In communications a popular method of transmitting information is by *amplitude modulation* (AM). In this process some signal is carried according to the expression $f(t)e^{i\omega_0 t}$, where ω_0 is the *carrier frequency* and $f(t)$ is some arbitrary function of time whose amplitude spectrum peaks at some frequency that is usually small compared to ω_0. We now

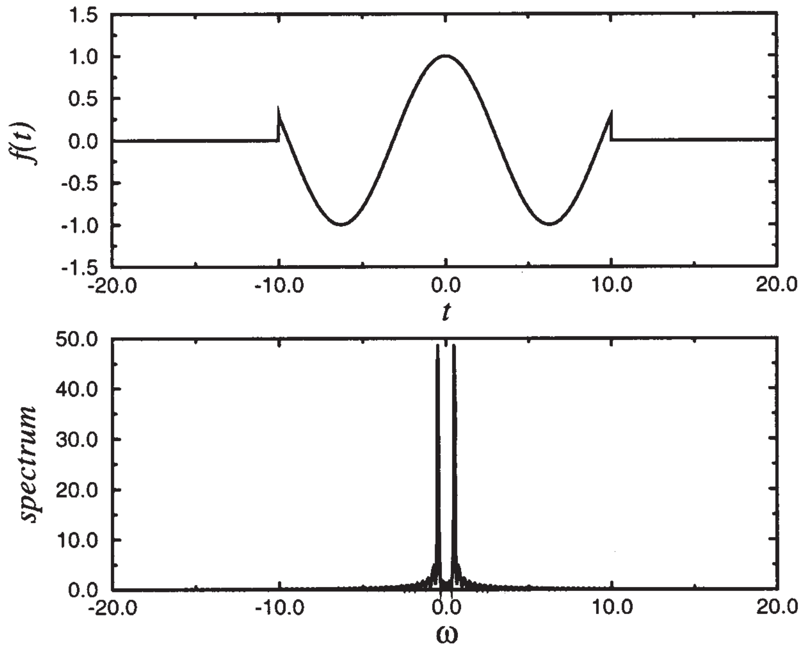


Figure 3.3.4: The (amplitude) spectrum of a rectangular pulse (3.1.9) with a half width $a = 10$ that has been modulated with $\cos(t/2)$.

want to show that the Fourier transform of $f(t)e^{i\omega_0 t}$ is $F(\omega - \omega_0)$, where $F(\omega)$ is the Fourier transform of $f(t)$.

We begin by using the definition of the Fourier transform:

$$\mathcal{F}[f(t)e^{i\omega_0 t}] = \int_{-\infty}^{\infty} f(t)e^{i\omega_0 t}e^{-i\omega t}dt = \int_{-\infty}^{\infty} f(t)e^{-i(\omega-\omega_0)t}dt \tag{3.3.22}$$

$$= F(\omega - \omega_0). \tag{3.3.23}$$

Therefore, if we have the spectrum of a particular function $f(t)$, then the Fourier transform of the modulated function $f(t)e^{i\omega_0 t}$ is the same as that for $f(t)$ except that it is now centered on the frequency ω_0 rather than on the zero frequency.

• **Example 3.3.5**

Let us determine the Fourier transform of a square pulse modulated by a cosine wave as shown in Figures 3.3.3 and 3.3.4. Because $\cos(\omega_0 t) = \frac{1}{2}[e^{i\omega_0 t} + e^{-i\omega_0 t}]$ and the Fourier transform of a square pulse is $F(\omega) = 2\sin(\omega a)/\omega$,

$$\mathcal{F}[f(t)\cos(\omega_0 t)] = \frac{\sin[(\omega - \omega_0)a]}{\omega - \omega_0} + \frac{\sin[(\omega + \omega_0)a]}{\omega + \omega_0}. \tag{3.3.24}$$

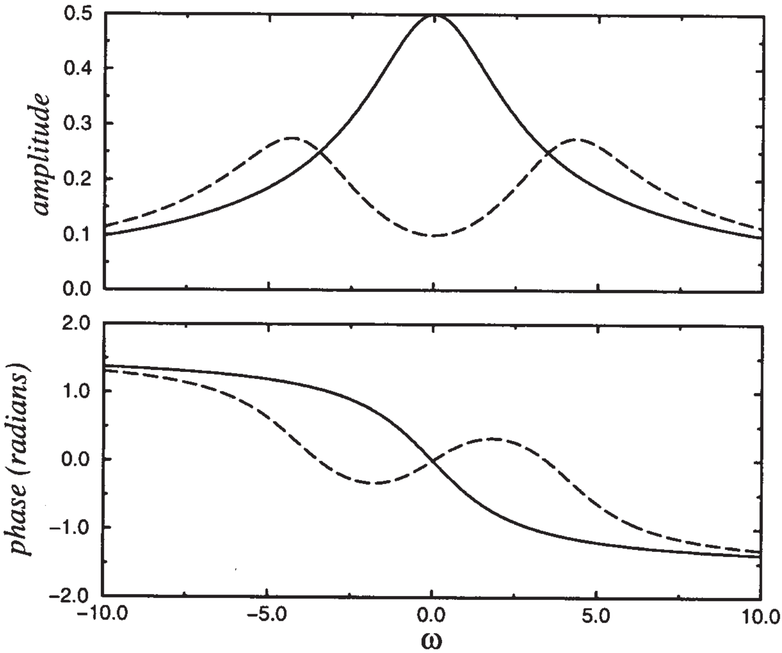


Figure 3.3.5: The amplitude and phase spectra of the Fourier transform for $e^{-2t}H(t)$ (solid line) and $e^{-2t}\cos(4t)H(t)$ (dashed line).

Therefore, the Fourier transform of the modulated pulse equals one half of the sum of the pulse centered on ω_0 and the other that of the pulse centered on $-\omega_0$. See Figures 3.3.3 and 3.3.4.

In many practical situations, $\omega_0 \gg \pi/a$. In this case we may consider that the two terms are completely independent from each other and the contribution from the peak at $\omega = \omega_0$ has a negligible effect on the peak at $\omega = -\omega_0$.

• **Example 3.3.6**

The Fourier transform of $f(t) = e^{-bt}H(t)$ is $F(\omega) = 1/(b + i\omega)$. Therefore,

$$\mathcal{F}[e^{-bt}\cos(at)H(t)] = \tfrac{1}{2}\mathcal{F}\left(e^{iat}e^{-bt} + e^{-iat}e^{-bt}\right) \tag{3.3.25}$$

$$= \frac{1}{2}\left(\left.\frac{1}{b+i\omega'}\right|_{\omega'=\omega-a} + \left.\frac{1}{b+i\omega'}\right|_{\omega'=\omega+a}\right) \tag{3.3.26}$$

$$\mathcal{F}[e^{-bt}\cos(at)H(t)] = \frac{1}{2}\left[\frac{1}{(b+i\omega)-ai} + \frac{1}{(b+i\omega)+ai}\right] \tag{3.3.27}$$

$$= \frac{b+i\omega}{(b+i\omega)^2 + a^2}. \tag{3.3.28}$$

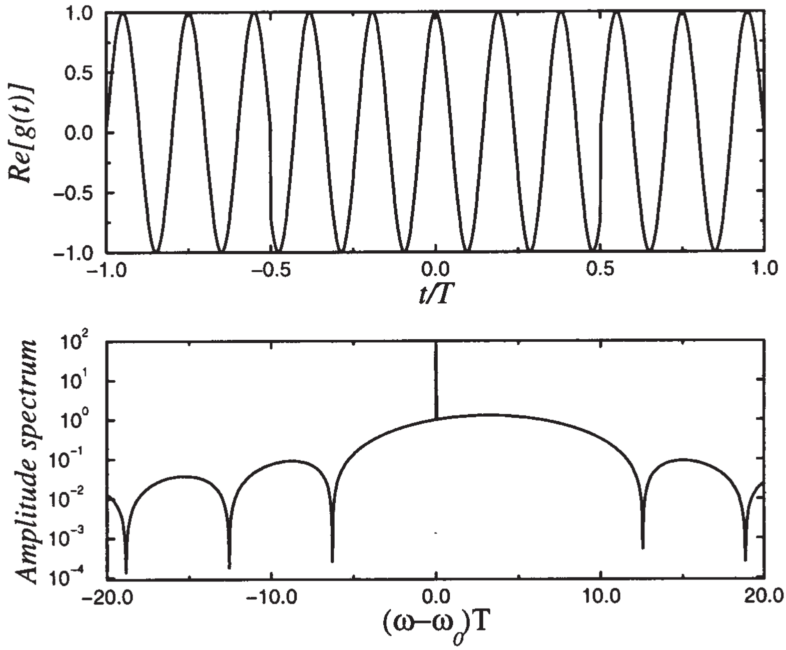


Figure 3.3.6: The (amplitude) spectrum $|G(\omega)|/T$ of a frequency-modulated signal (shown top) using the parameters $\omega_1 T = 2\pi$ and $\omega_0 T = 10\pi$. The transform becomes undefined at $\omega = \omega_0$.

We have illustrated this result using $e^{-2t}H(t)$ and $e^{-2t}\cos(4t)H(t)$ in Figure 3.3.5.

• **Example 3.3.7: Frequency modulation**

In contrast to amplitude modulation, *frequency modulation* (FM) transmits information by instantaneous variations of the carrier frequency. It may be expressed mathematically as $\exp\left[i\int_{-\infty}^{t} f(\tau)\,d\tau + iC\right]e^{i\omega_0 t}$, where C is a constant. To illustrate this concept, let us find the Fourier transform of a simple frequency modulation:

$$f(t) = \begin{cases} \omega_1, & |t| < T/2 \\ 0, & |t| > T/2 \end{cases} \tag{3.3.29}$$

and $C = -\omega_1 T/2$. In this case, the signal in the time domain is

$$g(t) = \exp\left[i\int_{-\infty}^{t} f(\tau)\,d\tau + iC\right]e^{i\omega_0 t} \tag{3.3.30}$$

$$= \begin{cases} e^{-i\omega_1 T/2}e^{i\omega_0 t}, & t < -T/2 \\ e^{i\omega_1 t}e^{i\omega_0 t}, & -T/2 < t < T/2 \\ e^{i\omega_1 T/2}e^{i\omega_0 t}, & t > T/2. \end{cases} \tag{3.3.31}$$

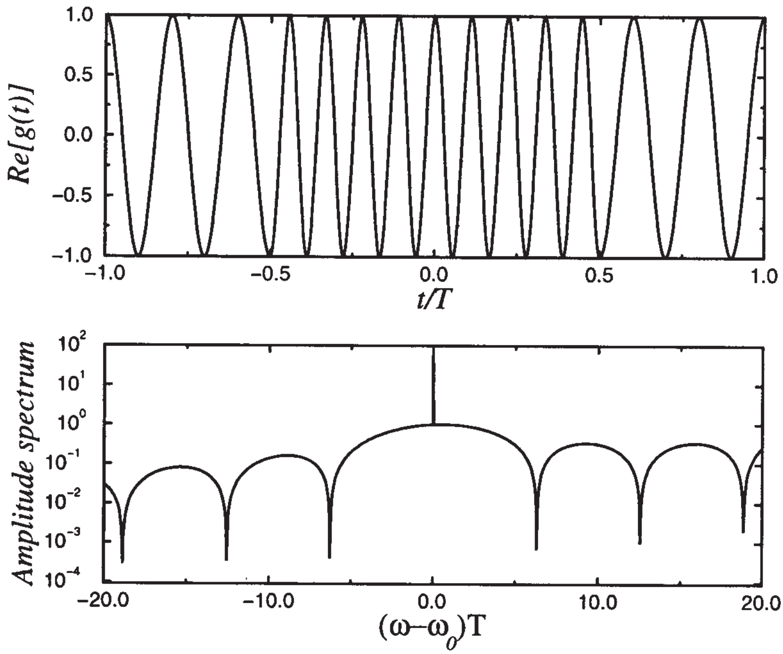


Figure 3.3.7: The (amplitude) spectrum $|G(\omega)|/T$ of a frequency-modulated signal (shown top) using the parameters $\omega_1 T = 8\pi$ and $\omega_0 T = 10\pi$. The transform becomes undefined at $\omega = \omega_0$.

We have illustrated this signal in Figures 3.3.6 and 3.3.7.

The Fourier transform of the signal $G(\omega)$ equals

$$G(\omega) = e^{-i\omega_1 T/2} \int_{-\infty}^{-T/2} e^{i(\omega_0-\omega)t}\,dt + \int_{-T/2}^{T/2} e^{i(\omega_0+\omega_1-\omega)t}\,dt + e^{i\omega_1 T/2} \int_{T/2}^{\infty} e^{i(\omega_0-\omega)t}\,dt \tag{3.3.32}$$

$$= e^{-i\omega_1 T/2} \int_{-\infty}^{0} e^{i(\omega_0-\omega)t}\,dt + e^{i\omega_1 T/2} \int_{0}^{\infty} e^{i(\omega_0-\omega)t}\,dt - e^{-i\omega_1 T/2} \int_{-T/2}^{0} e^{i(\omega_0-\omega)t}\,dt + \int_{-T/2}^{T/2} e^{i(\omega_0+\omega_1-\omega)t}\,dt - e^{i\omega_1 T/2} \int_{0}^{T/2} e^{i(\omega_0-\omega)t}\,dt. \tag{3.3.33}$$

Applying the fact that

$$\int_0^\infty e^{\pm i\alpha t}\,dt = \pi\delta(\alpha) \pm \frac{i}{\alpha}, \tag{3.3.34}$$

$$G(\omega) = \pi\delta(\omega - \omega_0)\left[e^{i\omega_1 T/2} + e^{-i\omega_1 T/2}\right] + \frac{\left[e^{i(\omega_0+\omega_1-\omega)T/2} - e^{-i(\omega_0+\omega_1-\omega)T/2}\right]}{i(\omega_0 + \omega_1 - \omega)} - \frac{\left[e^{i(\omega_0+\omega_1-\omega)T/2} - e^{-i(\omega_0+\omega_1-\omega)T/2}\right]}{i(\omega_0 - \omega)} \tag{3.3.35}$$

$$= 2\pi\delta(\omega - \omega_0)\cos(\omega_1 T/2) + \frac{2\omega_1 \sin[(\omega - \omega_0 - \omega_1)T/2]}{(\omega - \omega_0)(\omega - \omega_0 - \omega_1)}. \tag{3.3.36}$$

In Figures 3.3.6 and 3.3.7 we have illustrated the amplitude spectrum for various parameters. In general, the transform is not symmetric, with an increasing number of humped curves as $\omega_1 T$ increases.

Parseval's equality

In applying Fourier methods to practical problems we may encounter a situation where we are interested in computing the energy of a system. Energy is usually expressed by the integral $\int_{-\infty}^{\infty} |f(t)|^2\, dt$. Can we compute this integral if we only have the Fourier transform of $F(\omega)$?

From the definition of the inverse Fourier transform

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\, d\omega, \tag{3.3.37}$$

we have that

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\left[\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\, d\omega\right] dt. \tag{3.3.38}$$

Interchanging the order of integration on the right side of (3.3.38),

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\left[\int_{-\infty}^{\infty} f(t)e^{i\omega t}\, dt\right] d\omega. \tag{3.3.39}$$

However,

$$F^*(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}\, dt. \tag{3.3.40}$$

Therefore,

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega. \tag{3.3.41}$$

This is *Parseval's equality*[6] as it applies to Fourier transforms. The quantity $|F(\omega)|^2$ is called the *power spectrum*.

• **Example 3.3.8**

In Example 3.1.1, we showed that the Fourier transform for a unit rectangular pulse between $-a < t < a$ is $2\sin(\omega a)/\omega$. Therefore, by Parseval's equality,

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2(\omega a)}{\omega^2}\,d\omega = \int_{-a}^{a} 1^2\,dt = 2a \tag{3.3.42}$$

or

$$\int_{-\infty}^{\infty} \frac{\sin^2(\omega a)}{\omega^2}\,d\omega = \pi a. \tag{3.3.43}$$

Poisson's summation formula

If $f(x)$ is integrable over $(-\infty, \infty)$, there exists a relationship between the function and its Fourier transform, commonly called *Poisson's summation formula.*[7]

We begin by inventing a periodic function $g(x)$ defined by

$$g(x) = \sum_{k=-\infty}^{\infty} f(x + 2\pi k). \tag{3.3.44}$$

Because $g(x)$ is a periodic function of 2π, it can be represented by the complex Fourier series:

$$g(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{3.3.45}$$

or

$$g(0) = \sum_{k=-\infty}^{\infty} f(2\pi k) = \sum_{n=-\infty}^{\infty} c_n. \tag{3.3.46}$$

---

[6] Apparently first derived by Rayleigh, J. W., 1889: On the character of the complete radiation at a given temperature. *Philos. Mag., Ser. 5*, **27**, 460–469.

[7] Poisson, S. D., 1823: Suite du mémoire sur les intégrales définies et sur la sommation des séries. *J. École Polytech.*, **19**, 404–509. See page 451.

Computing c_n, we find that

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} f(x+2k\pi)e^{-inx}\,dx \tag{3.3.47}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi} f(x+2k\pi)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-inx}\,dx \tag{3.3.48}$$

$$= \frac{F(n)}{2\pi}, \tag{3.3.49}$$

where $F(\omega)$ is the Fourier transform of $f(x)$. Substituting (3.3.49) into (3.3.46), we obtain

$$\sum_{k=-\infty}^{\infty} f(2\pi k) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} F(n) \tag{3.3.50}$$

or

$$\boxed{\sum_{k=-\infty}^{\infty} f(\alpha k) = \frac{1}{\alpha}\sum_{n=-\infty}^{\infty} F\left(\frac{2\pi n}{\alpha}\right).} \tag{3.3.51}$$

• Example 3.3.9

One of the popular uses of Poisson's summation formula is the evaluation of infinite series. For example, let $f(x) = 1/(a^2+x^2)$ with a real and nonzero. Then, $F(\omega) = \pi e^{-|a\omega|}/|a|$ and

$$\sum_{k=-\infty}^{\infty} \frac{1}{a^2+(2\pi k)^2} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{1}{|a|}e^{-|an|} = \frac{1}{2|a|}\left(1+2\sum_{n=1}^{\infty}e^{-|a|n}\right) \tag{3.3.52}$$

$$= \frac{1}{2|a|}\left(-1+\frac{2}{1-e^{-|a|}}\right) = \frac{1}{2|a|}\coth\left(\frac{|a|}{2}\right). \tag{3.3.53}$$

Problems

1. Find the Fourier transform of $1/(1+a^2t^2)$, where a is real, given that $\mathcal{F}[1/(1+t^2)] = \pi e^{-|\omega|}$.

2. Find the Fourier transform of $\cos(at)/(1+t^2)$, where a is real, given that $\mathcal{F}[1/(1+t^2)] = \pi e^{-|\omega|}$.

3. Use the fact that $\mathcal{F}[e^{-at}H(t)] = 1/(a+i\omega)$ with $a > 0$ and Parseval's equality to show that

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \frac{\pi}{a}.$$

4. Use the fact that $\mathcal{F}[1/(1+t^2)] = \pi e^{-|\omega|}$ and Parseval's equality to show that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2}.$$

5. Use the function $f(t) = e^{-at}\sin(bt)H(t)$ with $a > 0$ and Parseval's equality to show that

$$2\int_{0}^{\infty} \frac{dx}{(x^2+a^2-b^2)^2+4a^2b^2} = \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2-b^2)^2+4a^2b^2} = \frac{\pi}{2a(a^2+b^2)}.$$

6. Using the modulation property and $\mathcal{F}[e^{-bt}H(t)] = 1/(b+i\omega)$, show that

$$\mathcal{F}\left[e^{-bt}\sin(at)H(t)\right] = \frac{a}{(b+i\omega)^2+a^2}.$$

Plot and compare the amplitude and phase spectra for $e^{-t}H(t)$ and $e^{-t}\sin(2t)\ H(t)$.

7. Use Poisson's summation formula to prove that

$$\sum_{n=-\infty}^{\infty} e^{-ianT} = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta\left(\frac{2\pi n}{T} - a\right),$$

where $\delta(\ )$ is the Dirac delta function.

3.4 INVERSION OF FOURIER TRANSFORMS

Having focused on the Fourier transform in the previous sections, we consider in detail the inverse Fourier transform in this section. Recall that the improper integral (3.1.6) defines the inverse. Consequently one method of inversion is direct integration.

• Example 3.4.1

Let us find the inverse of $F(\omega) = \pi e^{-|\omega|}$.
From the definition of the inverse Fourier transform,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi e^{-|\omega|} e^{i\omega t}\,d\omega \tag{3.4.1}$$

$$= \frac{1}{2}\int_{-\infty}^{0} e^{(1+it)\omega}\,d\omega + \frac{1}{2}\int_{0}^{\infty} e^{(-1+it)\omega}\,d\omega \tag{3.4.2}$$

$$= \frac{1}{2}\left[\frac{e^{(1+it)\omega}}{1+it}\bigg|_{-\infty}^{0} + \frac{e^{(-1+it)\omega}}{-1+it}\bigg|_{0}^{\infty}\right] \tag{3.4.3}$$

$$= \frac{1}{2}\left[\frac{1}{1+it} - \frac{1}{-1+it}\right] = \frac{1}{1+t^2}. \tag{3.4.4}$$

Another method for inverting Fourier transforms is rewriting the Fourier transform using partial fractions so that we can use transform tables. The following example illustrates this technique.

• Example 3.4.2

Let us invert the transform

$$F(\omega) = \frac{1}{(1+i\omega)(1-2i\omega)^2}. \tag{3.4.5}$$

We begin by rewriting (3.4.5) as

$$F(\omega) = \frac{1}{9}\left[\frac{1}{1+i\omega} + \frac{2}{1-2i\omega} + \frac{6}{(1-2i\omega)^2}\right] \tag{3.4.6}$$

$$= \frac{1}{9(1+i\omega)} + \frac{1}{\frac{1}{2}-i\omega} + \frac{1}{6(\frac{1}{2}-i\omega)^2}. \tag{3.4.7}$$

Using Table 3.1.1, we invert (3.4.7) term by term and find that

$$f(t) = \tfrac{1}{9}e^{-t}H(t) + \tfrac{1}{9}e^{t/2}H(-t) - \tfrac{1}{6}te^{t/2}H(-t). \tag{3.4.8}$$

Although we may find the inverse by direct integration or partial fractions, in many instances the Fourier transform does not lend itself to these techniques. On the other hand, if we view the inverse Fourier transform as a line integral along the real axis in the complex ω-plane, then perhaps some of the techniques that we developed in Chapter 1 might be applicable to this problem. To this end, we rewrite the inversion integral (3.1.6) as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{it\omega}\,d\omega = \frac{1}{2\pi}\oint_C F(z)e^{itz}\,dz - \frac{1}{2\pi}\int_{C_R} F(z)e^{itz}\,dz, \tag{3.4.9}$$

where C denotes a closed contour consisting of the entire real axis plus a new contour C_R that joins the point $(\infty, 0)$ to $(-\infty, 0)$. There are countless possibilities for C_R. For example, it could be the loop $(\infty, 0)$ to (∞, R) to $(-\infty, R)$ to $(-\infty, 0)$ with $R > 0$. However, any choice of C_R must be such that we can compute $\int_{C_R} F(z)e^{itz}\,dz$. When we take that constraint into account, the number of acceptable contours decrease to just a few. The best is given by *Jordan's lemma:*[8]

Jordan's lemma: *Suppose that, on a circular arc C_R with radius R and center at the origin, $f(z) \to 0$ uniformly as $R \to \infty$. Then*

(1) $$\lim_{R\to\infty}\int_{C_R} f(z)e^{imz}\,dz = 0, \qquad (m > 0) \tag{3.4.10}$$

if C_R lies in the first and/or second quadrant;

(2) $$\lim_{R\to\infty}\int_{C_R} f(z)e^{-imz}\,dz = 0, \qquad (m > 0) \tag{3.4.11}$$

if C_R lies in the third and/or fourth quadrant;

(3) $$\lim_{R\to\infty}\int_{C_R} f(z)e^{mz}\,dz = 0, \qquad (m > 0) \tag{3.4.12}$$

if C_R lies in the second and/or third quadrant; and

(4) $$\lim_{R\to\infty}\int_{C_R} f(z)e^{-mz}\,dz = 0, \qquad (m > 0) \tag{3.4.13}$$

if C_R lies in the first and/or fourth quadrant.

---

[8] Jordan, C., 1894: *Cours D'Analyse de l'École Polytechnique. Vol. 2*, Gauthier-Villars, Paris, pp. 285–286. See also Whittaker, E. T. and Watson, G. N., 1963: *A Course of Modern Analysis*, Cambridge University Press, Cambridge, p. 115.

Technically, only (1) is actually Jordan's lemma while the remaining points are variations.

Proof: We shall prove the first part; the remaining portions follow by analog. We begin by noting that

$$|I_R| = \left|\int_{C_R} f(z)e^{imz}\,dz\right| \le \int_{C_R} |f(z)|\left|e^{imz}\right||dz|. \quad (3.4.14)$$

Now

$$|dz| = R\,d\theta, \quad |f(z)| \le M_R, \quad (3.4.15)$$

$$\left|e^{imz}\right| = \left|\exp(imRe^{\theta i})\right| = |\exp\{imR[\cos(\theta) + i\sin(\theta)]\}| = e^{-mR\sin(\theta)}. \quad (3.4.16)$$

Therefore,

$$|I_R| \le RM_R \int_{\theta_0}^{\theta_1} \exp[-mR\sin(\theta)]\,d\theta, \quad (3.4.17)$$

where $0 \le \theta_0 < \theta_1 \le \pi$. Because the integrand is positive, the right side of (3.4.17) is largest if we take $\theta_0 = 0$ and $\theta_1 = \pi$. Then

$$|I_R| \le RM_R \int_0^{\pi} e^{-mR\sin(\theta)}d\theta = 2RM_R \int_0^{\pi/2} e^{-mR\sin(\theta)}d\theta. \quad (3.4.18)$$

We cannot evaluate the integrals in (3.4.18) as they stand. However, because $\sin(\theta) \ge 2\theta/\pi$ if $0 \le \theta \le \pi/2$, we can bound the value of the integral by

$$|I_R| \le 2RM_R \int_0^{\pi/2} e^{-2mR\theta/\pi}d\theta = \frac{\pi}{m}M_R\left(1 - e^{-mR}\right). \quad (3.4.19)$$

If $m > 0$, $|I_R|$ tends to zero with M_R as $R \to \infty$. □

Consider now the following inversions of Fourier transforms:

• **Example 3.4.3**

For our first example we find the inverse for

$$F(\omega) = \frac{1}{\omega^2 - 2ib\omega - a^2 - b^2}. \quad (3.4.20)$$

From the inversion integral,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{it\omega}}{\omega^2 - 2ib\omega - a^2 - b^2}\,d\omega \quad (3.4.21)$$

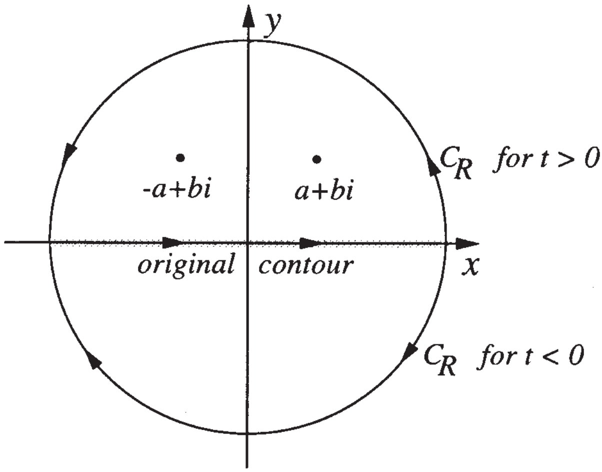


Figure 3.4.1: Contour used to find the inverse of the Fourier transform (3.4.20). The contour C consists of the line integral along the real axis plus C_R.

or

$$f(t) = \frac{1}{2\pi} \oint_C \frac{e^{itz}}{z^2 - 2ibz - a^2 - b^2}\, dz - \frac{1}{2\pi} \int_{C_R} \frac{e^{itz}}{z^2 - 2ibz - a^2 - b^2}\, dz, \tag{3.4.22}$$

where C denotes a closed contour consisting of the entire real axis plus C_R. Because $f(z) = 1/(z^2 - 2ibz - a^2 - b^2)$ tends to zero uniformly as $|z| \to \infty$ and $m = t$, the second integral in (3.4.22) will vanish by Jordan's lemma if C_R is a semicircle of infinite radius in the upper half of the z-plane when $t > 0$ and a semicircle in the lower half of the z-plane when $t < 0$.

Next we must find the location and nature of the singularities. They are located at

$$z^2 - 2ibz - a^2 - b^2 = 0 \tag{3.4.23}$$

or

$$z = \pm a + bi. \tag{3.4.24}$$

Therefore we can rewrite (3.4.22) as

$$f(t) = \frac{1}{2\pi} \oint_C \frac{e^{itz}}{(z - a - bi)(z + a - bi)}\, dz. \tag{3.4.25}$$

Thus, all of the singularities are simple poles.

Consider now $t > 0$. As stated earlier, we close the line integral with an infinite semicircle in the upper half-plane. See Figure 3.4.1.

Inside this closed contour there are two singularities: $z = \pm a + bi$. For these poles,

$$\operatorname{Res}\left(\frac{e^{itz}}{z^2 - 2ibz - a^2 - b^2}; a + bi\right)$$

$$= \lim_{z \to a+bi} (z - a - bi)\frac{e^{itz}}{(z - a - bi)(z + a - bi)} \tag{3.4.26}$$

$$= \frac{e^{iat}e^{-bt}}{2a} = \frac{1}{2a}e^{-bt}[\cos(at) + i\sin(at)], \tag{3.4.27}$$

where we have used Euler's formula to eliminate e^{iat}. Similarly,

$$\operatorname{Res}\left(\frac{e^{itz}}{z^2 - 2ibz - a^2 - b^2}; -a + bi\right) = -\frac{1}{2a}e^{-bt}[\cos(at) - i\sin(at)]. \tag{3.4.28}$$

Consequently the inverse Fourier transform follows from (3.4.25) after applying the residue theorem and equals

$$f(t) = -\frac{1}{2a}e^{-bt}\sin(at) \tag{3.4.29}$$

for $t > 0$.

For $t < 0$ the semicircle is in the lower half-plane because the contribution from the semicircle vanishes as $R \to \infty$. Because there are no singularities within the closed contour, $f(t) = 0$. Therefore, we can write in general that

$$f(t) = -\frac{1}{2a}e^{-bt}\sin(at)H(t). \tag{3.4.30}$$

• Example 3.4.4

Let us find the inverse of the Fourier transform

$$F(\omega) = \frac{e^{-\omega i}}{\omega^2 + a^2}, \tag{3.4.31}$$

where a is real and positive.

From the inversion integral,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i(t-1)\omega}}{\omega^2 + a^2}\,d\omega \tag{3.4.32}$$

$$= \frac{1}{2\pi}\oint_C \frac{e^{i(t-1)z}}{z^2 + a^2}\,dz - \frac{1}{2\pi}\int_{C_R} \frac{e^{i(t-1)z}}{z^2 + a^2}\,dz, \tag{3.4.33}$$

where C denotes a closed contour consisting of the entire real axis plus C_R. The contour C_R is determined by Jordan's lemma because $1/(z^2 + a^2) \to 0$ uniformly as $|z| \to \infty$. Since $m = t - 1$, the semicircle C_R of infinite radius lies in the upper half-plane if $t > 1$ and in the lower half-plane if $t < 1$. Thus, if $t > 1$,

$$f(t) = \frac{1}{2\pi}(2\pi i)\text{Res}\left[\frac{e^{i(t-1)z}}{z^2 + a^2}; ai\right] = \frac{e^{-a(t-1)}}{2a}, \tag{3.4.34}$$

whereas for $t < 1$,

$$f(t) = \frac{1}{2\pi}(-2\pi i)\text{Res}\left[\frac{e^{i(t-1)z}}{z^2 + a^2}; -ai\right] = \frac{e^{a(t-1)}}{2a}. \tag{3.4.35}$$

The minus sign in front of the $2\pi i$ arises from the negative sense of the contour. We may write the inverse as the single expression:

$$f(t) = \frac{1}{2a}e^{-a|t-1|}. \tag{3.4.36}$$

• Example 3.4.5

Let us evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos(kx)}{x^2 + a^2}\, dx, \tag{3.4.37}$$

where $a, k > 0$.

We begin by noting that

$$\int_{-\infty}^{\infty} \frac{\cos(kx)}{x^2 + a^2}\, dx = \text{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2 + a^2}\, dx\right) = \text{Re}\left(\oint_{C_1} \frac{e^{ikz}}{z^2 + a^2}\, dz\right), \tag{3.4.38}$$

where C_1 denotes a line integral along the real axis from $-\infty$ to ∞. A quick check shows that the integrand of the right side of (3.4.38) satisfies Jordan's lemma. Therefore,

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2 + a^2}\, dx = \oint_C \frac{e^{ikz}}{z^2 + a^2}\, dz = 2\pi i \text{ Res}\left(\frac{e^{ikz}}{z^2 + a^2}; ai\right) \tag{3.4.39}$$

$$= 2\pi i \lim_{z \to ai} \frac{(z - ai)e^{ikz}}{z^2 + a^2} = \frac{\pi}{a}e^{-ka}, \tag{3.4.40}$$

where C denotes the closed infinite semicircle in the upper half-plane. Taking the real and imaginary parts of (3.4.40),

$$\int_{-\infty}^{\infty} \frac{\cos(kx)}{x^2 + a^2}\, dx = \frac{\pi}{a}e^{-ka} \tag{3.4.41}$$

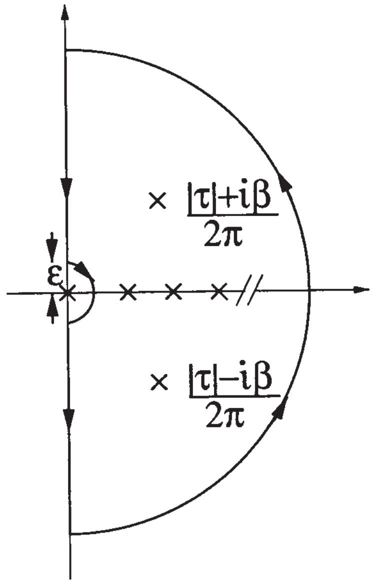


Figure 3.4.2: Contour used in Example 3.4.6.

and

$$\int_{-\infty}^{\infty} \frac{\sin(kx)}{x^2 + a^2}\, dx = 0. \tag{3.4.42}$$

• **Example 3.4.6**

So far we have used only the first two points of Jordan's lemma. In this example[9] we illustrate how the remaining two points may be applied.

Consider the contour integral

$$\oint_C \cot(\pi z) \left[\frac{e^{-cz}}{(\tau + 2\pi z)^2 + \beta^2} + \frac{e^{-cz}}{(\tau - 2\pi z)^2 + \beta^2} \right] dz,$$

where $c > 0$ and β, τ are real. Let us evaluate this contour integral where the contour is shown in Figure 3.4.2.

[9] Reprinted from *Int. J. Heat Mass Transfer*, **15**, Hsieh, T. C., and R. Greif, Theoretical determination of the absorption coefficient and the total band absorptance including a specific application to carbon monoxide, 1477–1487, ©1972, with kind permission from Elsevier Science Ltd., The Boulevard, Langford Lane, Kidlington OX5 1GB, UK.

From the residue theorem,

$$\oint_C \cot(\pi z)\left[\frac{e^{-cz}}{(\tau+2\pi z)^2+\beta^2}+\frac{e^{-cz}}{(\tau-2\pi z)^2+\beta^2}\right]dz$$
$$= 2\pi i\sum_{n=1}^{\infty}\operatorname{Res}\left\{\cot(\pi z)\left[\frac{e^{-cz}}{(\tau+2\pi z)^2+\beta^2}+\frac{e^{-cz}}{(\tau-2\pi z)^2+\beta^2}\right];n\right\}$$
$$+2\pi i\operatorname{Res}\left\{\cot(\pi z)\left[\frac{e^{-cz}}{(\tau+2\pi z)^2+\beta^2}+\frac{e^{-cz}}{(\tau-2\pi z)^2+\beta^2}\right];\frac{|\tau|+\beta i}{2\pi}\right\}$$
$$+2\pi i\operatorname{Res}\left\{\cot(\pi z)\left[\frac{e^{-cz}}{(\tau+2\pi z)^2+\beta^2}+\frac{e^{-cz}}{(\tau-2\pi z)^2+\beta^2}\right];\frac{|\tau|-\beta i}{2\pi}\right\}. \tag{3.4.43}$$

Now

$$\operatorname{Res}\left\{\cot(\pi z)\left[\frac{e^{-cz}}{(\tau+2\pi z)^2+\beta^2}+\frac{e^{-cz}}{(\tau-2\pi z)^2+\beta^2}\right];n\right\}$$
$$=\lim_{z\to n}\frac{(z-n)\cos(\pi z)}{\sin(\pi z)}\lim_{z\to n}\left[\frac{e^{-cz}}{(\tau+2\pi z)^2+\beta^2}+\frac{e^{-cz}}{(\tau-2\pi z)^2+\beta^2}\right] \tag{3.4.44}$$
$$=\frac{1}{\pi}\left[\frac{e^{-nc}}{(\tau+2n\pi)^2+\beta^2}+\frac{e^{-nc}}{(\tau-2n\pi)^2+\beta^2}\right], \tag{3.4.45}$$

$$\operatorname{Res}\left\{\cot(\pi z)\left[\frac{e^{-cz}}{(\tau+2\pi z)^2+\beta^2}+\frac{e^{-cz}}{(\tau-2\pi z)^2+\beta^2}\right];\frac{|\tau|+\beta i}{2\pi}\right\}$$
$$=\lim_{z\to(|\tau|+\beta i)/2\pi}\frac{\cot(\pi z)}{4\pi^2}$$
$$\times\left[\frac{(z-|\tau|-\beta i)e^{-cz}}{(z+\tau/2\pi)^2+\beta^2/4\pi^2}+\frac{(z-|\tau|-\beta i)e^{-cz}}{(z-\tau/2\pi)^2+\beta^2/4\pi^2}\right] \tag{3.4.46}$$
$$=\frac{\cot(|\tau|/2+\beta i/2)\exp(-c|\tau|/2\pi)[\cos(c\beta/2\pi)-i\sin(c\beta/2\pi)]}{4\pi\beta i} \tag{3.4.47}$$

and

$$\operatorname{Res}\left\{\cot(\pi z)\left[\frac{e^{-cz}}{(\tau+2\pi z)^2+\beta^2}+\frac{e^{-cz}}{(\tau-2\pi z)^2+\beta^2}\right];\frac{|\tau|-\beta i}{2\pi}\right\}$$
$$=\lim_{z\to(|\tau|-\beta i)/2\pi}\frac{\cot(\pi z)}{4\pi^2}$$
$$\times\left[\frac{(z-|\tau|+\beta i)e^{-cz}}{(z+\tau/2\pi)^2+\beta^2/4\pi^2}+\frac{(z-|\tau|+\beta i)e^{-cz}}{(z-\tau/2\pi)^2+\beta^2/4\pi^2}\right] \tag{3.4.48}$$
$$=\frac{\cot(|\tau|/2-\beta i/2)\exp(-c|\tau|/2\pi)[\cos(c\beta/2\pi)+i\sin(c\beta/2\pi)]}{-4\pi\beta i}. \tag{3.4.49}$$

Therefore,

$$\oint_C \cot(\pi z)\left[\frac{e^{-cz}}{(\tau+2\pi z)^2+\beta^2}+\frac{e^{-cz}}{(\tau-2\pi z)^2+\beta^2}\right]dz$$

$$= 2i\sum_{n=1}^{\infty}\left[\frac{e^{-nc}}{(\tau+2n\pi)^2+\beta^2}+\frac{e^{-nc}}{(\tau-2n\pi)^2+\beta^2}\right]$$

$$+\frac{i}{2\beta}\frac{e^{i|\tau|}+e^{\beta}}{e^{i|\tau|}-e^{\beta}}e^{-c|\tau|/2\pi}[\cos(c\beta/2\pi)-i\sin(c\beta/2\pi)]$$

$$-\frac{i}{2\beta}\frac{e^{i|\tau|}+e^{-\beta}}{e^{i|\tau|}-e^{-\beta}}e^{-c|\tau|/2\pi}[\cos(c\beta/2\pi)+i\sin(c\beta/2\pi)] \quad (3.4.50)$$

$$= 2i\sum_{n=1}^{\infty}\left[\frac{e^{-nc}}{(\tau+2n\pi)^2+\beta^2}+\frac{e^{-nc}}{(\tau-2n\pi)^2+\beta^2}\right]$$

$$-\frac{i}{\beta}\frac{\sinh(\beta)\cos(c\beta/2\pi)+\sin(|\tau|)\sin(c\beta/2\pi)}{\cosh(\beta)-\cos(\tau)}e^{-c|\tau|/2\pi}, \quad (3.4.51)$$

where $\cot(\alpha) = i(e^{2i\alpha}+1)/(e^{2i\alpha}-1)$ and we have made extensive use of Euler's formula.

Let us now evaluate the contour integral by direct integration. The contribution from the integration along the semicircle at infinity vanishes according to Jordan's lemma. Indeed that is why this particular contour was chosen. Therefore,

$$\oint_C \cot(\pi z)\left[\frac{e^{-cz}}{(\tau+2\pi z)^2+\beta^2}+\frac{e^{-cz}}{(\tau-2\pi z)^2+\beta^2}\right]dz$$

$$= \int_{i\infty}^{i\epsilon} \cot(\pi z)\left[\frac{e^{-cz}}{(\tau+2\pi z)^2+\beta^2}+\frac{e^{-cz}}{(\tau-2\pi z)^2+\beta^2}\right]dz$$

$$+\int_{C_\epsilon} \cot(\pi z)\left[\frac{e^{-cz}}{(\tau+2\pi z)^2+\beta^2}+\frac{e^{-cz}}{(\tau-2\pi z)^2+\beta^2}\right]dz$$

$$+\int_{-i\epsilon}^{-i\infty} \cot(\pi z)\left[\frac{e^{-cz}}{(\tau+2\pi z)^2+\beta^2}+\frac{e^{-cz}}{(\tau-2\pi z)^2+\beta^2}\right]dz. \quad (3.4.52)$$

Now, because $z = iy$,

$$\int_{i\infty}^{i\epsilon} \cot(\pi z)\left[\frac{e^{-cz}}{(\tau+2\pi z)^2+\beta^2}+\frac{e^{-cz}}{(\tau-2\pi z)^2+\beta^2}\right]dz$$

$$= \int_{\infty}^{\epsilon} \coth(\pi y)\left[\frac{e^{-icy}}{(\tau+2\pi iy)^2+\beta^2}+\frac{e^{-icy}}{(\tau-2\pi iy)^2+\beta^2}\right]dy \quad (3.4.53)$$

$$= -2\int_{\epsilon}^{\infty} \frac{\coth(\pi y)(\tau^2+\beta^2-4\pi^2y^2)e^{-icy}}{(\tau^2+\beta^2-4\pi^2y^2)^2+16\pi^2\tau^2y^2}\,dy, \quad (3.4.54)$$

$$\int_{-i\epsilon}^{-i\infty} \cot(\pi z)\left[\frac{e^{-cz}}{(\tau+2\pi z)^2+\beta^2}+\frac{e^{-cz}}{(\tau-2\pi z)^2+\beta^2}\right] dz$$

$$= \int_{-\epsilon}^{-\infty} \coth(\pi y)\left[\frac{e^{-icy}}{(\tau+2\pi iy)^2+\beta^2}+\frac{e^{-icy}}{(\tau-2\pi iy)^2+\beta^2}\right] dy \tag{3.4.55}$$

$$= 2\int_{\epsilon}^{\infty} \frac{\coth(\pi y)(\tau^2+\beta^2-4\pi^2y^2)e^{icy}}{(\tau^2+\beta^2-4\pi^2y^2)^2+16\pi^2\tau^2y^2}\, dy \tag{3.4.56}$$

and

$$\int_{C_\epsilon} \cot(\pi z)\left[\frac{e^{-cz}}{(\tau+2\pi z)^2+\beta^2}+\frac{e^{-cz}}{(\tau-2\pi z)^2+\beta^2}\right] dz$$

$$= \int_{\pi/2}^{-\pi/2}\left[\frac{1}{\pi\epsilon e^{\theta i}}-\frac{\pi\epsilon e^{\theta i}}{3}-\cdots\right]\epsilon i e^{\theta i}\, d\theta$$

$$\times\left[\frac{\exp(-c\epsilon e^{\theta i})}{(\tau+2\pi\epsilon e^{\theta i})^2+\beta^2}+\frac{\exp(-c\epsilon e^{\theta i})}{(\tau-2\pi\epsilon e^{\theta i})^2+\beta^2}\right]. \tag{3.4.57}$$

In the limit of $\epsilon \to 0$,

$$\oint_C \cot(\pi z)\left[\frac{e^{-cz}}{(\tau+2\pi z)^2+\beta^2}+\frac{e^{-cz}}{(\tau-2\pi z)^2+\beta^2}\right] dz$$

$$= 4i\int_0^{\infty} \frac{\coth(\pi y)(\tau^2+\beta^2-4\pi^2y^2)\sin(cy)}{(\tau^2+\beta^2-4\pi^2y^2)^2+16\pi^2\tau^2y^2}\, dy - \frac{2i}{\tau^2+\beta^2} \tag{3.4.58}$$

$$= 2i\sum_{n=1}^{\infty}\left[\frac{e^{-nc}}{(\tau+2n\pi)^2+\beta^2}+\frac{e^{-nc}}{(\tau-2n\pi)^2+\beta^2}\right]$$
$$-\frac{i}{\beta}\frac{\sinh(\beta)\cos(c\beta/2\pi)+\sin(|\tau|)\sin(c\beta/2\pi)}{\cosh(\beta)-\cos(\tau)}e^{-c|\tau|/2\pi} \tag{3.4.59}$$

or

$$4\int_0^{\infty} \frac{\coth(\pi y)(\tau^2+\beta^2-4\pi^2y^2)\sin(cy)}{(\tau^2+\beta^2-4\pi^2y^2)^2+16\pi^2\tau^2y^2}\, dy$$

$$= 2\sum_{n=1}^{\infty}\left[\frac{e^{-nc}}{(\tau+2n\pi)^2+\beta^2}+\frac{e^{-nc}}{(\tau-2n\pi)^2+\beta^2}\right]$$
$$-\frac{1}{\beta}\frac{\sinh(\beta)\cos(c\beta/2\pi)+\sin(|\tau|)\sin(c\beta/2\pi)}{\cosh(\beta)-\cos(\tau)}e^{-c|\tau|/2\pi}$$
$$+\frac{2}{\tau^2+\beta^2}. \tag{3.4.60}$$

If we let $y = x/2\pi$,

$$\frac{\beta}{\pi}\int_0^{\infty} \frac{\coth(x/2)(\tau^2+\beta^2-x^2)\sin(cx/2\pi)}{(\tau^2+\beta^2-x^2)^2+4\tau^2x^2}\, dx$$

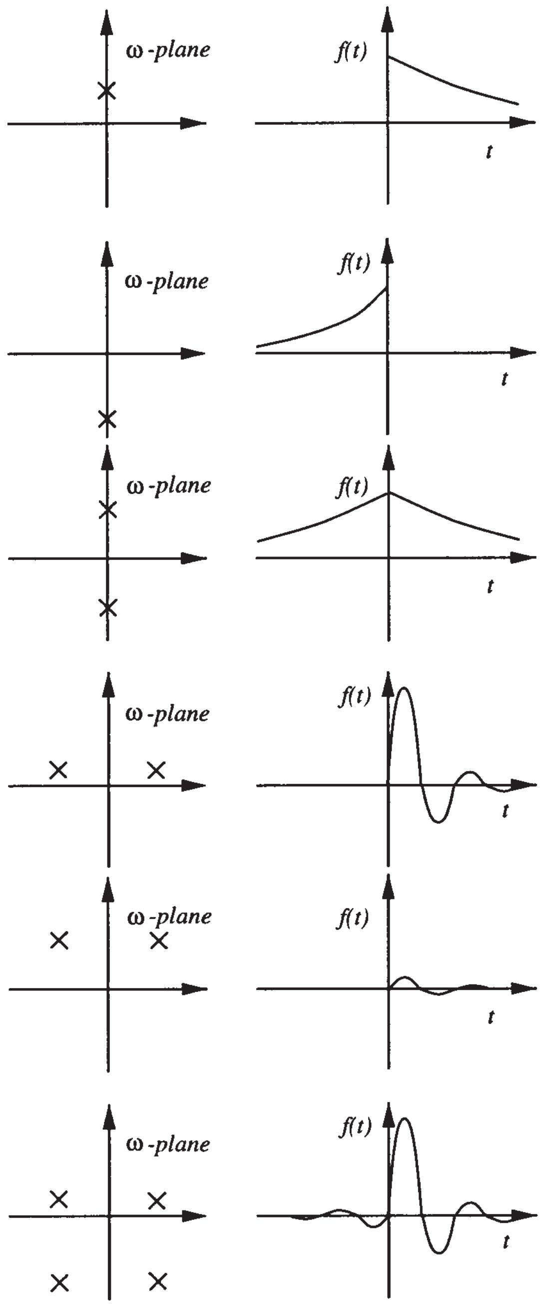


Figure 3.4.3: The correspondence between the location of the simple poles of the Fourier transform $F(\omega)$ and the behavior of $f(t)$.

$$= 2\beta \sum_{n=1}^{\infty} \left[\frac{e^{-nc}}{(\tau + 2n\pi)^2 + \beta^2} + \frac{e^{-nc}}{(\tau - 2n\pi)^2 + \beta^2} \right]$$
$$- \frac{\sinh(\beta)\cos(c\beta/2\pi) + \sin(|\tau|)\sin(c\beta/2\pi)}{\cosh(\beta) - \cos(\tau)} e^{-c|\tau|/2\pi}$$
$$+ \frac{2\beta}{\tau^2 + \beta^2}. \tag{3.4.61}$$

• **Example 3.4.7**

An additional benefit of understanding inversion by the residue method is the ability to qualitatively anticipate the inverse by knowing the location of the poles of $F(\omega)$. This intuition is important because many engineering analyses discuss stability and performance entirely in terms of the properties of the system's Fourier transform. In Figure 3.4.3 we have graphed the location of the poles of $F(\omega)$ and the corresponding $f(t)$. The student should go through the mental exercise of connecting the two pictures.

Problems

1. Use direct integration to find the inverse of the Fourier transform

$$F(\omega) = \frac{i\omega\pi}{2} e^{-|\omega|}.$$

Use partial fractions to invert the following Fourier transforms:

2. $\dfrac{1}{(1+i\omega)(1+2i\omega)}$

3. $\dfrac{1}{(1+i\omega)(1-i\omega)}$

4. $\dfrac{i\omega}{(1+i\omega)(1+2i\omega)}$

5. $\dfrac{1}{(1+i\omega)(1+2i\omega)^2}$

By taking the appropriate closed contour, find the inverse of the following Fourier transform by contour integration. The parameter a is real and positive.

6. $\dfrac{1}{\omega^2 + a^2}$

7. $\dfrac{\omega}{\omega^2 + a^2}$

8. $\dfrac{\omega}{(\omega^2 + a^2)^2}$

9. $\dfrac{\omega^2}{(\omega^2 + a^2)^2}$

10. $\dfrac{1}{\omega^2 - 3i\omega - 3}$

11. $\dfrac{1}{(\omega - ia)^{2n+2}}$

12. $\dfrac{\omega^2}{(\omega^2 - 1)^2 + 4a^2\omega^2}$

13. $\dfrac{3}{(2 - \omega i)(1 + \omega i)}$

14. Find the inverse of $F(\omega) = \cos(\omega)/(\omega^2+a^2)$, $a > 0$, by first rewriting the transform as

$$F(\omega) = \frac{e^{i\omega}}{2(\omega^2 + a^2)} + \frac{e^{-i\omega}}{2(\omega^2 + a^2)}$$

and then using the residue theorem on each term.

15. As we shall show shortly, Fourier transforms can be used to solve differential equations. During the solution of the heat equation, Taitel et al.[10] had to invert the Fourier transform

$$F(\omega) = \frac{\cosh(y\sqrt{\omega^2 + 1}\,)}{\sqrt{\omega^2 + 1}\,\sinh(p\sqrt{\omega^2 + 1}/2)},$$

where y and p are real. Show that they should have found

$$f(t) = \frac{e^{-|t|}}{p} + \frac{2}{p}\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{1 + 4n^2\pi^2/p^2}} \cos\left(\frac{2n\pi y}{p}\right) e^{-\sqrt{1+4n^2\pi^2/p^2}\,|t|}.$$

In this case, our time variable t was their spatial variable $x - \xi$.

16. Find the inverse of the Fourier transform

$$F(\omega) = \left[\cos\left\{\frac{\omega L}{\beta[1 + i\gamma\ \mathrm{sgn}(\omega)]}\right\}\right]^{-1},$$

where L, β, and γ are real and positive and $\mathrm{sgn}(z) = 1$ if $\mathrm{Re}(z) > 0$ and -1 if $\mathrm{Re}(z) < 0$.

Use the residue theorem to verify the following integrals:

17.

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x^2 + 4x + 5}\,dx = -\frac{\pi}{e}\sin(2)$$

18.

$$\int_0^{\infty} \frac{\cos(x)}{(x^2 + 1)^2}\,dx = \frac{\pi}{2e}$$

[10] Reprinted from *Int. J. Heat Mass Transfer*, **16**, Taitel, Y., M. Bentwich and A. Tamir, Effects of upstream and downstream boundary conditions on heat (mass) transfer with axial diffusion, 359–369, ©1973, with kind permission from Elsevier Science Ltd., The Boulevard, Langford Lane, Kidlington OX5 1GB, UK.

19.

$$\int_{-\infty}^{\infty} \frac{x \sin(ax)}{x^2+4}\, dx = \pi e^{-2a}, \qquad a > 0$$

20. The concept of forced convection is normally associated with heat streaming through a duct or past an obstacle. Bentwich[11] wanted to show a similar transport can exist when convection results from a wave traveling through an essentially stagnant fluid. In the process of computing the amount of heating he had to prove the following identity:

$$\int_{-\infty}^{\infty} \frac{\cosh(hx)-1}{x\sinh(hx)} \cos(ax)\, dx = \ln[\coth(|a|\pi/h)], \qquad h > 0.$$

Confirm his result.

3.5 CONVOLUTION

The most important property of Fourier transforms is convolution. We shall use it extensively in the solution of differential equations and the design of filters because it yields in time or space the effect of multiplying two transforms together.

The convolution operation is

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(x)g(t-x)\, dx = \int_{-\infty}^{\infty} f(t-x)g(x)\, dx. \qquad (3.5.1)$$

Then,

$$\mathcal{F}[f(t) * g(t)] = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\left[\int_{-\infty}^{\infty} g(t-x)e^{-i\omega(t-x)}dt\right]dx \qquad (3.5.2)$$

$$= \int_{-\infty}^{\infty} f(x)G(\omega)e^{-i\omega x}\, dx = F(\omega)G(\omega). \qquad (3.5.3)$$

Thus, the Fourier transform of the convolution of two functions equals the product of the Fourier transforms of each of the functions.

• Example 3.5.1

Verify the convolution theorem using the functions $f(t) = H(t+a) - H(t-a)$ and $g(t) = e^{-t}H(t)$, where $a > 0$.

[11] Reprinted from *Int. J. Heat Mass Transfer*, **9**, Bentwich, M., Convection enforced by surface and tidal waves, 663–670, ©1966, with kind permission from Elsevier Science Ltd., The Boulevard, Langford Lane, Kidlington OX5 1GB, UK.

The convolution of $f(t)$ with $g(t)$ is

$$f(t) * g(t) = \int_{-\infty}^{\infty} e^{-(t-x)} H(t-x)\,[H(x+a) - H(x-a)]\,dx \tag{3.5.4}$$

$$= e^{-t} \int_{-a}^{a} e^{x} H(t-x)\,dx. \tag{3.5.5}$$

If $t < -a$, then the integrand of (3.5.5) is always zero and $f(t)*g(t) = 0$. If $t > a$,

$$f(t) * g(t) = e^{-t} \int_{-a}^{a} e^{x}\,dx = e^{-(t-a)} - e^{-(t+a)}. \tag{3.5.6}$$

Finally, for $-a < t < a$,

$$f(t) * g(t) = e^{-t} \int_{-a}^{t} e^{x}\,dx = 1 - e^{-(t+a)}. \tag{3.5.7}$$

In summary,

$$f(t) * g(t) = \begin{cases} 0, & t < -a \\ 1 - e^{-(t+a)}, & -a < t < a \\ e^{-(t-a)} - e^{-(t+a)}, & t > a. \end{cases} \tag{3.5.8}$$

The Fourier transform of $f(t) * g(t)$ is

$$\begin{aligned} \mathcal{F}[f(t) * g(t)] &= \int_{-a}^{a} \left[1 - e^{-(t+a)}\right] e^{-i\omega t}\,dt \\ &\quad + \int_{a}^{\infty} \left[e^{-(t-a)} - e^{-(t+a)}\right] e^{-i\omega t}\,dt \qquad (3.5.9) \\ &= \frac{2\sin(\omega a)}{\omega} - \frac{2i\sin(\omega a)}{1+\omega i} \qquad (3.5.10) \\ &= \frac{2\sin(\omega a)}{\omega}\left(\frac{1}{1+\omega i}\right) = F(\omega)G(\omega) \qquad (3.5.11) \end{aligned}$$

and the convolution theorem is true for this special case.

• **Example 3.5.2**

Let us consider the convolution of $f(t) = f_+(t)H(t)$ with $g(t) = g_+H(t)$. Note that both of the functions are nonzero only for $t > 0$.

From the definition of convolution,

$$f(t) * g(t) = \int_{-\infty}^{\infty} f_+(t-x)H(t-x)g_+(x)H(x)\,dx \quad (3.5.12)$$

$$= \int_0^{\infty} f_+(t-x)H(t-x)g_+(x)\,dx. \quad (3.5.13)$$

For $t < 0$, the integrand is always zero and $f(t) * g(t) = 0$. For $t > 0$,

$$f(t) * g(t) = \int_0^t f_+(t-x)g_+(x)\,dx. \quad (3.5.14)$$

Therefore, in general,

$$f(t) * g(t) = \left[\int_0^t f_+(t-x)g_+(x)\,dx\right] H(t). \quad (3.5.15)$$

This is the definition of convolution that we will use for Laplace transforms where all of the functions equal zero for $t < 0$.

Problems

1. Show that

$$e^{-t}H(t) * e^{-t}H(t) = te^{-t}H(t).$$

2. Show that

$$e^{-t}H(t) * e^{t}H(-t) = \tfrac{1}{2}e^{-|t|}.$$

3. Show that

$$e^{-t}H(t) * e^{-2t}H(t) = \left(e^{-t} - e^{-2t}\right) H(t).$$

4. Show that

$$e^{t}H(-t) * [H(t) - H(t-2)] = \begin{cases} e^t - e^{t-2}, & t < 0 \\ 1 - e^{t-2}, & 0 < t < 2 \\ 0, & t > 2. \end{cases}$$

5. Show that

$$[H(t) - H(t-2)] * [H(t) - H(t-2)] = \begin{cases} 0, & t < 0 \\ t, & 0 < t < 2 \\ 4 - t, & 2 < t < 4 \\ 0, & t > 4. \end{cases}$$

6. Show that

$$e^{-|t|} * e^{-|t|} = (1 + |t|)e^{-|t|}.$$

7. Prove that the convolution of two Dirac delta functions is a Dirac delta function.

3.6 SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS BY FOURIER TRANSFORMS

As with Laplace transforms, we may use Fourier transforms to solve ordinary differential equations. However, this method gives only the particular solution and we must find the complementary solution separately.

Consider the differential equations

$$y' + y = \tfrac{1}{2}e^{-|t|}, \qquad -\infty < t < \infty. \tag{3.6.1}$$

Taking the Fourier transform of both sides of (3.6.1),

$$i\omega Y(\omega) + Y(\omega) = \frac{1}{\omega^2 + 1}, \tag{3.6.2}$$

where we have used the derivative rule (3.3.19) to obtain the transform of y' and $Y(\omega) = \mathcal{F}[y(t)]$. Therefore,

$$Y(\omega) = \frac{1}{(\omega^2 + 1)(1 + \omega i)}. \tag{3.6.3}$$

Applying the inversion integral to (3.6.3),

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{it\omega}}{(\omega^2 + 1)(1 + \omega i)}\, d\omega. \tag{3.6.4}$$

We evaluate (3.6.4) by contour integration. For $t > 0$ we close the line integral with an infinite semicircle in the upper half of the ω-plane. The integration along this arc equals zero by Jordan's lemma. Within this closed contour we have a second-order pole at $z = i$. Therefore,

$$\operatorname{Res}\left[\frac{e^{itz}}{(z^2 + 1)(1 + zi)}; i\right] = \lim_{z \to i} \frac{d}{dz}\left[(z - i)^2 \frac{e^{itz}}{i(z - i)^2(z + i)}\right] \tag{3.6.5}$$

$$= \frac{te^{-t}}{2i} + \frac{e^{-t}}{4i} \tag{3.6.6}$$

and

$$y(t) = \frac{1}{2\pi}(2\pi i)\left[\frac{te^{-t}}{2i} + \frac{e^{-t}}{4i}\right] = \frac{e^{-t}}{4}(2t + 1). \tag{3.6.7}$$

For $t < 0$, we again close the line integral with an infinite semicircle but this time it is in the lower half of the ω-plane. The contribution from the line integral along the arc vanishes by Jordan's lemma. Within the contour, we have a simple pole at $z = -i$. Therefore,

$$\text{Res}\left[\frac{e^{itz}}{(z^2+1)(1+zi)}; -i\right] = \lim_{z\to -i}(z+i)\frac{e^{itz}}{i(z+i)(z-i)^2} = -\frac{e^t}{4i} \tag{3.6.8}$$

and

$$y(t) = \frac{1}{2\pi}(-2\pi i)\left(-\frac{e^t}{4i}\right) = \frac{e^t}{4}. \tag{3.6.9}$$

The minus sign in front of the $2\pi i$ results from the contour being taken in the negative sense. Using the step function, we can combine (3.6.7) and (3.6.9) into the single expression

$$y(t) = \tfrac{1}{4}e^{-|t|} + \tfrac{1}{2}te^{-t}H(t). \tag{3.6.10}$$

Note that we have only found the particular or forced solution to (3.6.1). The most general solution therefore requires that we add the complementary solution Ae^{-t}, yielding

$$y(t) = Ae^{-t} + \tfrac{1}{4}e^{-|t|} + \tfrac{1}{2}te^{-t}H(t). \tag{3.6.11}$$

The arbitrary constant A would be determined by the initial condition which we have not specified.

Consider now a more general problem of

$$y' + y = f(t), \qquad -\infty < t < \infty, \tag{3.6.12}$$

where we assume that $f(t)$ has the Fourier transform $F(\omega)$. Then the Fourier-transformed solution to (3.6.12) is

$$Y(\omega) = \frac{1}{1+\omega i}F(\omega) = G(\omega)F(\omega) \tag{3.6.13}$$

or

$$y(t) = g(t) * f(t), \tag{3.6.14}$$

where $g(t) = \mathcal{F}^{-1}[1/(1+\omega i)] = e^{-t}H(t)$. Thus, we can obtain our solution in one of two ways. First, we can take the Fourier transform of $f(t)$, multiply this transform by $G(\omega)$, and finally compute the inverse. The second method requires a convolution of $f(t)$ with $g(t)$. Which method is easiest depends upon $f(t)$ and $g(t)$.

The function $g(t)$ may also be viewed as the particular solution of (3.6.12) resulting from the forcing function $\delta(t)$, the Dirac delta function, because $\mathcal{F}[\delta(t)] = 1$. Traditionally this forced solution $g(t)$ is called the

Green's function and $G(\omega)$ is called the *frequency response* or *steady-state transfer function* of our system. Engineers often extensively study the frequency response in their analysis rather than the Green's function because the frequency response is easier to obtain experimentally and the output from a linear system is just the product of two transforms [see (3.6.13)] rather than an integration.

In summary, we may use Fourier transforms to find particular solutions to differential equations. The complete solution consists of this particular solution plus any homogeneous solution that we need to satisfy the initial conditions. Convolution of the Green's function with the forcing function also gives the particular solution.

• **Example 3.6.1: Spectrum of a damped harmonic oscillator**

Second-order differential equations are ubiquitous in engineering. In electrical engineering many electrical circuits are governed by second-order, linear ordinary differential equations. In mechanical engineering they arise during the application of Newton's second law. For example, in mechanics the damped oscillations of a mass m attached to a spring with a spring constant k and damped with a velocity dependent resistance is govern by the equation

$$my'' + cy' + ky = f(t), \tag{3.6.15}$$

where $y(t)$ denotes the displacement of the oscillator from its equilibrium position, c denotes the damping coefficient and $f(t)$ denotes the forcing.

Assuming that both $f(t)$ and $y(t)$ have Fourier transforms, let us analyze this system by finding its frequency response. We begin our analysis by solving for the Green's function $g(t)$ which is given by

$$mg'' + cg' + kg = \delta(t), \tag{3.6.16}$$

because the Green's function is the response of a system to a delta function forcing. Taking the Fourier transform of both sides of (3.6.16), the frequency response is

$$G(\omega) = \frac{1}{k + ic\omega - m\omega^2} = \frac{1/m}{\omega_0^2 + ic\omega/m - \omega^2}, \tag{3.6.17}$$

where $\omega_0^2 = k/m$ is the natural frequency of the system. The most useful quantity to plot is the frequency response or

$$|G(\omega)| = \frac{\omega_0^2}{k\sqrt{(\omega^2 - \omega_0^2)^2 + \omega^2\omega_0^2(c^2/km)}} \tag{3.6.18}$$

$$= \frac{1}{k\sqrt{[(\omega/\omega_0)^2 - 1]^2 + (c^2/km)(\omega/\omega_0)^2}}. \tag{3.6.19}$$

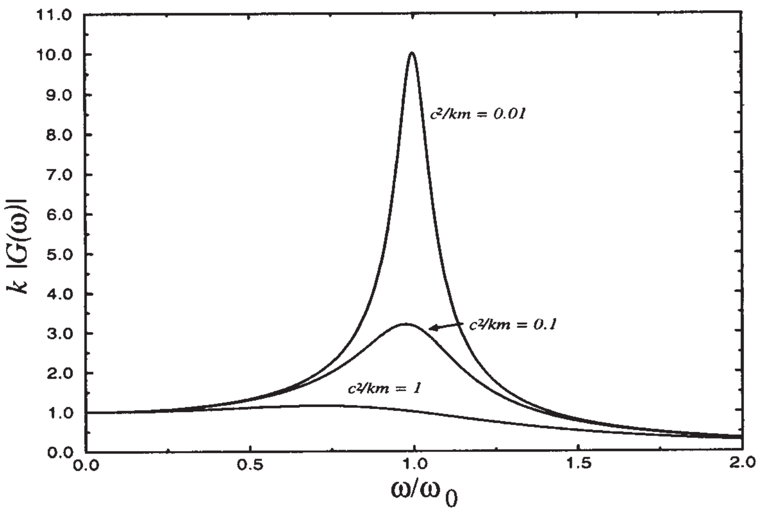


Figure 3.6.1: The variation of the frequency response for a damped harmonic oscillator as a function of driving frequency ω. See the text for the definition of the parameters.

In Figure 3.6.1 we have plotted with frequence response for different c^2/km's. Note that as the damping becomes larger, the sharp peak at $\omega = \omega_0$ essentially vanishes. As $c^2/km \to 0$, we obtain a very finely tuned response curve.

Let us now find the Green's function. From the definition of the inverse Fourier transform,

$$mg(t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega^2 - ic\omega/m - \omega_0^2}\, d\omega \tag{3.6.20}$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - \omega_1)(\omega - \omega_2)}\, d\omega, \tag{3.6.21}$$

where

$$\omega_{1,2} = \pm\sqrt{\omega_0^2 - \gamma^2} + \gamma i \tag{3.6.22}$$

and $\gamma = c/2m > 0$. We can evaluate (3.6.21) by residues. Clearly the poles always lie in the upper half of the ω-plane. Thus, if $t < 0$ in (3.6.21) we can close the line integration along the real axis with a semicircle of infinite radius in the lower half of the ω-plane by Jordan's lemma. Because the integrand is analytic within the closed contour, $g(t) = 0$ for $t < 0$. This is simply the causality condition,[12] the impulse

[12] The principle stating that an event cannot precede its cause.

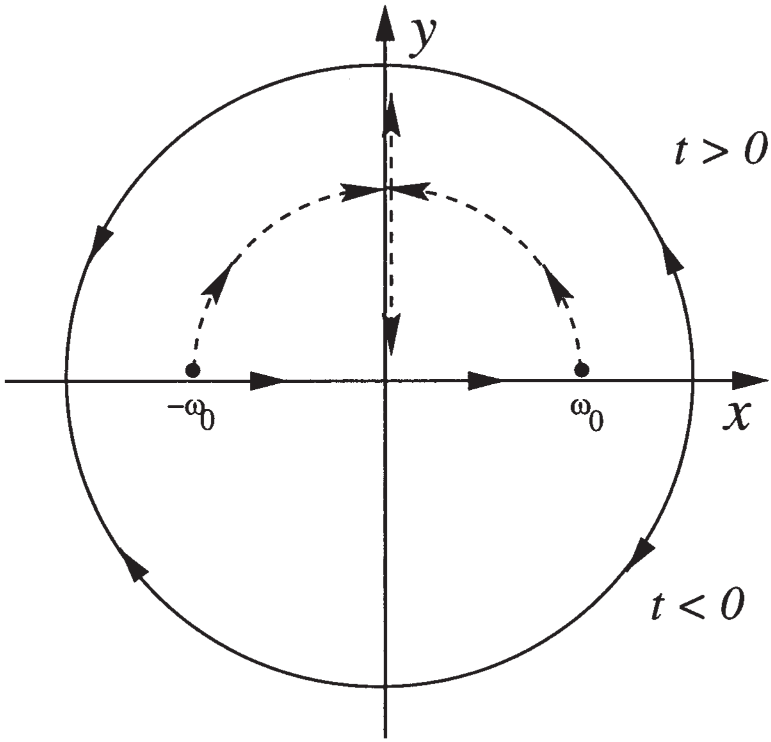


Figure 3.6.2: The migration of the poles of the frequency response of a damped harmonic oscillator as a function of γ.

forcing being the cause of the excitation. Clearly, causality is closely connected with the analyticity of the frequency response in the lower half of the ω-plane.

If $t > 0$, we close the line integration along the real axis with a semicircle of infinite radius in the upper half of the ω-plane and obtain

$$mg(t) = 2\pi i\left(-\frac{1}{2\pi}\right)\left\{\text{Res}\left[\frac{e^{izt}}{(z-\omega_1)(z-\omega_2)};\omega_1\right]\right.$$
$$\left.+\,\text{Res}\left[\frac{e^{izt}}{(z-\omega_1)(z-\omega_2)};\omega_2\right]\right\} \quad (3.6.23)$$

$$= \frac{-i}{\omega_1-\omega_2}\left(e^{i\omega_1 t}-e^{i\omega_2 t}\right) \quad (3.6.24)$$

$$= \frac{e^{-\gamma t}\sin\left(t\sqrt{\omega_0^2-\gamma^2}\right)}{\sqrt{\omega_0^2-\gamma^2}}H(t). \quad (3.6.25)$$

Let us now examine the damped harmonic oscillator by describing the migration of the poles $\omega_{1,2}$ in the complex ω-plane as γ increase from 0 to ∞. See Figure 3.6.2. For $\gamma \ll \omega_0$ (weak damping), the poles $\omega_{1,2}$ are very near to the real axis, above the points $\pm\omega_0$, respectively. This corresponds to the narrow resonance band discussed earlier and we have an underdamped harmonic oscillator. As γ increases from 0 to ω_0, the poles approach the positive imaginary axis, moving along a semicircle of radius ω_0 centered at the origin. They coalesce at the point $i\omega_0$ for $\gamma = \omega_0$, yielding repeated roots, and we have a critically damped oscillator. For $\gamma > \omega_0$, the poles move in opposite directions along the positive imaginary axis; one of them approaches the origin, while the other tends to $i\infty$ as $\gamma \to \infty$. The solution then has two purely decaying, overdamped solutions.

During the early 1950s, a similar diagram was invented by Evans[13] where the movement of closed-loop poles is plotted for all values of a system parameter, usually the gain. This *root-locus method* is very popular in system control theory for two reasons. First, the investigator can easily determine the contribution of a particular closed-loop pole to the transient response. Second, he may determine the manner in which open-loop poles or zeros should be introduced or their location modified so that he will achieve a desired performance characteristic for his system.

• **Example 3.6.2: Low frequency filter**

Consider the ordinary differential equation

$$Ry' + \frac{1}{C}y = f(t), \tag{3.6.26}$$

where R and C are real, positive constants. If $y(t)$ denotes current, then (3.6.26) would be the equation that gives the voltage across a capacitor in a RC circuit. Let us find the frequency response and Green's function for this system.

We begin by writing (3.6.26) as

$$Rg' + \frac{1}{C}g = \delta(t), \tag{3.6.27}$$

where $g(t)$ denotes the Green's function. If the Fourier transform of $g(t)$ is $G(\omega)$, the frequency response $G(\omega)$ is given by

$$i\omega RG(\omega) + \frac{G(\omega)}{C} = 1 \tag{3.6.28}$$

---

[13] Evans, W. R., 1948: Graphical analysis of control systems. *Trans. AIEE*, **67**, 547–551; Evans, W. R., 1954: *Control-System Dynamics*, McGraw-Hill, New York.

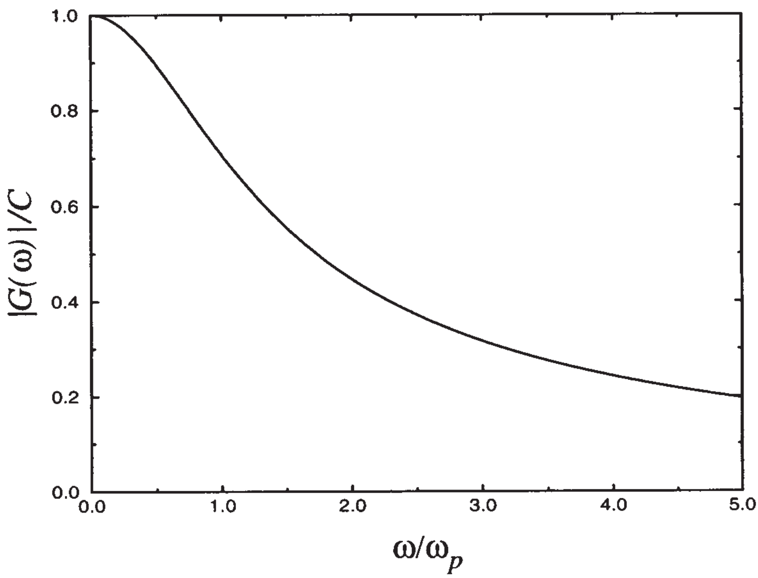


Figure 3.6.3: The variation of the frequency response (3.6.30) as a function of driving frequency ω. See the text for the definition of the parameters.

or

$$G(\omega) = \frac{1}{i\omega R + 1/C} = \frac{C}{1 + i\omega RC} \tag{3.6.29}$$

and

$$|G(\omega)| = \frac{C}{\sqrt{1 + \omega^2 R^2 C^2}} = \frac{C}{\sqrt{1 + \omega^2/\omega_p^2}}, \tag{3.6.30}$$

where $\omega_p = 1/(RC)$ is an intrinsic constant of the system. In Figure 3.6.3 we have plotted $|G(\omega)|$ as a function of ω. From this figure, we see that the response is largest for small ω and decreases as ω increases.

This is an example of a *low frequency filter* because relatively more signal passes through at lower frequencies than at higher frequencies. To understand this, let us drive the system with a forcing function that has the Fourier transform $F(\omega)$. The response of the system will be $G(\omega)F(\omega)$. Thus, that portion of the forcing function's spectrum at the lower frequencies will be relatively unaffected because $|G(\omega)|$ is near unity. However, at higher frequencies where $|G(\omega)|$ is smaller, the magnitude of the output will be greatly reduced.

Problems

Find the particular solutions for the following differential equations:

1. $y'' + 3y' + 2y = e^{-t}H(t)$ 2. $y'' + 4y' + 4y = \frac{1}{2}e^{-|t|}$

3. $y'' - 4y' + 4y = e^{-t}H(t)$ 4. $y^{iv} - \lambda^4 y = \delta(x)$,
where λ has a positive real part and a negative imaginary part.

Chapter 4
The Laplace Transform

The previous chapter introduced the concept of the Fourier integral. If the function is nonzero only when $t > 0$, a similar transform, the *Laplace transform*,[1] exists. It is particularly useful in solving initial-value problems involving linear, constant coefficient, ordinary and partial differential equations. The present chapter develops the general properties and techniques of Laplace transforms.

4.1 DEFINITION AND ELEMENTARY PROPERTIES

Consider a function $f(t)$ such that $f(t) = 0$ for $t < 0$. Then the *Laplace integral*

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt \tag{4.1.1}$$

[1] The standard reference for Laplace transforms is Doetsch, G., 1950: *Handbuch der Laplace-Transformation. Band 1. Theorie der Laplace-Transformation*, Birkhäuser Verlag, 581 pp.; Doetsch, G., 1955: *Handbuch der Laplace-Transformation. Band 2. Anwendungen der Laplace-Transformation. 1. Abteilung*, Birkhäuser Verlag, 433 pp.; Doetsch, G., 1956: *Handbuch der Laplace-Transformation. Band 3. Anwendungen der Laplace-Transformation. 2. Abteilung*, Birkhäuser Verlag, 298 pp.

defines the Laplace transform of $f(t)$, which we shall write $\mathcal{L}[f(t)]$ or $F(s)$. The Laplace transform converts a function of t into a function of the transform variable s.

Not all functions have a Laplace transform because the integral (4.1.1) may fail to exist. For example, the function may have infinite discontinuities. For this reason, $f(t) = \tan(t)$ does *not* have a Laplace transform. We may avoid this difficulty by requiring that $f(t)$ be *piece-wise continuous.* That is, we can divide a finite range into a finite number of intervals in such a manner that $f(t)$ is continuous inside each interval and approaches finite values as we approach either end of any interval from the interior.

Another unacceptable function is $f(t) = 1/t$ because the integral (4.1.1) fails to exist. This leads to the requirement that the product $t^n|f(t)|$ is bounded near $t = 0$ for some number $n < 1$.

Finally $|f(t)|$ cannot grow too rapidly or it could overwhelm the e^{-st} term. To express this, we introduce the concept of functions of *exponential order*. By exponential order we mean that there exists some constants, M and k, for which

$$|f(t)| \leq Me^{kt} \tag{4.1.2}$$

for all $t > 0$. Then, the Laplace transform of $f(t)$ exists if s, or just the real part of s, is greater than k.

In summary, the Laplace transform of $f(t)$ exists, for sufficiently large s, provided $f(t)$ satisfies the following conditions:

- $f(t) = 0$ for $t < 0$,
- $f(t)$ is continuous or piece-wise continuous in every interval,
- $t^n|f(t)| < \infty$ as $t \to 0$ for some number n, where $n < 1$,
- $e^{-s_0t}|f(t)| < \infty$ as $t \to \infty$, for some number s_0. The quantity s_0 is called the *abscissa of convergence.*

• Example 4.1.1

Let us find the Laplace transform of 1, e^{at}, $\sin(at)$, $\cos(at)$, and t^n from the definition of the Laplace transform. From (4.1.1), direct integration yields:

$$\mathcal{L}(1) = \int_0^\infty e^{-st}dt = -\left.\frac{e^{-st}}{s}\right|_0^\infty = \frac{1}{s}, \qquad s > 0, \tag{4.1.3}$$

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{-(s-a)t}dt \tag{4.1.4}$$

$$= -\left.\frac{e^{-(s-a)t}}{s-a}\right|_0^\infty = \frac{1}{s-a}, \qquad s > a, \tag{4.1.5}$$

$$\mathcal{L}[\sin(at)] = \int_0^\infty \sin(at)e^{-st}dt = -\frac{e^{-st}}{s^2+a^2}[s\sin(at)+a\cos(at)]\Big|_0^\infty \tag{4.1.6}$$

$$= \frac{a}{s^2+a^2}, \qquad s>0, \tag{4.1.7}$$

$$\mathcal{L}[\cos(at)] = \int_0^\infty \cos(at)e^{-st}dt = \frac{e^{-st}}{s^2+a^2}[-s\cos(at)+a\sin(at)]\Big|_0^\infty \tag{4.1.8}$$

$$= \frac{s}{s^2+a^2}, \qquad s>0 \tag{4.1.9}$$

and

$$\mathcal{L}(t^n) = \int_0^\infty t^n e^{-st}dt = n!e^{-st}\sum_{m=0}^{n}\frac{t^{n-m}}{(n-m)!s^{m+1}}\Big|_0^\infty = \frac{n!}{s^{n+1}}, \quad s>0, \tag{4.1.10}$$

where n is a positive integer.

The Laplace transform inherits two important properties from its integral definition. First, the transform of a sum equals the sum of the transforms:

$$\mathcal{L}[c_1 f(t) + c_2 g(t)] = c_1\mathcal{L}[f(t)] + c_2\mathcal{L}[g(t)]. \tag{4.1.11}$$

This linearity property holds with complex numbers and functions as well.

- **Example 4.1.2**

Success with Laplace transforms often rests with the ability to manipulate a given transform into a form which you can invert by inspection. Consider the following examples.

Given $F(s) = 4/s^3$, then

$$F(s) = 2\times\frac{2}{s^3} \quad \text{and} \quad f(t) = 2t^2 \tag{4.1.12}$$

from (4.1.10).

Table 4.1.1: The Laplace Transforms of Some Commonly Encountered Functions.

	$f(t),\ t \geq 0$	$F(s)$
1.	1	$\frac{1}{s}$
2.	e^{-at}	$\frac{1}{s+a}$
3.	$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$
4.	$\frac{1}{a-b}(e^{-bt}-e^{-at})$	$\frac{1}{(s+a)(s+b)}$
5.	$\frac{1}{b-a}(be^{-bt}-ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
6.	$\sin(at)$	$\frac{a}{s^2+a^2}$
7.	$\cos(at)$	$\frac{s}{s^2+a^2}$
8.	$\sinh(at)$	$\frac{a}{s^2-a^2}$
9.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
10.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
11.	$1-\cos(at)$	$\frac{a^2}{s(s^2+a^2)}$
12.	$at-\sin(at)$	$\frac{a^3}{s^2(s^2+a^2)}$
13.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
14.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$
15.	$t\sinh(at)$	$\frac{2as}{(s^2-a^2)^2}$
16.	$t\cosh(at)$	$\frac{s^2+a^2}{(s^2-a^2)^2}$
17.	$at\cosh(at)-\sinh(at)$	$\frac{2a^3}{(s^2-a^2)^2}$

Table 4.1.1 (contd.): The Laplace Transforms of Some Commonly Encountered Functions.

	$\mathbf{f(t),\ t \geq 0}$	$\mathbf{F(s)}$
18.	$e^{-bt}\sin(at)$	$\dfrac{a}{(s+b)^2+a^2}$
19.	$e^{-bt}\cos(at)$	$\dfrac{s+b}{(s+b)^2+a^2}$
20.	$(1+a^2t^2)\sin(at)-\cos(at)$	$\dfrac{8a^3s^2}{(s^2+a^2)^3}$
21.	$\sin(at)\cosh(at)-\cos(at)\sinh(at)$	$\dfrac{4a^3}{s^4+4a^4}$
22.	$\sin(at)\sinh(at)$	$\dfrac{2a^2s}{s^4+4a^4}$
23.	$\sinh(at)-\sin(at)$	$\dfrac{2a^3}{s^4-a^4}$
24.	$\cosh(at)-\cos(at)$	$\dfrac{2a^2s}{s^4-a^4}$
25.	$\dfrac{a\sin(at)-b\sin(bt)}{a^2-b^2}, a^2 \neq b^2$	$\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$
26.	$\dfrac{b\sin(at)-a\sin(bt)}{ab(b^2-a^2)}, a^2 \neq b^2$	$\dfrac{1}{(s^2+a^2)(s^2+b^2)}$
27.	$\dfrac{\cos(at)-\cos(bt)}{b^2-a^2}, a^2 \neq b^2$	$\dfrac{s}{(s^2+a^2)(s^2+b^2)}$
28.	$t^n, n \geq 0$	$\dfrac{n!}{s^{n+1}}$
29.	$\dfrac{t^{n-1}e^{-at}}{(n-1)!}, n > 0$	$\dfrac{1}{(s+a)^n}$
30.	$\dfrac{(n-1)-at}{(n-1)!}t^{n-2}e^{-at}, n > 1$	$\dfrac{s}{(s+a)^n}$
31.	$t^ne^{-at}, n \geq 0$	$\dfrac{n!}{(s+a)^{n+1}}$
32.	$\dfrac{2^nt^{n-(1/2)}}{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}, n \geq 1$	$s^{-[n+(1/2)]}$
33.	$J_0(at)$	$\dfrac{1}{\sqrt{s^2+a^2}}$

Table 4.1.1 (contd.): The Laplace Transforms of Some Commonly Encountered Functions.

	f(t), t ≥ 0	**F(s)**
34.	$I_0(at)$	$\dfrac{1}{\sqrt{s^2-a^2}}$
35.	$\dfrac{1}{\sqrt{a}}\operatorname{erf}(\sqrt{at})$	$\dfrac{1}{s\sqrt{s+a}}$
36.	$\dfrac{1}{\sqrt{\pi t}}e^{-at}+\sqrt{a}\operatorname{erf}(\sqrt{at})$	$\dfrac{\sqrt{s+a}}{s}$
37.	$\dfrac{1}{\sqrt{\pi t}}-ae^{a^2t}\operatorname{erfc}(a\sqrt{t})$	$\dfrac{1}{a+\sqrt{s}}$
38.	$e^{at}\operatorname{erfc}(\sqrt{at})$	$\dfrac{1}{s+\sqrt{as}}$
39.	$\dfrac{1}{2\sqrt{\pi t^3}}\left(e^{bt}-e^{at}\right)$	$\sqrt{s-a}-\sqrt{s-b}$
40.	$\dfrac{1}{\sqrt{\pi t}}+ae^{a^2t}\operatorname{erf}(a\sqrt{t})$	$\dfrac{\sqrt{s}}{s-a^2}$
41.	$\dfrac{1}{\sqrt{\pi t}}e^{at}(1+2at)$	$\dfrac{s}{(s-a)\sqrt{s-a}}$
42.	$\dfrac{1}{a}e^{a^2t}\operatorname{erf}(a\sqrt{t})$	$\dfrac{1}{(s-a^2)\sqrt{s}}$
43.	$\sqrt{\dfrac{a}{\pi t^3}}e^{-a/t}, a>0$	$e^{-2\sqrt{as}}$
44.	$\dfrac{1}{\sqrt{\pi t}}e^{-a/t}, a\geq 0$	$\dfrac{1}{\sqrt{s}}e^{-2\sqrt{as}}$
45.	$\operatorname{erfc}\left(\sqrt{\dfrac{a}{t}}\right), a\geq 0$	$\dfrac{1}{s}e^{-2\sqrt{as}}$
46.	$2\sqrt{\dfrac{t}{\pi}}\exp\left(-\dfrac{a^2}{4t}\right)-a\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right),\ a\geq 0$	$\dfrac{e^{-a\sqrt{s}}}{s\sqrt{s}}$
47.	$-e^{b^2t+ab}\operatorname{erfc}\left(b\sqrt{t}+\frac{a}{2\sqrt{t}}\right)+\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right), a\geq 0$	$\dfrac{be^{-a\sqrt{s}}}{s(b+\sqrt{s})}$
48.	$e^{ab}e^{b^2t}\operatorname{erfc}\left(b\sqrt{t}+\dfrac{a}{2\sqrt{t}}\right), a\geq 0$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}(b+\sqrt{s})}$

Notes: Error function: $\operatorname{erf}(x)=\dfrac{2}{\pi}\displaystyle\int_0^x e^{-y^2}\,dy$

Complementary error function: $\operatorname{erfc}(x)=1-\operatorname{erf}(x)$

Given

$$F(s) = \frac{s+2}{s^2+1} = \frac{s}{s^2+1} + \frac{2}{s^2+1}, \tag{4.1.13}$$

then

$$f(t) = \cos(t) + 2\sin(t) \tag{4.1.14}$$

by (4.1.7), (4.1.9), and (4.1.11).

Because

$$F(s) = \frac{1}{s(s-1)} = \frac{1}{s-1} - \frac{1}{s} \tag{4.1.15}$$

by partial fractions, then

$$f(t) = e^t - 1 \tag{4.1.16}$$

by (4.1.3), (4.1.5), and (4.1.11).

The second important property deals with derivatives. Suppose $f(t)$ is continuous and has a piece-wise continuous derivative $f'(t)$. Then

$$\mathcal{L}[f'(t)] = \int_0^\infty f'(t)e^{-st}dt = e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}dt \tag{4.1.17}$$

by integration by parts. If $f(t)$ is of exponential order, $e^{-st}f(t)$ tends to zero as $t \to \infty$, for large enough s, so that

$$\mathcal{L}[f'(t)] = sF(s) - f(0). \tag{4.1.18}$$

Similarly, if $f(t)$ and $f'(t)$ are continuous, $f''(t)$ is piece-wise continuous, and all three functions are of exponential order, then

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f'(t)] - f'(0) = s^2F(s) - sf(0) - f'(0). \tag{4.1.19}$$

In general,

$$\boxed{\mathcal{L}[f^{(n)}(t)] = s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)} \tag{4.1.20}$$

on the assumption that $f(t)$ and its first $n-1$ derivatives are continuous, $f^{(n)}(t)$ is piece-wise continuous, and all are of exponential order so that the Laplace transform exists.

The converse of (4.1.20) is also of some importance. If

$$u(t) = \int_0^t f(\tau)\, d\tau, \tag{4.1.21}$$

then

$$\mathcal{L}[u(t)] = \int_0^\infty e^{-st} \left[\int_0^t f(\tau)\, d\tau\right] dt \tag{4.1.22}$$

$$= -\frac{e^{-st}}{s} \int_0^t f(\tau)\, d\tau \bigg|_0^\infty + \frac{1}{s} \int_0^\infty f(t) e^{-st}\, dt \tag{4.1.23}$$

and

$$\mathcal{L}\left[\int_0^t f(\tau)\, d\tau\right] = \frac{F(s)}{s}, \tag{4.1.24}$$

where $u(0) = 0$.

Problems

Using the definition of the Laplace transform, find the Laplace transform of the following function:

1. $f(t) = \cosh(at)$

2. $f(t) = \cos^2(at)$

3. $f(t) = (t+1)^2$

4. $f(t) = (t+1)e^{-at}$

5. $f(t) = \begin{cases} e^t, & 0 < t < 2 \\ 0, & t > 2 \end{cases}$

6. $f(t) = \begin{cases} \sin(t), & 0 < t < \pi \\ 0, & t > \pi \end{cases}$

Using your knowledge of the transform for 1, e^{at}, $\sin(at)$, $\cos(at)$, and t^n, find the Laplace transform of

7. $f(t) = 2\sin(t) - \cos(2t) + \cos(3) - t$

8. $f(t) = t - 2 + e^{-5t} - \sin(5t) + \cos(2)$.

Find the inverse of the following transforms:

9. $F(s) = 1/(s+3)$

10. $F(s) = 1/s^4$

11. $F(s) = 1/(s^2+9)$ 12. $F(s) = (2s+3)/(s^2+9)$

13. $F(s) = 2/(s^2+1) - 15/s^3 + 2/(s+1) - 6s/(s^2+4)$

14. $F(s) = 3/s + 15/s^3 + (s+5)/(s^2+1) - 6/(s-2)$.

15. Verify the derivative rule for Laplace transforms using the function $f(t) = \sin(at)$.

16. Show that $\mathcal{L}[f(at)] = F(s/a)/a$, where $F(s) = \mathcal{L}[f(t)]$.

17. Using the trigonometric identity $\sin^2(x) = [1-\cos(2x)]/2$, find the Laplace transform of $f(t) = \sin^2[\pi t/(2T)]$.

4.2 THE HEAVISIDE STEP AND DIRAC DELTA FUNCTIONS

Change can occur abruptly. We throw a switch and electricity suddenly flows. In this section we introduce two functions, the Heaviside step and Dirac delta, that will give us the ability to construct complicated discontinuous functions to express these changes.

Heaviside step function

We define the *Heaviside step function* as

$$H(t-a) = \begin{cases} 1, & t > a \\ 0, & t < a, \end{cases} \tag{4.2.1}$$

where $a \geq 0$. From this definition,

$$\mathcal{L}[H(t-a)] = \int_a^\infty e^{-st}\,dt = \frac{e^{-as}}{s}, \qquad s > 0. \tag{4.2.2}$$

Note that this transform is identical to that for $f(t) = 1$ if $a = 0$. This should not surprise us. As pointed out earlier, the function $f(t)$ is zero for all $t < 0$ by definition. Thus, when dealing with Laplace transforms $f(t) = 1$ and $H(t)$ are identical. Generally we will take 1 rather than $H(t)$ as the inverse of $1/s$.

The Heaviside step function is essentially a bookkeeping device that gives us the ability to "switch on" and "switch off" a given function. For example, if we want a function $f(t)$ to become nonzero at time $t = a$, we represent this process by the product $f(t)H(t-a)$. On the other hand, if we only want the function to be "turned on" when $a < t < b$, the desired expression is then $f(t)[H(t-a) - H(t-b)]$. For $t < a$, both step

Figure 4.2.1: Largely a self-educated man, Oliver Heaviside (1850–1925) lived the life of a recluse. It was during his studies of the implications of Maxwell's theory of electricity and magnetism that he reinvented Laplace transforms. Initially rejected, it would require the work of Bromwich to justify its use. (Portrait courtesy of the Institution of Electrical Engineers, London.)

functions in the brackets have the value of zero. For $a < t < b$, the first step function has the value of unity and the second step function has the value of zero, so that we have $f(t)$. For $t > b$, both step functions equal unity so that their difference is zero.

• Example 4.2.1

Quite often we need to express the graphical representation of a function by a mathematical equation. We can conveniently do this through the use of step functions in a two-step procedure. The following example illustrates this procedure.

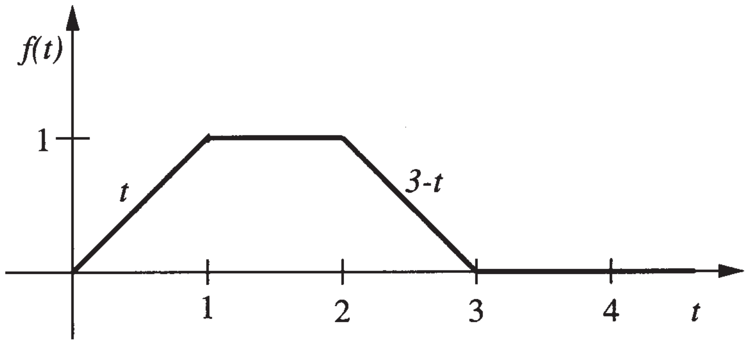


Figure 4.2.2: Graphical representation of (4.2.5).

Consider Figure 4.2.2. We would like to express this graph in terms of Heaviside step functions. We begin by introducing step functions at each point where there is a kink (discontinuity in the first derivative) or jump in the graph – in the present case at $t = 0$, $t = 1$, $t = 2$, and $t = 3$. Thus,

$$f(t) = a_0(t)H(t)+a_1(t)H(t-1)+a_2(t)H(t-2)+a_3(t)H(t-3), \quad \mathbf{(4.2.3)}$$

where the coefficients $a_0(t), a_1(t), \ldots$ are yet to be determined. Proceeding from left to right in Figure 4.2.2, the coefficient of each step function equals the mathematical expression that we want after the kink or jump minus the expression before the kink or jump. Thus, in the present example,

$$f(t) = (t-0)H(t)+(1-t)H(t-1)+[(3-t)-1]H(t-2)+[0-(3-t)]H(t-3) \quad \mathbf{(4.2.4)}$$

or

$$f(t) = tH(t)-(t-1)H(t-1)-(t-2)H(t-2)+(t-3)H(t-3). \quad \mathbf{(4.2.5)}$$

We can easily find the Laplace transform of (4.2.5) by the "second shifting" theorem introduced in the next section.

• **Example 4.2.2**

Laplace transforms are particularly useful in solving initial-value problems involving linear, constant coefficient, ordinary differential equations where the nonhomogeneous term is discontinuous. As we shall show in the next section, we must first rewrite the nonhomogeneous term using the Heaviside step function before we can use Laplace transforms. For example, given the nonhomogeneous ordinary differential equation:

$$y'' + 3y' + 2y = \begin{cases} t, & 0 < t < 1 \\ 0, & t > 1, \end{cases} \quad \mathbf{(4.2.6)}$$

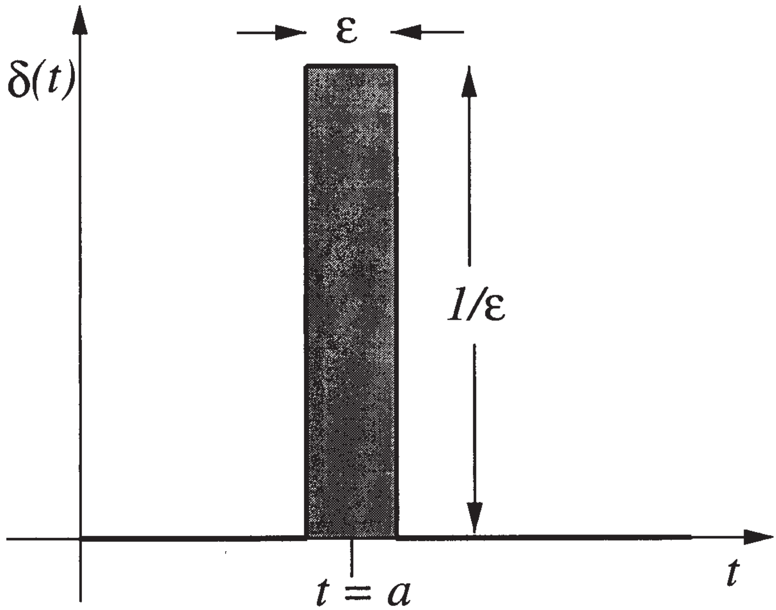


Figure 4.2.3: The Dirac delta function.

we can rewrite the right side of (4.2.6) as

$$y'' + 3y' + 2y = t - tH(t-1) \tag{4.2.7}$$
$$= t - (t-1)H(t-1) - H(t-1). \tag{4.2.8}$$

In Section 4.8 we will show how to solve this type of ordinary differential equation using Laplace transforms.

Dirac delta function

The second special function is the *Dirac delta function* or *impulse function*. We define it by

$$\delta(t-a) = \begin{cases} \infty, & t = a \\ 0, & t \neq a, \end{cases} \qquad \int_0^\infty \delta(t-a)\,dt = 1, \tag{4.2.9}$$

where $a \geq 0$.

A popular way of visualizing the delta function is as a very narrow rectangular pulse:

$$\delta(t-a) = \lim_{\epsilon \to 0} \begin{cases} 1/\epsilon, & 0 < |t-a| < \epsilon/2 \\ 0, & |t-a| > \epsilon/2, \end{cases} \tag{4.2.10}$$

where $\epsilon > 0$ is some small number and $a > 0$. This pulse has a width ϵ, height $1/\epsilon$, and centered at $t = a$ so that its area is unity. Now as this pulse shrinks in width ($\epsilon \to 0$), its height increases so that it remains centered at $t = a$ and its area equals unity. If we continue this process, always keeping the area unity and the pulse symmetric about $t = a$, eventually we obtain an extremely narrow, very large amplitude pulse at $t = a$. If we proceed to the limit, where the width approaches zero and the height approaches infinity (but still with unit area), we obtain the delta function $\delta(t-a)$.

The delta function was introduced earlier during our study of Fourier transforms. So what is the difference between the delta function introduced then and the delta function now? Simply put, the delta function can now only be used on the interval $[0, \infty)$. Outside of that, we shall use it very much as we did with Fourier transforms.

Using (4.2.10), the Laplace transform of the delta function is

$$\mathcal{L}[\delta(t-a)] = \int_0^\infty \delta(t-a)e^{-st}\,dt = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_{a-\epsilon/2}^{a+\epsilon/2} e^{-st}\,dt \tag{4.2.11}$$

$$= \lim_{\epsilon\to 0} \frac{1}{\epsilon s}\left(e^{-as+\epsilon s/2} - e^{-as-\epsilon s/2}\right) \tag{4.2.12}$$

$$= \lim_{\epsilon\to 0} \frac{e^{-as}}{\epsilon s}\left(1 + \frac{\epsilon s}{2} + \frac{\epsilon^2 s^2}{8} + \cdots - 1 + \frac{\epsilon s}{2} - \frac{\epsilon^2 s^2}{8} + \cdots\right) \tag{4.2.13}$$

$$= e^{-as}. \tag{4.2.14}$$

In the special case when $a = 0$, $\mathcal{L}[\delta(t)] = 1$, a property that we will use in Section 4.9. Note that this is exactly the result that we obtained for the Fourier transform of the delta function.

If we integrate the impulse function,

$$\int_0^t \delta(\tau - a)\,d\tau = \begin{cases} 0, & t < a \\ 1, & t > a, \end{cases} \tag{4.2.15}$$

according to whether the impulse does or does not come within the range of integration. This integral gives a result that is precisely the definition of the Heaviside step function so that we can rewrite (4.2.15)

$$\int_0^t \delta(\tau - a)\,d\tau = H(t-a). \tag{4.2.16}$$

Consequently the delta function behaves like the derivative of the step function or

$$\frac{d}{dt}\Big[H(t-a)\Big] = \delta(t-a). \tag{4.2.17}$$

Because the conventional derivative does not exist at a point of discontinuity, we can only make sense of (4.2.17) if we extend the definition of the derivative. Here we have extended the definition formally, but a richer and deeper understanding arises from the theory of generalized functions.[2]

Problems

Sketch the following functions and express them in terms of Heaviside's step functions:

1. $f(t) = \begin{cases} 0, & 0 \le t \le 2 \\ t-2, & 2 \le t < 3 \\ 0, & t > 3 \end{cases}$

2. $f(t) = \begin{cases} 0, & 0 < t < a \\ 1, & a < t < 2a \\ -1, & 2a < t < 3a \\ 0, & t > 3a \end{cases}$

Rewrite the following nonhomogeneous ordinary differential equations using Heaviside's step functions.

3. $y'' + 3y' + 2y = \begin{cases} 0, & 0 < t < 1 \\ 1, & t > 1 \end{cases}$

4. $y'' + 4y = \begin{cases} 0, & 0 < t < 4 \\ 3, & t > 4 \end{cases}$

5. $y'' + 4y' + 4y = \begin{cases} 0, & 0 < t < 2 \\ t, & t > 2 \end{cases}$

6. $y'' + 3y' + 2y = \begin{cases} 0, & 0 < t < 1 \\ e^t, & t > 1 \end{cases}$

7. $y'' - 3y' + 2y = \begin{cases} 0, & 0 < t < 2 \\ e^{-t}, & t > 2 \end{cases}$

8. $y'' - 3y' + 2y = \begin{cases} 0, & 0 < t < 1 \\ t^2, & t > 1 \end{cases}$

---

[2] The generalization of the definition of a function so that it can express in a mathematically correct form such idealized concepts as the density of a material point, a point charge or point dipole, the space charge of a simple or double layer, the intensity of an instantaneous source, etc.

9. $y'' + y = \begin{cases} \sin(t), & 0 \le t \le \pi \\ 0, & t \ge \pi \end{cases}$

10. $y'' + 3y' + 2y = \begin{cases} t, & 0 \le t \le a \\ ae^{-(t-a)}, & t \ge a \end{cases}$

4.3 SOME USEFUL THEOREMS

Although at first sight there would appear to be a bewildering number of transforms to either memorize or tabulate, there are several useful theorems which can extend the applicability of a given transform.

First shifting theorem

Consider the transform of the function $e^{-at}f(t)$, where a is any real number. Then, by definition,

$$\mathcal{L}\left[e^{-at}f(t)\right] = \int_0^\infty e^{-st}e^{-at}f(t)\,dt = \int_0^\infty e^{-(s+a)t}f(t)\,dt, \qquad (4.3.1)$$

or

$$\mathcal{L}\left[e^{-at}f(t)\right] = F(s+a). \qquad (4.3.2)$$

That is, if $F(s)$ is the transform of $f(t)$ and a is a constant, then $F(s+a)$ is the transform of $e^{-at}f(t)$.

• Example 4.3.1

Let us find the Laplace transform of $f(t) = e^{-at}\sin(bt)$. Because the Laplace transform of $\sin(bt)$ is $b/(s^2+b^2)$,

$$\mathcal{L}\left[e^{-at}\sin(bt)\right] = \frac{b}{(s+a)^2+b^2}, \qquad (4.3.3)$$

where we have simply replaced s by $s+a$ in the transform for $\sin(bt)$.

• Example 4.3.2

Let us find the inverse of the Laplace transform

$$F(s) = \frac{s+2}{s^2+6s+1}. \qquad (4.3.4)$$

Rearranging terms,

$$F(s) = \frac{s+2}{s^2+6s+1} = \frac{s+2}{(s+3)^2-8} \tag{4.3.5}$$

$$= \frac{s+3}{(s+3)^2-8} - \frac{1}{2\sqrt{2}}\frac{2\sqrt{2}}{(s+3)^2-8}. \tag{4.3.6}$$

Immediately, from the first shifting theorem,

$$f(t) = e^{-3t}\cosh(2\sqrt{2}t) - \frac{1}{2\sqrt{2}}e^{-3t}\sinh(2\sqrt{2}t). \tag{4.3.7}$$

Second shifting theorem

The *second shifting theorem* states that if $F(s)$ is the transform of $f(t)$, then $e^{-bs}F(s)$ is the transform of $f(t-b)H(t-b)$, where b is real and positive. To show this, consider the Laplace transform of $f(t-b)H(t-b)$. Then, from the definition,

$$\mathcal{L}[f(t-b)H(t-b)] = \int_0^\infty f(t-b)H(t-b)e^{-st}\,dt \tag{4.3.8}$$

$$= \int_b^\infty f(t-b)e^{-st}\,dt = \int_0^\infty e^{-bs}e^{-sx}f(x)\,dx \tag{4.3.9}$$

$$= e^{-bs}\int_0^\infty e^{-sx}f(x)\,dx \tag{4.3.10}$$

or

$$\mathcal{L}[f(t-b)H(t-b)] = e^{-bs}F(s), \tag{4.3.11}$$

where we have set $x = t-b$. This theorem is of fundamental importance because it allows us to write down the transforms for "delayed" time functions. That is, functions which "turn on" b units after the initial time.

• Example 4.3.3

Let us find the inverse of the transform $(1-e^{-s})/s$. Since

$$\frac{1-e^{-s}}{s} = \frac{1}{s} - \frac{e^{-s}}{s}, \tag{4.3.12}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s} - \frac{e^{-s}}{s}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{e^{-s}}{s}\right) = H(t) - H(t-1), \tag{4.3.13}$$

because $\mathcal{L}^{-1}(1/s) = f(t) = 1$ and $f(t-1) = 1$.

• Example 4.3.4

Let us find the Laplace transform of $f(t) = (t^2 - 1)H(t-1)$. We begin by noting that

$$(t^2 - 1)H(t-1) = [(t-1+1)^2 - 1]H(t-1) \tag{4.3.14}$$

$$= [(t-1)^2 + 2(t-1)]H(t-1) \tag{4.3.15}$$

$$= (t-1)^2 H(t-1) + 2(t-1)H(t-1). \tag{4.3.16}$$

A direct application of the second shifting theorem leads then to

$$\mathcal{L}[(t^2-1)H(t-1)] = \frac{2e^{-s}}{s^3} + \frac{2e^{-s}}{s^2}. \tag{4.3.17}$$

• Example 4.3.5

In Example 4.2.2 we discussed the use of Laplace transforms in solving ordinary differential equations. One further step along the road consists of finding $Y(s) = \mathcal{L}[y(t)]$. Now that we have the second shifting theorem, let us do this.

Continuing Example 4.2.2 with $y(0) = 0$ and $y'(0) = 1$, let us take the Laplace transform of (4.2.8). Employing the second shifting theorem and (4.1.20), we find that

$$s^2Y(s) - sy(0) - y'(0) + 3sY(s) - 3y(0) + 2Y(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}. \tag{4.3.18}$$

Substituting in the initial conditions and solving for $Y(s)$, we finally obtain

$$Y(s) = \frac{1}{(s+2)(s+1)} + \frac{1}{s^2(s+2)(s+1)} + \frac{e^{-s}}{s^2(s+2)(s+1)} + \frac{e^{-s}}{s(s+2)(s+1)}. \tag{4.3.19}$$

Laplace transform of $t^n f(t)$

In addition to the shifting theorems, there are two other particularly useful theorems that involve the derivative and integral of the transform $F(s)$. For example, if we write

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt \tag{4.3.20}$$

and differentiate with respect to s, then

$$F'(s) = \int_0^\infty -tf(t)e^{-st}\,dt = -\mathcal{L}[tf(t)]. \tag{4.3.21}$$

In general, we have that

$$F^{(n)}(s) = (-1)^n \mathcal{L}[t^n f(t)]. \tag{4.3.22}$$

Laplace transform of $f(t)/t$

Consider the following integration of the Laplace transform $F(s)$:

$$\int_s^\infty F(z)\,dz = \int_s^\infty \left[\int_0^\infty f(t)e^{-zt}\,dt\right] dz. \tag{4.3.23}$$

Upon interchanging the order of integration, we find that

$$\int_s^\infty F(z)\,dz = \int_0^\infty f(t)\left[\int_s^\infty e^{-zt}\,dz\right] dt \tag{4.3.24}$$

$$= -\int_0^\infty f(t)\left.\frac{e^{-zt}}{t}\right|_s^\infty dt = \int_0^\infty \frac{f(t)}{t}e^{-st}\,dt. \tag{4.3.25}$$

Therefore,

$$\int_s^\infty F(z)\,dz = \mathcal{L}\left[\frac{f(t)}{t}\right]. \tag{4.3.26}$$

• **Example 4.3.6**

Let us find the transform of $t\sin(at)$. From (4.3.21),

$$\mathcal{L}[t\sin(at)] = -\frac{d}{ds}\Big\{\mathcal{L}[\sin(at)]\Big\} = -\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right] \tag{4.3.27}$$

$$= \frac{2as}{(s^2+a^2)^2}. \tag{4.3.28}$$

• **Example 4.3.7**

Let us find the transform of $[1-\cos(at)]/t$. To solve this problem, we apply (4.3.26) and find that

$$\mathcal{L}\left[\frac{1-\cos(at)}{t}\right] = \int_s^\infty \mathcal{L}[1-\cos(at)]\bigg|_{s=z} dz = \int_s^\infty \left(\frac{1}{z} - \frac{z}{z^2+a^2}\right) dz \tag{4.3.29}$$

$$= \ln(z) - \tfrac{1}{2}\ln(z^2+a^2)\bigg|_s^\infty = \ln\left(\frac{z}{\sqrt{z^2+a^2}}\right)\bigg|_s^\infty \tag{4.3.30}$$

$$= \ln(1) - \ln\left(\frac{s}{\sqrt{s^2+a^2}}\right) = -\ln\left(\frac{s}{\sqrt{s^2+a^2}}\right). \tag{4.3.31}$$

Initial-value theorem

Let $f(t)$ and $f'(t)$ possess Laplace transforms. Then, from the definition of the Laplace transform,

$$\int_0^\infty f'(t)e^{-st}dt = sF(s) - f(0). \tag{4.3.32}$$

Because s is a parameter in (4.3.32) and the existence of the integral is implied by the derivative rule, we can let $s \to \infty$ before we integrate. In that case, the left side of (4.3.32) vanishes to zero, which leads to

$$\lim_{s\to\infty} sF(s) = f(0). \tag{4.3.33}$$

This is the *initial-value theorem.*

• **Example 4.3.8**

Let us verify the initial-value theorem using $f(t) = e^{3t}$. Because $F(s) = 1/(s-3)$, $\lim_{s\to\infty} s/(s-3) = 1$. This agrees with $f(0) = 1$.

Final-value theorem

Let $f(t)$ and $f'(t)$ possess Laplace transforms. Then, in the limit of $s \to 0$, (4.3.32) becomes

$$\int_0^\infty f'(t)\,dt = \lim_{t\to\infty}\int_0^t f'(\tau)\,d\tau = \lim_{t\to\infty} f(t) - f(0) = \lim_{s\to 0} sF(s) - f(0). \tag{4.3.34}$$

Because $f(0)$ is not a function of t or s, the quantity $f(0)$ cancels from the (4.3.34), leaving

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s). \tag{4.3.35}$$

Equation (4.3.35) is the *final-value theorem*. It should be noted that this theorem assumes that $\lim_{t\to\infty} f(t)$ exists. For example, it does not apply to sinusoidal functions. Thus, we must restrict ourselves to Laplace transforms that have singularities in the left half of the s-plane unless they occur at the origin.

• **Example 4.3.9**

Let us verify the final-value theorem using $f(t) = t$. Because $F(s) = 1/s^2$, $\lim_{s\to 0} sF(s) = \lim_{s\to 0} 1/s = \infty$. The limit of $f(t)$ as $t \to \infty$ is also undefined.

Problems

Find the Laplace transform of the following functions:

1. $f(t) = e^{-t}\sin(2t)$

2. $f(t) = e^{-2t}\cos(2t)$

3. $f(t) = te^t + \sin(3t)e^t + \cos(5t)e^{2t}$

4. $f(t) = t^4e^{-2t} + \sin(3t)e^t + \cos(4t)e^{2t}$

5. $f(t) = t^2e^{-t} + \sin(2t)e^t + \cos(3t)e^{-3t}$

6. $f(t) = t^2H(t-1)$

7. $f(t) = e^{2t}H(t-3)$

8. $f(t) = t^2H(t-1) + e^tH(t-2)$

9. $f(t) = (t^2+2)H(t-1) + H(t-2)$

10. $f(t) = (t+1)^2H(t-1) + e^tH(t-2)$

11. $f(t) = \begin{cases} \sin(t), & 0 \le t \le \pi \\ 0, & t > \pi \end{cases}$

12. $f(t) = \begin{cases} t, & 0 \le t \le 2 \\ 2, & t \ge 2 \end{cases}$

13. $f(t) = te^{-3t}\sin(2t)$

Find the inverse of the following Laplace transforms:

14. $F(s) = 1/(s+2)^4$

15. $F(s) = s/(s+2)^4$

16. $F(s) = s/(s^2+2s+2)$

17. $F(s) = (s+3)/(s^2+2s+2)$

18. $F(s) = s/(s+1)^3 + (s+1)/(s^2+2s+2)$

19. $F(s) = s/(s+2)^2 + (s+2)/(s^2+2s+2)$

20. $F(s) = s/(s+2)^3 + (s+4)/(s^2+4s+5)$

21. $F(s) = e^{-3s}/(s-1)$

22. $F(s) = e^{-2s}/(s+1)^2$

23. $F(s) = se^{-s}/(s^2+2s+2)$

24. $F(s) = e^{-4s}/(s^2+4s+5)$

25. $F(s) = se^{-s}/(s^2+4) + e^{-3s}/(s-2)^4$

26. $F(s) = e^{-s}/(s^2+4) + (s-1)e^{-3s}/s^4$

27. $F(s) = (s+1)e^{-s}/(s^2+4) + e^{-3s}/s^4$

28. Find the Laplace transform of $f(t) = te^t[H(t-1)-H(t-2)]$ by using (a) the definition of the Laplace transform, and (b) a joint application of the first and second shifting theorems.

29. Write the function

$$f(t) = \begin{cases} t, & 0 < t < a \\ 0, & t > a \end{cases}$$

in terms of Heaviside's step functions. Then find its transform using (a) the definition of the Laplace transform, and (b) the second shifting theorem.

In problems 30–33, write the function $f(t)$ in terms of Heaviside's step functions and then find its transform using the second shifting theorem.

30.

$$f(t) = \begin{cases} t/2, & 0 \le t < 2 \\ 0, & t > 2 \end{cases}$$

31.

$$f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 1, & 1 \le t < 2 \\ 0, & t > 2 \end{cases}$$

32.
$$f(t) = \begin{cases} t, & 0 \le t \le 2 \\ 4 - t, & 2 \le t \le 4 \\ 0, & t \ge 4 \end{cases}$$

33.
$$f(t) = \begin{cases} 0, & 0 \le t \le 1 \\ t - 1, & 1 \le t \le 2 \\ 1, & 2 \le t < 3 \\ 0, & t > 3 \end{cases}$$

Find $Y(s)$ for the following ordinary differential equations:

34. $y'' + 3y' + 2y = H(t-1); \qquad y(0) = y'(0) = 0$

35. $y'' + 4y = 3H(t-4); \qquad y(0) = 1, \ y'(0) = 0$

36. $y'' + 4y' + 4y = tH(t-2); \qquad y(0) = 0, \ y'(0) = 2$

37. $y'' + 3y' + 2y = e^t H(t-1); \qquad y(0) = y'(0) = 0$

38. $y'' - 3y' + 2y = e^{-t} H(t-2); \qquad y(0) = 2, \ y'(0) = 0$

39. $y'' - 3y' + 2y = t^2 H(t-1); \qquad y(0) = 0, \ y'(0) = 5$

40. $y'' + y = \sin(t)[1 - H(t-\pi)]; \qquad y(0) = y'(0) = 0$

41. $y'' + 3y' + 2y = t + \left[ae^{-(t-a)} - t\right] H(t-a); \qquad y(0) = y'(0) = 0.$

For each of the following functions, find its value at $t = 0$. Then check your answer using the initial-value theorem.

42. $f(t) = t$

43. $f(t) = \cos(at)$

44. $f(t) = te^{-t}$

45. $f(t) = e^t \sin(3t)$

For each of the following Laplace transforms, state whether you can or cannot apply the final-value theorem. If you can, find the final value. Check your result by finding the inverse and finding the limit as $t \to \infty$.

46. $F(s) = \dfrac{1}{s-1}$

47. $F(s) = \dfrac{1}{s}$

48. $F(s) = \dfrac{1}{s+1}$

49. $F(s) = \dfrac{s}{s^2+1}$

50. $F(s) = \dfrac{2}{s(s^2+3s+2)}$ 51. $F(s) = \dfrac{2}{s(s^2-3s+2)}$

4.4 THE LAPLACE TRANSFORM OF A PERIODIC FUNCTION

Periodic functions frequently occur in engineering problems and we shall now show how to calculate their transform. They possess the property that $f(t+T) = f(t)$ for $t > 0$ and equal zero for $t < 0$, where T is the period of the function.

For convenience let us define a function $x(t)$ which equals zero except over the interval $(0, T)$ where it equals $f(t)$:

$$x(t) = \begin{cases} f(t), & 0 < t < T \\ 0, & t > T. \end{cases} \tag{4.4.1}$$

By definition

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt \tag{4.4.2}$$

$$= \int_0^T f(t)e^{-st}\,dt + \int_T^{2T} f(t)e^{-st}\,dt + \cdots + \int_{kT}^{(k+1)T} f(t)e^{-st}\,dt + \cdots \tag{4.4.3}$$

Now let $z = t - kT$, where $k = 0, 1, 2, \ldots$, in the kth integral and $F(s)$ becomes

$$F(s) = \int_0^T f(z)e^{-sz}\,dz + \int_0^T f(z+T)e^{-s(z+T)}\,dz + \cdots$$
$$+ \int_0^T f(z+kT)e^{-s(z+kT)}\,dz + \cdots \tag{4.4.4}$$

However,

$$x(z) = f(z) = f(z+T) = \ldots = f(z+kT) = \ldots, \tag{4.4.5}$$

because the range of integration in each integral is from 0 to T. Thus, $F(s)$ becomes

$$F(s) = \int_0^T x(z)e^{-sz}\,dz + e^{-sT}\int_0^T x(z)e^{-sz}\,dz + \cdots$$
$$+ e^{-ksT}\int_0^T x(z)e^{-sz}\,dz + \cdots \tag{4.4.6}$$

or

$$F(s) = \left(1 + e^{-sT} + e^{-2sT} + \cdots + e^{-ksT} + \cdots\right)X(s). \tag{4.4.7}$$

The first term on the right side of (4.4.7) is a geometric series with common ratio e^{-sT}. If $|e^{-sT}| < 1$, then the series converges and

$$\boxed{F(s) = \frac{X(s)}{1 - e^{-sT}}.} \tag{4.4.8}$$

• **Example 4.4.1**

Let us find the Laplace transform of the square wave with period T:

$$f(t) = \begin{cases} h, & 0 < t < T/2 \\ -h, & T/2 < t < T. \end{cases} \tag{4.4.9}$$

By definition $x(t)$ is

$$x(t) = \begin{cases} h, & 0 < t < T/2 \\ -h, & T/2 < t < T \\ 0, & t > T. \end{cases} \tag{4.4.10}$$

Then

$$X(s) = \int_0^\infty x(t)e^{-st}\,dt = \int_0^{T/2} h\,e^{-st}\,dt + \int_{T/2}^T (-h)\,e^{-st}\,dt \tag{4.4.11}$$

$$= \frac{h}{s}\left(1 - 2e^{-sT/2} + e^{-sT}\right) = \frac{h}{s}\left(1 - e^{-sT/2}\right)^2 \tag{4.4.12}$$

and

$$F(s) = \frac{h\left(1 - e^{-sT/2}\right)^2}{s\left(1 - e^{-sT}\right)} = \frac{h\left(1 - e^{-sT/2}\right)}{s\left(1 + e^{-sT/2}\right)}. \tag{4.4.13}$$

If we multiply numerator and denominator by $\exp(sT/4)$ and recall that $\tanh(u) = (e^u - e^{-u})/(e^u + e^{-u})$, we have that

$$F(s) = \frac{h}{s}\tanh\left(\frac{sT}{4}\right). \tag{4.4.14}$$

• Example 4.4.2

Let us find the Laplace transform of the periodic function

$$f(t) = \begin{cases} \sin(2\pi t/T), & 0 < t < T/2 \\ 0, & T/2 < t < T. \end{cases} \tag{4.4.15}$$

By definition $x(t)$ is

$$x(t) = \begin{cases} \sin(2\pi t/T), & 0 < t < T/2 \\ 0, & t > T/2. \end{cases} \tag{4.4.16}$$

Then

$$X(s) = \int_0^{T/2} \sin\left(\frac{2\pi t}{T}\right) e^{-st}\, dt = \frac{2\pi T}{s^2T^2 + 4\pi^2}\left(1 + e^{-sT/2}\right). \tag{4.4.17}$$

Hence,

$$F(s) = \frac{X(s)}{1 - e^{-sT}} = \frac{2\pi T}{s^2T^2 + 4\pi^2} \times \frac{1 + e^{-sT/2}}{1 - e^{-sT}} \tag{4.4.18}$$

$$= \frac{2\pi T}{s^2T^2 + 4\pi^2} \times \frac{1}{1 - e^{-sT/2}}. \tag{4.4.19}$$

Problems

Find the Laplace transform for the following periodic functions:

1. $f(t) = \sin(t),\ 0 \le t \le \pi,$ $\qquad f(t) = f(t + \pi)$

2. $f(t) = \begin{cases} \sin(t), & 0 \le t \le \pi \\ 0, & \pi \le t \le 2\pi, \end{cases}$ $\qquad f(t) = f(t + 2\pi)$

3. $f(t) = \begin{cases} t, & 0 \le t < a \\ 0, & a < t \le 2a, \end{cases}$ $\qquad f(t) = f(t + 2a)$

4. $f(t) = \begin{cases} 1, & 0 < t < a \\ 0, & a < t < 2a \\ -1, & 2a < t < 3a \\ 0, & 3a < t < 4a, \end{cases}$ $\qquad f(t) = f(t + 4a).$

4.5 INVERSION BY PARTIAL FRACTIONS: HEAVISIDE'S EXPANSION THEOREM

In the previous sections, we have devoted our efforts to calculating the Laplace transform of a given function. Obviously we must have a method for going the other way. Given a transform, we must find the corresponding function. This is often a very formidable task. In the next few sections we shall present some general techniques for the inversion of a Laplace transform.

The first technique involves transforms that we can express as the ratio of two polynomials: $F(s) = q(s)/p(s)$. We shall assume that the order of $q(s)$ is *less* than $p(s)$ and we have divided out any common factor between them. In principle we know that $p(s)$ has n zeros, where n is the order of the $p(s)$ polynomial. Some of the zeros may be complex, some of them may be real, and some of them may be duplicates of other zeros. In the case when $p(s)$ has n simple zeros (nonrepeating roots), a simple method exists for inverting the transform.

We want to rewrite $F(s)$ in the form:

$$F(s) = \frac{a_1}{s - s_1} + \frac{a_2}{s - s_2} + \cdots + \frac{a_n}{s - s_n} = \frac{q(s)}{p(s)}, \tag{4.5.1}$$

where $s_1, s_2, \ldots, s_n$ are the n simple zeros of $p(s)$. We now multiply both sides of (4.5.1) by $s - s_1$ so that

$$\frac{(s - s_1)q(s)}{p(s)} = a_1 + \frac{(s - s_1)a_2}{s - s_2} + \cdots + \frac{(s - s_1)a_n}{s - s_n}. \tag{4.5.2}$$

If we set $s = s_1$, the right side of (4.5.2) becomes simply a_1. The left side takes the form 0/0 and there are two cases. If $p(s) = (s - s_1)g(s)$, then $a_1 = q(s_1)/g(s_1)$. If we cannot explicitly factor out $s - s_1$, l'Hôspital's rule gives

$$a_1 = \lim_{s \to s_1} \frac{(s - s_1)q(s)}{p(s)} = \lim_{s \to s_1} \frac{(s - s_1)q'(s) + q(s)}{p'(s)} = \frac{q(s_1)}{p'(s_1)}. \tag{4.5.3}$$

In a similar manner, we can compute all of the a_k's, where $k = 1, 2, \ldots, n$. Therefore,

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{q(s)}{p(s)}\right] = \mathcal{L}^{-1}\left(\frac{a_1}{s - s_1} + \frac{a_2}{s - s_2} + \cdots + \frac{a_n}{s - s_n}\right) \tag{4.5.4}$$

$$= a_1 e^{s_1 t} + a_2 e^{s_2 t} + \cdots + a_n e^{s_n t}. \tag{4.5.5}$$

This is *Heaviside's expansion theorem*, applicable when $p(s)$ has only simple poles.

- **Example 4.5.1**

Let us invert the transform $s/[(s+2)(s^2+1)]$. It has three simple poles at $s=-2$ and $s=\pm i$. From our earlier discussion, $q(s)=s$, $p(s)=(s+2)(s^2+1)$, and $p'(s)=3s^2+4s+1$. Therefore,

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{s}{(s+2)(s^2+1)}\right] &= \frac{-2}{12-8+1}e^{-2t} + \frac{i}{-3+4i+1}e^{it} \\
&\quad + \frac{-i}{-3-4i+1}e^{-it} && (4.5.6) \\
&= -\frac{2}{5}e^{-2t} + \frac{i}{-2+4i}e^{it} - \frac{i}{-2-4i}e^{-it} && (4.5.7) \\
&= -\frac{2}{5}e^{-2t} + i\frac{-2-4i}{4+16}e^{it} - i\frac{-2+4i}{4+16}e^{-it} && (4.5.8) \\
&= -\frac{2}{5}e^{-2t} + \frac{1}{5}\sin(t) + \frac{2}{5}\cos(t), && (4.5.9)
\end{aligned}$$

where we have used $\sin(t)=\frac{1}{2i}(e^{it}-e^{-it})$ and $\cos(t)=\frac{1}{2}(e^{it}+e^{-it})$.

- **Example 4.5.2**

Let us invert the transform $1/[(s-1)(s-2)(s-3)]$. There are three simple poles at $s_1=1$, $s_2=2$, and $s_3=3$. In this case, the easiest method for computing a_1, a_2, and a_3 is

$$a_1 = \lim_{s\to 1}\frac{s-1}{(s-1)(s-2)(s-3)} = \frac{1}{2}, \tag{4.5.10}$$

$$a_2 = \lim_{s\to 2}\frac{s-2}{(s-1)(s-2)(s-3)} = -1 \tag{4.5.11}$$

and

$$a_3 = \lim_{s\to 3}\frac{s-3}{(s-1)(s-2)(s-3)} = \frac{1}{2}. \tag{4.5.12}$$

Therefore,

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{(s-1)(s-2)(s-3)}\right] &= \mathcal{L}^{-1}\left[\frac{a_1}{s-1}+\frac{a_2}{s-2}+\frac{a_3}{s-3}\right] \\
&= \tfrac{1}{2}e^{t} - e^{2t} + \tfrac{1}{2}e^{3t}. && (4.5.13)
\end{aligned}$$

Note that for inverting transforms of the form $F(s)e^{-as}$ with $a>0$, you should use Heaviside's expansion theorem to first invert $F(s)$ and then apply the second shifting theorem.

Let us now find the expansion when we have multiple roots, namely

$$F(s) = \frac{q(s)}{p(s)} = \frac{q(s)}{(s-s_1)^{m_1}(s-s_2)^{m_2}\cdots(s-s_n)^{m_n}}, \tag{4.5.14}$$

where the order of the denominator, $m_1 + m_2 + \cdots + m_n$, is greater than that for the numerator. Once again we have eliminated any common factor between the numerator and denominator. Now we can write $F(s)$ as

$$F(s) = \sum_{k=1}^{n}\sum_{j=1}^{m_k} \frac{a_{kj}}{(s-s_k)^{m_k-j+1}}. \tag{4.5.15}$$

Multiplying (4.5.15) by $(s-s_k)^{m_k}$,

$$\begin{aligned}\frac{(s-s_k)^{m_k}q(s)}{p(s)} &= a_{k1} + a_{k2}(s-s_k) + \cdots + a_{km_k}(s-s_k)^{m_k-1}\\ &\quad + (s-s_k)^{m_k}\left[\frac{a_{11}}{(s-s_1)^{m_1}} + \cdots + \frac{a_{nm_n}}{s-s_n}\right],\end{aligned} \tag{4.5.16}$$

where we have grouped together into the square-bracketed term all of the terms except for those with a_{kj} coefficients. Taking the limit as $s \to s_k$,

$$a_{k1} = \lim_{s\to s_k} \frac{(s-s_k)^{m_k}q(s)}{p(s)}. \tag{4.5.17}$$

Let us now take the derivative of (4.5.16),

$$\begin{aligned}\frac{d}{ds}&\left[\frac{(s-s_k)^{m_k}q(s)}{p(s)}\right]\\ &= a_{k2} + 2a_{k3}(s-s_k) + \cdots + (m_k-1)a_{km_k}(s-s_k)^{m_k-2}\\ &+ \frac{d}{ds}\left\{(s-s_k)^{m_k}\left[\frac{a_{11}}{(s-s_1)^{m_1}} + \cdots + \frac{a_{nm_n}}{s-s_n}\right]\right\}.\end{aligned} \tag{4.5.18}$$

Taking the limit as $s \to s_k$,

$$a_{k2} = \lim_{s\to s_k} \frac{d}{ds}\left[\frac{(s-s_k)^{m_k}q(s)}{p(s)}\right]. \tag{4.5.19}$$

In general,

$$a_{kj} = \lim_{s\to s_k} \frac{1}{(j-1)!}\frac{d^{j-1}}{ds^{j-1}}\left[\frac{(s-s_k)^{m_k}q(s)}{p(s)}\right] \tag{4.5.20}$$

and by direct inversion,

$$f(t) = \sum_{k=1}^{n}\sum_{j=1}^{m_k} \frac{a_{kj}}{(m_k-j)!}t^{m_k-j}e^{s_k t}. \tag{4.5.21}$$

• **Example 4.5.3**

Let us find the inverse of

$$F(s) = \frac{s}{(s+2)^2(s^2+1)}. \tag{4.5.22}$$

We first note that the denominator has simple zeros at $s = \pm i$ and a repeated root at $s = -2$. Therefore,

$$F(s) = \frac{A}{s-i} + \frac{B}{s+i} + \frac{C}{s+2} + \frac{D}{(s+2)^2}, \tag{4.5.23}$$

where

$$A = \lim_{s\to i} (s-i)F(s) = \tfrac{1}{6+8i}, \tag{4.5.24}$$

$$B = \lim_{s\to -i} (s+i)F(s) = \tfrac{1}{6-8i}, \tag{4.5.25}$$

$$C = \lim_{s\to -2} \frac{d}{ds}\left[(s+2)^2F(s)\right] = \lim_{s\to -2} \frac{d}{ds}\left[\frac{s}{s^2+1}\right] = -\tfrac{3}{25} \tag{4.5.26}$$

and

$$D = \lim_{s\to -2} (s+2)^2F(s) = -\tfrac{2}{5}. \tag{4.5.27}$$

Thus,

$$f(t) = \tfrac{1}{6+8i}e^{it} + \tfrac{1}{6-8i}e^{-it} - \tfrac{3}{25}e^{-2t} - \tfrac{2}{5}te^{-2t} \tag{4.5.28}$$

$$= \tfrac{3}{25}\cos(t) + \tfrac{4}{25}\sin(t) - \tfrac{3}{25}e^{-2t} - \tfrac{10}{25}te^{-2t}. \tag{4.5.29}$$

In Section 4.10 we shall see that we can invert transforms just as easily with the residue theorem.

Let us now find the inverse of

$$F(s) = \frac{cs + (ca - \omega d)}{(s+a)^2 + \omega^2} = \frac{cs + (ca - \omega d)}{(s+a-\omega i)(s+a+\omega i)} \tag{4.5.30}$$

by Heaviside's expansion theorem. Then

$$F(s) = \frac{c+di}{2(s+a-\omega i)} + \frac{c-di}{2(s+a+\omega i)} \tag{4.5.31}$$

$$= \frac{\sqrt{c^2+d^2}e^{\theta i}}{2(s+a-\omega i)} + \frac{\sqrt{c^2+d^2}e^{-\theta i}}{2(s+a+\omega i)}, \tag{4.5.32}$$

where $\theta = \tan^{-1}(d/c)$. Note that we must choose θ so that it gives the correct sign for c and d.

Taking the inverse of (4.5.32),

$$f(t) = \tfrac{1}{2}\sqrt{c^2+d^2}e^{-at+\omega ti+\theta i} + \tfrac{1}{2}\sqrt{c^2+d^2}e^{-at-\omega ti-\theta i} \tag{4.5.33}$$

$$= \sqrt{c^2+d^2}e^{-at}\cos(\omega t+\theta). \tag{4.5.34}$$

Equation (4.5.34) is the amplitude/phase form of the inverse of (4.5.30). It is particularly popular with electrical engineers.

• Example 4.5.4

Let us express the inverse of

$$F(s) = \frac{8s-3}{s^2+4s+13} \tag{4.5.35}$$

in the amplitude/phase form.

Starting with

$$F(s) = \frac{8s-3}{(s+2-3i)(s+2+3i)} \tag{4.5.36}$$

$$= \frac{4+19i/6}{s+2-3i} + \frac{4-19i/6}{s+2+3i} \tag{4.5.37}$$

$$= \frac{5.1017e^{38.3675^\circ i}}{s+2-3i} + \frac{5.1017e^{-38.3675^\circ i}}{s+2+3i} \tag{4.5.38}$$

or

$$f(t) = 5.1017e^{-2t+3it+38.3675^\circ i} + 5.1017e^{-2t-3it-38.3675^\circ i} \tag{4.5.39}$$

$$= 10.2034e^{-2t}\cos(3t+38.3675^\circ). \tag{4.5.40}$$

• Example 4.5.5: The design of film projectors

For our final example we anticipate future work. The primary use of Laplace transforms is the solution of differential equations. In this example we illustrate this technique that includes Heaviside's expansion theorem in the form of amplitude and phase.

This problem[3] arose in the design of projectors for motion pictures. An early problem was ensuring that the speed at which the film passed the electric eye remained essentially constant; otherwise, a frequency modulation of the reproduced sound resulted. Figure 4.5.1(A) shows a diagram of the projector. Many will remember this design from their

[3] Cook, E. D., 1935: The technical aspects of the high-fidelity reproducer. *J. Soc. Motion Pict. Eng.*, **25**, 289–312.

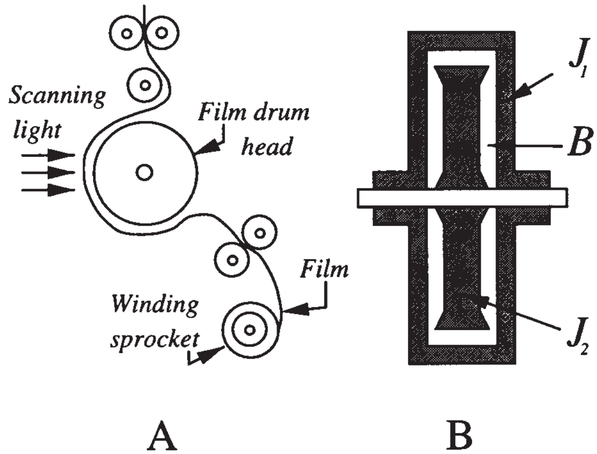


Figure 4.5.1: (A) The schematic for the scanning light in a motion-picture projector and (B) interior of the film drum head.

days as a school projectist. In this section we shall show that this particular design filters out variations in the film speed caused by irregularities either in the driving-gear trains or in the engagement of the sprocket teeth with the holes in the film.

Let us now focus on the film head – a hollow drum of small moment of inertia J_1. See Figure 4.5.1(B). Within it there is a concentric inner flywheel of moment of inertia J_2, where $J_2 \gg J_1$. The remainder of the space within the drum is filled with oil. The inner flywheel rotates on precision ball bearings on the drum shaft. The only coupling between the drum and flywheel is through fluid friction and the very small friction in the ball bearings. The flection of the film loops between the drum head and idler pulleys provides the spring restoring force for the system as the film runs rapidly through the system.

From Figure 4.5.1 the dynamical equations governing the outer case and inner flywheel are (1) the rate of change of the outer casing of the film head equals the frictional torque given to the casing from the inner flywheel plus the restoring torque due to the flection of the film, and (2) the rate of change of the inner flywheel equals the negative of the frictional torque given to the outer casing by the inner flywheel.

Assuming that the frictional torque between the two flywheels is proportional to the difference in their angular velocities, the frictional torque given to the casing from the inner flywheel is $B(\omega_2 - \omega_1)$, where B is the frictional resistance, ω_1 and ω_2 are the deviations of the drum and inner flywheel from their normal angular velocities, respectively. If r is the ratio of the diameter of the winding sprocket to the diameter of the drum, the restoring torque due to the flection of the film and its corresponding angular twist equals $K\int_0^t (r\omega_0 - \omega_1)\,d\tau$, where K is the rotational stiffness and ω_0 is the deviation of the winding sprocket from its normal angular velocity. The quantity $r\omega_0$ gives the angular velocity at which the film is running through the projector because the winding sprocket is the mechanism that pulls the film. Consequently the equations governing this mechanical system are

$$J_1 \frac{d\omega_1}{dt} = K \int_0^t (r\omega_0 - \omega_1)\, d\tau + B(\omega_2 - \omega_1) \tag{4.5.41}$$

and

$$J_2 \frac{d\omega_2}{dt} = -B(\omega_2 - \omega_1). \tag{4.5.42}$$

With the winding sprocket, the drum, and the flywheel running at their normal uniform angular velocities, let us assume that the winding sprocket introduces a disturbance equivalent to an unit increase in its angular velocity for 0.15 seconds, followed by the resumption of its normal velocity. It is assumed that the film in contact with the drum cannot slip. The initial conditions are $\omega_1(0) = \omega_2(0) = 0$.

Taking the Laplace transform of (4.5.41)–(4.5.42) using (4.1.18),

$$\left(J_1 s + B + \frac{K}{s}\right) \Omega_1(s) - B\Omega_2(s) = \frac{rK}{s}\Omega_0(s) = rK\mathcal{L}\left[\int_0^t \omega_0(\tau)\, d\tau\right] \tag{4.5.43}$$

and

$$-B\Omega_1(s) + (J_2 s + B)\Omega_2(s) = 0. \tag{4.5.44}$$

The solution of (4.5.43)–(4.5.44) for $\Omega_1(s)$ is

$$\Omega_1(s) = \frac{rK}{J_1} \frac{(s + a_0)\Omega_0(s)}{s^3 + b_2 s^2 + b_1 s + b_0}, \tag{4.5.45}$$

where typical values[4] are

$$\frac{rK}{J_1} = 90.8, \quad a_0 = \frac{B}{J_2} = 1.47, \quad b_0 = \frac{BK}{J_1 J_2} = 231, \tag{4.5.46}$$

$$b_1 = \frac{K}{J_1} = 157 \quad \text{and} \quad b_2 = \frac{B(J_1 + J_2)}{J_1 J_2} = 8.20. \tag{4.5.47}$$

The transform $\Omega_1(s)$ has three simple poles located at $s_1 = -1.58, s_2 = -3.32 + 11.6i$, and $s_3 = -3.32 - 11.6i$.

---

[4] $J_1 = 1.84 \times 10^4$ dyne cm sec^2 per radian, $J_2 = 8.43 \times 10^4$ dyne cm sec^2 per radian, $B = 12.4 \times 10^4$ dyne cm sec per radian, $K = 2.89 \times 10^6$ dyne cm per radian, and $r = 0.578$

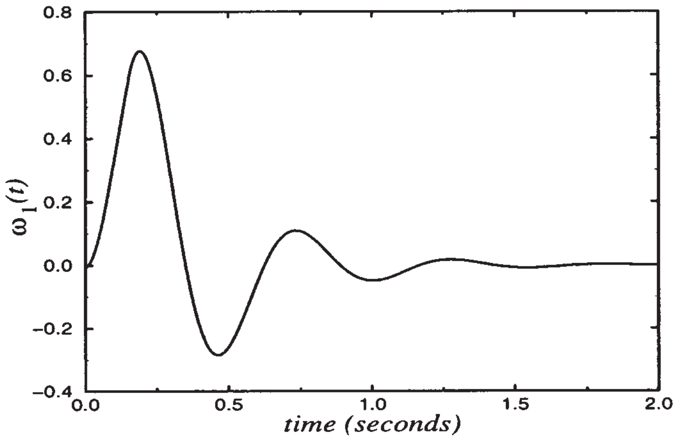


Figure 4.5.2: The deviation ω_1 of a film drum head from its uniform angular velocity when the sprocket angular velocity is perturbed by a unit amount for the duration of 0.15 seconds.

Because the sprocket angular velocity deviation $\omega_0(t)$ is a pulse of unit amplitude and 0.15 second duration, we express it as the difference of two Heaviside step functions:

$$\omega_0(t) = H(t) - H(t - 0.15). \tag{4.5.48}$$

Its Laplace transform is

$$\Omega_0(s) = \frac{1}{s} - \frac{1}{s}e^{-0.15s} \tag{4.5.49}$$

so that (4.5.45) becomes

$$\Omega_1(s) = \frac{rK}{J_1} \frac{(s + a_0)}{s(s - s_1)(s - s_2)(s - s_3)} \left(1 - e^{-0.15s}\right). \tag{4.5.50}$$

The inversion of (4.5.50) follows directly from the second shifting theorem and Heaviside's expansion theorem:

$$\begin{aligned}\omega_1(t) &= K_0 + K_1 e^{s_1 t} + K_2 e^{s_2 t} + K_3 e^{s_3 t} \\ &\quad - [K_0 + K_1 e^{s_1(t-0.15)} + K_2 e^{s_2(t-0.15)} + K_3 e^{s_3(t-0.15)}]H(t - 0.15),\end{aligned} \tag{4.5.51}$$

where

$$K_0 = \frac{rK}{J_1} \left. \frac{s + a_0}{(s - s_1)(s - s_2)(s - s_3)} \right|_{s=0} = 0.578, \tag{4.5.52}$$

$$K_1 = \frac{rK}{J_1} \left. \frac{s+a_0}{s(s-s_2)(s-s_3)} \right|_{s=s_1} = 0.046, \tag{4.5.53}$$

$$K_2 = \frac{rK}{J_1} \left. \frac{s+a_0}{s(s-s_1)(s-s_3)} \right|_{s=s_2} = 0.326e^{165^\circ i} \tag{4.5.54}$$

and

$$K_3 = \frac{rK}{J_1} \left. \frac{s+a_0}{s(s-s_1)(s-s_2)} \right|_{s=s_3} = 0.326e^{-165^\circ i}. \tag{4.5.55}$$

Using Euler's identity $\cos(t) = (e^{it} + e^{-it})/2$, we can write (4.5.51) as

$$\begin{aligned} \omega_1(t) = {} & 0.578 + 0.046e^{-1.58t} + 0.652e^{-3.32t}\cos(11.6t + 165^\circ) \\ & - \{0.578 + 0.046e^{-1.58(t-0.15)} + 0.652e^{-3.32(t-0.15)} \\ & \times \cos[11.6(t-0.15) + 165^\circ]\}H(t-0.15). \end{aligned} \tag{4.5.56}$$

Equation (4.5.56) is plotted in Figure 4.5.2. Note that fluctuations in $\omega_1(t)$ are damped out by the particular design of this film projector. Because this mechanical device dampens unwanted fluctuations (or noise) in the motion-picture projector, this particular device is an example of a *mechanical filter*.

Problems

Use Heaviside's expansion theorem to find the inverse of the following Laplace transforms:

1. $F(s) = \dfrac{1}{s^2 + 3s + 2}$
2. $F(s) = \dfrac{s+3}{(s+4)(s-2)}$
3. $F(s) = \dfrac{s-4}{(s+2)(s+1)(s-3)}$
4. $F(s) = \dfrac{s-3}{(s^2+4)(s+1)}$.

Find the inverse of the following transforms and express them in amplitude/phase form:

5. $F(s) = \dfrac{1}{s^2 + 4s + 5}$
6. $F(s) = \dfrac{1}{s^2 + 6s + 13}$
7. $F(s) = \dfrac{2s-5}{s^2 + 16}$
8. $F(s) = \dfrac{1}{s(s^2 + 2s + 2)}$
9. $F(s) = \dfrac{s+2}{s(s^2 + 4)}$

4.6 CONVOLUTION

In this section we turn to a fundamental concept in Laplace transforms: convolution. We shall restrict ourselves to its use in finding the inverse of a transform when that transform consists of the *product* of two simpler transforms. In subsequent sections we will use it to solve ordinary differential equations.

We begin by formally introducing the mathematical operation of the *convolution product*:

$$f(t) * g(t) = \int_0^t f(t-x)g(x)\,dx = \int_0^t f(x)g(t-x)\,dx. \tag{4.6.1}$$

In most cases the operations required by (4.6.1) are straightforward.

• **Example 4.6.1**

Let us find the convolution between $\cos(t)$ and $\sin(t)$.

$$\cos(t) * \sin(t) = \int_0^t \sin(t-x)\cos(x)\,dx \tag{4.6.2}$$

$$= \tfrac{1}{2}\int_0^t [\sin(t) + \sin(t-2x)]\,dx \tag{4.6.3}$$

$$= \tfrac{1}{2}\int_0^t \sin(t)\,dx + \tfrac{1}{2}\int_0^t \sin(t-2x)\,dx \tag{4.6.4}$$

$$= \tfrac{1}{2}\sin(t)\,x\Big|_0^t + \tfrac{1}{4}\cos(t-2x)\Big|_0^t = \tfrac{1}{2}t\sin(t). \tag{4.6.5}$$

• **Example 4.6.2**

Similarly, the convolution between t^2 and $\sin(t)$ is

$$t^2 * \sin(t) = \int_0^t (t-x)^2\sin(x)\,dx \tag{4.6.6}$$

$$= -(t-x)^2\cos(x)\Big|_0^t - 2\int_0^t (t-x)\cos(x)\,dx \tag{4.6.7}$$

$$= t^2 - 2(t-x)\sin(x)\Big|_0^t - 2\int_0^t \sin(x)\,dx \tag{4.6.8}$$

$$= t^2 + 2\cos(t) - 2 \tag{4.6.9}$$

by integration by parts.

• **Example 4.6.3**

Consider now the convolution between e^t and the discontinuous function $H(t-1) - H(t-2)$:

$$e^t * [H(t-1) - H(t-2)] = \int_0^t e^{t-x}[H(x-1) - H(x-2)]\,dx \quad \textbf{(4.6.10)}$$

$$= e^t \int_0^t e^{-x}[H(x-1) - H(x-2)]\,dx. \quad \textbf{(4.6.11)}$$

In order to evaluate the integral (4.6.11) we must examine various cases. If $t < 1$, then both of the step functions equal zero and the convolution equals zero. However, when $1 < t < 2$, the first step function equals one while the second equals zero. Therefore,

$$e^t * [H(t-1) - H(t-2)] = e^t \int_1^t e^{-x}\,dx = e^{t-1} - 1, \quad \textbf{(4.6.12)}$$

because the portion of the integral from zero to one equals zero. Finally, when $t > 2$, the integrand is only nonzero for that portion of the integration when $1 < x < 2$. Consequently,

$$e^t * [H(t-1) - H(t-2)] = e^t \int_1^2 e^{-x}\,dx = e^{t-1} - e^{t-2}. \quad \textbf{(4.6.13)}$$

Thus, the convolution of e^t with the pulse $H(t-1) - H(t-2)$ is

$$e^t * [H(t-1) - H(t-2)] = \begin{cases} 0, & 0 < t < 1 \\ e^{t-1} - 1, & 1 < t < 2 \\ e^{t-1} - e^{t-2}, & t > 2. \end{cases} \quad \textbf{(4.6.14)}$$

The reason why we have introduced convolution follows from the following fundamental theorem (often called *Borel's theorem*[5]). If

$$w(t) = u(t) * v(t) \quad \textbf{(4.6.15)}$$

then

$$W(s) = U(s)V(s). \quad \textbf{(4.6.16)}$$

---

[5] Borel, É., 1901: *Leçons sur les séries divergentes.* Gauthier-Villars, Paris, p. 104.

In other words, we can invert a complicated transform by convoluting the inverses to two simpler functions. The proof is as follows:

$$W(s) = \int_0^\infty \left[\int_0^t u(x)v(t-x)\,dx\right] e^{-st}dt \tag{4.6.17}$$

$$= \int_0^\infty \left[\int_x^\infty u(x)v(t-x)e^{-st}dt\right] dx \tag{4.6.18}$$

$$= \int_0^\infty u(x)\left[\int_0^\infty v(r)e^{-s(r+x)}dr\right] dx \tag{4.6.19}$$

$$= \left[\int_0^\infty u(x)e^{-sx}dx\right]\left[\int_0^\infty v(r)e^{-sr}\,dr\right] = U(s)V(s), \tag{4.6.20}$$

where $t = r + x$. □

• **Example 4.6.4**

Let us find the inverse of the transform

$$\frac{s}{(s^2+1)^2} = \frac{s}{s^2+1} \times \frac{1}{s^2+1} = \mathcal{L}[\cos(t)]\mathcal{L}[\sin(t)] \tag{4.6.21}$$

$$= \mathcal{L}[\cos(t) * \sin(t)] = \mathcal{L}[\tfrac{1}{2}t\sin(t)] \tag{4.6.22}$$

from Example 4.6.1.

• **Example 4.6.5**

Let us find the inverse of the transform

$$\frac{1}{(s^2+a^2)^2} = \frac{1}{a^2}\left(\frac{a}{s^2+a^2} \times \frac{a}{s^2+a^2}\right) \tag{4.6.23}$$

$$= \frac{1}{a^2}\mathcal{L}[\sin(at)]\mathcal{L}[\sin(at)]. \tag{4.6.24}$$

Therefore,

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{a^2}\int_0^t \sin[a(t-x)]\sin(ax)\,dx \tag{4.6.25}$$

$$= \frac{1}{2a^2}\int_0^t \cos[a(t-2x)]\,dx - \frac{1}{2a^2}\int_0^t \cos(at)\,dx \tag{4.6.26}$$

$$= -\frac{1}{4a^3}\sin[a(t-2x)]\bigg|_0^t - \frac{1}{2a^2}\cos(at)\,x\bigg|_0^t \tag{4.6.27}$$

$$= \frac{1}{2a^3}[\sin(at) - at\cos(at)]. \tag{4.6.28}$$

• **Example 4.6.6**

Let us use the results from Example 4.6.3 to verify the convolution theorem.

We begin by rewriting (4.6.14) in terms of Heaviside's step functions. Using the method outline in Example 4.2.1,

$$f(t) * g(t) = \left(e^{t-1} - 1\right) H(t-1) + \left(1 - e^{t-2}\right) H(t-2). \tag{4.6.29}$$

Employing the second shifting theorem,

$$\mathcal{L}[f * g] = \frac{e^{-s}}{s-1} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-2s}}{s-1} \tag{4.6.30}$$

$$= \frac{e^{-s}}{s(s-1)} - \frac{e^{-2s}}{s(s-1)} = \frac{1}{s-1}\left(\frac{e^{-s}}{s} - \frac{e^{-2s}}{s}\right) \tag{4.6.31}$$

$$= \mathcal{L}[e^t]\mathcal{L}[H(t-1) - H(t-2)] \tag{4.6.32}$$

and the convolution theorem holds true. If we had not rewritten (4.6.14) in terms of step functions, we could still have found $\mathcal{L}[f * g]$ from the definition of the Laplace transform.

Problems

Verify the following convolutions and then show that the convolution theorem is true.

1. $1 * 1 = t$

2. $1 * \cos(at) = \sin(at)/a$

3. $1 * e^t = e^t - 1$

4. $t * t = t^3/6$

5. $t * \sin(t) = t - \sin(t)$

6. $t * e^t = e^t - t - 1$

7.
$$t^2 * \sin(at) = \frac{t^2}{a} - \frac{4}{a^3}\sin^2\left(\frac{at}{2}\right)$$

8.
$$t * H(t-1) = \tfrac{1}{2}(t-1)^2 H(t-1)$$

9.
$$H(t-a) * H(t-b) = (t-a-b)H(t-a-b)$$

10.
$$t * [H(t) - H(t-2)] = \frac{t^2}{2} - \frac{(t-2)^2}{2}H(t-2)$$

Use the convolution theorem to invert the following functions:

11.

$$F(s) = \frac{1}{s^2(s-1)}$$

12.

$$F(s) = \frac{1}{s^2(s+a)^2}$$

13. Prove that the convolution of two Dirac delta functions is a Dirac delta function.

4.7 INTEGRAL EQUATIONS

An *integral equation* contains the dependent variable under an integral sign. The convolution theorem provides an excellent tool for solving a very special class of these equations, *Volterra equation of the second kind* :[6]

$$f(t) - \int_0^t K[t, x, f(x)]\, dx = g(t), \qquad 0 \leq t \leq T. \tag{4.7.1}$$

These equations appear in history-dependent problems, such as epidemics,[7] vibration problems,[8] and viscoelasticity.[9]

• **Example 4.7.1**

Let us find $f(t)$ from the integral equation

$$f(t) = 4t - 3\int_0^t f(x)\sin(t-x)\, dx. \tag{4.7.2}$$

---

[6] Fock, V., 1924: Über eine Klasse von Integralgleichungen. *Math. Z.*, **21**, 161–173; Koizumi, S., 1931: On Heaviside's operational solution of a Volterra's integral equation when its nucleus is a function of $(x-\xi)$. *Philos. Mag., Ser. 7*, **11**, 432–441.

[7] Wang, F. J. S., 1978: Asymptotic behavior of some deterministic epidemic models. *SIAM J. Math. Anal.*, **9**, 529–534.

[8] Lin, S. P., 1975: Damped vibration of a string. *J. Fluid Mech.*, **72**, 787–797.

[9] Rogers, T. G. and Lee, E. H., 1964: The cylinder problem in viscoelastic stress analysis. *Q. Appl. Math.*, **22**, 117–131.

The integral in (4.7.2) is such that we can use the convolution theorem to find its Laplace transform. Then, because $\mathcal{L}[\sin(t)] = 1/(s^2+1)$, the convolution theorem yields

$$\mathcal{L}\left[\int_0^t f(x)\sin(t-x)\,dx\right] = \frac{F(s)}{s^2+1}. \tag{4.7.3}$$

Therefore, the Laplace transform converts (4.7.2) into

$$F(s) = \frac{4}{s^2} - \frac{3F(s)}{s^2+1}. \tag{4.7.4}$$

Solving for $F(s)$,

$$F(s) = \frac{4(s^2+1)}{s^2(s^2+4)}. \tag{4.7.5}$$

By partial fractions, or by inspection,

$$F(s) = \frac{1}{s^2} + \frac{3}{s^2+4}. \tag{4.7.6}$$

Therefore, inverting term by term,

$$f(t) = t + \tfrac{3}{2}\sin(2t). \tag{4.7.7}$$

Note that the integral equation

$$f(t) = 4t - 3\int_0^t f(t-x)\sin(x)\,dx \tag{4.7.8}$$

also has the same solution.

- **Example 4.7.2**

Let us solve the equation

$$g(t) = \frac{t^2}{2} - \int_0^t (t-x)g(x)\,dx. \tag{4.7.9}$$

Again the integral is one of the convolution type. Taking the Laplace transform of (4.7.9),

$$G(s) = \frac{1}{s^3} - \frac{G(s)}{s^2}, \tag{4.7.10}$$

which yields

$$\left(1 + \frac{1}{s^2}\right)G(s) = \frac{1}{s^3} \tag{4.7.11}$$

or

$$G(s) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}. \tag{4.7.12}$$

Then

$$g(t) = 1 - \cos(t). \tag{4.7.13}$$

Problems

Solve the following integral equations:

1.
$$f(t) = 1 + 2\int_0^t f(t-x)e^{-2x}\,dx$$

2.
$$f(t) = 1 + \int_0^t f(x)\sin(t-x)\,dx$$

3.
$$f(t) = t + \int_0^t f(t-x)e^{-x}\,dx$$

4.
$$f(t) = 4t^2 - \int_0^t f(t-x)e^{-x}\,dx$$

5.
$$f(t) = t^3 + \int_0^t f(x)\sin(t-x)\,dx$$

6.
$$f(t) = 8t^2 - 3\int_0^t f(x)\sin(t-x)\,dx$$

7.
$$f(t) = t^2 - 2\int_0^t f(t-x)\sinh(2x)\,dx$$

8.
$$f(t) = 1 + 2\int_0^t f(t-x)\cos(x)\,dx$$

9.
$$f(t) = e^{2t} - 2\int_0^t f(t-x)\cos(x)\,dx$$

10.
$$f(t) = t^2 + \int_0^t f(x)\sin(t-x)\,dx$$

11.

$$f(t) = e^{-t} - 2\int_0^t f(x)\cos(t-x)\,dx$$

12.

$$f(t) = 6t + 4\int_0^t f(x)(x-t)^2\,dx$$

13. Solve the following equation for $f(t)$ with the condition that $f(0) = 4$:

$$f'(t) = t + \int_0^t f(t-x)\cos(x)\,dx.$$

14. Solve the following equation for $f(t)$ with the condition that $f(0) = 0$:

$$f'(t) = \sin(t) + \int_0^t f(t-x)\cos(x)\,dx.$$

15. During a study of nucleation involving idealized active sites along a boiling surface, Marto and Rohsenow[10] had to solve the integral equation

$$A = B\sqrt{t} + C\int_0^t \frac{x'(\tau)}{\sqrt{t-\tau}}\,d\tau$$

to find the position $x(t)$ of the liquid/vapor interface. If A, B, and C are constants and $x(0) = 0$, find the solution for them.

16. Solve the following equation for $x(t)$ with the condition that $x(0) = 0$:

$$x(t) + t = \frac{1}{c\sqrt{\pi}}\int_0^t \frac{x'(\tau)}{\sqrt{t-\tau}}\,d\tau,$$

where c is constant.

17. During a study of the temperature $f(t)$ of a heat reservoir attached to a semi-infinite heat-conducting rod, Huber[11] had to solve the integral equation

$$f'(t) = \alpha - \frac{\beta}{\sqrt{\pi}}\int_0^t \frac{f'(\tau)}{\sqrt{t-\tau}}\,d\tau,$$

where α and β are constants and $f(0) = 0$. Find $f(t)$ for him. Hint:

$$\frac{\alpha}{s^{3/2}(s^{1/2}+\beta)} = \frac{\alpha}{s(s-\beta^2)} - \frac{\alpha\beta}{s^{3/2}(s-\beta^2)}.$$

---

[10] From Marto, P. J. and Rohsenow, W. M., 1966: Nucleate boiling instability of alkali metals. *J. Heat Transfer*, **88**, 183–193 with permission.

[11] From Huber, A., 1934: Eine Methode zur Bestimmung der Wärme- und Temperaturleitfähigkeit. *Monatsh. Math. Phys.*, **41**, 35–42.

18. During the solution of a diffusion problem, Zhdanov, Chikhachev, and Yavlinskii[12] solved an integral equation similar to

$$\int_0^t f(\tau)\left[1 - \operatorname{erf}\left(a\sqrt{t-\tau}\right)\right]\, d\tau = at,$$

where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\, dy$ is the error function. What should have they found? Hint: You will need to prove that

$$\mathcal{L}\left[t\ \operatorname{erf}(a\sqrt{t}\,) - \frac{1}{2a^2}\operatorname{erf}(a\sqrt{t}\,) + \frac{\sqrt{t}}{a\sqrt{\pi}}e^{-a^2t}\right] = \frac{a}{s^2\sqrt{s+a^2}}.$$

4.8 SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

For the engineer, as it was for Oliver Heaviside, the primary use of Laplace transforms is the solution of ordinary, constant coefficient, linear differential equations. These equations are important not only because they appear in many engineering problems but also because they may serve as approximations, even if locally, to ordinary differential equations with nonconstant coefficients or to nonlinear ordinary differential equations.

For all of these reasons, we wish to solve the *initial-value problem*

$$\frac{d^n y}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_{n-1}\frac{dy}{dt} + a_n y = f(t), \quad t > 0 \qquad \textbf{(4.8.1)}$$

by Laplace transforms, where $a_1, a_2, \ldots$ are constants and we know the value of $y, y', \ldots, y^{(n-1)}$ at $t = 0$. The procedure is as follows. Applying the derivative rule (4.1.20) to (4.8.1), we reduce the *differential* equation to an *algebraic* one involving the constants $a_1, a_2, \ldots, a_n$, the parameter s, the Laplace transform of $f(t)$, and the values of the initial conditions. We then solve for the Laplace transform of $y(t)$, $Y(s)$. Finally, we apply one of the many techniques of inverting a Laplace transform to find $y(t)$.

Similar considerations hold with *systems* of ordinary differential equations. The Laplace transform of the system of ordinary differential equations results in an algebraic set of equations containing $Y_1(s), Y_2(s), \ldots, Y_n(s)$. By some method we solve this set of equations and invert each transform $Y_1(s), Y_2(s), \ldots, Y_n(s)$ in turn to give $y_1(t), y_2(t), \ldots, y_n(t)$.

[12] Zhdanov, S. K., Chikhachev, A. S., and Yavlinskii, Yu. N., 1976: Diffusion boundary-value problem for regions with moving boundaries and conservation of particles. *Sov. Phys. Tech. Phys.*, **21**, 883–884.

The following examples will illustrate the details of the process.

• Example 4.8.1

Let us solve the ordinary differential equation

$$y'' + 2y' = 8t \tag{4.8.2}$$

subject to the initial conditions that $y'(0) = y(0) = 0$. Taking the Laplace transform of both sides of (4.8.2),

$$\mathcal{L}(y'') + 2\mathcal{L}(y') = 8\mathcal{L}(t) \tag{4.8.3}$$

or

$$s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) = \frac{8}{s^2}, \tag{4.8.4}$$

where $Y(s) = \mathcal{L}[y(t)]$. Substituting the initial conditions into (4.8.4) and solving for $Y(s)$,

$$Y(s) = \frac{8}{s^3(s+2)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s+2} \tag{4.8.5}$$

$$= \frac{8}{s^3(s+2)} = \frac{(s+2)A + s(s+2)B + s^2(s+2)C + s^3D}{s^3(s+2)}. \tag{4.8.6}$$

Matching powers of s in the numerators of (4.8.6), $C + D = 0$, $B + 2C = 0$, $A + 2B = 0$, and $2A = 8$ or $A = 4$, $B = -2$, $C = 1$, and $D = -1$. Therefore,

$$Y(s) = \frac{4}{s^3} - \frac{2}{s^2} + \frac{1}{s} - \frac{1}{s+2}. \tag{4.8.7}$$

Finally, performing term-by-term inversion of (4.8.7), the final solution is

$$y(t) = 2t^2 - 2t + 1 - e^{-2t}. \tag{4.8.8}$$

• Example 4.8.2

Let us solve the ordinary differential equation

$$y'' + y = H(t) - H(t-1) \tag{4.8.9}$$

with the initial conditions that $y'(0) = y(0) = 0$. Taking the Laplace transform of both sides of (4.8.9),

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s} - \frac{e^{-s}}{s}, \tag{4.8.10}$$

where $Y(s) = \mathcal{L}[y(t)]$. Substituting the initial conditions into (4.8.10) and solving for $Y(s)$,

$$Y(s) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right) - \left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-s}. \tag{4.8.11}$$

Using the second shifting theorem, the final solution is

$$y(t) = 1 - \cos(t) - [1 - \cos(t-1)]H(t-1). \tag{4.8.12}$$

• Example 4.8.3

Let us solve the ordinary differential equation

$$y'' + 2y' + y = f(t) \tag{4.8.13}$$

with the initial conditions that $y'(0) = y(0) = 0$, where $f(t)$ is an unknown function whose Laplace transform exists. Taking the Laplace transform of both sides of (4.8.13),

$$s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + Y(s) = F(s), \tag{4.8.14}$$

where $Y(s) = \mathcal{L}[y(t)]$. Substituting the initial conditions into (4.8.14) and solving for $Y(s)$,

$$Y(s) = \frac{1}{(s+1)^2}F(s). \tag{4.8.15}$$

We have written (4.8.15) in this form because the transform $Y(s)$ equals the product of two transforms $1/(s+1)^2$ and $F(s)$. Therefore, by the convolution theorem we can immediately write

$$y(t) = te^{-t} * f(t) = \int_0^t xe^{-x} f(t-x)\,dx. \tag{4.8.16}$$

Without knowing $f(t)$, this is as far as we can go.

• Example 4.8.4: Forced harmonic oscillator

Let us solve the *simple harmonic oscillator* forced by a harmonic forcing:

$$y'' + \omega^2 y = \cos(\omega t) \tag{4.8.17}$$

subject to the initial conditions that $y'(0) = y(0) = 0$. Although the complete solution could be found by summing the complementary solution and a particular solution obtained, say, from the method of undetermined coefficients, we will now illustrate how we can use Laplace transforms to solve this problem.

Taking the Laplace transform of both sides of (4.8.17), substituting in the initial conditions, and solving for $Y(s)$,

$$Y(s) = \frac{s}{(s^2+\omega^2)^2} \tag{4.8.18}$$

and

$$y(t) = \frac{1}{\omega}\sin(\omega t) * \cos(\omega t) = \frac{t}{2\omega}\sin(\omega t). \tag{4.8.19}$$

Equation (4.8.19) gives an oscillation that grows linearly with time although the forcing function is simply periodic. Why does this occur? Recall that our simple harmonic oscillator has the natural frequency ω. But that is exactly the frequency at which we drive the system. Consequently, our choice of forcing has resulted in *resonance* where energy continuously feeds into the oscillator.

- **Example 4.8.5**

Let us solve the *system* of ordinary differential equations:

$$2x' + y = \cos(t) \tag{4.8.20}$$

and

$$y' - 2x = \sin(t) \tag{4.8.21}$$

subject to the initial conditions that $x(0) = 0$ and $y(0) = 1$. Taking the Laplace transform of (4.8.20) and (4.8.21),

$$2sX(s) + Y(s) = \frac{s}{s^2+1} \tag{4.8.22}$$

and

$$-2X(s) + sY(s) = 1 + \frac{1}{s^2+1}, \tag{4.8.23}$$

after introducing the initial conditions. Solving for $X(s)$ and $Y(s)$,

$$X(s) = -\frac{1}{(s^2+1)^2} \tag{4.8.24}$$

and

$$Y(s) = \frac{s}{s^2+1} + \frac{2s}{(s^2+1)^2}. \tag{4.8.25}$$

Taking the inverse of (4.8.24)–(4.8.25) term by term,

$$x(t) = \tfrac{1}{2}[t\cos(t) - \sin(t)] \tag{4.8.26}$$

and

$$y(t) = t\sin(t) + \cos(t). \tag{4.8.27}$$

- **Example 4.8.6**

Let us determine the displacement of a mass m attached to a spring and excited by the driving force:

$$F(t) = mA\left(1 - \frac{t}{T}\right)e^{-t/T}. \tag{4.8.28}$$

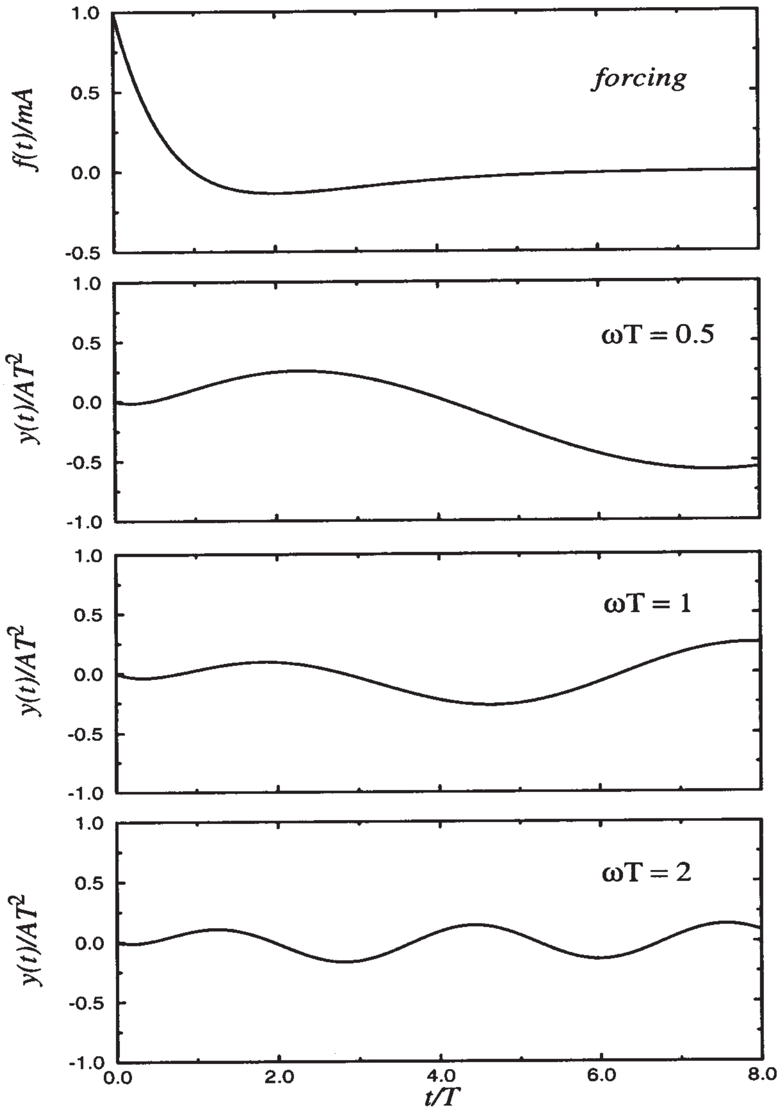


Figure 4.8.1: Displacement of a simple harmonic oscillator with nondimensional frequency ωT as a function of time t/T. The top frame shows the forcing function.

The dynamical equation governing this system is

$$y'' + \omega^2 y = A\left(1 - \frac{t}{T}\right)e^{-t/T}, \tag{4.8.29}$$

where $\omega^2 = k/m$ and k is the spring constant. Assuming that the system is initially at rest, the Laplace transform of the dynamical system is

$$(s^2+\omega^2)Y(s) = \frac{A}{s+1/T} - \frac{A}{T(s+1/T)^2} \tag{4.8.30}$$

or

$$Y(s) = \frac{A}{(s^2+\omega^2)(s+1/T)} - \frac{A}{T(s^2+\omega^2)(s+1/T)^2}. \tag{4.8.31}$$

Partial fractions yield

$$\begin{aligned} Y(s) = {} & \frac{A}{\omega^2+1/T^2}\left(\frac{1}{s+1/T} - \frac{s-1/T}{s^2+\omega^2}\right) - \frac{A}{T(\omega^2+1/T^2)^2} \\ & \times \left[\frac{1/T^2-\omega^2}{s^2+\omega^2} - \frac{2s/T}{s^2+\omega^2} + \frac{\omega^2+1/T^2}{(s+1/T)^2} + \frac{2/T}{s+1/T}\right]. \end{aligned} \tag{4.8.32}$$

Inverting (4.8.32) term by term,

$$\begin{aligned} y(t) = {} & \frac{AT^2}{1+\omega^2T^2}\left[e^{-t/T} - \cos(\omega t) + \frac{\sin(\omega t)}{\omega T}\right] \\ & - \frac{AT^2}{(1+\omega^2T^2)^2}\left\{(1-\omega^2T^2)\frac{\sin(\omega t)}{\omega T} + 2\left[e^{-t/T} - \cos(\omega t)\right]\right. \\ & \left. + (1+\omega^2T^2)(t/T)e^{-t/T}\right\}. \end{aligned} \tag{4.8.33}$$

The solution to this problem consists of two parts. The exponential terms result from the forcing and will die away with time. This is the *transient* portion of the solution. The sinusoidal terms are those natural oscillations that are necessary so that the solution satisfies the initial conditions. They are the *steady-state* portion of the solution. They endure forever. Figure 4.8.1 illustrates the solution when $\omega T = 0.1$, 1, and 2. Note that the displacement decreases in magnitude as the nondimensional frequency of the oscillator increases.

• **Example 4.8.7**

Let us solve the equation

$$y'' + 16y = \delta(t-\pi/4) \tag{4.8.34}$$

with the initial conditions that $y(0)=1$ and $y'(0)=0$.

Taking the Laplace transform of (4.8.34) and inserting the initial conditions,

$$(s^2+16)Y(s) = s + e^{-s\pi/4} \tag{4.8.35}$$

or

$$Y(s) = \frac{s}{s^2+16} + \frac{e^{-s\pi/4}}{s^2+16}. \tag{4.8.36}$$

Applying the second shifting theorem,

$$y(t) = \cos(4t) + \tfrac{1}{4}\sin[4(t-\pi/4)]H(t-\pi/4) \tag{4.8.37}$$

$$= \cos(4t) - \tfrac{1}{4}\sin(4t)H(t-\pi/4). \tag{4.8.38}$$

• **Example 4.8.8: Oscillations in electric circuits**

During the middle of the nineteenth century, Lord Kelvin[13] analyzed the LCR electrical circuit shown in Figure 4.8.2 which contains resistance R, capacitance C, and inductance L. For reasons that we shall shortly show, this LCR circuit has become one of the quintessential circuits for electrical engineers. In this example, we shall solve the problem by Laplace transforms.

Because we can add the potential differences across the elements, the equation governing the LCR circuit is

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I\,d\tau = E(t), \tag{4.8.39}$$

where I denotes the current in the circuit. Let us solve (4.8.39) when we close the circuit and the initial conditions are $I(0)=0$ and $Q(0)=-Q_0$. Taking the Laplace transform of (4.8.39),

$$\left(Ls + R + \frac{1}{Cs}\right)\overline{I}(s) = LI(0) - \frac{Q(0)}{Cs}. \tag{4.8.40}$$

Solving for $\overline{I}(s)$,

$$\overline{I}(s) = \frac{Q_0}{Cs(Ls+R+1/Cs)} = \frac{\omega_0^2 Q_0}{s^2+2\alpha s+\omega_0^2} \tag{4.8.41}$$

$$= \frac{\omega_0^2 Q_0}{(s+\alpha)^2+\omega_0^2-\alpha^2}, \tag{4.8.42}$$

where $\alpha = R/2L$ and $\omega_0^2 = 1/(LC)$. From the first shifting theorem,

$$I(t) = \frac{\omega_0^2 Q_0}{\omega} e^{-\alpha t}\sin(\omega t), \tag{4.8.43}$$

[13] Thomson, W., 1853: On transient electric currents. *Philos. Mag., Ser. 4*, **5**, 393–405.

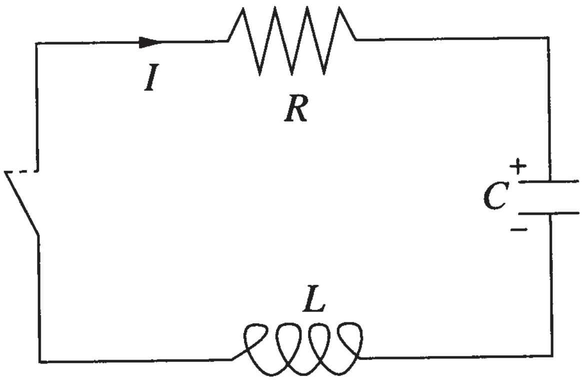


Figure 4.8.2: Schematic of a LCR circuit.

where $\omega^2 = \omega_0^2 - \alpha^2 > 0$. The quantity ω is the natural frequency of the circuit, which is lower than the free frequency ω_0 of a circuit formed by a condenser and coil. Most importantly, the solution decays in amplitude with time.

Although Kelvin's solution was of academic interest when he originally published it, this radically changed with the advent of radio telegraphy[14] because the LCR circuit described the fundamental physical properties of wireless transmitters and receivers.[15] The inescapable conclusion from this analysis was that no matter how clever the receiver was designed, eventually the resistance in the circuit would rapidly dampen the electrical oscillations and thus limit the strength of the received signal.

This technical problem was overcome by Armstrong[16] who invented an electrical circuit that used De Forest's audion (the first vacuum tube) for generating electrical oscillations and for amplifying externally impressed oscillations by "regenerative action". The effect of adding the "thermionic amplifier" is seen by again considering the LRC circuit as shown in Figure 4.8.3 with the modification suggested by Armstrong.[17]

The governing equations of this new circuit are

$$L_1 \frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I\, d\tau + M\frac{dI_p}{dt} = 0 \tag{4.8.44}$$

[14] Stone, J S., 1914: The resistance of the spark and its effect on the oscillations of electrical oscillators. *Proc. IRE*, **2**, 307–324.

[15] See Hogan, J. L., 1916: Physical aspects of radio telegraphy. *Proc. IRE*, **4**, 397–420.

[16] Armstrong, E. H., 1915: Some recent developments in the audion receiver. *Proc. IRE*, **3**, 215–247.

[17] From Ballantine, S., 1919: The operational characteristics of thermionic amplifiers. *Proc. IRE*, **7**, 129–161. ©IRE (now IEEE).

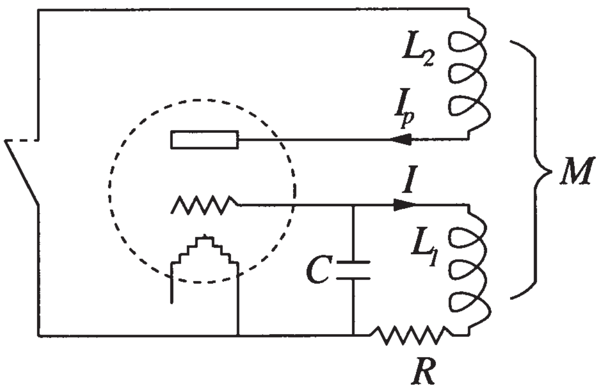


Figure 4.8.3: Schematic of a LCR circuit with the addition of a thermionic amplifier. [From Ballantine, S., 1919: The operational characteristics of thermionic amplifiers. *Proc. IRE*, **7**, 155. ©IRE (now IEEE).]

and

$$L_2\frac{dI_p}{dt} + R_0 I_p + M\frac{dI}{dt} + \frac{\mu}{C}\int_0^t I\,d\tau = 0, \tag{4.8.45}$$

where the plate circuit has the current I_p, the resistance R_0, the inductance L_2, and the electromotive force (emf) of $\mu\int_0^t I\,d\tau/C$. The mutual inductance between the two circuits is given by M. Taking the Laplace transform of (4.8.44)–(4.8.45),

$$L_1 s\overline{I}(s) + R\overline{I}(s) + \frac{\overline{I}(s)}{sC} + Ms\overline{I}_p(s) = \frac{Q_0}{sC} \tag{4.8.46}$$

and

$$L_2 s\overline{I}_p(s) + R_0\overline{I}_p(s) + Ms\overline{I}(s) + \frac{\mu}{sC}\overline{I}(s) = 0. \tag{4.8.47}$$

Eliminating $\overline{I}_p(s)$ between (4.8.46)–(4.8.47) and solving for $\overline{I}(s)$,

$$\overline{I}(s) = \frac{(L_2 s + R_0)Q_0}{(L_1L_2-M^2)Cs^3+(RL_2+R_0L_1)Cs^2+(L_2+CRR_0-\mu M)s+R_0}. \tag{4.8.48}$$

For high-frequency radio circuits, we can approximate the roots of the denominator of (4.8.48) as

$$s_1 \approx -\frac{R_0}{L_2 + CRR_0 - \mu M} \tag{4.8.49}$$

and

$$s_{2,3} \approx \frac{R_0}{2(L_2 + CRR_0 - \mu M)} - \frac{R_0L_1 + RL_2}{2(L_1L_2 - M^2)} \pm i\omega. \tag{4.8.50}$$

In the limit of M and R_0 vanishing, we recover our previous result for the LRC circuit. However, in reality, R_0 is very large and our solution has three terms. The term associated with s_1 is a rapidly decaying transient while the s_2 and s_3 roots yield oscillatory solutions with a *slight* amount of damping. Thus, our analysis has shown that in the ordinary regenerative circuit, the tube effectively introduces sufficient "negative" resistance so that the resultant positive resistance of the equivalent LCR circuit is relatively low, and the response of an applied signal voltage at the resonant frequency of the circuit is therefore relatively great. Later, Armstrong[18] extended his work on regeneration by introducing an electrical circuit – the superregenerative circuit – where the regeneration is made large enough so that the resultant resistance is negative, and self-sustained oscillations can occur.[19] It was this circuit[20] which led to the explosive development of radio in the 1920s and 1930s.

• **Example 4.8.9: Resonance transformer circuit**

One of the fundamental electrical circuits of early radio telegraphy[21] is the resonance transformer circuit shown in Figure 4.8.4. Its development gave transmitters and receivers the ability to tune to each other.

The governing equations follow from Kirchhoff's law and are

$$L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} + \frac{1}{C_1} \int_0^t I_1 \, d\tau = E(t) \tag{4.8.51}$$

and

$$M \frac{dI_1}{dt} + L_2 \frac{dI_2}{dt} + RI_2 + \frac{1}{C_2} \int_0^t I_2 \, d\tau = 0. \tag{4.8.52}$$

Let us examine the oscillations generated if initially the system has no currents or charges and the forcing function is $E(t) = \delta(t)$.

Taking the Laplace transform of (4.8.51)–(4.8.52),

$$L_1 s \overline{I}_1 + M s \overline{I}_2 + \frac{\overline{I}_2}{sC_1} = 1 \tag{4.8.53}$$

---

[18] Armstrong, E. H., 1922: Some recent developments of regenerative circuits. *Proc. IRE*, **10**, 244–260.

[19] See Frink, F. W., 1938: The basic principles of superregenerative reception. *Proc. IRE*, **26**, 76–106.

[20] Lewis, T., 1991: *Empire of the Air: The Men Who Made Radio,* HarperCollins Publishers, New York.

[21] Fleming, J. A., 1919: *The Principles of Electric Wave Telegraphy and Telephony,* Longmans, Green, Chicago.

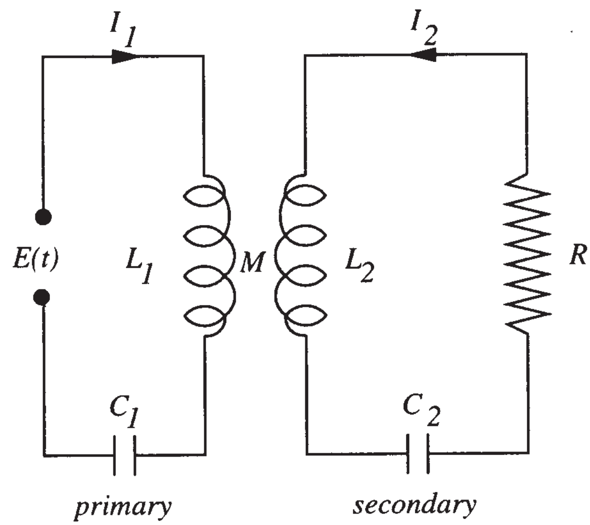


Figure 4.8.4: Schematic of a resonance transformer circuit.

and

$$Ms\overline{I}_1 + L_2 s\overline{I}_2 + R\overline{I}_2 + \frac{\overline{I}_2}{sC_2} = 0. \tag{4.8.54}$$

Because the current in the second circuit is of greater interest, we solve for $\overline{I}_2$ and find that

$$\overline{I}_2(s) = -\frac{Ms^3}{L_1 L_2[(1-k^2)s^4 + 2\alpha\omega_2^2 s^3 + (\omega_1^2+\omega_2^2)s^2 + 2\alpha\omega_1^2 s + \omega_1^2\omega_2^2]}, \tag{4.8.55}$$

where $\alpha = R/2L_2$, $\omega_1^2 = 1/L_1C_1$, $\omega_2^2 = 1/L_2C_2$, and $k^2 = M^2/L_1L_2$, the so-called coefficient of coupling.

We can obtain analytic solutions if we assume that the coupling is weak ($k^2 \ll 1$). Equation (4.8.55) becomes

$$\overline{I}_2 = -\frac{Ms^3}{L_1L_2(s^2+\omega_1^2)(s^2+2\alpha s+\omega_2^2)}. \tag{4.8.56}$$

Using partial fractions and inverting term by term, we find that

$$\begin{aligned} I_2(t) = \frac{M}{L_1L_2}\Bigg[&\frac{2\alpha\omega_1^3\sin(\omega_1 t)}{(\omega_2^2-\omega_1^2)^2+4\alpha^2\omega_1^2} + \frac{\omega_1^2(\omega_2^2-\omega_1^2)\cos(\omega_1 t)}{(\omega_2^2-\omega_1^2)^2+4\alpha^2\omega_1^2} \\ &+ \frac{\alpha\omega_2^4 - 3\alpha\omega_1^2\omega_2^2 + 4\alpha^3\omega_1^2}{(\omega_2^2-\omega_1^2)^2+4\alpha^2\omega_1^2} e^{-\alpha t}\frac{\sin(\omega t)}{\omega} \\ &- \frac{\omega_2^2(\omega_2^2-\omega_1^2)+4\alpha^2\omega_1^2}{(\omega_2^2-\omega_1^2)^2+4\alpha^2\omega_1^2} e^{-\alpha t}\cos(\omega t)\Bigg], \end{aligned} \tag{4.8.57}$$

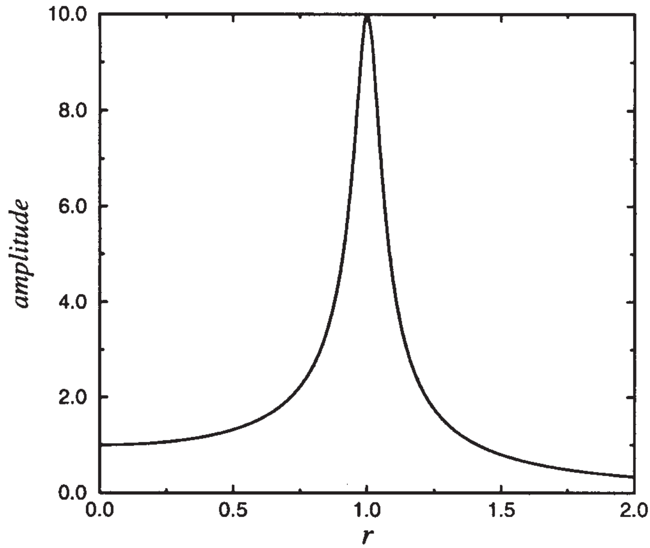


Figure 4.8.5: The resonance curve $1/\sqrt{(r^2-1)^2+0.01}$ for a resonance transformer circuit with $r = \omega_2/\omega_1$.

where $\omega^2 = \omega_2^2 - \alpha^2$.

The exponentially damped solutions will eventually disappear, leaving only the steady-state oscillations which vibrate with the angular frequency ω_1, the natural frequency of the primary circuit. If we rewrite this steady-state solution in amplitude/phase form, the amplitude is

$$\frac{M}{L_1 L_2 \sqrt{(r^2-1)^2 + 4\alpha^2/\omega_1^2}}, \tag{4.8.58}$$

where $r = \omega_2/\omega_1$. As Figure 4.8.5 shows, as r increases from zero to two, the amplitude rises until a very sharp peak occurs at $r = 1$ and then decreases just as rapidly as we approach $r = 2$. Thus, the resonance transformer circuit provides a convenient way to tune a transmitter or receiver to the frequency ω_1.

Problems

Solve the following ordinary differential equations by Laplace transforms:

1. $y' - 2y = 1 - t; \quad y(0) = 1$

2. $y'' - 4y' + 3y = e^t; \quad y(0) = 0, y'(0) = 0$

3. $y'' - 4y' + 3y = e^{2t}; \quad y(0) = 0, y'(0) = 1$

4. $y'' - 6y' + 8y = e^{t}; \quad y(0) = 3, y'(0) = 9$

5. $y'' + 4y' + 3y = e^{-t}; \quad y(0) = 1, y'(0) = 1$

6. $y'' + y = t; \quad y(0) = 1, y'(0) = 0$

7. $y'' + 4y' + 3y = e^{t}; \quad y(0) = 0, y'(0) = 2$

8. $y'' - 4y' + 5y = 0; \quad y(0) = 2, y'(0) = 4$

9. $y' + y = tH(t-1); \quad y(0) = 0$

10. $y'' + 3y' + 2y = H(t-1); \quad y(0) = 0, y'(0) = 1$

11. $y'' - 3y' + 2y = H(t-1); \quad y(0) = 0, y'(0) = 1$

12. $y'' + 4y = 3H(t-4); \quad y(0) = 1, y'(0) = 0$

13. $y'' + 4y' + 4y = 4H(t-2); \quad y(0) = 0, y'(0) = 0$

14. $y'' + 3y' + 2y = e^{t-1}H(t-1); \quad y(0) = 0, y'(0) = 1$

15. $y'' - 3y' + 2y = e^{-(t-2)}H(t-2); \quad y(0) = 0, y'(0) = 0$

16. $y'' - 3y' + 2y = H(t-1) - H(t-2); \quad y(0) = 0, y'(0) = 0$

17. $y'' + y = 1 - H(t-T); \quad y(0) = 0, y'(0) = 0$

18. $y'' + y = \begin{cases} \sin(t), & 0 \le t \le \pi \\ 0, & t \ge \pi; \end{cases} \quad y(0) = 0, y'(0) = 0$

19. $y'' + 3y' + 2y = \begin{cases} t, & 0 \le t \le a \\ ae^{-(t-a)}, & t \ge a; \end{cases} \quad y(0) = 0, y'(0) = 0$

20. $y'' + \omega^2 y = \begin{cases} t/a, & 0 \le t \le a \\ 1 - (t-a)/(b-a), & a \le t \le b \\ 0, & t \ge b; \end{cases}$
$y(0) = 0, y'(0) = 0$

21. $y'' - 2y' + y = 3\delta(t-2); \quad y(0) = 0, y'(0) = 1$

22. $y'' - 5y' + 4y = \delta(t-1); \quad y(0) = 0, y'(0) = 0$

23. $y'' + 5y' + 6y = 3\delta(t-2) - 4\delta(t-5); \quad y(0) = y'(0) = 0$

24. $x' - 2x + y = 0, y' - 3x - 4y = 0; \quad x(0) = 1, y(0) = 0$

25. $x' - 2y' = 1, x' + y - x = 0; \quad x(0) = y(0) = 0$

26. $x' + 2x - y' = 0, x' + y + x = t^2; \quad x(0) = y(0) = 0$

27. $x' + 3x - y = 1, x' + y' + 3x = 0; \quad x(0) = 2, y(0) = 0$

28. Forster, Escobal, and Lieske[22] used Laplace transforms to solve the linearized equations of motion of a vehicle in a gravitational field created by two other bodies. A simplified form of this problem involves solving the following system of ordinary differential equations:

$$x'' - 2y' = F_1 + x + 2y, \qquad 2x' + y'' = F_2 + 2x + 3y$$

subject to the initial conditions that $x(0) = y(0) = x'(0) = y'(0) = 0$. Find the solution to this system.

4.9 TRANSFER FUNCTIONS, GREEN'S FUNCTION, AND INDICIAL ADMITTANCE

One of the drawbacks of using Laplace transforms to solve ordinary differential equations with a forcing term is its lack of generality. Each new forcing function requires a repetition of the entire process. In this section we give some methods for finding the solution in a somewhat more general manner for stationary systems where the forcing, not any initially stored energy (i.e., nonzero initial conditions), produces the total output. Unfortunately, the solution must be written as an integral.

In Example 4.8.3 we solved the linear differential equation

$$y'' + 2y' + y = f(t) \tag{4.9.1}$$

subject to the initial conditions $y(0) = y'(0) = 0$. At that time we wrote the Laplace transform of $y(t)$, $Y(s)$, as the product of two Laplace transforms:

$$Y(s) = \frac{1}{(s+1)^2} F(s). \tag{4.9.2}$$

---

[22] Reprinted from *Astronaut. Acta*, **14**, Forster, K., P. R. Escobal and H. A. Lieske, Motion of a vehicle in the transition region of the three-body problem, 1–10, ©1968, with kind permission from Elsevier Science Ltd, The Boulevard, Langford Lane, Kidlington OX5 1GB, UK.

One drawback in using (4.9.2) is its dependence upon an unspecified Laplace transform $F(s)$. Is there a way to eliminate this dependence and yet retain the essence of the solution?

One way of obtaining a quantity that is independent of the forcing is to consider the ratio:

$$\frac{Y(s)}{F(s)} = G(s) = \frac{1}{(s+1)^2}. \tag{4.9.3}$$

This ratio is called the *transfer function* because we can transfer the input $F(s)$ into the output $Y(s)$ by multiplying $F(s)$ by $G(s)$. It depends only upon the properties of the system.

Let us now consider a related problem to (4.9.1), namely

$$g'' + 2g' + g = \delta(t), \qquad t > 0 \tag{4.9.4}$$

with $g(0) = g'(0) = 0$. Because the forcing equals the Dirac delta function, $g(t)$ is called the *impulse response* or *Green's function*.[23] Computing $G(s)$,

$$G(s) = \frac{1}{(s+1)^2}. \tag{4.9.5}$$

From (4.9.3) we see that $G(s)$ is also the transfer function. Thus, an alternative method for computing the transfer function is to subject the system to impulse forcing and the Laplace transform of the response is the transfer function.

From (4.9.3),

$$Y(s) = G(s)F(s) \tag{4.9.6}$$

or

$$y(t) = g(t) * f(t). \tag{4.9.7}$$

That is, the convolution of the impulse response with the particular forcing gives the response of the system. Thus, we may describe a stationary system in one of two ways: (1) in the transform domain we have the transfer function, and (2) in the time domain there is the impulse response.

Despite the fundamental importance of the impulse response or Green's function for a given linear system, it is often quite difficult to determine, especially experimentally, and a more convenient practice is to deal with the response to the unit step $H(t)$. This response is called the *indicial admittance* or *step response*, which we shall denote by $a(t)$.

---

[23] For the origin of the Green's function, see Farina, J. E. G., 1976: The work and significance of George Green, the miller mathematician, 1793–1841. *Bull. Inst. Math. Appl.*, **12**, 98–105.

Because $\mathcal{L}[H(t)] = 1/s$, we can determine the transfer function from the indicial admittance because $\mathcal{L}[a(t)] = G(s)\mathcal{L}[H(t)]$ or $sA(s) = G(s)$. Furthermore, because

$$\mathcal{L}[g(t)] = G(s) = \frac{\mathcal{L}[a(t)]}{\mathcal{L}[H(t)]}, \tag{4.9.8}$$

then

$$g(t) = \frac{da(t)}{dt} \tag{4.9.9}$$

from (4.1.18).

- **Example 4.9.1**

Let us find the transfer function, impulse response, and step response for the system

$$y'' - 3y' + 2y = f(t) \tag{4.9.10}$$

with $y(0) = y'(0) = 0$. To find the impulse response, we solve

$$g'' - 3g' + 2g = \delta(t) \tag{4.9.11}$$

with $g(0) = g'(0) = 0$. Taking the Laplace transform of (4.9.11), we find that

$$G(s) = \frac{1}{s^2 - 3s + 2}, \tag{4.9.12}$$

which is the transfer function for this system. The impulse response equals the inverse of $G(s)$ or

$$g(t) = e^{2t} - e^t. \tag{4.9.13}$$

To find the step response, we solve

$$a'' - 3a' + 2a = H(t) \tag{4.9.14}$$

with $a(0) = a'(0) = 0$. Taking the Laplace transform of (4.9.14),

$$A(s) = \frac{1}{s(s-1)(s-2)} \tag{4.9.15}$$

or

$$a(t) = \tfrac{1}{2} + \tfrac{1}{2}e^{2t} - e^t. \tag{4.9.16}$$

Note that $a'(t) = g(t)$.

• **Example 4.9.2**

There is an old joke about a man who took his car into a garage because of a terrible knocking sound. Upon his arrival the mechanic took one look at it and gave it a hefty kick.[24] Then, without a moment's hesitation he opened the hood, bent over, and tightened up a loose bolt. Turning to the owner, he said, "Your car is fine. That'll be $50." The owner felt that the charge was somewhat excessive, and demanded an itemized account. The mechanic said, "The kicking of the car and tightening one bolt, cost you a buck. The remaining $49 comes from knowing where to kick the car and finding the loose bolt."

Although the moral of the story may be about expertise as a marketable commodity, it also illustrates the concept of transfer function.[25] Let us model the car as a linear system where the equation

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = f(t) \tag{4.9.17}$$

governs the response $y(t)$ to a forcing $f(t)$. Assuming that the car has been sitting still, the initial conditions are zero and the Laplace transform of (4.9.17) is

$$K(s)Y(s) = F(s), \tag{4.9.18}$$

where

$$K(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0. \tag{4.9.19}$$

Hence

$$Y(s) = \frac{F(s)}{K(s)} = G(s)F(s), \tag{4.9.20}$$

where the transfer function $G(s)$ clearly depends only on the internal workings of the car. So if we know the transfer function, we understand how the car vibrates because

$$y(t) = \int_0^t g(t-x) f(x)\, dx. \tag{4.9.21}$$

But what does this have to do with our mechanic? He realized that a short sharp kick mimics an impulse forcing with $f(t) = \delta(t)$ and

---

[24] This is obviously a very old joke.

[25] Originally suggested by Stern, M. D., 1987: Why the mechanic kicked the car – A teaching aid for transfer functions. *Math. Gaz.*, **71**, 62–64.

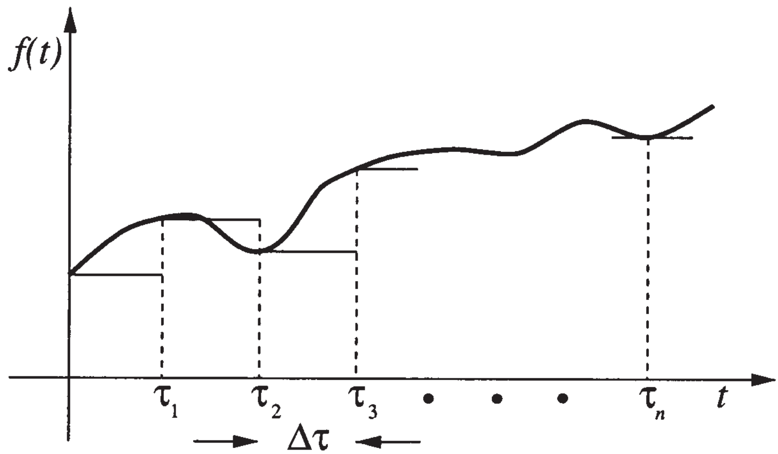


Figure 4.9.1: Diagram used in the derivation of Duhamel's integral.

$y(t) = g(t)$. Therefore, by observing the response of the car to his kick, he diagnosed the loose bolt and fixed the car.

In this section we have shown how the response of any system may be expressed in terms of its Green's function and the arbitrary forcing. Can we also determine the response using the indicial admittance $a(t)$?

Consider first a system that is dormant until a certain time $t = \tau_1$. At that instant we subject the system to a forcing $H(t - \tau_1)$. Then the response will be zero if $t < \tau_1$ and will equal the indicial admittance $a(t - \tau_1)$ when $t > \tau_1$ because the indicial admittance is the response of a system to the step function. Here $t - \tau_1$ is the time measured from the instant of change.

Next, suppose that we now force the system with the value $f(0)$ when $t = 0$ and hold that value until $t = \tau_1$. We then abruptly change the forcing by an amount $f(\tau_1) - f(0)$ to the value $f(\tau_1)$ at the time τ_1 and hold it at that value until $t = \tau_2$. Then we again abruptly change the forcing by an amount $f(\tau_2) - f(\tau_1)$ at the time τ_2, and so forth (see Figure 4.9.1). From the *linearity* of the problem the response after the instant $t = \tau_n$ equals the sum

$$
\begin{aligned}
y(t) = f(0)a(t) + [f(\tau_1) - f(0)]a(t - \tau_1) + [f(\tau_2) - f(\tau_1)]a(t - \tau_2) \\
+ \cdots + [f(\tau_n) - f(\tau_{n-1})]a(t - \tau_n). \qquad \textbf{(4.9.22)}
\end{aligned}
$$

If we write $f(\tau_k) - f(\tau_{k-1}) = \Delta f_k$ and $\tau_k - \tau_{k-1} = \Delta\tau_k$, (4.9.22) becomes

$$y(t) = f(0)a(t) + \sum_{k=1}^{n} a(t - \tau_k)\frac{\Delta f_k}{\Delta\tau_k}\,\Delta\tau_k. \qquad \textbf{(4.9.23)}$$

Finally, proceeding to the limit as the number n of jumps becomes infinite, in such a manner that all jumps and intervals between successive jumps tend to zero, this sum has the limit

$$y(t) = f(0)a(t) + \int_0^t f'(\tau)a(t-\tau)\,d\tau. \tag{4.9.24}$$

Because the total response of the system equals the weighted sum [the weights being $a(t)$] of the forcing from the initial moment up to the time t, we refer to (4.9.24) as the *superposition integral*, or *Duhamel's integral*.[26]

We can also express (4.9.24) in several different forms. Integration by parts yields

$$y(t) = f(t)a(0) + \int_0^t f(\tau)a'(t-\tau)\,d\tau \tag{4.9.25}$$

$$= \frac{d}{dt}\left[\int_0^t f(\tau)a(t-\tau)\,d\tau\right]. \tag{4.9.26}$$

• **Example 4.9.3**

Suppose that a system has the step response of $a(t) = A[1-e^{-t/T}]$, where A and T are positive constants. Let us find the response if we force this system by $f(t) = kt$, where k is a constant.

From the superposition integral (4.9.24),

$$y(t) = 0 + \int_0^t kA[1-e^{-(t-\tau)/T}]\,d\tau \tag{4.9.27}$$

$$= kA[t - T(1-e^{-t/T})]. \tag{4.9.28}$$

Problems

For the following nonhomogeneous differential equations, find the transfer function, impulse response, and step response. Assume that all of the necessary initial conditions are zero.

1. $y' + ky = f(t)$

2. $y'' - 2y' - 3y = f(t)$

---

[26] Duhamel, J.-M.-C., 1833: Mémoire sur la méthode générale relative au mouvement de la chaleur dans les corps solides plongés dans des milieux dont la température varie avec le temps, *J. École Polytech.*, **22**, 20–77.

3. $y'' + 4y' + 3y = f(t)$

4. $y'' - 2y' + 5y = f(t)$

5. $y'' - 3y' + 2y = f(t)$

6. $y'' + 4y' + 4y = f(t)$

7. $y'' - 9y = f(t)$

8. $y'' + y = f(t)$

9. $y'' - y' = f(t)$

4.10 INVERSION BY CONTOUR INTEGRATION

In Sections 4.5 and 4.6 we showed how we may use partial fractions and convolution to find the inverse of the Laplace transform $F(s)$. In many instances these methods fail simply because of the complexity of the transform to be inverted. In this section we shall show how we may invert transforms through the powerful method of contour integration. Of course, the student must be proficient in the use of complex variables.

Consider the piece-wise differentiable function $f(x)$ which vanishes for $x < 0$. We can express the function $e^{-cx}f(x)$ by the complex Fourier representation of

$$f(x)e^{-cx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\left[\int_0^{\infty} e^{-ct}f(t)e^{-i\omega t}\,dt\right]d\omega, \tag{4.10.1}$$

for any value of the real constant c, where the integral

$$I = \int_0^{\infty} e^{-ct}|f(t)|\,dt \tag{4.10.2}$$

exists. By multiplying both sides of (4.10.1) by e^{cx} and bringing it inside the first integral,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(c+\omega i)x}\left[\int_0^{\infty} f(t)e^{-(c+\omega i)t}\,dt\right]d\omega. \tag{4.10.3}$$

With the substitution $z = c + \omega i$, where z is a new, complex variable of integration,

$$f(x) = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} e^{zx}\left[\int_0^{\infty} f(t)e^{-zt}\,dt\right]dz. \tag{4.10.4}$$

The quantity inside the square brackets is the Laplace transform $F(z)$. Therefore, we can express $f(t)$ in terms of its transform by the complex contour integral:

$$\boxed{f(t) = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} F(z)e^{tz}\,dz.} \tag{4.10.5}$$

Figure 4.10.1: An outstanding mathematician at Cambridge University at the turn of the twentieth century, Thomas John I'Anson Bromwich (1875–1929) came to Heaviside's operational calculus through his interest in divergent series. Beginning a correspondence with Heaviside, Bromwich was able to justify operational calculus through the use of contour integrals by 1915. After his premature death, individuals such as J. R. Carson and Sir H. Jeffreys brought Laplace transforms to the increasing attention of scientists and engineers. (Portrait courtesy of the Royal Society of London.)

This line integral, *Bromwich's integral*,[27] runs along the line $x = c$ parallel to the imaginary axis and c units to the right of it, the so-called *Bromwich contour*. We select the value of c sufficiently large so that the integral (4.10.2) exists; subsequent analysis shows that this occurs when c is larger than the real part of any of the singularities of $F(z)$.

[27] Bromwich, T. J. I'A., 1916: Normal coordinates in dynamical systems. *Proc. London Math. Soc., Ser. 2*, **15**, 401–448.

We must now evaluate the contour integral. Because of the power of the *residue* theorem in complex variables, the contour integral is usually transformed into a closed contour through the use of *Jordan's lemma.* See Section 3.4, Equations (3.4.12) and (3.4.13). The following examples will illustrate the proper use of (4.10.5).

• Example 4.10.1

Let us invert

$$F(s) = \frac{e^{-3s}}{s^2(s-1)}. \tag{4.10.6}$$

From Bromwich's integral,

$$f(t) = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{e^{(t-3)z}}{z^2(z-1)}\, dz \tag{4.10.7}$$

$$= \frac{1}{2\pi i}\oint_C \frac{e^{(t-3)z}}{z^2(z-1)}\, dz - \frac{1}{2\pi i}\int_{C_R} \frac{e^{(t-3)z}}{z^2(z-1)}\, dz, \tag{4.10.8}$$

where C_R is a semicircle of infinite radius in either the right or left half of the z-plane and C is the closed contour that includes C_R and Bromwich's contour. See Figure 4.10.2.

Our first task is to choose an appropriate contour so that the integral along C_R vanishes. By Jordan's lemma this requires a semicircle in the right half-plane if $t-3<0$ and a semicircle in the left half-plane if $t-3>0$. Consequently, by considering these two separate cases, we have forced the second integral in (4.10.8) to zero and the inversion simply equals the closed contour.

Consider the case $t<3$ first. Because Bromwich's contour lies to the right of any singularities, there are no singularities within the closed contour and $f(t)=0$.

Consider now the case $t>3$. Within the closed contour in the left half-plane, there is a second-order pole at $z=0$ and a simple pole at $z=1$. Therefore,

$$f(t) = \text{Res}\left[\frac{e^{(t-3)z}}{z^2(z-1)};0\right] + \text{Res}\left[\frac{e^{(t-3)z}}{z^2(z-1)};1\right], \tag{4.10.9}$$

where

$$\text{Res}\left[\frac{e^{(t-3)z}}{z^2(z-1)};0\right] = \lim_{z\to 0}\frac{d}{dz}\left[z^2\frac{e^{(t-3)z}}{z^2(z-1)}\right] \tag{4.10.10}$$

$$= \lim_{z\to 0}\left[\frac{(t-3)e^{(t-3)z}}{z-1} - \frac{e^{(t-3)z}}{(z-1)^2}\right] \tag{4.10.11}$$

$$= 2-t \tag{4.10.12}$$

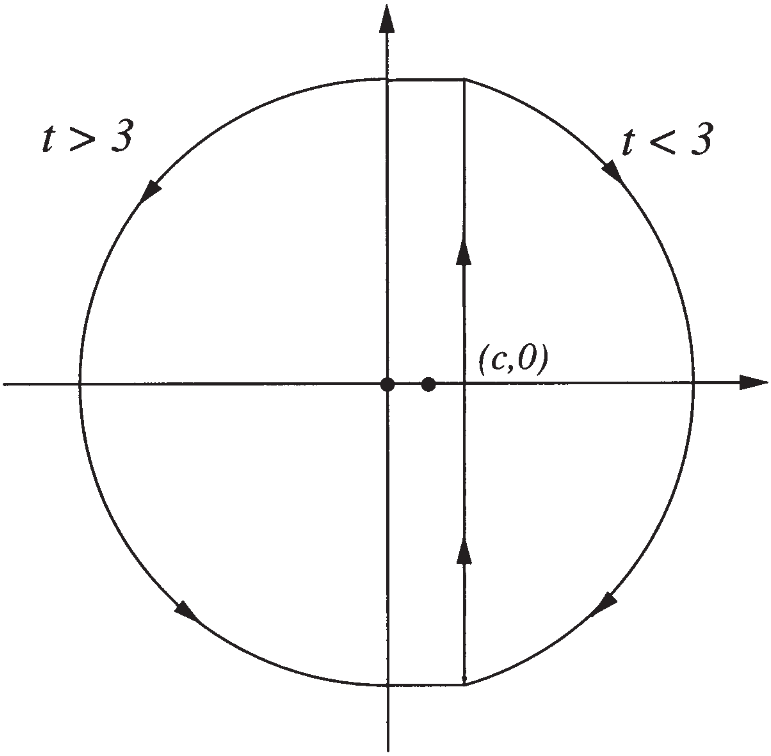


Figure 4.10.2: Contours used in the inversion of (4.10.6).

and

$$\operatorname{Res}\left[\frac{e^{(t-3)z}}{z^2(z-1)};1\right] = \lim_{z\to 1}(z-1)\frac{e^{(t-3)z}}{z^2(z-1)} = e^{t-3}. \qquad \textbf{(4.10.13)}$$

Taking our earlier results into account, the inverse equals

$$f(t) = \left[e^{t-3} - (t-3) - 1\right] H(t-3) \qquad \textbf{(4.10.14)}$$

which we would have obtained from the second shifting theorem and tables.

• **Example 4.10.2**

For our second example of the inversion of Laplace transforms by complex integration, let us find the inverse of

$$F(s) = \frac{1}{s\sinh(as)}, \qquad \textbf{(4.10.15)}$$

where a is real. From Bromwich's integral,

$$f(t) = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{e^{tz}}{z\sinh(az)}\,dz. \qquad (4.10.16)$$

Here c is greater than the real part of any of the singularities in (4.10.15). Using the infinite product for the hyperbolic sine,[28]

$$\frac{e^{tz}}{z\sinh(az)} = \frac{e^{tz}}{az^2[1+a^2z^2/\pi^2][1+a^2z^2/(4\pi^2)][1+a^2z^2/(9\pi^2)]\cdots}. \qquad (4.10.17)$$

Thus, we have a second-order pole at $z=0$ and simple poles at $z_n = \pm n\pi i/a$, where $n = 1, 2, 3, \ldots$

We may convert the line integral (4.10.16), with the Bromwich contour lying parallel and slightly to the right of the imaginary axis, into a closed contour using Jordan's lemma through the addition of an infinite semicircle joining $i\infty$ to $-i\infty$ as shown in Figure 4.10.3. We now apply the residue theorem. For the second-order pole at $z=0$,

$$\operatorname{Res}\left[\frac{e^{tz}}{z\sinh(az)}; 0\right] = \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left[\frac{(z-0)^2e^{tz}}{z\sinh(az)}\right] \qquad (4.10.18)$$

$$= \lim_{z\to 0}\frac{d}{dz}\left[\frac{ze^{tz}}{\sinh(az)}\right] \qquad (4.10.19)$$

$$= \lim_{z\to 0}\left[\frac{e^{tz}}{\sinh(az)} + \frac{zte^{tz}}{\sinh(az)} - \frac{az\cosh(az)e^{tz}}{\sinh^2(az)}\right] \qquad (4.10.20)$$

$$= \frac{t}{a} \qquad (4.10.21)$$

after using $\sinh(az) = az + O(z^3)$. For the simple poles $z_n = \pm n\pi i/a$,

$$\operatorname{Res}\left[\frac{e^{tz}}{z\sinh(az)}; z_n\right] = \lim_{z\to z_n}\frac{(z-z_n)e^{tz}}{z\sinh(az)} \qquad (4.10.22)$$

$$= \lim_{z\to z_n}\frac{e^{tz}}{\sinh(az)+az\cosh(az)} \qquad (4.10.23)$$

$$= \frac{\exp(\pm n\pi it/a)}{(-1)^n(\pm n\pi i)}, \qquad (4.10.24)$$

---

28 Gradshteyn, I. S. and Ryzhik, I. M., 1965: *Table of Integrals, Series and Products*, Academic Press, New York. See Section 1.431, formula 2.

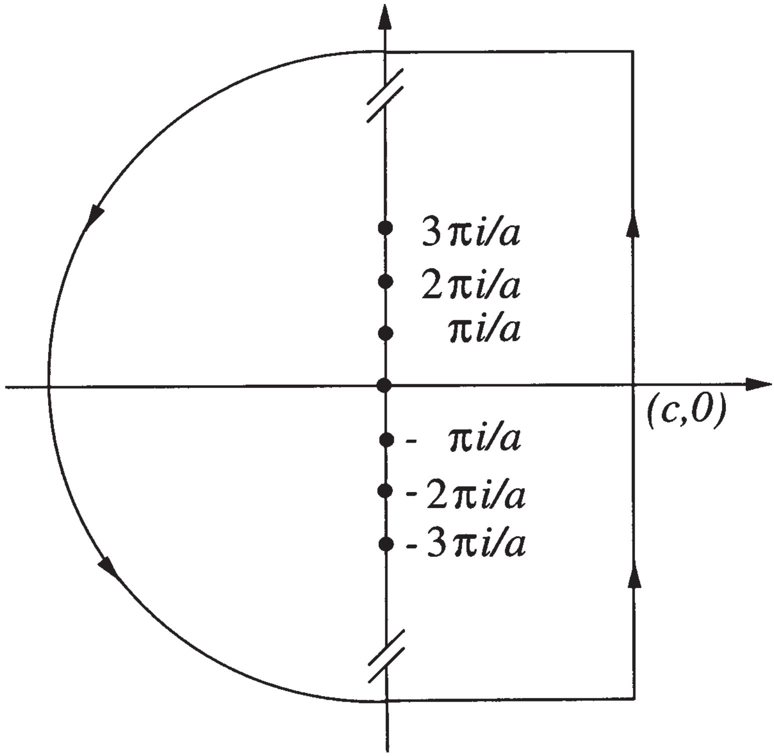


Figure 4.10.3: Contours used in the inversion of (4.10.15).

because $\cosh(\pm n\pi i) = \cos(n\pi) = (-1)^n$. Thus, summing up all of the residues gives

$$f(t) = \frac{t}{a} + \sum_{n=1}^{\infty} \frac{(-1)^n \exp(n\pi i t/a)}{n\pi i} - \sum_{n=1}^{\infty} \frac{(-1)^n \exp(-n\pi i t/a)}{n\pi i} \tag{4.10.25}$$

$$= \frac{t}{a} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(n\pi t/a). \tag{4.10.26}$$

In addition to computing the inverse of Laplace transforms, Bromwich's integral places certain restrictions on $F(s)$ in order that an inverse exists. If α denotes the minimum value that c may possess, the restrictions are threefold.[29] First, $F(z)$ must be analytic in the half-plane $x \geq \alpha$, where $z = x + iy$. Second, in the same half-plane it must behave as z^{-k}, where $k > 1$. Finally, $F(x)$ must be real when $x \geq \alpha$.

[29] For the proof, see Churchill, R. V., 1972: *Operational Mathematics*, McGraw-Hill, New York, Section 67.

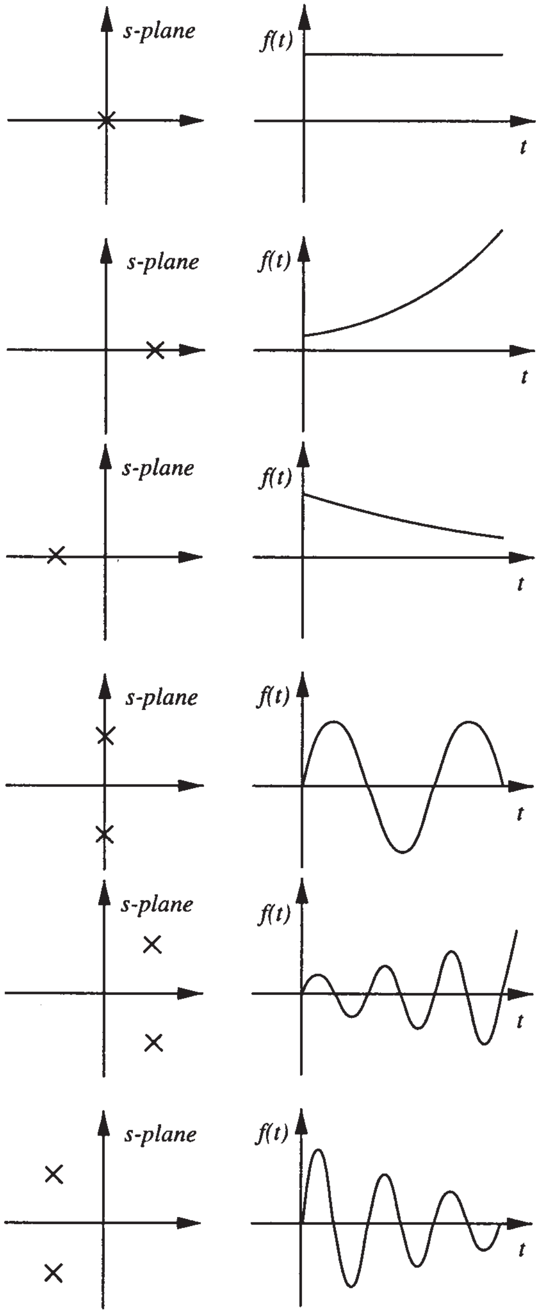


Figure 4.10.4: The correspondence between the location of the simple poles of the Laplace transform $F(s)$ and the behavior of $f(t)$.

- **Example 4.10.3**

Is the function $\sin(s)/(s^2+4)$ a proper Laplace transform? Although the function satisfies the first and third criteria listed in the previous paragraph on the half-plane $x > 2$, the function becomes unbounded as $y \to \pm\infty$ for any fixed $x > 2$. Thus, $\sin(s)/(s^2+4)$ cannot be a Laplace transform.

- **Example 4.10.4**

An additional benefit of understanding inversion by the residue method is the ability to qualitatively anticipate the inverse by knowing the location of the poles of $F(s)$. This intuition is important because many engineering analyses discuss stability and performance entirely in terms of the properties of the system's Laplace transform. In Figure 4.10.4 we have graphed the location of the poles of $F(s)$ and the corresponding $f(t)$. The student should go through the mental exercise of connecting the two pictures.

Problems

Use Bromwich's integral to invert the following Laplace transform:

1. $F(s) = \dfrac{s+1}{(s+2)^2(s+3)}$

2. $F(s) = \dfrac{1}{s^2(s+a)^2}$

3. $F(s) = \dfrac{1}{s(s-2)^3}$

4. $F(s) = \dfrac{1}{s(s+a)^2(s^2+b^2)}$

5. $F(s) = \dfrac{e^{-s}}{s^2(s+2)}$

6. $F(s) = \dfrac{1}{s(1+e^{-as})}$

7. $F(s) = \dfrac{1}{(s+b)\cosh(as)}$

8. $F(s) = \dfrac{1}{s(1-e^{-as})}$

9. Consider a function $f(t)$ which has the Laplace transform $F(z)$ which is analytic in the half-plane $\mathrm{Re}(z) > s_0$. Can we use this knowledge to find $g(t)$ whose Laplace transform $G(z)$ equals $F[\varphi(z)]$, where $\varphi(z)$ is also analytic for $\mathrm{Re}(z) > s_0$? The answer to this question leads to the Schouten[30] – Van der Pol[31] theorem.

---

[30] Schouten, J. P., 1935: A new theorem in operational calculus together with an application of it. *Physica*, **2**, 75–80.

[31] Van der Pol, B., 1934: A theorem on electrical networks with applications to filters. *Physica*, **1**, 521–530.

Step 1: Show that the following relationships hold true:

$$G(z) = F[\varphi(z)] = \int_0^\infty f(\tau)e^{-\varphi(z)\tau}\,d\tau$$

and

$$g(t) = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} F[\varphi(z)]e^{tz}\,dz.$$

Step 2: Using the results from Step 1, show that

$$g(t) = \int_0^\infty f(\tau)\left[\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} e^{-\varphi(z)\tau}e^{tz}\,dz\right]\,d\tau.$$

This is the Schouten-Van der Pol theorem.

Step 3: If $G(z) = F(\sqrt{z})$ show that

$$g(t) = \frac{1}{2\sqrt{\pi t^3}}\int_0^\infty \tau f(\tau)\exp\left(-\frac{\tau^2}{4t}\right)\,d\tau.$$

Hint: Do not evaluate the contour integral. Instead, ask yourself: What function of time has a Laplace transform that equals $e^{-\varphi(z)\tau}$, where τ is a parameter? Then use tables.

Chapter 5
The Z-Transform

Since the Second World War, the rise of digital technology has resulted in a corresponding demand for designing and understanding discrete-time (data sampled) systems. These systems are governed by *difference equations* in which members of the sequence y_n are coupled to each other.

One source of difference equations is the numerical evaluation of integrals on a digital computer. Because we can only have values at discrete time points $t_k = kT$ for $k = 0, 1, 2, \ldots$, the value of the integral $y(t) = \int_0^t f(\tau)\,d\tau$ is

$$y(kT) = \int_0^{kT} f(\tau)\,d\tau = \int_0^{(k-1)T} f(\tau)\,d\tau + \int_{(k-1)T}^{kT} f(\tau)\,d\tau \qquad (5.0.1)$$

$$= y[(k-1)T] + \int_{(k-1)T}^{kT} f(\tau)\,d\tau \qquad (5.0.2)$$

$$= y[(k-1)T] + Tf(kT), \qquad (5.0.3)$$

because $\int_{(k-1)T}^{kT} f(\tau)\,d\tau \approx Tf(kT)$. Equation (5.0.3) is an example of a first-order difference equation because the numerical scheme couples the sequence value $y(kT)$ directly to the previous sequence value $y[(k-1)T]$. If (5.0.3) had contained $y[(k-2)T]$, then it would have been a second-order difference equation, and so forth.

Although we could use the conventional Laplace transform to solve these difference equations, the use of z-transforms can greatly facilitate the analysis, especially when we only desire responses at the sampling instants. Often the entire analysis can be done using only the transforms and the analyst does not actually find the sequence $y(kT)$.

In this chapter we shall first define the z-transform and discuss its properties. Then we will show how to find its inverse. Finally we shall use them to solve difference equations.

5.1 THE RELATIONSHIP OF THE Z-TRANSFORM TO THE LAPLACE TRANSFORM

Let $f(t)$ be a continuous function that an instrument samples every T units of time. We denote this data-sampled function by $f_S^*(t)$. See Figure 5.1.1. Taking ϵ, the duration of an individual sampling event, to be small, we may approximate the narrow-width pulse in Figure 5.1.1 by flat-topped pulses. Then $f_S^*(t)$ approximately equals

$$f_S^*(t) \approx \frac{1}{\epsilon} \sum_{n=0}^{\infty} f(nT) \left[H(t - nT + \epsilon/2) - H(t - nT - \epsilon/2)\right] \tag{5.1.1}$$

if $\epsilon \ll T$.

Clearly the presence of ϵ is troublesome in (5.1.1); it adds one more parameter to our problem. For this reason we introduce the concept of the *ideal sampler*, where the sampling time becomes infinitesimally small so that

$$f_S(t) = \lim_{\epsilon \to 0} \sum_{n=0}^{\infty} f(nT) \left[\frac{H(t - nT + \epsilon/2) - H(t - nT - \epsilon/2)}{\epsilon}\right] \tag{5.1.2}$$

$$= \sum_{n=0}^{\infty} f(nT)\delta(t - nT). \tag{5.1.3}$$

Let us now find the Laplace transform of this data-sampled function. We find from the linearity property of Laplace transforms that

$$F_S(s) = \mathcal{L}[f_S(t)] = \mathcal{L}\left[\sum_{n=0}^{\infty} f(nT)\delta(t - nT)\right] \tag{5.1.4}$$

$$= \sum_{n=0}^{\infty} f(nT)\mathcal{L}[\delta(t - nT)]. \tag{5.1.5}$$

Because $\mathcal{L}[\delta(t - nT)] = e^{-nsT}$, (5.1.5) simplifies to

$$F_S(s) = \sum_{n=0}^{\infty} f(nT)e^{-nsT}. \tag{5.1.6}$$

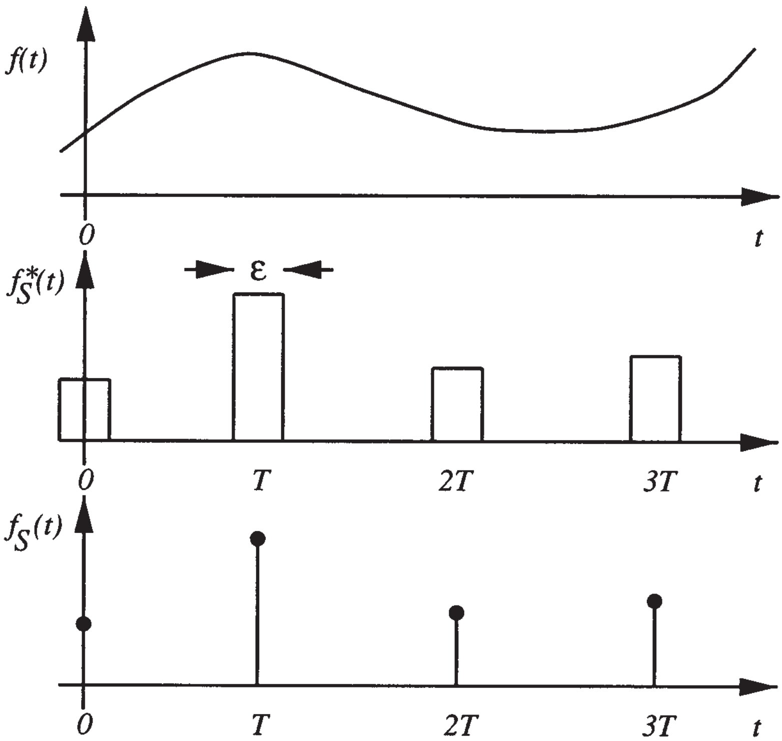


Figure 5.1.1: Schematic of how a continuous function $f(t)$ is sampled by a narrow-width pulse sampler $f_S^*(t)$ and an ideal sampler $f_S(t)$.

If we now make the substitution that $z = e^{sT}$, then $F_S(s)$ becomes

$$F(z) = \mathcal{Z}(f_n) = \sum_{n=0}^{\infty} f_n z^{-n}, \tag{5.1.7}$$

where $F(z)$ is the one-sided z-transform[1] of the sequence $f(nT)$, which we shall denote from now on by f_n. Here $\mathcal{Z}$ denotes the operation of taking the z-transform while $\mathcal{Z}^{-1}$ represents the inverse z-transformation. We will consider methods for finding the inverse z-transform in Section 5.3.

[1] The standard reference is Jury, E. I., 1964: *Theory and Application of the z-Transform Method,* John Wiley & Sons, New York.

Just as the Laplace transform was defined by an integration in t, the z-transform is defined by a power series (Laurent series) in z. Consequently, every z-transform has a region of convergence which must be implicitly understood if not explicitly stated. Furthermore, just as the Laplace integral diverged for certain functions, there are sequences where the associated power series will diverge and its z-transform does not exist.

Consider now the following examples of how to find the z-transform.

• Example 5.1.1

Given the unit sequence $f_n = 1$, $n \geq 0$, let us find $F(z)$. Substituting f_n into the definition of the z-transform leads to

$$F(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{z}{z-1}, \tag{5.1.8}$$

because $\sum_{n=0}^{\infty} z^{-n}$ is a complex-valued *geometric series* with common ratio z^{-1}. This series converges if $|z^{-1}| < 1$ or $|z| > 1$, which gives the region of convergence of $F(z)$.

• Example 5.1.2

Let us find the z-transform of the sequence

$$f_n = e^{-anT}, \qquad n \geq 0, \tag{5.1.9}$$

for a real and a imaginary.

For a real, substitution of the sequence into the definition of the z-transform yields

$$F(z) = \sum_{n=0}^{\infty} e^{-anT} z^{-n} = \sum_{n=0}^{\infty} \left(e^{-aT} z^{-1}\right)^n. \tag{5.1.10}$$

If $u = e^{-aT} z^{-1}$, then (5.1.10) is a geometric series so that

$$F(z) = \sum_{n=0}^{\infty} u^n = \frac{1}{1-u}. \tag{5.1.11}$$

Because $|u| = e^{-aT}|z^{-1}|$, the condition for convergence is that $|z| > e^{-aT}$. Thus,

$$F(z) = \frac{z}{z - e^{-aT}}, \qquad |z| > e^{-aT}. \tag{5.1.12}$$

For imaginary a, the infinite series in (5.1.10) converges if $|z| > 1$, because $|u| = |z^{-1}|$ when a is imaginary. Thus,

$$F(z) = \frac{z}{z - e^{-aT}}, \qquad |z| > 1. \tag{5.1.13}$$

Although the z-transforms in (5.1.12) and (5.1.13) are the same in these two cases, the corresponding regions of convergence are different. If a is a complex number, then

$$F(z) = \frac{z}{z - e^{-aT}}, \qquad |z| > |e^{-aT}|. \tag{5.1.14}$$

- **Example 5.1.3**

Let us find the z-transform of the sinusoidal sequence

$$f_n = \cos(n\omega T), \qquad n \geq 0. \tag{5.1.15}$$

Substituting (5.1.15) into the definition of the z-transform results in

$$F(z) = \sum_{n=0}^{\infty} \cos(n\omega T) z^{-n}. \tag{5.1.16}$$

From Euler's formula,

$$\cos(n\omega T) = \tfrac{1}{2}(e^{in\omega T} + e^{-in\omega T}), \tag{5.1.17}$$

so that (5.1.16) becomes

$$F(z) = \frac{1}{2} \sum_{n=0}^{\infty} \left(e^{in\omega T} z^{-n} + e^{-in\omega T} z^{-n} \right) \tag{5.1.18}$$

or

$$F(z) = \tfrac{1}{2}\left[\mathcal{Z}(e^{in\omega T}) + \mathcal{Z}(e^{-in\omega T})\right]. \tag{5.1.19}$$

From (5.1.13),

$$\mathcal{Z}(e^{\pm in\omega T}) = \frac{z}{z - e^{\pm i\omega T}}, \qquad |z| > 1. \tag{5.1.20}$$

Substituting (5.1.20) into (5.1.19) and simplifying yields

$$F(z) = \frac{z[z - \cos(\omega T)]}{z^2 - 2z\cos(\omega T) + 1}, \qquad |z| > 1. \tag{5.1.21}$$

Table 5.1.1: Z-Transforms of Some Commonly Used Sequences.

	f_n, $n \geq 0$	$F(z)$	Region of convergence
1.	$f_0 = k =$ const. $f_n = 0$, $n \geq 1$	k	$\|z\| > 0$
2.	$f_m = k =$ const. $f_n = 0$, all other n's	kz^{-m}	$\|z\| > 0$
3.	$k =$ constant	$kz/(z-1)$	$\|z\| > 1$
4.	kn	$kz/(z-1)^2$	$\|z\| > 1$
5.	kn^2	$kz(z+1)/(z-1)^3$	$\|z\| > 1$
6.	ke^{-anT}, a complex	$kz/\left(z - e^{-aT}\right)$	$\|z\| > \|e^{-aT}\|$
7.	kne^{-anT}, a complex	$\frac{kze^{-aT}}{(z-e^{-aT})^2}$	$\|z\| > \|e^{-aT}\|$
8.	$\sin(\omega_0 nT)$	$\frac{z\sin(\omega_0 T)}{z^2-2z\cos(\omega_0 T)+1}$	$\|z\| > 1$
9.	$\cos(\omega_0 nT)$	$\frac{z[z-\cos(\omega_0 T)]}{z^2-2z\cos(\omega_0 T)+1}$	$\|z\| > 1$
10.	$e^{-anT}\sin(\omega_0 nT)$	$\frac{ze^{-aT}\sin(\omega_0 T)}{z^2-2ze^{-aT}\cos(\omega_0 T)+e^{-2aT}}$	$\|z\| > e^{-aT}$
11.	$e^{-anT}\cos(\omega_0 nT)$	$\frac{ze^{-aT}[ze^{aT}-\cos(\omega_0 T)]}{z^2-2ze^{-aT}\cos(\omega_0 T)+e^{-2aT}}$	$\|z\| > e^{-aT}$
12.	α^n, α constant	$z/(z-\alpha)$	$\|z\| > \alpha$
13.	$n\alpha^n$	$\alpha z/(z-\alpha)^2$	$\|z\| > \alpha$
14.	$n^2\alpha^n$	$\alpha z(z+\alpha)/(z-\alpha)^3$	$\|z\| > \alpha$
15.	$\sinh(\omega_0 nT)$	$\frac{z\sinh(\omega_0 T)}{z^2-2z\cosh(\omega_0 T)+1}$	$\|z\| > \cosh(\omega_0 T)$
16.	$\cosh(\omega_0 nT)$	$\frac{z[z-\cosh(\omega_0 T)]}{z^2-2z\cosh(\omega_0 T)+1}$	$\|z\| > \sinh(\omega_0 T)$
17.	$a^n/n!$	$e^{a/z}$	$\|z\| > 0$
18.	$[\ln(a)]^n/n!$	$a^{1/z}$	$\|z\| > 0$

• Example 5.1.4

Let us find the z-transform for the sequence

$$f_n = \begin{cases} 1, & 0 \le n \le 5 \\ (\frac{1}{2})^n, & n \ge 6. \end{cases} \tag{5.1.22}$$

From the definition of the z-transform,

$$\mathcal{Z}(f_n) = F(z) = \sum_{n=0}^{5} z^{-n} + \sum_{n=6}^{\infty} \left(\frac{1}{2z}\right)^n. \tag{5.1.23}$$

Because

$$\sum_{n=0}^{N} q^n = \frac{1-q^{N+1}}{1-q}, \tag{5.1.24}$$

$$F(z) = \frac{1-z^{-6}}{1-z^{-1}} + \left(\frac{1}{2z}\right)^6 \sum_{m=0}^{\infty} \left(\frac{1}{2z}\right)^m \tag{5.1.25}$$

$$= \frac{z^6-1}{z^6-z^5} + \left(\frac{1}{2z}\right)^6 \frac{1}{1-\frac{1}{2z}} \tag{5.1.26}$$

$$= \frac{z^6-1}{z^6-z^5} + \frac{1}{(2z)^6-(2z)^5}, \tag{5.1.27}$$

if $n = m+6$ and $|z| > 1/2$. We summarize some of the more commonly encountered sequences and their transforms in Table 5.1.1 along with their regions of convergence.

• Example 5.1.5

In many engineering studies, the analysis is done entirely using transforms without actually finding any inverses. Consequently, it is useful to compare and contrast how various transforms behave in very simple test problems.

Consider the simple time function $f(t) = ae^{-at}H(t)$, $a > 0$. Its Laplace and Fourier transform are identical, namely $a/(a+i\omega)$, if we set $s = i\omega$. In Figure 5.1.2 we have illustrated its behavior as a function of positive ω.

Let us now generate the sequence of observations that we would measure if we sampled $f(t)$ every T units of time apart: $f_n = ae^{-anT}$. Taking the z-transform of this sequence, it equals $az/\left(z - e^{-aT}\right)$. Recalling that $z = e^{sT} = e^{i\omega T}$, we can also plot this transform as a function of positive ω. For small ω, the transforms agree, but as ω becomes larger they diverge markedly. Why does this occur?

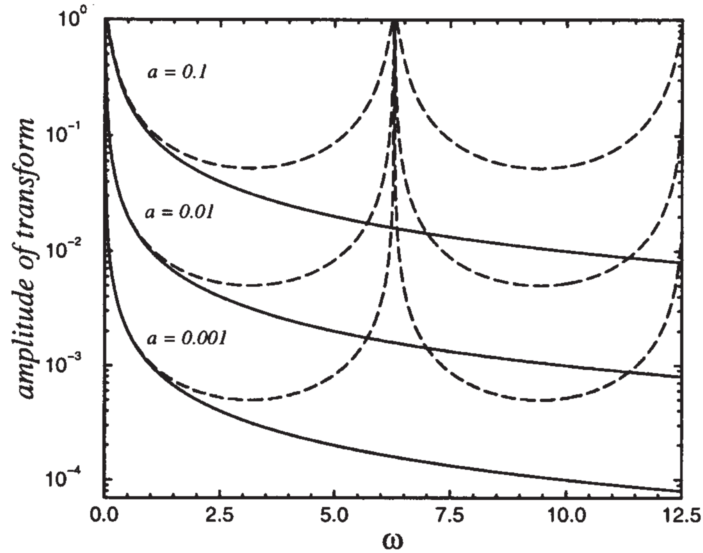


Figure 5.1.2: The amplitude of the Laplace or Fourier transform (solid line) for $ae^{-at}H(t)$ and the z-transform (dashed line) for $f_n = ae^{-anT}$ as a function of frequency ω for various positive a's and $T = 1$.

Recall that the z-transform is computed from a sequence comprised of samples from a continuous signal. One very important flaw in sampled data is the possible misrepresentation of high-frequency effects as lower-frequency phenomena. It is this *aliasing* or *folding* effect that we are observing here. Consequently, the z-transform of a sampled record can differ markedly from the corresponding Laplace or Fourier transforms of the continuous record at frequencies above one half of the sampling frequency. This also suggests that care should be exercised in interpolating between sampling instants. Indeed, in those applications where the output between sampling instants is very important, such as in a hybrid mixture of digital and analog systems, we must apply the so-called "modified z-transform".

Problems

From the fundamental definition of the z-transform, find the transform of the following sequences, where $n \geq 0$:

1. $f_n = \left(\frac{1}{2}\right)^n$

2. $f_n = e^{in\theta}$

3. $f_n = \begin{cases} 1, & 0 \leq n \leq 5 \\ 0, & n > 5 \end{cases}$

4. $f_n = \begin{cases} \left(\frac{1}{2}\right)^n, & n = 0, 1, \ldots, 10 \\ \left(\frac{1}{4}\right)^n, & n \geq 11 \end{cases}$

5. $f_n = \begin{cases} 0, & n = 0 \\ -1, & n = 1 \\ a^n, & n \geq 2 \end{cases}$

5.2 SOME USEFUL PROPERTIES

In principle we could construct any desired transform from the definition of the z-transform. However, there are several general theorems that are much more effective in finding new transforms.

Linearity

From the definition of the z-transform, it immediately follows that

$$\text{if} \quad h_n = c_1 f_n + c_2 g_n, \quad \text{then} \quad H(z) = c_1 F(z) + c_2 G(z), \tag{5.2.1}$$

where $F(z) = \mathcal{Z}(f_n)$, $G(z) = \mathcal{Z}(g_n)$, $H(z) = \mathcal{Z}(h_n)$, and c_1, c_2 are arbitrary constants.

Multiplication by an exponential sequence

$$\text{If} \quad g_n = e^{-anT} f_n, n \geq 0, \quad \text{then} \quad G(z) = F(ze^{aT}). \tag{5.2.2}$$

This follows from

$$G(z) = \mathcal{Z}(g_n) = \sum_{n=0}^{\infty} g_n z^{-n} = \sum_{n=0}^{\infty} e^{-anT} f_n z^{-n} \tag{5.2.3}$$

$$= \sum_{n=0}^{\infty} f_n (ze^{aT})^{-n} = F(ze^{aT}). \tag{5.2.4}$$

This is the z-transform analog to the first shifting theorem in Laplace transforms.

Table 5.2.1: Examples of Shifting Involving Sequences.

n	f_n	f_{n-2}	f_{n+2}
0	1	0	4
1	2	0	8
2	4	1	16
3	8	2	64
4	16	4	128
⋮	⋮	⋮	⋮

Shifting

The effect of shifting depends upon whether it is to the right or to the left, as Table 5.2.1 illustrates. For the sequence f_{n-2}, no values from the sequence f_n are lost; thus, we anticipate that the z-transform of f_{n-2} only involves $F(z)$. However, in forming the sequence f_{n+2}, the first two values of f_n are lost, and we anticipate that the z-transform of f_{n+2} cannot be expressed solely in terms of $F(z)$ but must include those two lost pieces of information.

Let us now confirm these conjectures by finding the z-transform of f_{n+1} which is a sequence that has been shifted one step to the left. From the definition of the z-transform, it follows that

$$\mathcal{Z}(f_{n+1}) = \sum_{n=0}^{\infty} f_{n+1} z^{-n} = z \sum_{n=0}^{\infty} f_{n+1} z^{-(n+1)} \tag{5.2.5}$$

$$= z \sum_{k=1}^{\infty} f_k z^{-k} - z f_0 + z f_0, \tag{5.2.6}$$

where we have added zero in (5.2.6). This algebraic trick allows us to collapse the first two terms on the right side of (5.2.6) to

$$\mathcal{Z}(f_{n+1}) = zF(z) - zf_0. \tag{5.2.7}$$

In a similar manner, repeated applications of (5.2.7) yield

$$\mathcal{Z}(f_{n+m}) = z^m F(z) - z^m f_0 - z^{m-1} f_1 - \ldots - z f_{m-1}, \tag{5.2.8}$$

where $m > 0$. This shifting operation transforms f_{n+m} into an algebraic expression involving m. Furthermore, we have introduced initial sequence values, just as we introduced initial conditions when we took the Laplace transform of the nth derivative of $f(t)$. We will make frequent use of this property in solving difference equations in Section 5.4.

Consider now shifting to the right by the positive integer k,

$$g_n = f_{n-k}H_{n-k}, \quad n \geq 0, \tag{5.2.9}$$

where $H_{n-k} = 0$ for $n < k$ and 1 for $n \geq k$. Then the z-transform of (5.2.9) is

$$G(z) = z^{-k}F(z), \tag{5.2.10}$$

where $G(z) = \mathcal{Z}(g_n)$ and $F(z) = \mathcal{Z}(f_n)$. This follows from

$$G(z) = \sum_{n=0}^{\infty} g_n z^{-n} = \sum_{n=0}^{\infty} f_{n-k}H_{n-k}z^{-n} \tag{5.2.11}$$

$$= z^{-k}\sum_{n=k}^{\infty} f_{n-k}z^{-(n-k)} = z^{-k}\sum_{m=0}^{\infty} f_m z^{-m} \tag{5.2.12}$$

$$= z^{-k}F(z). \tag{5.2.13}$$

This result is the z-transform analog to the second shifting theorem in Laplace transforms.

Initial-value theorem

The initial value of the sequence f_n, f_0, can be computed from $F(z)$ using the initial-value theorem:

$$f_0 = \lim_{z\to\infty} F(z). \tag{5.2.14}$$

From the definition of the z-transform,

$$F(z) = \sum_{n=0}^{\infty} f_n z^{-n} = f_0 + f_1 z^{-1} + f_2 z^{-2} + \ldots \tag{5.2.15}$$

In the limit of $z \to \infty$, we obtain the desired result.

Final-value theorem

The value of f_n, as $n \to \infty$, is given by the final-value theorem:

$$f_\infty = \lim_{z\to 1} (z-1)F(z), \tag{5.2.16}$$

where $F(z)$ is the z-transform of f_n.

We begin by noting that

$$\mathcal{Z}(f_{n+1} - f_n) = \lim_{n\to\infty} \sum_{k=0}^{n} (f_{k+1} - f_k)z^{-k}. \tag{5.2.17}$$

Using the shifting theorem on the left side of (5.2.17),

$$zF(z) - zf_0 - F(z) = \lim_{n\to\infty} \sum_{k=0}^{n} (f_{k+1} - f_k)z^{-k}. \tag{5.2.18}$$

Applying the limit as z approaches 1 to both sides of (5.2.18):

$$\lim_{z\to 1} (z-1)F(z) - f_0 = \lim_{n\to\infty} \sum_{k=0}^{n} (f_{k+1} - f_k) \tag{5.2.19}$$

$$= \lim_{n\to\infty} \big[(f_1 - f_0) + (f_2 - f_1) + \dots + (f_n - f_{n-1}) + (f_{n+1} - f_n) + \dots\big] \tag{5.2.20}$$

$$= \lim_{n\to\infty} (-f_0 + f_{n+1}) \tag{5.2.21}$$

$$= -f_0 + f_\infty. \tag{5.2.22}$$

Consequently,

$$f_\infty = \lim_{z\to 1} (z-1)F(z). \tag{5.2.23}$$

Note that this limit has meaning only if f_∞ exists. This occurs if $F(z)$ has no second-order or higher poles on the unit circle and no poles outside the unit circle.

Multiplication by n

Given

$$g_n = nf_n, \qquad n \ge 0, \tag{5.2.24}$$

this theorem states that

$$G(z) = -z\frac{dF(z)}{dz}, \tag{5.2.25}$$

where $G(z) = \mathcal{Z}(g_n)$ and $F(z) = \mathcal{Z}(f_n)$.

This follows from

$$G(z) = \sum_{n=0}^{\infty} g_n z^{-n} = \sum_{n=0}^{\infty} n f_n z^{-n} = z\sum_{n=0}^{\infty} n f_n z^{-n-1} = -z\frac{dF(z)}{dz}. \tag{5.2.26}$$

Periodic sequence theorem

Consider the N-periodic sequence:

$$f_n = \{\underbrace{f_0 f_1 f_2 \dots f_{N-1}}_{\text{first period}} f_0 f_1 \dots\} \tag{5.2.27}$$

and the related sequence:

$$x_n = \begin{cases} f_n, & 0 \le n \le N-1 \\ 0, & n \ge N. \end{cases} \tag{5.2.28}$$

This theorem allows us to find the z-transform of f_n if we can find the z-transform of x_n via the relationship

$$F(z) = \frac{X(z)}{1 - z^{-N}}, \qquad |z^N| > 1, \tag{5.2.29}$$

where $X(z) = \mathcal{Z}(x_n)$.

This follows from

$$F(z) = \sum_{n=0}^{\infty} f_n z^{-n} \tag{5.2.30}$$

$$= \sum_{n=0}^{N-1} x_n z^{-n} + \sum_{n=N}^{2N-1} x_{n-N} z^{-n} + \sum_{n=2N}^{3N-1} x_{n-2N} z^{-n} + \dots \tag{5.2.31}$$

Application of the shifting theorem in (5.2.31) leads to

$$F(z) = X(z) + z^{-N}X(z) + z^{-2N}X(z) + \dots \tag{5.2.32}$$

$$= X(z)\left[1 + z^{-N} + z^{-2N} + \dots\right]. \tag{5.2.33}$$

Equation (5.2.33) contains an infinite geometric series with common ratio z^{-N}, which converges if $|z^{-N}| < 1$. Thus,

$$F(z) = \frac{X(z)}{1 - z^{-N}}, \qquad |z^N| > 1. \tag{5.2.34}$$

Convolution

Given the sequences f_n and g_n, the convolution product of these two sequences is

$$w_n = f_n * g_n = \sum_{k=0}^{n} f_k g_{n-k} = \sum_{k=0}^{n} f_{n-k} g_k. \tag{5.2.35}$$

Given $F(z)$ and $G(z)$, we then have that $W(z) = F(z)G(z)$.

This follows from

$$W(z) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} f_k g_{n-k} \right] z^{-n} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f_k g_{n-k} z^{-n}, \tag{5.2.36}$$

because $g_{n-k} = 0$ for $k > n$. Reversing the order of summation and letting $m = n - k$,

$$W(z) = \sum_{k=0}^{\infty} \sum_{m=-k}^{\infty} f_k g_m z^{-(m+k)} \tag{5.2.37}$$

$$= \left[\sum_{k=0}^{\infty} f_k z^{-k} \right] \left[\sum_{m=0}^{\infty} g_m z^{-m} \right] = F(z)G(z). \tag{5.2.38}$$

Consider now the following examples of the properties discussed in this section.

• Example 5.2.1

From

$$\mathcal{Z}(a^n) = \frac{1}{1 - az^{-1}} \tag{5.2.39}$$

for $n \geq 0$ and $|z| < a$, we have that

$$\mathcal{Z}\left(e^{inx}\right) = \frac{1}{1 - e^{ix} z^{-1}} \tag{5.2.40}$$

and

$$\mathcal{Z}\left(e^{-inx}\right) = \frac{1}{1 - e^{-ix}z^{-1}}, \tag{5.2.41}$$

if $n \geq 0$ and $|z| < 1$. Therefore, the sequence $f_n = \cos(nx)$ has the z-transform

$$F(z) = \mathcal{Z}[\cos(nx)] = \tfrac{1}{2}\mathcal{Z}\left(e^{inx}\right) + \tfrac{1}{2}\mathcal{Z}\left(e^{-inx}\right) \tag{5.2.42}$$

$$= \frac{1}{2}\frac{1}{1 - e^{ix}z^{-1}} + \frac{1}{2}\frac{1}{1 - e^{-ix}z^{-1}} = \frac{1 - \cos(x)z^{-1}}{1 - 2\cos(x)z^{-1} + z^{-2}}. \tag{5.2.43}$$

- **Example 5.2.2**

Using the z-transform,

$$\mathcal{Z}\left(a^n\right) = \frac{1}{1 - az^{-1}}, \qquad n \geq 0, \tag{5.2.44}$$

we find that

$$\mathcal{Z}\left(na^n\right) = -z\frac{d}{dz}\left[\left(1 - az^{-1}\right)^{-1}\right] \tag{5.2.45}$$

$$= (-z)(-1)\left(1 - az^{-1}\right)^{-2}(-a)(-1)z^{-2} \tag{5.2.46}$$

$$= \frac{az^{-1}}{\left(1 - az^{-1}\right)^2} = \frac{az}{(z - a)^2}. \tag{5.2.47}$$

- **Example 5.2.3**

Consider $F(z) = 2az^{-1}/(1 - az^{-1})^3$, where $|a| < |z|$ and $|a| < 1$. Here we have that

$$f_0 = \lim_{z \to \infty} F(z) = \lim_{z \to \infty} \frac{2az^{-1}}{(1 - az^{-1})^3} = 0 \tag{5.2.48}$$

from the initial-value theorem. This agrees with the inverse of $X(z)$:

$$F(z) = \mathcal{Z}\left[n(n + 1)a^n\right], \qquad n \geq 0. \tag{5.2.49}$$

- **Example 5.2.4**

Given the z-transform $F(z) = (1 - a)z/[(z - 1)(z - a)]$, where $|z| > 1 > a > 0$, then from the final-value theorem we have that

$$\lim_{n \to \infty} f_n = \lim_{z \to 1}(z - 1)F(z) = \lim_{z \to 1} \frac{1 - a}{1 - az^{-1}} = 1. \tag{5.2.50}$$

This is consistent with the inverse transform $f_n = 1 - a^n$ with $n \geq 0$.

- **Example 5.2.5**

Using the sequences $f_n = 1$ and $g_n = a^n$, where a is real, verify the convolution theorem.

We first compute the convolution of f_n with g_n, namely

$$w_n = f_n * g_n = \sum_{k=0}^{n} a^k = \frac{1}{1-a} - \frac{a^{n+1}}{1-a}. \tag{5.2.51}$$

Taking the z-transform of w_n,

$$W(z) = \frac{z}{(1-a)(z-1)} - \frac{az}{(1-a)(z-a)} = \frac{z^2}{(z-1)(z-a)} = F(z)G(z) \tag{5.2.52}$$

and convolution theorem holds true for this special case.

Problems

Use the properties and Table 5.1.1 to find the z-transform of the following sequences:

1. $f_n = nTe^{-anT}$

2. $f_n = \begin{cases} 0, & n = 0 \\ na^{n-1}, & n \geq 1 \end{cases}$

3. $f_n = \begin{cases} 0, & n = 0 \\ n^2a^{n-1}, & n \geq 1 \end{cases}$

4. $f_n = a^n \cos(n)$

   [Use $\cos(n) = \frac{1}{2}(e^{in} + e^{-in})$]

5. $f_n = \cos(n-2)H_{n-2}$

6. $f_n = 3 + e^{-2nT}$

7. $f_n = \sin(n\omega_0 T + \theta)$

8. $f_n = \begin{cases} 0, & n = 0 \\ 1, & n = 1 \\ 2, & n = 2 \\ 1, & n = 3, \end{cases} \quad f_{n+4} = f_n$

9. $f_n = (-1)^n$

   (Hint: It's periodic.)

10. Using the property stated in (5.2.24)–(5.2.25) *twice*, find the z-transform of $n^2 = n[n(1)^n]$.

11. Verify the convolution theorem using the sequences $f_n = g_n = 1$.

12. Verify the convolution theorem using the sequences $f_n = 1$ and $g_n = n$.

13. Verify the convolution theorem using the sequences $f_n = g_n = 1/(n!)$. [Hint: Use the binomial theorem with $x = 1$ to evaluate the summation.]

14. If a is a real, show that $\mathcal{Z}(a^n f_n) = F(z/a)$, where $\mathcal{Z}(f_n) = F(z)$.

5.3 INVERSE Z-TRANSFORMS

In the previous two sections we have dealt with finding the z-transform. In this section we find f_n by inverting the z-transform $F(z)$. There are four methods for finding the inverse: (1) power series, (2) recursion, (3) partial fractions, and (4) the residue method. We will discuss each technique individually. The first three apply only to those $F(z)$'s that are *rational* functions while the residue method is more general.

Power series

By means of the long-division process, we can always rewrite $F(z)$ as the Laurent expansion:

$$F(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots \tag{5.3.1}$$

From the definition of the z-transform

$$F(z) = \sum_{n=0}^{\infty} f_n z^{-n} = f_0 + f_1 z^{-1} + f_2 z^{-2} + \dots, \tag{5.3.2}$$

the desired sequence f_n is given by a_n.

- **Example 5.3.1**

Let

$$F(z) = \frac{z+1}{2z-2} = \frac{N(z)}{D(z)}. \tag{5.3.3}$$

Using long division, $N(z)$ is divided by $D(z)$ and we obtain

$$F(z) = \tfrac{1}{2} + z^{-1} + z^{-2} + z^{-3} + z^{-4} + \dots \tag{5.3.4}$$

Therefore,

$$a_0 = \tfrac{1}{2},\ a_1 = 1,\ a_2 = 1,\ a_3 = 1,\ a_4 = 1,\ \text{etc.} \tag{5.3.5}$$

which suggests that $f_0 = \frac{1}{2}$ and $f_n = 1$ for $n \geq 1$ is the inverse of $F(z)$.

• **Example 5.3.2**

Let us find the inverse of the z-transform:

$$F(z) = \frac{2z^2 - 1.5z}{z^2 - 1.5z + 0.5}. \tag{5.3.6}$$

By the long-division process, we have that

$$\begin{array}{r|lllll}
 & 2 & +\,1.5z^{-1} & +\,1.25z^{-2} & +\,1.125z^{-3} & +\cdots \\ \hline
z^2 - 1.5z + 0.5 & 2z^2 - 1.5z & & & & \\
 & 2z^2 - 3z & +\,1 & & & \\ \cline{2-3}
 & & 1.5z - 1 & & & \\
 & & 1.5z - 2.25 & +\,0.75z^{-1} & & \\ \cline{3-4}
 & & & 1.25 - 0.75z^{-1} & & \\
 & & & 1.25 - 1.87z^{-1} & +\cdots & \\ \cline{4-5}
 & & & & 1.125z^{-1} +\cdots &
\end{array}$$

Thus, $f_0 = 2$, $f_1 = 1.5$, $f_2 = 1.25$, $f_3 = 1.125$, and so forth, or $f_n = 1 + (\frac{1}{2})^n$. In general, this technique only produces numerical values for some of the elements of the sequence. Note also that our long division must always yield the power series (5.3.1) in order for this method to be of any use.

Recursive method

An alternative to long division was suggested[2] several years ago. It obtains the inverse recursively.

We begin by assuming that the z-transform is of the form

$$F(z) = \frac{a_0 z^m + a_1 z^{m-1} + a_2 z^{m-2} + \cdots + a_{m-1} z + a_m}{b_0 z^m + b_1 z^{m-1} + b_2 z^{m-2} + \cdots + b_{m-1} z + b_m}, \tag{5.3.7}$$

where some of the coefficients a_i and b_i may be zero and $b_0 \neq 0$. Applying the final-value theorem,

$$f_0 = \lim_{z \to \infty} F(z) = a_0/b_0. \tag{5.3.8}$$

---

[2] Jury, E. I., 1964: *Theory and Application of the z-Transform Method*, John Wiley & Sons, New York, p. 41; Pierre, D. A., 1963: A tabular algorithm for z-transform inversion. *Control Eng.*, **10(9)**, 110–111. The present derivation is by Jenkins, L. B., 1967: A useful recursive form for obtaining inverse z-transforms. *Proc. IEEE*, **55**, 574–575. ©IEEE.

Next, we apply the final-value theorem to $z[F(z) - f_0]$ and find that

$$f_1 = \lim_{z\to\infty} z[F(z) - f_0] \tag{5.3.9}$$

$$= \lim_{z\to\infty} z\frac{(a_0 - b_0 f_0)z^m + (a_1 - b_1 f_0)z^{m-1} + \cdots + (a_m - b_m f_0)}{b_0 z^m + b_1 z^{m-1} + b_2 z^{m-2} + \cdots + b_{m-1} z + b_m} \tag{5.3.10}$$

$$= (a_1 - b_1 f_0)/b_0. \tag{5.3.11}$$

Note that the coefficient $a_0 - b_0 f_0 = 0$ from (5.3.8). Similarly,

$$f_2 = \lim_{z\to\infty} z[zF(z) - zf_0 - f_1] \tag{5.3.12}$$

$$= \lim_{z\to\infty} z\frac{(a_0-b_0f_0)z^{m+1}+(a_1-b_1f_0-b_0f_1)z^m + (a_2-b_2f_0-b_1f_1)z^{m-1}+\cdots-b_mf_1}{b_0 z^m + b_1 z^{m-1} + b_2 z^{m-2} + \cdots + b_{m-1} z + b_m} \tag{5.3.13}$$

$$= (a_2 - b_2 f_0 - b_1 f_1)/b_0 \tag{5.3.14}$$

because $a_0 - b_0 f_0 = a_1 - b_1 f_0 - f_1 b_0 = 0$. Continuing this process, we finally have that

$$f_n = (a_n - b_n f_0 - b_{n-1} f_1 - \cdots - b_1 f_{n-1})/b_0, \tag{5.3.15}$$

where $a_n = b_n \equiv 0$ for $n > m$.

• Example 5.3.3

Let us redo Example 5.3.2 using the recursive method. Comparing (5.3.7) to (5.3.6), $a_0 = 2$, $a_1 = -1.5$, $a_2 = 0$, $b_0 = 1$, $b_1 = -1.5$, $b_2 = 0.5$ and $a_n = b_n = 0$ if $n \geq 3$. From (5.3.15),

$$f_0 = a_0/b_0 = 2/1 = 2, \tag{5.3.16}$$

$$f_1 = (a_1 - b_1 f_0)/b_0 = [-1.5 - (-1.5)(2)]/1 = 1.5, \tag{5.3.17}$$

$$f_2 = (a_2 - b_2 f_0 - b_1 f_1)/b_0 \tag{5.3.18}$$

$$= [0 - (0.5)(2) - (-1.5)(1.5)]/1 = 1.25 \tag{5.3.19}$$

and

$$f_3 = (a_3 - b_3 f_0 - b_2 f_1 - b_1 f_2)/b_0 \tag{5.3.20}$$

$$= [0 - (0)(2) - (0.5)(1.5) - (-1.5)(1.25)]/1 = 1.125. \tag{5.3.21}$$

Partial fraction expansion

One of the popular methods for inverting Laplace transforms is partial fractions. A similar, but slightly different scheme works here.

• Example 5.3.4

Given $F(z) = z/\left(z^2 - 1\right)$, let us find f_n. The first step is to obtain the partial fraction expansion of $F(z)/z$. Why we want $F(z)/z$ rather than $F(z)$ will be made clear in a moment. Thus,

$$\frac{F(z)}{z} = \frac{1}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1}, \tag{5.3.22}$$

where

$$A = (z-1)\left.\frac{F(z)}{z}\right|_{z=1} = \frac{1}{2} \tag{5.3.23}$$

and

$$B = (z+1)\left.\frac{F(z)}{z}\right|_{z=-1} = -\frac{1}{2}. \tag{5.3.24}$$

Multiplying (5.3.22) by z,

$$F(z) = \frac{1}{2}\left(\frac{z}{z-1} - \frac{z}{z+1}\right). \tag{5.3.25}$$

Next, we find the inverse z-transform of each of the terms $z/(z-1)$ and $z/(z+1)$ in Table 5.1.1. This yields

$$\mathcal{Z}^{-1}\left(\frac{z}{z-1}\right) = 1 \quad \text{and} \quad \mathcal{Z}^{-1}\left(\frac{z}{z+1}\right) = (-1)^n. \tag{5.3.26}$$

Thus, the inverse is

$$f_n = \tfrac{1}{2}\left[1 - (-1)^n\right], \; n \geq 0. \tag{5.3.27}$$

From this example it is clear that there are two steps involved: (1) obtain the partial fraction expansion of $F(z)/z$, and (2) finding the inverse z-transform by referring to Table 5.1.1.

• Example 5.3.5

Given $F(z) = 2z^2/[(z+2)(z+1)^2]$, let us find f_n. We begin by expanding $F(z)/z$ as

$$\frac{F(z)}{z} = \frac{2z}{(z+2)(z+1)^2} = \frac{A}{z+2} + \frac{B}{z+1} + \frac{C}{(z+1)^2}, \tag{5.3.28}$$

where

$$A = (z+2)\,\frac{F(z)}{z}\bigg|_{z=-2} = -4, \tag{5.3.29}$$

$$B = \frac{d}{dz}\left[(z+1)^2\frac{F(z)}{z}\right]\bigg|_{z=-1} = 4 \tag{5.3.30}$$

and

$$C = (z+1)^2\,\frac{F(z)}{z}\bigg|_{z=-1} = -2 \tag{5.3.31}$$

so that

$$F(z) = \frac{4z}{z+1} - \frac{4z}{z+2} - \frac{2z}{(z+1)^2} \tag{5.3.32}$$

or

$$f_n = \mathcal{Z}^{-1}\left[\frac{4z}{z+1}\right] - \mathcal{Z}^{-1}\left[\frac{4z}{z+2}\right] - \mathcal{Z}^{-1}\left[\frac{2z}{(z+1)^2}\right]. \tag{5.3.33}$$

From Table 5.1.1,

$$\mathcal{Z}^{-1}\left(\frac{z}{z+1}\right) = (-1)^n, \tag{5.3.34}$$

$$\mathcal{Z}^{-1}\left(\frac{z}{z+2}\right) = (-2)^n \tag{5.3.35}$$

and

$$\mathcal{Z}^{-1}\left[\frac{z}{(z+1)^2}\right] = -\ \mathcal{Z}^{-1}\left[\frac{-z}{(z+1)^2}\right] = -n(-1)^n = n(-1)^{n+1}. \tag{5.3.36}$$

Applying (5.3.34)–(5.3.36) to (5.3.33),

$$f_n = 4(-1)^n - 4(-2)^n + 2n(-1)^n,\ n \geq 0. \tag{5.3.37}$$

• Example 5.3.6

Given $F(z) = (z^2+z)/(z-2)^2$, let us determine f_n. Because

$$\frac{F(z)}{z} = \frac{z+1}{(z-2)^2} = \frac{1}{z-2} + \frac{3}{(z-2)^2}, \tag{5.3.38}$$

$$f_n = \mathcal{Z}^{-1}\left[\frac{z}{z-2}\right] + \mathcal{Z}^{-1}\left[\frac{3z}{(z-2)^2}\right]. \tag{5.3.39}$$

Referring to Table 5.1.1,

$$\mathcal{Z}^{-1}\left(\frac{z}{z-2}\right) = 2^n \quad\text{and}\quad \mathcal{Z}^{-1}\left[\frac{3z}{(z-2)^2}\right] = \tfrac{3}{2}n2^n. \tag{5.3.40}$$

Substituting (5.3.40) into (5.3.39) yields

$$f_n = \left(\tfrac{3}{2}n + 1\right) 2^n, \quad n \geq 0. \tag{5.3.41}$$

Residue method

The power series, recursive, and partial fraction expansion methods are rather limited. We will now prove that f_n may be computed from the following *inverse integral formula*:

$$f_n = \frac{1}{2\pi i} \oint_C z^{n-1} F(z)\, dz, \quad n \geq 0, \tag{5.3.42}$$

where C is any simple curve, taken in the positive sense, that encloses all of the singularities of $F(z)$. It is readily shown that the power series and partial fraction methods are *special cases* of the residue method.

Proof: Starting with the definition of the z-transform

$$F(z) = \sum_{n=0}^{\infty} f_n z^{-n}, \qquad |z| > R_1, \tag{5.3.43}$$

we multiply (5.3.43) by z^{n-1} and integrating both sides around any contour C which includes all of the singularities,

$$\frac{1}{2\pi i} \oint_C z^{n-1} F(z)\, dz = \sum_{m=0}^{\infty} f_m \frac{1}{2\pi i} \oint_C z^{n-m} \frac{dz}{z}. \tag{5.3.44}$$

Let C be a circle of radius R, where $R > R_1$. Then, changing variables to $z = R\, e^{i\theta}$ and $dz = iz\, d\theta$,

$$\frac{1}{2\pi i} \oint_C z^{n-m} \frac{dz}{z} = \frac{R^{n-m}}{2\pi} \int_0^{2\pi} e^{i(n-m)\theta}\, d\theta = \begin{cases} 1, & m = n \\ 0, & \text{otherwise.} \end{cases} \tag{5.3.45}$$

Substituting (5.3.45) into (5.3.44) yields the desired result that

$$\frac{1}{2\pi i} \oint_C z^{n-1} F(z)\, dz = f_n. \tag{5.3.46}$$

□

We can easily evaluate the inversion integral (5.3.42) using Cauchy's residue theorem.

• **Example 5.3.7**

Let us find the inverse z-transform of

$$F(z) = \frac{1}{(z-1)(z-2)}. \tag{5.3.47}$$

From the inversion integral,

$$f_n = \frac{1}{2\pi i} \oint_C \frac{z^{n-1}}{(z-1)(z-2)}\, dz. \tag{5.3.48}$$

Clearly the integral has simple poles at $z = 1$ and $z = 2$. However, when $n = 0$ we also have a simple pole at $z = 0$. Thus the cases $n = 0$ and $n > 0$ must be considered separately.

Case 1: $n = 0$. The residue theorem yields

$$f_0 = \text{Res}\left[\frac{1}{z(z-1)(z-2)}; 0\right] + \text{Res}\left[\frac{1}{z(z-1)(z-2)}; 1\right] + \text{Res}\left[\frac{1}{z(z-1)(z-2)}; 2\right]. \tag{5.3.49}$$

Evaluating these residues,

$$\text{Res}\left[\frac{1}{z(z-1)(z-2)}; 0\right] = \left.\frac{1}{(z-1)(z-2)}\right|_{z=0} = \frac{1}{2}, \tag{5.3.50}$$

$$\text{Res}\left[\frac{1}{z(z-1)(z-2)}; 1\right] = \left.\frac{1}{z(z-2)}\right|_{z=1} = -1 \tag{5.3.51}$$

and

$$\text{Res}\left[\frac{1}{z(z-1)(z-2)}; 2\right] = \left.\frac{1}{z(z-1)}\right|_{z=2} = \frac{1}{2}. \tag{5.3.52}$$

Substituting (5.3.50)–(5.3.52) into (5.3.49) yields $f_0 = 0$.

Case 2: $n > 0$. Here we only have contributions from $z = 1$ and $z = 2$.

$$f_n = \text{Res}\left[\frac{z^{n-1}}{(z-1)(z-2)}; 1\right] + \text{Res}\left[\frac{z^{n-1}}{(z-1)(z-2)}; 2\right], n > 0, \tag{5.3.53}$$

where

$$\text{Res}\left[\frac{z^{n-1}}{(z-1)(z-2)}; 1\right] = \left.\frac{z^{n-1}}{z-2}\right|_{z=1} = -1 \tag{5.3.54}$$

and

$$\operatorname{Res}\left[\frac{z^{n-1}}{(z-1)(z-2)};2\right] = \left.\frac{z^{n-1}}{z-1}\right|_{z=2} = 2^{n-1}, \quad n > 0. \tag{5.3.55}$$

Thus,

$$f_n = 2^{n-1} - 1, \quad n > 0. \tag{5.3.56}$$

Combining our results,

$$f_n = \begin{cases} 0, & n = 0 \\ \frac{1}{2}\left(2^n - 2\right), & n > 0. \end{cases} \tag{5.3.57}$$

• Example 5.3.8

Let us use the inversion integral to find the inverse of

$$F(z) = \frac{z^2 + 2z}{(z-1)^2}. \tag{5.3.58}$$

The inversion theorem gives

$$f_n = \frac{1}{2\pi i}\oint_C \frac{z^{n+1} + 2z^n}{(z-1)^2}\,dz = \operatorname{Res}\left[\frac{z^{n+1} + 2z^n}{(z-1)^2};1\right], \tag{5.3.59}$$

where the pole at $z = 1$ is second order. Consequently, the corresponding residue is

$$\operatorname{Res}\left[\frac{z^{n+1} + 2z^n}{(z-1)^2};1\right] = \left.\frac{d}{dz}\left(z^{n+1} + 2z^n\right)\right|_{z=1} = 3n + 1. \tag{5.3.60}$$

Thus, the inverse z-transform of (5.3.58) is

$$f_n = 3n + 1, \quad n \geq 0. \tag{5.3.61}$$

• Example 5.3.9

Let $F(z)$ be a z-transform whose poles lie within the unit circle $|z| = 1$. Then

$$F(z) = \sum_{n=0}^{\infty} f_n z^{-n}, \qquad |z| > 1 \tag{5.3.62}$$

and

$$F(z)F(z^{-1}) = \sum_{n=0}^{\infty} f_n^2 + \sum_{\substack{n=0 \\ n \neq m}}^{\infty} \sum_{m=0}^{\infty} f_m f_n z^{m-n}. \tag{5.3.63}$$

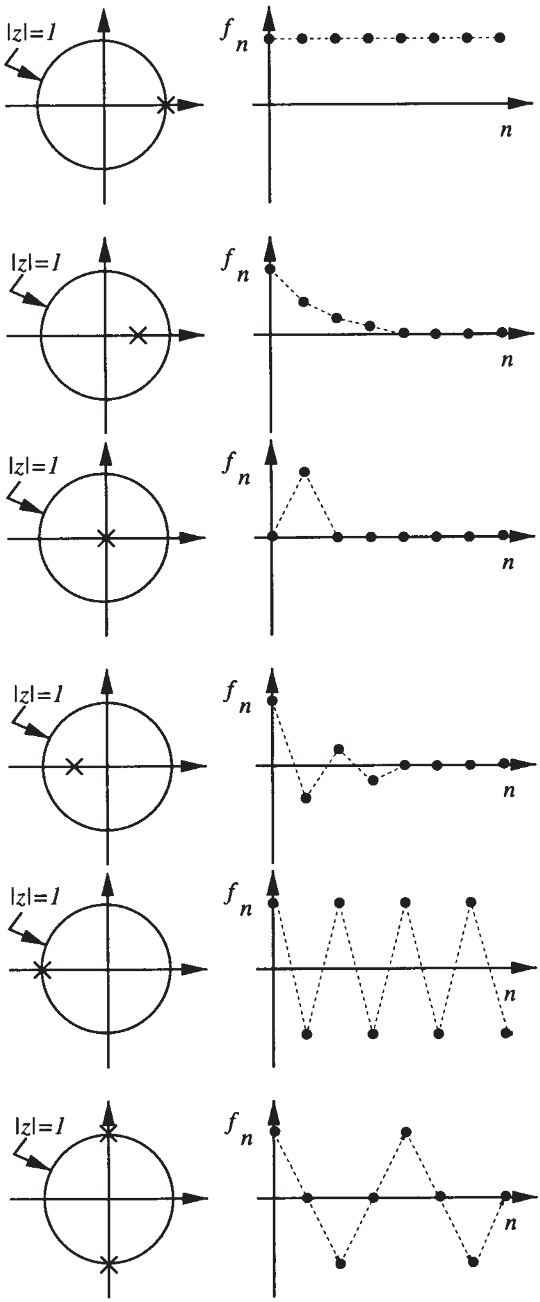


Figure 5.3.1: The correspondence between the location of the simple poles of the z-transform $F(z)$ and the behavior of f_n.

We now multiply both sides of (5.3.63) by z^{-1} and integrate around the unit circle C. Therefore,

$$\oint_{|z|=1} F(z)F(z^{-1})z^{-1}\,dz = \sum_{n=0}^{\infty} \oint_{|z|=1} f_n^2 z^{-1}\,dz + \sum_{\substack{n=0\\ n\neq m}}^{\infty} \sum_{m=0}^{\infty} f_m f_n \oint_{|z|=1} z^{m-n-1}\,dz \quad \textbf{(5.3.64)}$$

after interchanging the order of integration and summation. Performing the integration,

$$\sum_{n=0}^{\infty} f_n^2 = \frac{1}{2\pi i} \oint_{|z|=1} F(z)F(z^{-1})z^{-1}\,dz, \quad \textbf{(5.3.65)}$$

which is *Parseval's theorem* for one-sided z-transforms. Recall that there are similar theorems for Fourier series and transforms.

• Example 5.3.10

An additional benefit of understanding inversion by the residue method is the ability to *qualitatively* anticipate the inverse by knowing the location of the poles of $F(z)$. This intuition is important because many engineering analyses discuss stability and performance entirely in terms of the properties of the system's z-transform. In Figure 5.3.1 we have graphed the location of the poles of $F(z)$ and the corresponding f_n. The student should go through the mental exercise of connecting the two pictures.

Problems

Use the power series or recursive method to compute the first few f_n's of the following z-transforms:

1. $F(z) = \dfrac{0.09z^2 + 0.9z + 0.09}{12.6z^2 - 24z + 11.4}$

2. $F(z) = \dfrac{z+1}{2z^4 - 2z^3 + 2z - 2}$

3. $F(z) = \dfrac{1.5z^2 + 1.5z}{15.25z^2 - 36.75z + 30.75}$

4. $F(z) = \dfrac{6z^2 + 6z}{19z^3 - 33z^2 + 21z - 7}$

Use partial fractions to find the inverse of the following z-transforms:

5. $F(z) = \dfrac{z(z+1)}{(z-1)(z^2 - z + 1/4)}$

6. $F(z) = \dfrac{(1 - e^{-aT})z}{(z-1)(z - e^{-aT})}$

7. $F(z) = \dfrac{z^2}{(z-1)(z-\alpha)}$

8. $F(z) = \dfrac{(2z - a - b)z}{(z-a)(z-b)}$

9. Using the property that the z-transform of $g_n = f_{n-k}H_{n-k}$ if $n \geq 0$ is $G(z) = z^{-k}F(z)$, find the inverse of

$$F(z) = \frac{z+1}{z^{10}(z-1/2)}.$$

Use the residue method to find the inverse z-transform of the following z-transforms:

10. $F(z) = \dfrac{z^2+3z}{(z-1/2)^3}$

11. $F(z) = \dfrac{z}{(z+1)^2(z-2)}$

12. $F(z) = \dfrac{z}{(z+1)^2(z-1)^2}$

13. $F(z) = e^{a/z}$

5.4 SOLUTION OF DIFFERENCE EQUATIONS

Having reached the point where we can take a z-transform and then find its inverse, we are ready to use it to solve difference equations. The procedure parallels that of solving ordinary differential equations by Laplace transforms. Essentially we reduce the difference equation to an algebraic problem. We then find the solution by inverting $Y(z)$.

• Example 5.4.1

Let us solve the second-order difference equation

$$2y_{n+2} - 3y_{n+1} + y_n = 5\ 3^n, \quad n \geq 0, \tag{5.4.1}$$

where $y_0 = 0$ and $y_1 = 1$.

Taking the z-transform of both sides of (5.4.1), we obtain

$$2\mathcal{Z}(y_{n+2}) - 3\mathcal{Z}(y_{n+1}) + \mathcal{Z}(y_n) = 5\,\mathcal{Z}(3^n). \tag{5.4.2}$$

From the shifting theorem and Table 5.1.1,

$$2z^2Y(z) - 2z^2y_0 - 2zy_1 - 3[zY(z) - zy_0] + Y(z) = \frac{5z}{z-3}. \tag{5.4.3}$$

Substituting $y_0 = 0$ and $y_1 = 1$ into (5.4.3) and simplifying yields

$$(2z-1)(z-1)Y(z) = \frac{z(2z-1)}{z-3}. \tag{5.4.4}$$

or

$$Y(z) = \frac{z}{(z-3)(z-1)}. \tag{5.4.5}$$

To obtain y_n from $Y(z)$ we can employ partial fractions or the residue method. Applying partial fractions yields

$$\frac{Y(z)}{z} = \frac{A}{z-1} + \frac{B}{z-3}, \tag{5.4.6}$$

where

$$A = (z-1)\left.\frac{Y(z)}{z}\right|_{z=1} = -\frac{1}{2} \tag{5.4.7}$$

and

$$B = (z-3)\left.\frac{Y(z)}{z}\right|_{z=3} = \frac{1}{2}. \tag{5.4.8}$$

Thus,

$$Y(z) = -\frac{1}{2}\frac{z}{z-1} + \frac{1}{2}\frac{z}{z-3} \tag{5.4.9}$$

or

$$y_n = -\frac{1}{2}\mathcal{Z}^{-1}\left(\frac{z}{z-1}\right) + \frac{1}{2}\mathcal{Z}^{-1}\left(\frac{z}{z-3}\right). \tag{5.4.10}$$

From (5.4.10) and Table 5.1.1,

$$y_n = \tfrac{1}{2}\left(3^n - 1\right), \qquad n \ge 0. \tag{5.4.11}$$

Two checks confirm that we have the *correct* solution. First, our solution must satisfy the initial values of the sequence. Computing y_0 and y_1,

$$y_0 = \tfrac{1}{2}(3^0 - 1) = \tfrac{1}{2}(1-1) = 0 \tag{5.4.12}$$

and

$$y_1 = \tfrac{1}{2}(3^1 - 1) = \tfrac{1}{2}(3-1) = 1. \tag{5.4.13}$$

Thus, our solution gives the correct initial values.

Our sequence y_n must also satisfy the difference equation. Now

$$y_{n+2} = \tfrac{1}{2}(3^{n+2} - 1) = \tfrac{1}{2}(9\ 3^n - 1) \tag{5.4.14}$$

and

$$y_{n+1} = \tfrac{1}{2}(3^{n+1} - 1) = \tfrac{1}{2}(3\ 3^n - 1). \tag{5.4.15}$$

Therefore,

$$2y_{n+2} - 3y_{n+1} + y_n = \left(9 - \tfrac{9}{2} + \tfrac{1}{2}\right)3^n - 1 + \tfrac{3}{2} - \tfrac{1}{2} = 5\ 3^n \tag{5.4.16}$$

and our solution is correct.

Finally, we note that the term $3^n/2$ is necessary to give the right side of (5.4.1); it is the particular solution. The $-1/2$ term is necessary so that the sequence satisfies the initial values; it is the complementary solution.

• Example 5.4.2

Let us find the y_n in the difference equation

$$y_{n+2} - 2y_{n+1} + y_n = 1, \quad n \geq 0 \tag{5.4.17}$$

with the initial conditions $y_0 = 0$ and $y_1 = 3/2$.

From (5.4.17),

$$\mathcal{Z}(y_{n+2}) - 2\mathcal{Z}(y_{n+1}) + \mathcal{Z}(y_n) = \mathcal{Z}(1). \tag{5.4.18}$$

The z-transform of the left side of (5.4.18) is obtained from the shifting theorem and Table 5.1.1 yields $\mathcal{Z}(1)$. Thus,

$$z^2Y(z) - z^2y_0 - zy_1 - 2zY(z) + 2zy_0 + Y(z) = \frac{z}{z-1}. \tag{5.4.19}$$

Substituting $y_0 = 0$ and $y_1 = 3/2$ in (5.4.19) and simplifying yields

$$Y(z) = \frac{3z^2 - z}{2(z-1)^3} \tag{5.4.20}$$

or

$$y_n = \mathcal{Z}^{-1}\left[\frac{3z^2 - z}{2(z-1)^3}\right]. \tag{5.4.21}$$

We find the inverse z-transform of (5.4.21) by the residue method or

$$y_n = \frac{1}{2\pi i}\oint_C \frac{3z^{n+1} - z^n}{2(z-1)^3}\,dz = \frac{1}{2!}\frac{d^2}{dz^2}\left[\frac{3z^{n+1}}{2} - \frac{z^n}{2}\right]\bigg|_{z=1} \tag{5.4.22}$$

$$= \tfrac{1}{2}n^2 + n. \tag{5.4.23}$$

Thus,

$$y_n = \tfrac{1}{2}n^2 + n, \quad n \geq 0. \tag{5.4.24}$$

Note that $n^2/2$ gives the particular solution to (5.4.17), while n is there so that y_n satisfies the initial conditions. This problem is particularly interesting because our constant forcing produces a response that grows as n^2, just as in the case of resonance in a time-continuous system when a finite forcing such as $\sin(\omega_0 t)$ results in a response whose amplitude grows as t^m.

• **Example 5.4.3**

Let us solve the difference equation

$$b^2 y_n + y_{n+2} = 0, \tag{5.4.25}$$

where $|b| < 1$ and the initial conditions are $y_0 = b^2$ and $y_1 = 0$.

We begin by taking the z-transform of each term in (5.4.25). This yields

$$b^2 \mathcal{Z}(y_n) + \mathcal{Z}(y_{n+2}) = 0. \tag{5.4.26}$$

From the shifting theorem, it follows that

$$b^2 Y(z) + z^2 Y(z) - z^2 y_0 - z y_1 = 0. \tag{5.4.27}$$

Substituting $y_0 = b^2$ and $y_1 = 0$ into (5.4.27),

$$b^2 Y(z) + z^2 Y(z) - b^2 z^2 = 0 \tag{5.4.28}$$

or

$$Y(z) = \frac{b^2 z^2}{z^2 + b^2}. \tag{5.4.29}$$

To find y_n we employ the residue method or

$$y_n = \frac{1}{2\pi i} \oint_C \frac{b^2 z^{n+1}}{(z - ib)(z + ib)}\, dz. \tag{5.4.30}$$

Thus,

$$y_n = \left.\frac{b^2 z^{n+1}}{z + ib}\right|_{z=ib} + \left.\frac{b^2 z^{n+1}}{z - ib}\right|_{z=-ib} = \frac{b^{n+2} i^n}{2} + \frac{b^{n+2}(-i)^n}{2} \tag{5.4.31}$$

$$= \frac{b^{n+2} e^{in\pi/2}}{2} + \frac{b^{n+2} e^{-in\pi/2}}{2} = b^{n+2} \cos\left(\frac{n\pi}{2}\right), \tag{5.4.32}$$

because $\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$. Consequently, we obtain the desired result that

$$y_n = b^{n+2} \cos\left(\frac{n\pi}{2}\right) \quad \text{for } n \geq 0. \tag{5.4.33}$$

• **Example 5.4.4: Compound interest**

Finite difference equations arise in finance because the increase or decrease in an account occurs in discrete steps. For example, the amount of money in a compound interest saving account after $n + 1$ conversion periods (the time period between interest payments) is

$$y_{n+1} = y_n + r y_n, \tag{5.4.34}$$

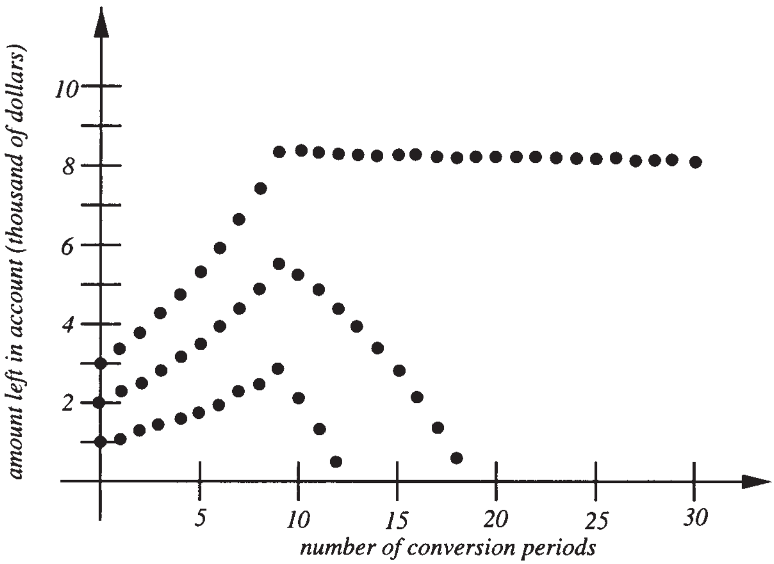


Figure 5.4.1: The amount in a saving account as a function of an annual conversion period when interest is compounded at the annual rate of 12% and a \$1000 is taken from the account every period starting with period 10.

where r is the interest rate per conversion period. The second term on the right side of (5.4.34) is the amount of interest paid at the end of each period.

Let us ask a somewhat more difficult question of how much money we will have if we withdraw the amount A at the end of every period starting after the period ℓ. Now the difference equation becomes

$$y_{n+1} = y_n + ry_n - AH_{n-\ell-1}. \tag{5.4.35}$$

Taking the z-transform of (5.4.35),

$$zY(z) - zy_0 = (1+r)Y(z) - \frac{Az^{2-\ell}}{z-1} \tag{5.4.36}$$

after using (5.2.10) or

$$Y(z) = \frac{y_0 z}{z-(1+r)} - \frac{Az^{2-\ell}}{(z-1)[z-(1+r)]}. \tag{5.4.37}$$

Taking the inverse of (5.4.37),

$$y_n = y_0(1+r)^n - \frac{A}{r}\left[(1+r)^{n-\ell+1} - 1\right] H_{n-\ell}. \tag{5.4.38}$$

The first term in (5.4.38) represents the growth of money by compound interest while the second term gives the depletion of the account by withdrawals. Figure 5.4.1 gives the values of y_n for various starting amounts assuming an annual conversion period with $r = 0.12$, $\ell = 10$ years, and $A = \$1000$. It shows that if an investor places an initial amount of \$3000 in an account bearing 12% annually, after 10 years he can withdraw \$1000 annually, essentially forever. This is because the amount that he removes every year is replaced by the interest on the funds that remain in the account.

• **Example 5.4.5**

Let us solve the following system of difference equations:

$$x_{n+1} = 4x_n + 2y_n \tag{5.4.39}$$

and

$$y_{n+1} = 3x_n + 3y_n \tag{5.4.40}$$

with the initial values of $x_0 = 0$ and $y_0 = 5$.

Taking the z-transform of (5.4.39)–(5.4.40),

$$zX(z) - x_0 z = 4X(z) + 2Y(z) \tag{5.4.41}$$

$$zY(z) - y_0 z = 3X(z) + 3Y(z) \tag{5.4.42}$$

or

$$(z-4)X(z) - 2Y(z) = 0 \tag{5.4.43}$$

$$3X(z) - (z-3)Y(z) = -5z. \tag{5.4.44}$$

Solving for $X(z)$ and $Y(z)$,

$$X(z) = -\frac{10z}{(z-6)(z-1)} = \frac{2z}{z-1} - \frac{2z}{z-6} \tag{5.4.45}$$

and

$$Y(z) = \frac{5z(z-4)}{(z-6)(z-1)} = \frac{2z}{z-6} + \frac{3z}{z-1}. \tag{5.4.46}$$

Taking the inverse of (5.4.45)–(5.4.46) term by term,

$$x_n = 2 - 2\,6^n \qquad \text{and} \qquad y_n = 3 + 2\,6^n. \tag{5.4.47}$$

Problems

Solve the following difference equations using z-transforms, where $n \geq 0$.

1. $y_{n+1} - y_n = n^2, \quad y_0 = 1.$

2. $y_{n+2} - 2y_{n+1} + y_n = 0, \quad y_0 = y_1 = 1.$

3. $y_{n+2} - 2y_{n+1} + y_n = 1, \quad y_0 = y_1 = 0.$

4. $y_{n+1} + 3y_n = n, \quad y_0 = 0.$

5. $y_{n+1} - 5y_n = \cos(n\pi), \quad y_0 = 0.$

6. $y_{n+2} - 4y_n = 1, \quad y_0 = 1, y_1 = 0.$

7. $y_{n+2} - \frac{1}{4}y_n = (\frac{1}{2})^n, \quad y_0 = y_1 = 0.$

8. $y_{n+2} - 5y_{n+1} + 6y_n = 0, \quad y_0 = y_1 = 1.$

9. $y_{n+2} - 3y_{n+1} + 2y_n = 1, \quad y_0 = y_1 = 0.$

10. $y_{n+2} - 2y_{n+1} + y_n = 2, \quad y_0 = 0, \quad y_1 = 2.$

11. $x_{n+1} = 3x_n - 4y_n, \; y_{n+1} = 2x_n - 3y_n, \quad x_0 = 3, \; y_0 = 2.$

12. $x_{n+1} = 2x_n - 10y_n, \; y_{n+1} = -x_n - y_n, \quad x_0 = 3, \; y_0 = -2.$

13. $x_{n+1} = x_n - 2y_n, \; y_{n+1} = -6y_n, \quad x_0 = -1, \; y_0 = -7.$

14. $x_{n+1} = 4x_n - 5y_n, \; y_{n+1} = x_n - 2y_n, \quad x_0 = 6, \; y_0 = 2.$

5.5 STABILITY OF DISCRETE-TIME SYSTEMS

When we discussed the solution of ordinary differential equations by Laplace transforms, we introduced the concept of transfer function and impulse response. In the case of discrete-time systems, similar considerations come into play.

Consider the recursive system

$$y_n = a_1 y_{n-1} H_{n-1} + a_2 y_{n-2} H_{n-2} + x_n, \quad n \geq 0, \tag{5.5.1}$$

where H_{n-k} is the unit step function. It equals 0 for $n < k$ and 1 for $n \geq k$. Equation (5.5.1) is called a *recursive system* because future values of the sequence depend upon all of the previous values. At present, a_1 and a_2 are free parameters which we shall vary.

Using (5.2.10),

$$z^2Y(z) - a_1zY(z) - a_2Y(z) = z^2X(z) \tag{5.5.2}$$

or

$$G(z) = \frac{Y(z)}{X(z)} = \frac{z^2}{z^2 - a_1z - a_2}. \tag{5.5.3}$$

As in the case of Laplace transforms, the ratio $Y(z)/X(z)$ is the transfer function. The inverse of the transfer function gives the impulse response for our discrete-time system. This particular transfer function has two poles, namely

$$z_{1,2} = \frac{a_1}{2} \pm \sqrt{\frac{a_1^2}{4} + a_2}. \tag{5.5.4}$$

At this point, we consider three cases.

Case 1: $a_1^2/4 + a_2 < 0$. In this case z_1 and z_2 are complex conjugates. Let us write them as $z_{1,2} = re^{\pm i\omega_0 T}$. Then

$$G(z) = \frac{z^2}{(z - re^{i\omega_0 T})(z - re^{-i\omega_0 T})} = \frac{z^2}{z^2 - 2r\cos(\omega_0 T)z + r^2}, \tag{5.5.5}$$

where $r^2 = -a_2$ and $\omega_0 T = \cos^{-1}(a_1/2r)$. From the inversion integral,

$$\begin{aligned} g_n = \operatorname{Res}&\left[\frac{z^{n+1}}{z^2 - 2r\cos(\omega_0 T)z + r^2}; z_1\right] \\ + \operatorname{Res}&\left[\frac{z^{n+1}}{z^2 - 2r\cos(\omega_0 T)z + r^2}; z_2\right], \end{aligned} \tag{5.5.6}$$

where g_n denotes the impulse response. Now

$$\operatorname{Res}\left[\frac{z^{n+1}}{z^2 - 2r\cos(\omega_0 T)z + r^2}; z_1\right] = \lim_{z\to z_1} \frac{(z - z_1)z^{n+1}}{(z - z_1)(z - z_2)} \tag{5.5.7}$$

$$= r^n \frac{\exp[i(n+1)\omega_0 T]}{e^{i\omega_0 T} - e^{-i\omega_0 T}} \tag{5.5.8}$$

$$= \frac{r^n \exp[i(n+1)\omega_0 T]}{2i\sin(\omega_0 T)}. \tag{5.5.9}$$

Similarly,

$$\operatorname{Res}\left[\frac{z^{n+1}}{z^2 - 2r\cos(\omega_0 T)z + r^2}; z_2\right] = -\frac{r^n \exp[-i(n+1)\omega_0 T]}{2i\sin(\omega_0 T)} \tag{5.5.10}$$

and

$$g_n = \frac{r^n \sin[(n+1)\omega_0 T]}{\sin(\omega_0 T)}. \tag{5.5.11}$$

A graph of $\sin[(n+1)\omega_0 T]/\sin(\omega_0 T)$ with respect to n gives a sinusoidal envelope. More importantly, if $|r| < 1$ these oscillations will vanish as $n \to \infty$ and the system is stable. On the other hand, if $|r| > 1$ the oscillation will grow without bound as $n \to \infty$ and the system is unstable.

Recall that $|r| > 1$ corresponds to poles that lie outside the unit circle while $|r| < 1$ is exactly the opposite. Our example suggests that for discrete-time systems to be stable, all of the poles of the transfer function must lie within the unit circle while an unstable system has at least one pole that lies outside of this circle.

Case 2: $a_1^2/4 + a_2 > 0$. This case leads to two real roots, z_1 and z_2. From the inversion integral, the sum of the residues gives the impulse response

$$g_n = \frac{z_1^{n+1} - z_2^{n+1}}{z_1 - z_2}. \tag{5.5.12}$$

Once again, if the poles lie within the unit circle, $|z_1| < 1$ and $|z_2| < 1$, the system is stable.

Case 3: $a_1^2/4 + a_2 = 0$. This case yields $z_1 = z_2$,

$$G(z) = \frac{z^2}{(z - a_1/2)^2} \tag{5.5.13}$$

and

$$g_n = \frac{1}{2\pi i} \oint_C \frac{z^{n+1}}{(z - a_1/2)^2} dz = \left(\frac{a_1}{2}\right)^n (n+1). \tag{5.5.14}$$

This system is obviously stable if $|a_1/2| < 1$ and the pole of the transfer function lies within the unit circle.

In summary, finding the transfer function of a discrete-time system is important in determining its stability. Because the location of the poles of $G(z)$ determines the response of the system, a stable system will have all of its poles within the unit circle. Conversely, if any of the poles of $G(z)$ lie outside of the unit circle, the system is unstable. Finally, if $\lim_{n\to\infty} g_n = c$, the system is marginally stable. For example, if $G(z)$ has simple poles, some of the poles must lie *on* the unit circle.

• **Example 5.5.1**

Numerical methods of integration provide some of the simplest, yet most important, difference equations in the literature. In this example,[3]

[3] From Salzer, J. M., 1954: Frequency analysis of digital computers operating in real time. *Proc. IRE*, **42**, 457–466. ©IRE (now IEEE).

we show how z-transforms can be used to highlight the strengths and weaknesses of such schemes.

Consider the trapezoidal integration rule in numerical analysis. The integral y_n is updated by adding the latest trapezoidal approximation of the continuous curve. Thus, the integral is computed by

$$y_n = \tfrac{1}{2}T(x_n + x_{n-1}H_{n-1}) + y_{n-1}H_{n-1}, \tag{5.5.15}$$

where T is the interval between evaluations of the integrand.

We first determine the stability of this rule because it is of little value if it is not stable. Using (5.2.10), the transfer function is

$$G(z) = \frac{Y(z)}{X(z)} = \frac{T}{2}\left(\frac{z+1}{z-1}\right). \tag{5.5.16}$$

To find the impulse response, we use the inversion integral and find that

$$g_n = \frac{T}{4\pi i}\oint_C z^{n-1}\frac{z+1}{z-1}\,dz. \tag{5.5.17}$$

At this point, we must consider two cases: $n = 0$ and $n > 0$. For $n = 0$,

$$g_0 = \frac{T}{2}\text{Res}\left[\frac{z+1}{z(z-1)};0\right] + \frac{T}{2}\text{Res}\left[\frac{z+1}{z(z-1)};1\right] = \frac{T}{2}. \tag{5.5.18}$$

For $n > 0$,

$$g_0 = \frac{T}{2}\text{Res}\left[\frac{z^{n-1}(z+1)}{z-1};1\right] = T. \tag{5.5.19}$$

Therefore, the impulse response for this numerical scheme is $g_0 = \frac{T}{2}$ and $g_n = T$ for $n > 0$. Note that this is a marginally stable system (the solution neither grows nor decays with n) because the pole associated with the transfer function lies *on* the unit circle.

Having discovered that the system is not unstable, let us continue and explore some of its properties. Recall now that $z = e^{sT} = e^{i\omega T}$ if $s = i\omega$. Then the transfer function becomes

$$G(\omega) = \frac{T}{2}\frac{1+e^{-i\omega T}}{1-e^{-i\omega T}} = -\frac{iT}{2}\cot\left(\frac{\omega T}{2}\right). \tag{5.5.20}$$

On the other hand, the transfer function of an ideal integrator is $1/s$ or $-i/\omega$. Thus, the trapezoidal rule has ideal phase but its shortcoming lies in its amplitude characteristic; it lies below the ideal integrator for

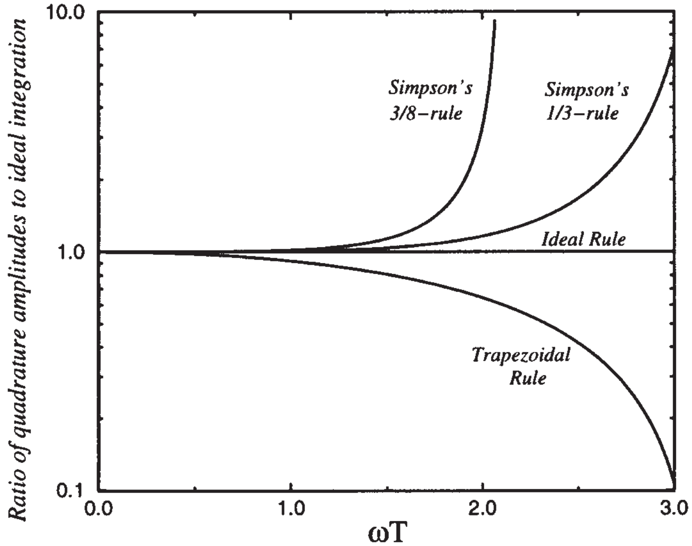


Figure 5.5.1: Comparison of various quadrature formulas by ratios of their amplitudes to that of an ideal integrator. [From Salzer, J. M., 1954: Frequency analysis of digital computers operating in real time. *Proc. IRE*, **42**, p. 463. ©IRE (now IEEE).]

$0 < \omega T < \pi$. We show this behavior, along with that for Simpson's $\frac{1}{3}$rd-rule and Simpson's $\frac{3}{8}$th-rule, in Figure 5.5.1.

Figure 5.5.1 confirms the superiority of Simpson's $\frac{1}{3}$rd rule over his $\frac{3}{8}$th rule. The figure also shows that certain schemes are better at suppressing noise at higher frequencies; an effect not generally emphasized in numerical calculus but often important in system design. For example, the trapezoidal rule is inferior to all others at low frequencies but only to Simpson's $\frac{1}{3}$rd rule at higher frequencies. Furthermore, the trapezoidal rule might actually be preferred not only because of its simplicity but also because it attenuates at higher frequencies, thereby counteracting the effect of noise.

• **Example 5.5.2**

Given the transfer function

$$G(z) = \frac{z^2}{(z-1)(z-1/2)}, \qquad \textbf{(5.5.21)}$$

is this discrete-time system stable or marginally stable?

This transfer function has two simple poles. The pole at $z = 1/2$ gives rise to a term that varies as $(\frac{1}{2})^n$ in the impulse response while the $z = 1$ pole will give a constant. Because this constant will neither grow nor decay with n, the system is marginally stable.

Problems

For the following time-discrete systems, find the transfer function and determine whether the systems are unstable, marginally stable, or stable.

1. $y_n = y_{n-1}H_{n-1} + x_n$

2. $y_n = 2y_{n-1}H_{n-1} - y_{n-2}H_{n-2} + x_n$

3. $y_n = 3y_{n-1}H_{n-1} + x_n$

4. $y_n = \frac{1}{4}y_{n-2}H_{n-2} + x_n$

Chapter 6

The Sturm-Liouville Problem

In the next three chapters we shall be solving partial differential equations using the technique of separation of variables. This technique requires that we expand a piece-wise continuous function $f(x)$ as a linear sum of *eigenfunctions*, much as we used sines and cosines to reexpress $f(x)$ in a Fourier series. The purpose of this chapter is to explain and illustrate these eigenfunction expansions.

6.1 EIGENVALUES AND EIGENFUNCTIONS

Repeatedly, in the next three chapters on partial differential equations, we will solve the following second-order linear differential equation:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]y = 0, \quad a \le x \le b, \tag{6.1.1}$$

together with the boundary conditions:

$$\boxed{\alpha y(a)+\beta y'(a)=0 \qquad \text{and} \qquad \gamma y(b)+\delta y'(b)=0.} \tag{6.1.2}$$

In (6.1.1), $p(x)$, $q(x)$ and $r(x)$ are real functions of x; λ is a parameter; and $p(x)$ and $r(x)$ are functions that are continuous and positive on the interval $a \le x \le b$. Taken together, (6.1.1) and (6.1.2) constitute a regular *Sturm-Liouville problem.* This name honors the French mathematicians Sturm and Liouville[1] who first studied these equations in the 1830s. In the case when $p(x)$ or $r(x)$ vanishes at one of the endpoints of the interval $[a, b]$ or when the interval is of infinite length, the problem is a *singular Sturm-Liouville problem.*

Consider now the solutions of the Sturm-Liouville problem. Clearly there is the trivial solution $y = 0$ for all λ. However, nontrivial solutions will exist only if λ takes on specific values; these values are called *characteristic values* or *eigenvalues*. The corresponding nontrivial solutions are called the *characteristic functions* or *eigenfunctions*. In particular, we have the following theorems.

Theorem: *For a regular Sturm-Liouville problem with* $p(x) > 0$, *all of the eigenvalues are real if* $p(x)$, $q(x)$, *and* $r(x)$ *are real functions and the eigenfunctions are differentiable and continuous.*

Proof: Let $y(x) = u(x) + iv(x)$ be an eigenfunction corresponding to an eigenvalue $\lambda = \lambda_r + i\lambda_i$, where λ_r, λ_i are real numbers and $u(x), v(x)$ are real functions of x. Substituting into the Sturm-Liouville equation yields

$$\{p(x)[u'(x)+iv'(x)]\}'+[q(x)+(\lambda_r+i\lambda_i)r(x)][u(x)+iv(x)]=0. \tag{6.1.3}$$

Separating the real and imaginary parts yields

$$[p(x)u'(x)]' + [q(x) + \lambda_r]u(x) - \lambda_i r(x)v(x) = 0 \tag{6.1.4}$$

and

$$[p(x)v'(x)]' + [q(x) + \lambda_r]v(x) + \lambda_i r(x)u(x) = 0. \tag{6.1.5}$$

---

[1] For the complete history as well as the relevant papers, see Lützen, J., 1984: Sturm and Liouville's work on ordinary linear differential equations. The emergence of Sturm-Liouville theory. *Arch. Hist. Exact Sci.*, **29**, 309–376.

Figure 6.1.1: By the time that Charles-François Sturm (1803–1855) met Joseph Liouville in the early 1830s, he had already gained fame for his work on the compression of fluids and his celebrated theorem on the number of real roots of a polynomial. An eminent teacher, Sturm spent most of his career teaching at various Parisian colleges. (Portrait courtesy of the Archives de l'Académie des sciences, Paris.)

If we multiply (6.1.4) by v and (6.1.5) by u and subtract the results, we find that

$$u(x)[p(x)v'(x)]' - v(x)[p(x)u'(x)]' + \lambda_i r(x)[u^2(x) + v^2(x)] = 0. \quad (6.1.6)$$

The derivative terms in (6.1.6) can be rewritten in such a manner that it becomes

$$\frac{d}{dx}\{[p(x)v'(x)]u(x) - [p(x)u'(x)]v(x)\} + \lambda_i r(x)[u^2(x) + v^2(x)] = 0. \quad (6.1.7)$$

Figure 6.1.2: Although educated as an engineer, Joseph Liouville (1809–1882) would devote his life to teaching pure and applied mathematics in the leading Parisian institutions of higher education. Today he is most famous for founding and editing for almost 40 years the *Journal de Liouville*. (Portrait courtesy of the Archives de l'Académie des sciences, Paris.)

Integrating from a to b, we find that

$$-\lambda_i \int_a^b r(x)[u^2(x) + v^2(x)]\, dx = \{p(x)[u(x)v'(x) - v(x)u'(x)]\}\big|_a^b . \tag{6.1.8}$$

From the boundary conditions (6.1.2),

$$\alpha[u(a) + iv(a)] + \beta[u'(a) + iv'(a)] = 0 \tag{6.1.9}$$

and

$$\gamma[u(b) + iv(b)] + \delta[u'(b) + iv'(b)] = 0. \tag{6.1.10}$$

Separating the real and imaginary parts yields

$$\alpha u(a) + \beta u'(a) = 0 \quad \text{and} \quad \alpha v(a) + \beta v'(a) = 0 \tag{6.1.11}$$

and

$$\gamma u(b) + \delta u'(b) = 0 \quad \text{and} \quad \gamma v(b) + \delta v'(b) = 0. \tag{6.1.12}$$

Both α and β cannot be zero; otherwise, there would be no boundary condition at $x = a$. Similar considerations hold for γ and δ. Therefore,

$$u(a)v'(a) - u'(a)v(a) = 0 \quad \text{and} \quad u(b)v'(b) - u'(b)v(b) = 0, \tag{6.1.13}$$

if we treat α, β, γ, and δ as unknowns in a system of homogeneous equations (6.1.11)–(6.1.12) and require that the corresponding determinants equal zero. Applying (6.1.13) to the right side of (6.1.8), we obtain

$$\lambda_i \int_a^b r(x)[u^2(x) + v^2(x)]\,dx = 0. \tag{6.1.14}$$

Because $r(x) > 0$, the integral is positive and $\lambda_i = 0$. Since $\lambda_i = 0$, λ is purely real. This implies that the eigenvalues are real. .□

If there is only one independent eigenfunction for each eigenvalue, that eigenvalue is *simple*. When more than one eigenfunction belongs to a single eigenvalue, the problem is *degenerate*.

Theorem: *The regular Sturm-Liouville problem has infinitely many real and simple eigenvalues* λ_n, $n = 0, 1, 2, \ldots$, *which can be arranged in a monotonically increasing sequence* $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ *such that* $\lim_{n\to\infty} \lambda_n = \infty$. *Every eigenfunction* $y_n(x)$ *associated with the corresponding eigenvalue* λ_n *has exactly* n *zeros in the interval* (a, b). *For each eigenvalue there exists only one eigenfunction (up to a multiplicative constant).*

The proof is beyond the scope of this book but may be found in more advanced treatises.[2]

In the following examples we will illustrate how to find these real eigenvalues and their corresponding eigenfunctions.

---

[2] See, for example, Birkhoff, G. and Rota, G.-C., 1989: *Ordinary Differential Equations*, John Wiley & Sons, New York, chaps. 10 and 11; Sagan, H., 1961: *Boundary and Eigenvalue Problems in Mathematical Physics*, John Wiley & Sons, New York, chap. 5.

• **Example 6.1.1**

Let us find the eigenvalues and eigenfunctions of

$$y'' + \lambda y = 0 \tag{6.1.15}$$

subject to the boundary conditions

$$y(0) = 0 \qquad \text{and} \qquad y(\pi) - y'(\pi) = 0. \tag{6.1.16}$$

Our first task is to check to see whether the problem is indeed a regular Sturm-Liouville problem. A comparison between (6.1.1) and (6.1.15) shows that they are the same if $p(x) = 1$, $q(x) = 0$, and $r(x) = 1$. Similarly, the boundary conditions (6.1.16) are identical to (6.1.2) if $\alpha = \gamma = 1$, $\delta = -1$, $\beta = 0$, $a = 0$, and $b = \pi$.

Because the form of the solution to (6.1.15) depends on λ, we consider three cases: λ negative, positive, or equal to zero. The general solution of the differential equation is

$$y(x) = A\cosh(mx) + B\sinh(mx) \quad \text{if} \quad \lambda < 0, \tag{6.1.17}$$

$$y(x) = C + Dx \quad \text{if} \quad \lambda = 0 \tag{6.1.18}$$

and

$$y(x) = E\cos(kx) + F\sin(kx) \quad \text{if} \quad \lambda > 0, \tag{6.1.19}$$

where for convenience $\lambda = -m^2 < 0$ in (6.1.17) and $\lambda = k^2 > 0$ in (6.1.19). Both k and m are real and positive by these definitions.[3]

---

[3] In many differential equations courses, the solution to

$$y'' - m^2 y = 0, \qquad m > 0$$

is written

$$y(x) = c_1 e^{mx} + c_2 e^{-mx}.$$

However, we can rewrite this solution as

$$y(x) = (c_1 + c_2)\tfrac{1}{2}(e^{mx} + e^{-mx}) + (c_1 - c_2)\tfrac{1}{2}(e^{mx} - e^{-mx})$$

$$= A\cosh(mx) + B\sinh(mx),$$

where $\cosh(mx) = (e^{mx} + e^{-mx})/2$ and $\sinh(mx) = (e^{mx} - e^{-mx})/2$. The advantage of using these hyperbolic functions over exponentials is the simplification that occurs when we substitute the hyperbolic functions into the boundary conditions.

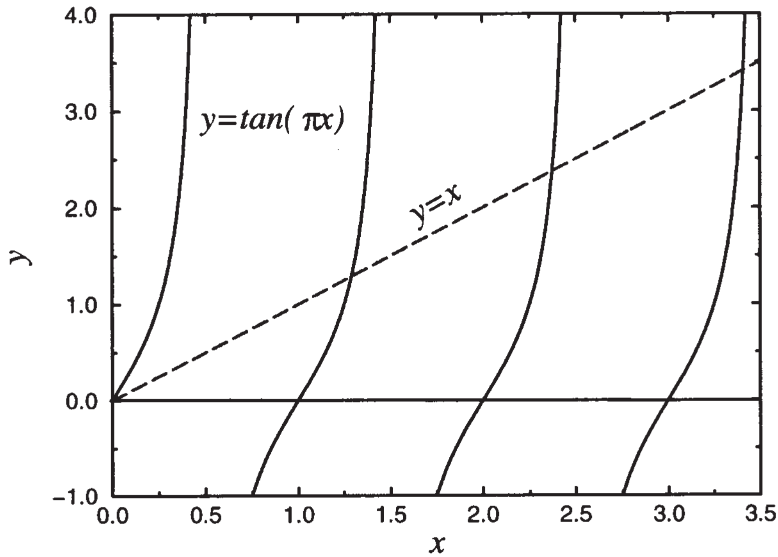


Figure 6.1.3: Graphical solution of $\tan(\pi x) = x$.

Turning to the condition that $y(0) = 0$, we find that $A = C = E = 0$. The other boundary condition $y(\pi) - y(\pi) = 0$ gives

$$B[\sinh(m\pi) - m\cosh(m\pi)] = 0, \tag{6.1.20}$$

$$D = 0 \tag{6.1.21}$$

and

$$F[\sin(k\pi) - k\cos(k\pi)] = 0. \tag{6.1.22}$$

If we graph $\sinh(m\pi) - m\cosh(m\pi)$ for all positive m, this quantity is always negative. Consequently, $B = 0$. However, in (6.1.22), a nontrivial solution (i.e., $F \neq 0$) occurs if

$$F\cos(k\pi)[\tan(k\pi) - k] = 0 \quad \text{or} \quad \tan(k\pi) = k. \tag{6.1.23}$$

In summary, we have found nontrivial solutions only when $\lambda_n = k_n^2 > 0$, where k_n is the nth root of the transcendental Equation (6.1.23). We may find the roots either graphically or through the use of a numerical algorithm. Figure 6.1.3 illustrates the graphical solution to the problem. We exclude the root $k = 0$ because λ must be greater than zero.

Let us now find the corresponding eigenfunctions. Because $A = B = C = D = E = 0$, we are left with $y(x) = F\sin(kx)$. Consequently, the eigenfunction, traditionally written without the arbitrary amplitude constant, is

$$y_n(x) = \sin(k_n x), \tag{6.1.24}$$

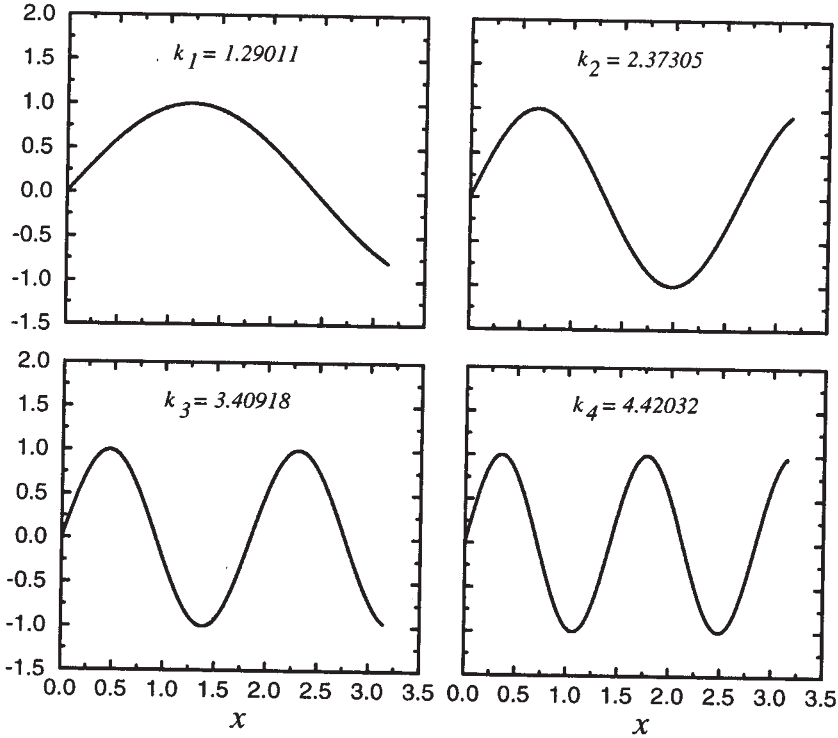


Figure 6.1.4: The first four eigenfunctions $\sin(k_n x)$ corresponding to the eigenvalue problem $\tan(k\pi) = k$.

because k must equal k_n. Figure 6.1.4 shows the first four eigenfunctions.

• Example 6.1.2

For our second example let us solve the Sturm-Liouville problem,

$$y'' + \lambda y = 0 \tag{6.1.25}$$

with the boundary conditions

$$y(0) - y'(0) = 0 \quad \text{and} \quad y(\pi) - y'(\pi) = 0. \tag{6.1.26}$$

Once again the three possible solutions to (6.1.25) are

$$y(x) = A\cosh(mx) + B\sinh(mx) \quad \text{if} \quad \lambda = -m^2 < 0, \tag{6.1.27}$$

$$y(x) = C + Dx \quad \text{if} \quad \lambda = 0 \tag{6.1.28}$$

and

$$y(x) = E\cos(kx) + F\sin(kx) \quad \text{if} \quad \lambda = k^2 > 0. \tag{6.1.29}$$

Let us first check and see if there are any nontrivial solutions for $\lambda < 0$. Two simultaneous equations result from the substitution of (6.1.27) into (6.1.26):

$$A - mB = 0 \tag{6.1.30}$$

$$[\cosh(m\pi) - m\sinh(m\pi)]A + [\sinh(m\pi) - m\cosh(m\pi)]B = 0. \tag{6.1.31}$$

The elimination of A between the two equations yields

$$\sinh(m\pi)(1 - m^2)B = 0. \tag{6.1.32}$$

If (6.1.27) is a nontrivial solution, then $B \neq 0$ and

$$\sinh(m\pi) = 0 \tag{6.1.33}$$

or

$$m^2 = 1. \tag{6.1.34}$$

Equation (6.1.33) cannot hold because it implies $m = \lambda = 0$ which contradicts the assumption used in deriving (6.1.27) that $\lambda < 0$. On the other hand, (6.1.34) is quite acceptable. It corresponds to the eigenvalue $\lambda = -1$ and the eigenfunction is

$$y_0 = \cosh(x) + \sinh(x) = e^x, \tag{6.1.35}$$

because it satisfies the differential equation

$$y_0'' - y_0 = 0 \tag{6.1.36}$$

and the boundary conditions

$$y_0(0) - y_0'(0) = 0 \tag{6.1.37}$$

and

$$y_0(\pi) - y_0'(\pi) = 0. \tag{6.1.38}$$

An alternative method of finding m, which is quite popular because of its use in more difficult problems, follows from viewing (6.1.30) and (6.1.31) as a system of homogeneous linear equations, where A and B are the unknowns. It is well known[4] that in order for (6.1.30)–(6.1.31)

[4] See Chapter 11.

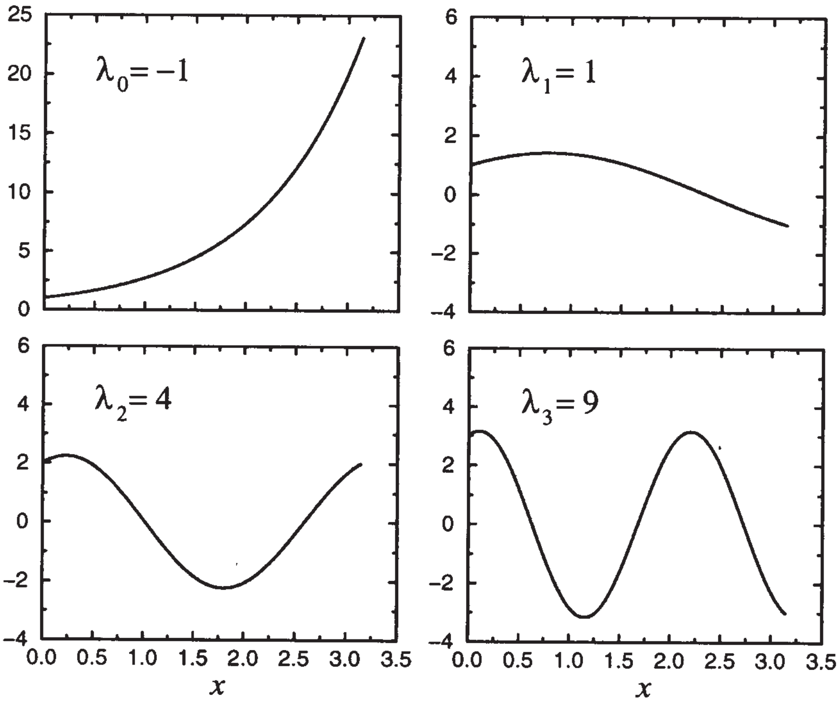


Figure 6.1.5: The first four eigenfunctions for the Sturm-Liouville problem (6.1.25)–(6.1.26).

to have a nontrivial solution (i.e., $A \neq 0$ and/or $B \neq 0$) the determinant of the coefficients must vanish:

$$\begin{vmatrix} 1 & -m \\ \cosh(m\pi) - m\sinh(m\pi) & \sinh(m\pi) - m\cosh(m\pi) \end{vmatrix} = 0. \quad \textbf{(6.1.39)}$$

Expanding the determinant,

$$\sinh(m\pi)(1 - m^2) = 0, \quad \textbf{(6.1.40)}$$

which leads directly to (6.1.33) and (6.1.34).

We consider next the case of $\lambda = 0$. Substituting (6.1.28) into (6.1.26), we find that

$$C - D = 0 \quad \textbf{(6.1.41)}$$

and

$$C + D\pi - D = 0. \quad \textbf{(6.1.42)}$$

This set of simultaneous equations yields $C = D = 0$ and we have only trivial solutions for $\lambda = 0$.

Finally, we examine the case when $\lambda > 0$. Substituting (6.1.29) into (6.1.26), we obtain

$$E - kF = 0 \tag{6.1.43}$$

and

$$[\cos(k\pi) + k\sin(k\pi)]E + [\sin(k\pi) - k\cos(k\pi)]F = 0. \tag{6.1.44}$$

The elimination of E from (6.1.43) and (6.1.44) gives

$$F(1 + k^2)\sin(k\pi) = 0. \tag{6.1.45}$$

In order that (6.1.29) be nontrivial, $F \neq 0$ and

$$k^2 = -1 \tag{6.1.46}$$

or

$$\sin(k\pi) = 0. \tag{6.1.47}$$

Condition (6.1.46) violates the assumption that k is real, which follows from the fact that $\lambda = k^2 > 0$. On the other hand, we can satisfy (6.1.47) if $k = 1, 2, 3, \ldots$; a negative k yields the same λ. Consequently we have the additional eigenvalues $\lambda_n = n^2$.

Let us now find the corresponding eigenfunctions. Because $E = kF$, $y(x) = F\sin(kx) + Fk\cos(kx)$ from (6.1.29). Thus, the eigenfunctions for $\lambda > 0$ are

$$y_n(x) = \sin(nx) + n\cos(nx). \tag{6.1.48}$$

Figure 6.1.4 illustrates some of the eigenfunctions given by (6.1.35) and (6.1.48).

• Example 6.1.3

Consider now the Sturm-Liouville problem

$$y'' + \lambda y = 0 \tag{6.1.49}$$

with

$$y(\pi) = y(-\pi) \quad \text{and} \quad y'(\pi) = y'(-\pi). \tag{6.1.50}$$

This is *not* a regular Sturm-Liouville problem because the boundary conditions are periodic and do not conform to the canonical boundary condition (6.1.2).

The general solution to (6.1.49) is

$$y(x) = A\cosh(mx) + B\sinh(mx) \quad \text{if} \quad \lambda = -m^2 < 0, \tag{6.1.51}$$

$$y(x) = C + Dx \quad \text{if} \quad \lambda = 0 \tag{6.1.52}$$

and

$$y(x) = E\cos(kx) + F\sin(kx) \quad \text{if} \quad \lambda = k^2 > 0. \tag{6.1.53}$$

Substituting these solutions into the boundary condition (6.1.50),

$$A\cosh(m\pi) + B\sinh(m\pi) = A\cosh(-m\pi) + B\sinh(-m\pi), \tag{6.1.54}$$

$$C + D\pi = C - D\pi \tag{6.1.55}$$

and

$$E\cos(k\pi) + F\sin(k\pi) = E\cos(-k\pi) + F\sin(-k\pi) \tag{6.1.56}$$

or

$$B\sinh(m\pi) = 0, \quad D = 0 \quad \text{and} \quad F\sin(k\pi) = 0, \tag{6.1.57}$$

because $\cosh(-m\pi) = \cosh(m\pi)$, $\sinh(-m\pi) = -\sinh(m\pi)$, $\cos(-k\pi) = \cos(k\pi)$, and $\sin(-k\pi) = -\sin(k\pi)$. Because m must be positive, $\sinh(m\pi)$ cannot equal zero and $B = 0$. On the other hand, if $\sin(k\pi) = 0$ or $k = n$, $n = 1, 2, 3, \ldots$, we have a nontrivial solution for positive λ and $\lambda_n = n^2$. Note that we still have A, C, E, and F as free constants.

From the boundary condition (6.1.50),

$$A\sinh(m\pi) = A\sinh(-m\pi) \tag{6.1.58}$$

and

$$-E\sin(k\pi) + F\cos(k\pi) = -E\sin(-k\pi) + F\cos(-k\pi). \tag{6.1.59}$$

The solution $y_0(x) = C$ identically satisfies the boundary condition (6.1.50) for all C. Because m and $\sinh(m\pi)$ must be positive, $A = 0$. From (6.1.57), we once again have $\sin(k\pi) = 0$ and $k = n$. Consequently, the eigenfunction solutions to (6.1.49)–(6.1.50) are

$$\lambda_0 = 0, \qquad y_0(x) = 1 \tag{6.1.60}$$

and

$$\lambda_n = n^2, \qquad y_n(x) = \begin{cases} \sin(nx) \\ \cos(nx) \end{cases} \tag{6.1.61}$$

and we have a degenerate set of eigenfunctions to the Sturm-Liouville problem (6.1.49) with the periodic boundary condition (6.1.50).

Problems

Find the eigenvalues and eigenfunctions for each of the following:

1. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(L) = 0$

2. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(\pi) = 0$

3. $y'' + \lambda y = 0, \quad y(0) + y'(0) = 0, \quad y(\pi) + y'(\pi) = 0$

4. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(\pi) - y'(\pi) = 0$

5. $y^{(iv)} + \lambda y = 0, \quad y(0) = y''(0) = 0, \quad y(L) = y''(L) = 0$

Find an equation from which you could find λ and give the form of the eigenfunction for each of the following:

6. $y'' + \lambda y = 0, \quad y(0) + y'(0) = 0, \quad y(1) = 0$

7. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) + y'(\pi) = 0$

8. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) - y'(1) = 0$

9. $y'' + \lambda y = 0, \quad y(0) + y'(0) = 0, \quad y'(\pi) = 0$

10. $y'' + \lambda y = 0, \quad y(0) + y'(0) = 0, \quad y(\pi) - y'(\pi) = 0$

11. Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] + \frac{\lambda}{x}y = 0, \quad 1 \le x \le e$$

for each of the following boundary conditions: (a) $u(1) = u(e) = 0$, (b) $u(1) = u'(e) = 0$, and (c) $u'(1) = u'(e) = 0$.

Find the eigenvalues and eigenfunctions of the following Sturm-Liouville problems:

12.

$$x^2y'' + 2xy' + \lambda y = 0, \quad y(1) = y(e) = 0, \quad 1 \le x \le e.$$

13.

$$\frac{d}{dx}\left[x^3y'\right] + \lambda xy = 0, \quad y(1) = y(e^{\pi}) = 0, \quad 1 \le x \le e^{\pi}.$$

14.

$$\frac{d}{dx}\left[\frac{1}{x}y'\right] + \frac{\lambda}{x}y = 0, \quad y(1) = y(e) = 0, \quad 1 \le x \le e.$$

6.2 ORTHOGONALITY OF EIGENFUNCTIONS

In the previous section we saw how nontrivial solutions to the regular Sturm-Liouville problem consist of eigenvalues and eigenfunctions. The most important property of eigenfunctions is orthogonality.

Theorem: *Let the functions $p(x)$, $q(x)$, and $r(x)$ of the regular Sturm-Liouville problem (6.1.1)–(6.1.2) be real and continuous on the interval $[a, b]$. If $y_n(x)$ and $y_m(x)$ are continuously differentiable eigenfunctions corresponding to the distinct eigenvalues λ_n and λ_m, respectively, then $y_n(x)$ and $y_m(x)$ satisfy the* orthogonality *condition:*

$$\int_a^b r(x)y_n(x)y_m(x)\,dx = 0, \qquad (6.2.1)$$

if $\lambda_n \neq \lambda_m$. When (6.2.1) is satisfied, the eigenfunction $y_n(x)$ and $y_m(x)$ are said to be *orthogonal* to each other with respect to the *weight function* $r(x)$. The term *orthogonality* appears to be borrowed from linear algebra where a similar relationship holds between two perpendicular or orthogonal vectors.

Proof: Let y_n and y_m denote the eigenfunctions associated with two different eigenvalues λ_n and λ_m. Then

$$\frac{d}{dx}\left[p(x)\frac{dy_n}{dx}\right] + [q(x) + \lambda_n r(x)]y_n(x) = 0, \qquad (6.2.2)$$

$$\frac{d}{dx}\left[p(x)\frac{dy_m}{dx}\right] + [q(x) + \lambda_m r(x)]y_m(x) = 0 \qquad (6.2.3)$$

and both solutions satisfy the boundary conditions. Let us multiply the first differential equation by y_m; the second by y_n. Next, we subtract these two equations and move the terms containing $y_n y_m$ to the right side. The resulting equation is

$$y_n\frac{d}{dx}\left[p(x)\frac{dy_m}{dx}\right] - y_m\frac{d}{dx}\left[p(x)\frac{dy_n}{dx}\right] = (\lambda_n - \lambda_m)r(x)y_n y_m. \quad (6.2.4)$$

Integrating (6.2.4) from a to b yields

$$\int_a^b \left\{y_n\frac{d}{dx}\left[p(x)\frac{dy_m}{dx}\right] - y_m\frac{d}{dx}\left[p(x)\frac{dy_n}{dx}\right]\right\}dx$$

$$= (\lambda_n - \lambda_m)\int_a^b r(x)y_n y_m\,dx. \quad (6.2.5)$$

We may simplify the left side of (6.2.5) by integrating by parts to give

$$\int_a^b \left\{ y_n \frac{d}{dx}\left[p(x)\frac{dy_m}{dx}\right] - y_n \frac{d}{dx}\left[p(x)\frac{dy_m}{dx}\right]\right\} dx$$
$$= [p(x)y_m' y_n - p(x)y_n' y_m]_a^b - \int_a^b p(x)[y_n' y_m' - y_n' y_m']dx. \quad (6.2.6)$$

The second integral equals zero since the integrand vanishes identically. Because $y_n(x)$ and $y_m(x)$ satisfy the boundary condition at $x = a$,

$$\alpha y_n(a) + \beta y_n'(a) = 0 \quad (6.2.7)$$

and

$$\alpha y_m(a) + \beta y_m'(a) = 0. \quad (6.2.8)$$

These two equations are simultaneous equations in α and β. Hence, the determinant of the equations must be zero:

$$y_n'(a)y_m(a) - y_m'(a)y_n(a) = 0. \quad (6.2.9)$$

Similarly, at the other end,

$$y_n'(b)y_m(b) - y_m'(b)y_n(b) = 0. \quad (6.2.10)$$

Consequently, the right side of (6.2.6) vanishes and (6.2.5) reduces to (6.2.1). □

- **Example 6.2.1**

Let us verify the orthogonality condition for the eigenfunctions that we found in Example 6.1.1.

Because $r(x) = 1$, $a = 0$, $b = \pi$, and $y_n(x) = \sin(k_n x)$, we find that

$$\int_a^b r(x) y_n y_m \, dx = \int_0^\pi \sin(k_n x)\sin(k_m x)\, dx \quad (6.2.11)$$

$$= \tfrac{1}{2}\int_0^\pi \{\cos[(k_n - k_m)x] - \cos[(k_n + k_m)x]\, dx \quad (6.2.12)$$

$$= \left.\frac{\sin[(k_n - k_m)x]}{2(k_n - k_m)}\right|_0^\pi - \left.\frac{\sin[(k_n + k_m)x]}{2(k_n + k_m)}\right|_0^\pi \quad (6.2.13)$$

$$= \frac{\sin[(k_n - k_m)\pi]}{2(k_n - k_m)} - \frac{\sin[(k_n + k_m)\pi]}{2(k_n + k_m)} \quad (6.2.14)$$

$$= \frac{\sin(k_n\pi)\cos(k_m\pi) - \cos(k_n\pi)\sin(k_m\pi)}{2(k_n - k_m)}$$

$$- \frac{\sin(k_n\pi)\cos(k_m\pi) + \cos(k_n\pi)\sin(k_m\pi)}{2(k_n + k_m)} \quad (6.2.15)$$

$$\int_a^b r(x) y_n y_m \, dx = \frac{k_n\cos(k_n\pi)\cos(k_m\pi) - k_m\cos(k_n\pi)\cos(k_m\pi)}{2(k_n - k_m)} - \frac{k_n\cos(k_n\pi)\cos(k_m\pi) + k_m\cos(k_n\pi)\cos(k_m\pi)}{2(k_n + k_m)} \quad (6.2.16)$$

$$= \frac{(k_n - k_m)\cos(k_n\pi)\cos(k_m\pi)}{2(k_n - k_m)} - \frac{(k_n + k_m)\cos(k_n\pi)\cos(k_m\pi)}{2(k_n + k_m)} = 0. \quad (6.2.17)$$

We have used the relationships $k_n = \tan(k_n\pi)$ and $k_m = \tan(k_m\pi)$ to simplify (6.2.15). Note, however, that if $n = m$,

$$\int_0^\pi \sin(k_n x)\sin(k_n x)\, dx = \tfrac{1}{2}\int_0^\pi [1 - \cos(2k_n x)]\, dx \quad (6.2.18)$$

$$= \frac{\pi}{2} - \frac{\sin(2k_n\pi)}{4k_n} \quad (6.2.19)$$

$$= \tfrac{1}{2}[\pi - \cos^2(k_n\pi)] > 0 \quad (6.2.20)$$

because $\sin(2A) = 2\sin(A)\cos(A)$ and $k_n = \tan(k_n\pi)$. That is, any eigenfunction *cannot* be orthogonal to itself.

In closing, we note that had we defined the eigenfunction in our example as

$$y_n(x) = \frac{\sin(k_n x)}{\sqrt{[\pi - \cos^2(k_n\pi)]/2}} \quad (6.2.21)$$

rather than $y_n(x) = \sin(k_n x)$, the orthogonality condition would read

$$\int_0^\pi y_n(x) y_m(x)\, dx = \begin{cases} 0, & m \neq n \\ 1, & m = n. \end{cases} \quad (6.2.22)$$

This process of *normalizing* an eigenfunction so that the orthogonality condition becomes

$$\int_a^b r(x) y_n(x) y_m(x)\, dx = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} \quad (6.2.23)$$

generates *orthonormal* eigenfunctions. We will see the convenience of doing this in the next section.

Problems

1. The Sturm-Liouville problem $y'' + \lambda y = 0$, $y(0) = y(L) = 0$ has the eigenfunction solution $y_n(x) = \sin(n\pi x/L)$. By direct integration verify the orthogonality condition (6.2.1).

2. The Sturm-Liouville problem $y'' + \lambda y = 0$, $y'(0) = y'(L) = 0$ has the eigenfunction solutions $y_0(x) = 1$ and $y_n(x) = \cos(n\pi x/L)$. By direct integration verify the orthogonality condition (6.2.1).

3. The Sturm-Liouville problem $y'' + \lambda y = 0$, $y(0) = y'(L) = 0$ has the eigenfunction solution $y_n(x) = \sin[(2n-1)\pi x/(2L)]$. By direct integration verify the orthogonality condition (6.2.1).

4. The Sturm-Liouville problem $y'' + \lambda y = 0$, $y'(0) = y(L) = 0$ has the eigenfunction solution $y_n(x) = \cos[(2n-1)\pi x/(2L)]$. By direct integration verify the orthogonality condition (6.2.1).

6.3 EXPANSION IN SERIES OF EIGENFUNCTIONS

In calculus we learned that under certain conditions we could represent a function $f(x)$ by a linear and infinite sum of polynomials $(x-x_0)^n$. In this section we show that an analogous procedure exists for representing a piece-wise continuous function by a linear sum of eigenfunctions. These *eigenfunction expansions* will be used in the next three chapters to solve partial differential equations.

Let the function $f(x)$ be defined in the interval $a < x < b$. We wish to reexpress $f(x)$ in terms of the eigenfunctions $y_n(x)$ given by a regular Sturm-Liouville problem. Assuming that the function $f(x)$ can be represented by a uniformly convergent series,[5] we write

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x). \tag{6.3.1}$$

The orthogonality relation (6.2.1) gives us the method for computing the coefficients c_n. First we multiply both sides of (6.3.1) by $r(x)y_m(x)$, where m is a fixed integer, and then integrate from a to b. Because this

[5] If $S_n(x) = \sum_{k=1}^{n} u_k(x)$, $S(x) = \lim_{n\to\infty} S_n(x)$ and $0 < |S_n(x) - S(x)| < \epsilon$ for all $n > M > 0$, the series $\sum_{k=1}^{\infty} u_k(x)$ is uniformly convergent if M is dependent on ϵ alone and not x.

series is uniformly convergent and $y_n(x)$ is continuous, we can integrate the series term by term or

$$\int_a^b r(x)f(x)y_m(x)\,dx = \sum_{n=1}^{\infty} c_n \int_a^b r(x)y_n(x)y_m(x)\,dx. \tag{6.3.2}$$

The orthogonality relationship states that all of the terms on the right side of (6.3.2) must disappear except the one for which $n = m$. Thus, we are left with

$$\int_a^b r(x)f(x)y_m(x)\,dx = c_m \int_a^b r(x)y_m(x)y_m(x)\,dx \tag{6.3.3}$$

or

$$\boxed{c_n = \frac{\int_a^b r(x)f(x)y_n(x)\,dx}{\int_a^b r(x)y_n^2(x)\,dx},} \tag{6.3.4}$$

if we replace m by n in (6.3.3).

The series (6.3.1) with the coefficients found by (6.3.4) is a *generalized Fourier series* of the function $f(x)$ with respect to the eigenfunction $y_n(x)$. It is called a generalized Fourier series because we have generalized the procedure of reexpressing a function $f(x)$ by sines and cosines into one involving solutions to regular Sturm-Liouville problems. Note that if we had used an orthonormal set of eigenfunctions, then the denominator of (6.3.4) would equal one and we reduce our work by half. The coefficients c_n are the *Fourier coefficients.*

One of the most remarkable facts about generalized Fourier series is their applicability even when the function has a finite number of bounded discontinuities in the range $[a, b]$. We may formally express this fact by the following theorem:

Theorem: *If both $f(x)$ and $f'(x)$ are piece-wise continuous in $a \le x \le b$, then $f(x)$ can be expanded in a uniformly convergent Fourier series (6.3.1), whose coefficients c_n are given by (6.3.4). It converges to $[f(x^+) + f(x^-)]/2$ at any point x in the open interval $a < x < b$.*

The proof is beyond the scope of this book but may be found in more advanced treatises.[6] If we are willing to include stronger constraints,

---

[6] For example, Titchmarsh, E. C., 1962: *Eigenfunction Expansions Associated with Second-Order Differential Equations. Part I*, Oxford University Press, Oxford, pp. 12–16.

we can make even stronger statements about convergence. For example,[7] if we require that $f(x)$ be a continuous function with a piece-wise continuous first derivative, then the eigenfunction expansion (6.3.1) will converge to $f(x)$ uniformly and absolutely in $[a,b]$ if $f(x)$ satisfies the same boundary conditions as does $y_n(x)$.

In the case when $f(x)$ is discontinuous, we are not merely rewriting $f(x)$ in a new form. We are actually choosing the c_n's so that the eigenfunctions fit $f(x)$ in the "least squares" sense that

$$\int_a^b r(x)\left|f(x)-\sum_{n=1}^{\infty} c_n y_n(x)\right|^2 dx = 0. \tag{6.3.5}$$

Consequently we should expect peculiar things, such as spurious oscillations, to occur in the neighborhood of the discontinuity. This is *Gibbs phenomena*,[8] the same phenomena discovered with Fourier series. See Section 2.2.

• Example 6.3.1

To illustrate the concept of an eigenfunction expansion, let us find the expansion for $f(x) = x$ over the interval $0 < x < \pi$ using the solution to the regular Sturm-Liouville problem of

$$y'' + \lambda y = 0, \qquad y(0) = y(\pi) = 0. \tag{6.3.6}$$

This problem will arise when we solve the wave or heat equation by separation of variables in the next two chapters.

Because the eigenfunctions are $y_n(x) = \sin(nx)$, $n = 1, 2, 3, \ldots$, $r(x) = 1$, $a = 0$, and $b = \pi$, (6.3.4) gives

$$c_n = \frac{\int_0^\pi x\sin(nx)\,dx}{\int_0^\pi \sin^2(nx)\,dx} = \frac{-x\cos(nx)/n + \sin(nx)/n^2\big|_0^\pi}{x/2 - \sin(2nx)/(4n)\big|_0^\pi} \tag{6.3.7}$$

$$= -\frac{2}{n}\cos(n\pi) = \frac{2}{n}(-1)^n. \tag{6.3.8}$$

Equation (6.3.1) then gives

$$f(x) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nx). \tag{6.3.9}$$

---

[7] Tolstov, G. P., 1962: *Fourier Series*, Dover Publishers, Mineola, NY, p. 255.

[8] Apparently first discussed by Weyl, H., 1910: Die Gibbs'sche Erscheinung in der Theorie der Sturm-Liouvilleschen Reihen. *Rend. Circ. Mat. Palermo*, **29**, 321–323.

This particular example is in fact an example of a half-range sine expansion.

Finally we must state the values of x for which (6.3.9) is valid. At $x = \pi$ the series converges to zero while $f(\pi) = \pi$. At $x = 0$ both the series and the function converge to zero. Hence the series expansion (6.3.9) is valid for $0 \le x < \pi$.

• **Example 6.3.2**

For our second example let us find the expansion for $f(x) = x$ over the interval $0 \le x < \pi$ using the solution to the regular Sturm-Liouville problem of

$$y'' + \lambda y = 0, \qquad y(0) = y(\pi) - y'(\pi) = 0. \tag{6.3.10}$$

We will encounter this problem when we solve the heat equation with radiative boundary conditions by separation of variables.

Because $r(x) = 1$, $a = 0$, $b = \pi$ and the eigenfunctions are $y_n(x) = \sin(k_n x)$, where $k_n = \tan(k_n\pi)$, (6.3.4) give

$$c_n = \frac{\int_0^\pi x\sin(k_n x)\,dx}{\int_0^\pi \sin^2(k_n x)\,dx} = \frac{\int_0^\pi x\sin(k_n x)\,dx}{\frac{1}{2}\int_0^\pi [1-\cos(2k_n x)]\,dx} \tag{6.3.11}$$

$$= \frac{2\sin(k_n x)/k_n^2 - 2x\cos(k_n x)/k_n\big|_0^\pi}{x - \sin(2k_n x)/(2k_n)\big|_0^\pi} \tag{6.3.12}$$

$$= \frac{2\sin(k_n\pi)/k_n^2 - 2\pi\cos(k_n\pi)/k_n}{\pi - \sin(2k_n\pi)/(2k_n)} \tag{6.3.13}$$

$$= \frac{2[\cos(k_n\pi) - \pi\cos(k_n\pi)]/k_n}{\pi - \cos^2(k_n\pi)}, \tag{6.3.14}$$

where we have used the property that $\sin(k_n\pi) = k_n\cos(k_n\pi)$. Equation (6.3.1) then gives

$$f(x) = 2(1-\pi)\sum_{n=1}^{\infty} \frac{\cos(k_n\pi)}{k_n[\pi - \cos^2(k_n\pi)]}\sin(k_n x). \tag{6.3.15}$$

Problems

1. The Sturm-Liouville problem $y'' + \lambda y = 0$, $y(0) = y(L) = 0$ has the eigenfunction solution $y_n(x) = \sin(n\pi x/L)$. Find the eigenfunction expansion for $f(x) = x$ using this eigenfunction.

2. The Sturm-Liouville problem $y'' + \lambda y = 0$, $y'(0) = y'(L) = 0$ has the eigenfunction solutions $y_0(x) = 1$ and $y_n(x) = \cos(n\pi x/L)$. Find the eigenfunction expansion for $f(x) = x$ using these eigenfunctions.

3. The Sturm-Liouville problem $y'' + \lambda y = 0$, $y(0) = y'(L) = 0$ has the eigenfunction solution $y_n(x) = \sin[(2n-1)\pi x/(2L)]$. Find the eigenfunction expansion for $f(x) = x$ using this eigenfunction.

4. The Sturm-Liouville problem $y'' + \lambda y = 0$, $y'(0) = y(L) = 0$ has the eigenfunction solution $y_n(x) = \cos[(2n-1)\pi x/(2L)]$. Find the eigenfunction expansion for $f(x) = x$ using this eigenfunction.

6.4 A SINGULAR STURM-LIOUVILLE PROBLEM: LEGENDRE'S EQUATION

In the previous sections we used solutions to a regular Sturm-Liouville problem in the eigenfunction expansion of the function $f(x)$. The fundamental reason why we could form such an expansion was the orthogonality condition (6.2.1). This crucial property allowed us to solve for the Fourier coefficient c_n given by (6.3.4).

In the next few chapters, when we solve partial differential equations in cylindrical and spherical coordinates, we will find that $f(x)$ must be expanded in terms of eigenfunctions from singular Sturm-Liouville problems. Is this permissible? How do we compute the Fourier coefficients in this case? The final two sections of this chapter deal with these questions by examining the two most frequently encountered singular Sturm-Liouville problems, those involving Legendre's and Bessel's equations.

We begin by determining the orthogonality condition for singular Sturm-Liouville problems. Returning to the beginning portions of Section 6.2, we combine (6.2.5) and (6.2.6) to obtain

$$(\lambda_n - \lambda_m)\int_a^b r(x) y_n y_m \, dx = [p(b)y_m'(b)y_n(b) - p(b)y_n'(b)y_m(b) - p(a)y_m'(a)y_n(a) + p(a)y_n'(a)y_m(a)]. \tag{6.4.1}$$

From (6.4.1) the right side vanishes and we preserve orthogonality if $y_n(x)$ is finite and $p(x)y_n'(x)$ tends to zero at both endpoints. This is not the only choice but let us see where it leads.

Consider now Legendre's equation:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \tag{6.4.2}$$

or

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0, \tag{6.4.3}$$

where we set $a = -1$, $b = 1$, $\lambda = n(n+1)$, $p(x) = 1 - x^2$, $q(x) = 0$, and $r(x) = 1$. This equation arises in the solution of partial differential

Figure 6.4.1: Born into an affluent family, Adrien-Marie Legendre's (1752–1833) modest family fortune was sufficient to allow him to devote his life to research in celestial mechanics, number theory, and the theory of elliptic functions. In July 1784 he read before the *Académie des sciences* his *Recherches sur la figure des planètes*. It is in this paper that Legendre polynomials first appeared. (Portrait courtesy of the Archives de l'Académie des sciences, Paris.)

equations involving spherical geometry. Because $p(-1) = p(1) = 0$, we are faced with a singular Sturm-Liouville problem. Before we can determine if any of its solutions can be used in an eigenfunction expansion, we must find them.

Equation (6.4.2) does not have a simple general solution. [If $n = 0$, then $y(x) = 1$ is a solution.] Consequently we try to solve it with the power series:

$$y(x) = \sum_{k=0}^{\infty} A_k x^k, \tag{6.4.4}$$

$$y'(x) = \sum_{k=0}^{\infty} kA_k x^{k-1} \tag{6.4.5}$$

and

$$y''(x) = \sum_{k=0}^{\infty} k(k-1)A_k x^{k-2}. \tag{6.4.6}$$

Substituting into (6.4.2),

$$\sum_{k=0}^{\infty} k(k-1)A_k x^{k-2} + \sum_{k=0}^{\infty} [n(n+1) - 2k - k(k-1)]\, A_k x^k = 0, \tag{6.4.7}$$

which equals

$$\sum_{m=2}^{\infty} m(m-1)A_m x^{m-2} + \sum_{k=0}^{\infty} [n(n+1) - k(k+1)]\, A_k x^k = 0. \tag{6.4.8}$$

If we define $k = m + 2$ in the first summation, then

$$\sum_{k=0}^{\infty} (k+2)(k+1)A_{k+2} x^k + \sum_{k=0}^{\infty} [n(n+1) - k(k+1)]\, A_k x^k = 0. \tag{6.4.9}$$

Because (6.4.9) must be true for any x, each power of x must vanish separately. It then follows that

$$(k+2)(k+1)A_{k+2} = [k(k+1) - n(n+1)]A_k \tag{6.4.10}$$

or

$$A_{k+2} = \frac{[k(k+1) - n(n+1)]}{(k+1)(k+2)} A_k, \tag{6.4.11}$$

where $k = 0, 1, 2, \ldots$ Note that we still have the two arbitrary constants A_0 and A_1 that are necessary for the general solution of (6.4.2).

The first few terms of the solution associated with A_0 are

$$u_p(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \frac{n(n-2)(n-4)(n+1)(n+3)(n+5)}{6!}x^6 + \cdots \tag{6.4.12}$$

while the first few terms associated with the A_1 coefficient are

$$v_p(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \frac{(n-1)(n-3)(n-5)(n+2)(n+4)(n+6)}{7!}x^7 + \cdots \tag{6.4.13}$$

If n is an *even* positive integer (including $n = 0$), then the series (6.6.12) terminates with the term involving x^n: the solution is a polynomial of degree n. Similarly, if n is an *odd* integer, the series (6.4.13) terminates with the term involving x^n. Otherwise, for n noninteger the expressions are infinite series.

For reasons that will become apparent, we restrict ourselves to positive integers n. Actually, this includes all possible integers because the negative integer $-n-1$ has the same Legendre's equation and solution as the positive integer n. These polynomials are *Legendre polynomials*[9] and we may compute them by the power series:

$$P_n(x) = \sum_{k=0}^{m} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}, \qquad \textbf{(6.4.14)}$$

where $m = n/2$ or $m = (n-1)/2$, depending upon which is an integer. We have chosen to use (6.4.14) over (6.4.12) or (6.4.13) because (6.4.14) has the advantage that $P_n(1) = 1$. Table 6.4.1 gives the first ten Legendre polynomials.

The other solution, the infinite series, is the Legendre function of the second kind, $Q_n(x)$. Figure 6.4.2 illustrates the first four Legendre polynomials $P_n(x)$ while Figure 6.4.3 gives the first four Legendre functions of the second kind Q_n. From this figure we see that $Q_n(x)$ becomes infinite at the points $x = \pm 1$. As shown earlier, this is important because we are only interested in solutions to Legendre's equation that are finite over the interval $[-1, 1]$. On the other hand, in problems where we exclude the points $x = \pm 1$, Legendre functions of the second kind will appear in the general solution.[10]

In the case that n is not an integer, we can construct a solution[11] that remains finite at $x = 1$ but not at $x = -1$. Furthermore, we can

---

[9] Legendre, A. M., 1785: Sur l'attraction des sphéroïdes homogénes. *Mém. math. phys. présentés à l'Acad. sci. pars divers savants*, **10**, 411–434. The best reference on Legendre polynomials is given by Hobson, E. W., 1965: *The Theory of Spherical and Ellipsoidal Harmonics,* Chelsea Publishing Co., New York.

[10] See Smythe, W. R., 1950: *Static and Dynamic Electricity,* McGraw-Hill, New York, Section 5.215 for an example.

[11] See Carrier, G. F., Krook, M., and Pearson, C. E., 1966: *Functions of the Complex Variable: Theory and Technique*, McGraw-Hill, New York, pp. 212–213.

Table 6.4.1: The First Ten Legendre Polynomials.

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \tfrac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$P_6(x) = \tfrac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$$

$$P_7(x) = \tfrac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$$

$$P_8(x) = \tfrac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)$$

$$P_9(x) = \tfrac{1}{128}(12155x^9 - 25740x^7 + 18018x^5 - 4620x^3 + 315x)$$

$$P_{10}(x) = \tfrac{1}{256}(46189x^{10} - 109395x^8 + 90090x^6 - 30030x^4 + 3465x^2 - 63)$$

construct a solution which is finite at $x = -1$ but not at $x = 1$. Because our solutions must be finite at both endpoints so that we can use them in an eigenfunction expansion, we must reject these solutions from further consideration and are left only with Legendre polynomials. From now on, we will only consider the properties and uses of these polynomials.

Although we have the series (6.4.14) to compute $P_n(x)$, there are several alternative methods. We obtain the first method, known as *Rodrigues' formula*,[12] by writing (6.4.14) in the form

$$P_n(x) = \frac{1}{2^n n!} \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2n-2k)!}{(n-2k)!} x^{n-2k} \tag{6.4.15}$$

$$= \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[\sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} x^{2n-2k} \right]. \tag{6.4.16}$$

The last summation is the binomial expansion of $(x^2 - 1)^n$ so that

[12] Rodriques, O., 1816: Mémoire sur l'attraction des sphéroïdes. *Correspond. l'Ecole Polytech.*, **3**, 361–385.

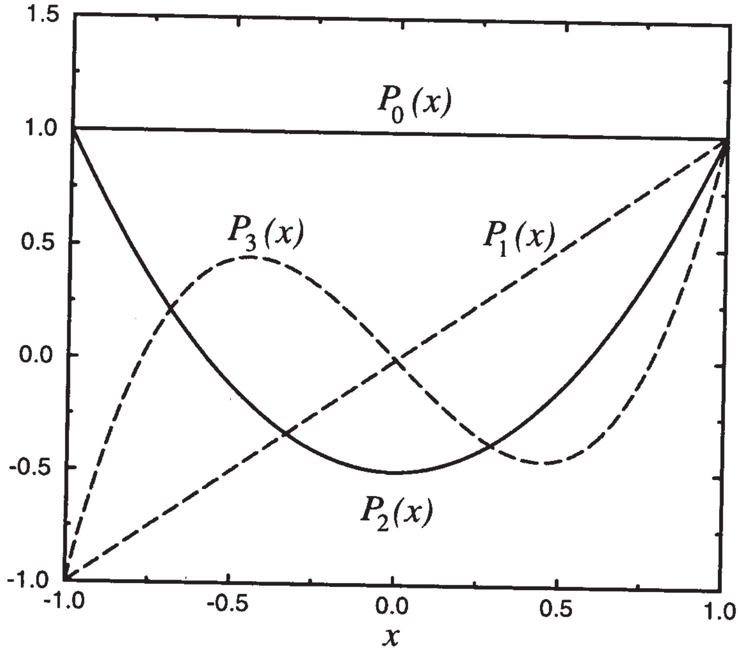


Figure 6.4.2: The first four Legendre functions of the first kind.

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n. \tag{6.4.17}$$

Another method for computing $P_n(x)$ involves the use of recurrence formulas. The first step in finding these formulas is to establish the fact that

$$(1 + h^2 - 2xh)^{-1/2} = P_0(x) + hP_1(x) + h^2P_2(x) + \cdots \tag{6.4.18}$$

The function $(1 + h^2 - 2xh)^{-1/2}$ is the *generating function* for $P_n(x)$. We obtain the expansion via the formal binomial expansion

$$(1+h^2-2xh)^{-1/2} = 1 + \tfrac{1}{2}(2xh - h^2) + \tfrac{1}{2}\tfrac{3}{2}\tfrac{1}{2!}(2xh - h^2)^2 + \cdots \tag{6.4.19}$$

Upon expanding the terms contained in $2x - h^2$ and grouping like powers of h,

$$(1 + h^2 - 2xh)^{-1/2} = 1 + xh + (\tfrac{3}{2}x^2 - \tfrac{1}{2})h^2 + \cdots \tag{6.4.20}$$

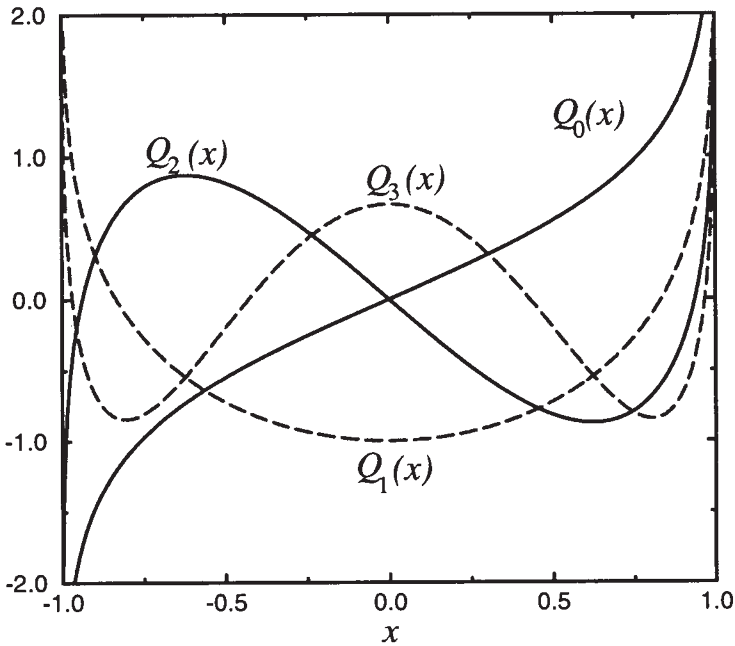


Figure 6.4.3: The first four Legendre functions of the second kind.

A direct comparison between the coefficients of each power of h and the Legendre polynomial $P_n(x)$ completes the demonstration. Note that these results hold only if $|x|$ and $|h| < 1$.

Next we define $W(x, h) = (1 + h^2 - 2xh)^{-1/2}$. A quick check shows that $W(x, h)$ satisfies the first-order partial differential equation

$$(1 - 2xh + h^2)\frac{\partial W}{\partial h} + (h - x)W = 0. \tag{6.4.21}$$

The substitution of (6.4.18) into (6.4.21) yields

$$(1 - 2xh + h^2)\sum_{n=0}^{\infty} nP_n(x)h^{n-1} + (h - x)\sum_{n=0}^{\infty} P_n(x)h^n = 0. \tag{6.4.22}$$

Setting the coefficients of h^n equal to zero, we find that

$$(n + 1)P_{n+1}(x) - 2nxP_n(x) + (n - 1)P_{n-1}(x) + P_{n-1}(x) - xP_n(x) = 0 \tag{6.4.23}$$

or

$$\boxed{(n + 1)P_{n+1}(x) - (2n + 1)xP_n(x) + nP_{n-1}(x) = 0} \tag{6.4.24}$$

with $n = 1, 2, 3, \ldots$

Similarly, the first-order partial differential equation

$$(1 - 2xh + h^2)\frac{\partial W}{\partial x} - hW = 0 \tag{6.4.25}$$

leads to

$$(1 - 2xh + h^2)\sum_{n=0}^{\infty} P_n'(x)h^n - \sum_{n=0}^{\infty} P_n(x)h^{n+1} = 0, \tag{6.4.26}$$

which implies

$$P_{n+1}'(x) - 2xP_n'(x) + P_{n-1}'(x) - P_n(x) = 0. \tag{6.4.27}$$

Differentiating (6.4.24), we first eliminate $P_{n-1}'(x)$ and then $P_{n+1}'(x)$ from the resulting equations and (6.4.27). This gives two further recurrence relationships:

$$P_{n+1}'(x) - xP_n'(x) - (n+1)P_n(x) = 0, \quad n = 0, 1, 2, \ldots \tag{6.4.28}$$

and

$$xP_n'(x) - P_{n-1}'(x) - nP_n(x) = 0, \quad n = 1, 2, 3, \ldots \tag{6.4.29}$$

Adding (6.4.28) and (6.4.29), we obtain the more symmetric formula

$$\boxed{P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x), \qquad n = 1, 2, 3, \ldots} \tag{6.4.30}$$

Given any two of the polynomials $P_{n+1}(x)$, $P_n(x)$ and $P_{n-1}(x)$, (6.4.24) or (6.4.30) yields the third.

Having determined several methods for finding the Legendre polynomial $P_n(x)$, we now turn to the actual orthogonality condition.[13] Consider the integral

$$J = \int_{-1}^{1} \frac{dx}{\sqrt{1 + h^2 - 2xh}\,\sqrt{1 + t^2 - 2xt}}, \qquad |h|, |t| < 1 \tag{6.4.31}$$

$$= \int_{-1}^{1} [P_0(x) + hP_1(x) + \cdots + h^n P_n(x) + \cdots] \times [P_0(x) + tP_1(x) + \cdots + t^n P_n(x) + \cdots]\, dx \tag{6.4.32}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h^n t^m \int_{-1}^{1} P_n(x)P_m(x)\, dx. \tag{6.4.33}$$

---

[13] From Symons, B., 1982: Legendre polynomials and their orthogonality. *Math. Gaz.*, **66**, 152–154 with permission.

On the other hand, if $a = (1+h^2)/2h$ and $b = (1+t^2)/2t$, the integral J is

$$J = \int_{-1}^{1} \frac{dx}{\sqrt{1+h^2-2xh}\,\sqrt{1+t^2-2xt}} \tag{6.4.34}$$

$$= \frac{1}{2\sqrt{ht}} \int_{-1}^{1} \frac{dx}{\sqrt{a-x}\,\sqrt{b-x}} = \frac{1}{\sqrt{ht}} \int_{-1}^{1} \frac{\frac{1}{2}\left(\frac{1}{\sqrt{a-x}} + \frac{1}{\sqrt{b-x}}\right)}{\sqrt{a-x}+\sqrt{b-x}}\, dx \tag{6.4.35}$$

$$= -\frac{1}{\sqrt{ht}} \ln(\sqrt{a-x}+\sqrt{b-x})\Big|_{-1}^{1} = \frac{1}{\sqrt{ht}} \ln\left(\frac{\sqrt{a+1}+\sqrt{b+1}}{\sqrt{a-1}+\sqrt{b-1}}\right). \tag{6.4.36}$$

But $a+1 = (1+h^2+2h)/2h = (1+h)^2/2h$ and $a-1 = (1-h)^2/2h$. After a little algebra,

$$J = \frac{1}{\sqrt{ht}} \ln\left(\frac{1+\sqrt{ht}}{1-\sqrt{ht}}\right) = \frac{2}{\sqrt{ht}}\left(\sqrt{ht} + \frac{1}{3}\sqrt{(ht)^3} + \frac{1}{5}\sqrt{(ht)^5} + \cdots\right) \tag{6.4.37}$$

$$= 2\left(1 + \frac{ht}{3} + \frac{h^2t^2}{5} + \cdots + \frac{h^n t^n}{2n+1} + \cdots\right). \tag{6.4.38}$$

As we noted earlier, the coefficients of $h^n t^m$ in this series is $\int_{-1}^{1} P_n(x)$ $P_m(x)\,dx$. If we match the powers of $h^n t^m$, the orthogonality condition is

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n. \end{cases} \tag{6.4.39}$$

With the orthogonality condition (6.4.39) we are ready to show that we can represent a function $f(x)$, which is piece-wise differentiable in the interval $(-1,1)$, by the series:

$$f(x) = \sum_{m=0}^{\infty} A_m P_m(x), \qquad -1 \le x \le 1. \tag{6.4.40}$$

To find A_m we multiply both sides of (6.4.40) by $P_n(x)$ and integrate from -1 to 1:

$$\int_{-1}^{1} f(x)P_n(x)\,dx = \sum_{m=0}^{\infty} A_m \int_{-1}^{1} P_n(x)P_m(x)\,dx. \qquad \textbf{(6.4.41)}$$

All of the terms on the right side vanish except for $n = m$ because of the orthogonality condition (6.4.39). Consequently, the coefficient A_n is

$$A_n \int_{-1}^{1} P_n^2(x)\,dx = \int_{-1}^{1} f(x)P_n(x)\,dx \qquad \textbf{(6.4.42)}$$

or

$$A_n = \frac{2n+1}{2} \int_{-1}^{1} f(x)P_n(x)\,dx. \qquad \textbf{(6.4.43)}$$

In the special case when $f(x)$ and its first n derivatives are continuous throughout the interval $(-1,1)$, we may use Rodrigues' formula to evaluate

$$\int_{-1}^{1} f(x)P_n(x)\,dx = \frac{1}{2^n n!} \int_{-1}^{1} f(x)\frac{d^n(x^2-1)^n}{dx^n}\,dx \qquad \textbf{(6.4.44)}$$

$$= \frac{(-1)^n}{2^n n!} \int_{-1}^{1} (x^2-1)^n f^{(n)}(x)\,dx \qquad \textbf{(6.4.45)}$$

by integrating by parts n times. Consequently,

$$A_n = \frac{2n+1}{2^{n+1} n!} \int_{-1}^{1} (1-x^2)^n f^{(n)}(x)\,dx. \qquad \textbf{(6.4.46)}$$

A particularly useful result follows from (6.4.46) if $f(x)$ is a polynomial of degree k. Because all derivatives of $f(x)$ of order n vanish identically when $n > k$, $A_n = 0$ if $n > k$. Consequently, any polynomial of degree k can be expressed as a linear combination of the first $k+1$ Legendre polynomials $[P_0(x), \ldots, P_k(x)]$. Another way of viewing this result is to recognize that any polynomial of degree k is an expansion in powers of x. When we expand in Legendre polynomials we are merely regrouping these powers of x into new groups that can be identified as $P_0(x), P_1(x), P_2(x), \ldots, P_k(x)$.

- **Example 6.4.1**

Let us use Rodrigues' formula to compute $P_2(x)$. From (6.4.17) with $n = 2$,

$$P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}[(x^2-1)^2] = \frac{1}{8}\frac{d^2}{dx^2}(x^4 - 2x^2 - 1) = \frac{1}{2}(3x^2 - 1). \tag{6.4.47}$$

- **Example 6.4.2**

Let us compute $P_3(x)$ from a recurrence relation. From (6.4.24) with $n = 2$,

$$3P_3(x) - 5xP_2(x) + 2P_1(x) = 0. \tag{6.4.48}$$

But $P_2(x) = (3x^2 - 1)/2$ and $P_1(x) = x$, so that

$$3P_3(x) = 5xP_2(x) - 2P_1(x) = 5x[(3x^2-1)/2] - 2x = \tfrac{15}{2}x^3 - \tfrac{9}{2}x \tag{6.4.49}$$

or

$$P_3(x) = (5x^3 - 3x)/2. \tag{6.4.50}$$

- **Example 6.4.3**

We want to show that

$$\int_{-1}^{1} P_n(x)\,dx = 0. \tag{6.4.51}$$

From (6.4.30),

$$(2n+1)\int_{-1}^{1} P_n(x)\,dx = \int_{-1}^{1} [P'_{n+1}(x) - P'_{n-1}(x)]\,dx \tag{6.4.52}$$

$$= P_{n+1}(x) - P_{n-1}(x)\big|_{-1}^{1} \tag{6.4.53}$$

$$= P_{n+1}(1) - P_{n-1}(1) - P_{n+1}(-1) + P_{n-1}(-1) = 0, \tag{6.4.54}$$

because $P_n(1) = 1$ and $P_n(-1) = (-1)^n$.

- **Example 6.4.4**

Let us express $f(x) = x^2$ in terms of Legendre polynomials. The results from (6.4.46) mean that we need only worry about $P_0(x)$, $P_1(x)$, and $P_2(x)$:

$$x^2 = A_0P_0(x) + A_1P_1(x) + A_2P_2(x). \tag{6.4.55}$$

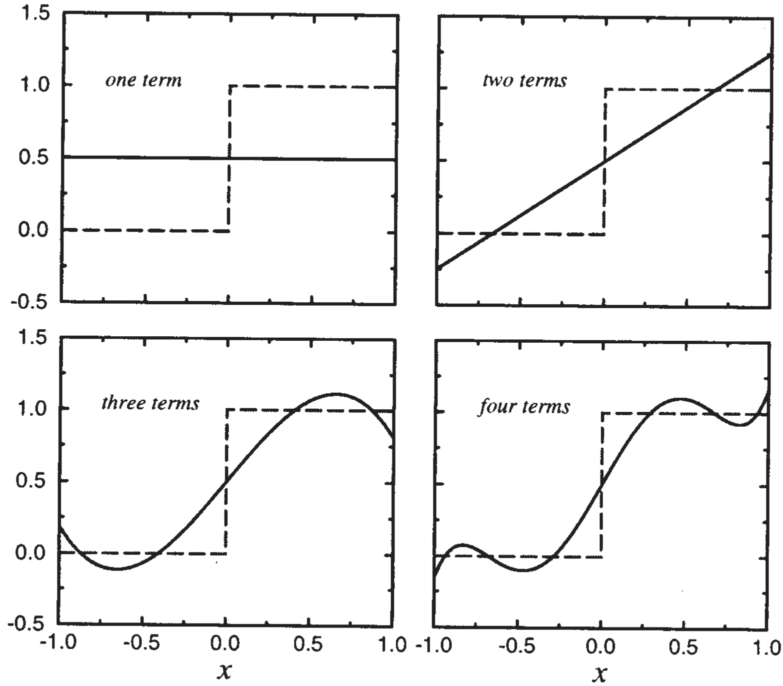


Figure 6.4.4: Representation of the function $f(x) = 1$ for $0 < x < 1$ and 0 for $-1 < x < 0$ by various partial summations of its Legendre polynomial expansion. The dashed lines denote the exact function.

Substituting for the Legendre polynomials,

$$x^2 = A_0 + A_1 x + \tfrac{1}{2}A_2(3x^2 - 1) \tag{6.4.56}$$

and

$$A_0 = \tfrac{1}{3}, \quad A_1 = 0 \quad \text{and} \quad A_2 = \tfrac{2}{3}. \tag{6.4.57}$$

• **Example 6.4.5**

Let us find the expansion in Legendre polynomials of the function:

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 1. \end{cases} \tag{6.4.58}$$

We could have done this expansion as a Fourier series but in the solution of partial differential equations on a sphere we must make the expansion in Legendre polynomials.

In this problem, we find that

$$A_n = \frac{2n+1}{2} \int_0^1 P_n(x)\, dx. \tag{6.4.59}$$

Therefore,

$$A_0 = \tfrac{1}{2}\int_0^1 1\,dx = \tfrac{1}{2}, \qquad A_1 = \tfrac{3}{2}\int_0^1 x\,dx = \tfrac{3}{4}, \tag{6.4.60}$$

$$A_2 = \tfrac{5}{2}\int_0^1 \tfrac{1}{2}(3x^2-1)\,dx = 0 \quad \text{and} \quad A_3 = \tfrac{7}{2}\int_0^1 \tfrac{1}{2}(5x^3-3x)\,dx = -\tfrac{7}{16} \tag{6.4.61}$$

so that

$$f(x) = \tfrac{1}{2}P_0(x) + \tfrac{3}{4}P_1(x) - \tfrac{7}{16}P_3(x) + \tfrac{11}{32}P_5(x) + \cdots \tag{6.4.62}$$

Figure 6.4.4 illustrates the expansion (6.4.62) where we have used only the first four terms. As we add each additional term in the orthogonal expansion, the expansion fits $f(x)$ better in the "least squares" sense of (6.3.5). The spurious oscillations arise from trying to represent a discontinuous function by four continuous, oscillatory functions. Even if we add additional terms, the spurious oscillations will persist although located nearer to the discontinuity. This is another example of *Gibbs phenomena*.[14] See Section 2.2.

• Example 6.4.6: Iterative solution of the radiative transfer equation

One of the fundamental equations of astrophysics is the integro-differential equation that describes radiative transfer (the propagation of energy by radiative, rather than conductive or convective, processes) in a gas.

Consider a gas which varies in only one spatial direction that we divide into infinitesimally thin slabs. As radiation enters a slab, it is absorbed and scattered. If we assume that all of the radiation undergoes isotropic scattering, the radiative transfer equation is

$$\mu\frac{dI}{d\tau} = I - \tfrac{1}{2}\int_{-1}^{1} I\,d\mu, \tag{6.4.63}$$

where I is the intensity of the radiation, τ is the optical depth (a measure of the absorbing power of the gas and related to the distance that you have traveled within the gas), $\mu = \cos(\theta)$, and θ is the angle at which radiation enters the slab. In this example, we show how the Fourier-Legendre expansion[15]

$$I(\tau,\mu) = \sum_{n=0}^{\infty} I_n(\tau)P_n(\mu) \tag{6.4.64}$$

---

[14] Weyl, H., 1910: Die Gibbs'sche Erscheinung in der Theorie der Kugelfunktionen. *Rend. Circ. Mat. Palermo*, **29**, 308–321.

[15] Chandrasekhar, S., 1944: On the radiative equilibrium of a stellar atmosphere. *Astrophys. J.*, **99**, 180–190. Published by University of Chicago Press, ©1944.

may be used to solve (6.4.63). Here $I_n(\tau)$ is the Fourier coefficient in the Fourier-Legendre expansion involving the Legendre polynomial $P_n(\mu)$.

We begin by substituting (6.4.64) into (6.4.63),

$$\sum_{n=0}^{\infty} \frac{[(n+1)P_{n+1}(\mu) + nP_{n-1}(\mu)]}{2n+1} \frac{dI_n}{d\tau} = \sum_{n=0}^{\infty} I_n P_n(\mu) - I_0, \quad (6.4.65)$$

where we have used (6.4.24) to eliminate $\mu P_n(\mu)$. Note that only the $I_0(\tau)$ term remains after integrating because of the orthogonality condition:

$$\int_{-1}^{1} 1 \cdot P_n(\mu)\, d\mu = \int_{-1}^{1} P_0(\mu) P_n(\mu)\, d\mu = 0, \quad (6.4.66)$$

if $n > 0$. Equating the coefficients of the various Legendre polynomials,

$$\frac{n}{2n-1} \frac{dI_{n-1}}{d\tau} + \frac{n+1}{2n+3} \frac{dI_{n+1}}{d\tau} = I_n \quad (6.4.67)$$

for $n = 1, 2, \ldots$ and

$$\frac{dI_1}{d\tau} = 0. \quad (6.4.68)$$

Thus, the solution for I_1 is I_1 = constant $= 3F/4$, where F is the net integrated flux and an observable quantity.

For $n = 1$,

$$\frac{dI_0}{d\tau} + \frac{2}{5} \frac{dI_2}{d\tau} = I_1 = \frac{3F}{4}. \quad (6.4.69)$$

Therefore,

$$I_0 + \tfrac{2}{5} I_2 = \tfrac{3}{4} F\tau + A. \quad (6.4.70)$$

The next differential equation arises from $n = 2$ and equals

$$\frac{2}{3} \frac{dI_1}{d\tau} + \frac{3}{7} \frac{dI_3}{d\tau} = I_2. \quad (6.4.71)$$

Because I_1 is a constant and we only retain I_0, I_1, and I_2 in the simplest approximation, we neglect $dI_3/d\tau$ and $I_2 = 0$. Thus, the simplest approximate solution is

$$I_0 = \tfrac{3}{4} F\tau + A, \quad I_1 = \tfrac{3}{4} F \quad \text{and} \quad I_2 = 0. \quad (6.4.72)$$

To complete our approximate solution, we must evaluate A. If we are dealing with a stellar atmosphere where we assume no external radiation incident on the star, $I(0, \mu) = 0$ for $-1 \le \mu < 0$. Therefore,

$$\int_{-1}^{1} I(\tau, \mu) P_n(\mu)\, d\mu = \sum_{m=0}^{\infty} I_m(\tau) \int_{-1}^{1} P_m(\mu) P_n(\mu)\, d\mu = \frac{2}{2n+1} I_n(\tau). \quad (6.4.73)$$

Taking the limit $\tau \to 0$ and using the boundary condition,

$$\frac{2}{2n+1} I_n(0) = \int_0^1 I(0,\mu) P_n(\mu)\, d\mu = \sum_{m=0}^{\infty} I_m(0) \int_0^1 P_n(\mu) P_m(\mu)\, d\mu. \tag{6.4.74}$$

Thus, we must satisfy, in principle, an infinite set of equations. For example, for $n = 0$, 1, and 2,

$$2I_0(0) = I_0(0) + \tfrac{1}{2}I_1(0) - \tfrac{1}{8}I_3(0) + \tfrac{1}{16}I_5(0) + \cdots \tag{6.4.75}$$

$$\tfrac{2}{3}I_1(0) = \tfrac{1}{2}I_0(0) + \tfrac{1}{3}I_1(0) + \tfrac{1}{8}I_2(0) - \tfrac{1}{48}I_4(0) + \cdots \tag{6.4.76}$$

and

$$\tfrac{2}{5}I_2(0) = \tfrac{1}{8}I_1(0) + \tfrac{1}{5}I_2(0) + \tfrac{1}{8}I_3(0) - \tfrac{5}{128}I_5(0) + \cdots. \tag{6.4.77}$$

Using $I_1(0) = 3F/4$,

$$\tfrac{1}{2}I_0(0) + \tfrac{1}{16}I_3(0) - \tfrac{1}{32}I_5(0) + \cdots = \tfrac{3}{16}F, \tag{6.4.78}$$

$$\tfrac{1}{2}I_0(0) + \tfrac{1}{8}I_2(0) - \tfrac{1}{48}I_4(0) + \cdots = \tfrac{1}{4}F \tag{6.4.79}$$

and

$$\tfrac{2}{5}I_2(0) - \tfrac{1}{4}I_3(0) + \tfrac{5}{64}I_5(0) + \cdots = \tfrac{3}{16}F. \tag{6.4.80}$$

Of the two possible Equations (6.4.78)–(6.4.79), Chandrasekhar chose (6.4.79) from physical considerations. Thus, to first approximation, the solution is

$$I(\mu,\tau) = \tfrac{3}{4}F\left(\tau + \tfrac{2}{3}\right) + \tfrac{3}{4}F\mu + \cdots. \tag{6.4.81}$$

Better approximations can be obtained by including more terms; the interested reader is referred to the original article. In the early 1950s, Wang and Guth[16] improved the procedure for finding the successive approximations and formulating the approximate boundary conditions.

Problems

Find the first three nonvanishing coefficients in the Legendre polynomial expansion for the following functions:

1. $f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$

2. $f(x) = \begin{cases} 1/(2\epsilon), & |x| < \epsilon \\ 0, & \epsilon < |x| < 1 \end{cases}$

3. $f(x) = |x|, \qquad |x| < 1$

4. $f(x) = x^3, \qquad |x| < 1$

5. $f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$

6. $f(x) = \begin{cases} -1, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$

---

[16] Wang, M. C. and Guth, E., 1951: On the theory of multiple scattering, particularly of charged particles. *Phys. Rev., Ser. 2*, **84**, 1092–1111.

7. Use Rodrigues' formula to show that $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$.

8. Given $P_5(x) = \frac{63}{8}x^5 - \frac{70}{8}x^3 + \frac{15}{8}x$ and $P_4(x)$ from problem 7, use the recurrence formula for $P_{n+1}(x)$ to find $P_6(x)$.

9. Show that (a) $P_n(1) = 1$, (b) $P_n(-1) = (-1)^n$, (c) $P_{2n+1}(0) = 0$ and (d) $P_{2n}(0) = (-1)^n(2n)!/(2^{2n}n!n!)$.

10. Prove that

$$\int_x^1 P_n(t)\,dt = \frac{1}{2n+1}[P_{n-1}(x) - P_{n+1}(x)].$$

11. Given[17]

$$P_n[\cos(\theta)] = \frac{2}{\pi}\int_0^\theta \frac{\cos[(n+\frac{1}{2})x]}{\sqrt{2[\cos(x) - \cos(\theta)]}}\,dx$$
$$= \frac{2}{\pi}\int_\theta^\pi \frac{\sin[(n+\frac{1}{2})x]}{\sqrt{2[\cos(\theta) - \cos(x)]}}\,dx,$$

show that the following generalized Fourier series hold:

$$\frac{H(\theta - t)}{\sqrt{2\cos(t) - 2\cos(\theta)}} = \sum_{n=0}^{\infty} P_n[\cos(\theta)]\cos\left[\left(n+\tfrac{1}{2}\right)t\right], \quad 0 \le t < \theta \le \pi,$$

if we use the eigenfunction $y_n(x) = \cos\left[\left(n+\frac{1}{2}\right)x\right]$, $0 < x < \pi$, $r(x) = 1$ and $H(\ )$ is Heaviside's step function, and

$$\frac{H(t - \theta)}{\sqrt{2\cos(\theta) - 2\cos(t)}} = \sum_{n=0}^{\infty} P_n[\cos(\theta)]\sin\left[\left(n+\tfrac{1}{2}\right)t\right], \quad 0 \le \theta < t \le \pi,$$

if we use the eigenfunction $y_n(x) = \sin\left[\left(n+\frac{1}{2}\right)x\right]$, $0 < x < \pi$, $r(x) = 1$ and $H(\ )$ is Heaviside's step function.

12. The series given in problem 11 are also expansions in Legendre polynomials. In that light, show that

$$\int_0^t \frac{P_n[\cos(\theta)]\sin(\theta)}{\sqrt{2\cos(\theta) - 2\cos(t)}}\,d\theta = \frac{\sin\left[\left(n+\frac{1}{2}\right)t\right]}{n+\frac{1}{2}}$$

and

$$\int_t^\pi \frac{P_n[\cos(\theta)]\sin(\theta)}{\sqrt{2\cos(t) - 2\cos(\theta)}}\,d\theta = \frac{\cos\left[\left(n+\frac{1}{2}\right)t\right]}{n+\frac{1}{2}},$$

where $0 < t < \pi$.

---

[17] Hobson, E. W., 1965: *The Theory of Spherical and Ellipsoidal Harmonics*, Chelsea Publishing Co., New York, pp. 26–27.

6.5 ANOTHER SINGULAR STURM-LIOUVILLE PROBLEM: BESSEL'S EQUATION

In the previous section we discussed the solutions to Legendre's equation, especially with regard to their use in orthogonal expansions. In the section we consider another classic equation, Bessel's equation[18]

$$x^2y'' + xy' + (\mu^2x^2 - n^2)y = 0 \tag{6.5.1}$$

or

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \left(\mu^2 x - \frac{n^2}{x}\right)y = 0. \tag{6.5.2}$$

Once again, our ultimate goal is the use of its solutions in orthogonal expansions. These orthogonal expansions, in turn, are used in the solution of partial differential equations in cylindrical coordinates.

A quick check of Bessel's equation shows that it conforms to the canonical form of the Sturm-Liouville problem: $p(x) = x$, $q(x) = -n^2/x$, $r(x) = x$, and $\lambda = \mu^2$. Restricting our attention to the interval $[0, L]$, the Sturm-Liouville problem involving (6.5.2) is singular because $p(0) = 0$. From (6.4.1) in the previous section, the eigenfunctions to a singular Sturm-Liouville problem will still be orthogonal over the interval $[0, L]$ if (1) $y(x)$ is finite and $xy'(x)$ is zero at $x = 0$, and (2) $y(x)$ satisfies the homogeneous boundary condition (6.1.2) at $x = L$. Consequently, we will only seek solutions that satisfy these conditions.

We cannot write down the solution to Bessel's equation in a simple closed form; as in the case with Legendre's equation, we must find the solution by power series. Because we intend to make the expansion about $x = 0$ and this point is a regular singular point, we must use the method of Frobenius, where n is an integer.[19] Moreover, because the quantity n^2 appears in (6.5.2), we may take n to be nonnegative without any loss of generality.

To simplify matters, we first find the solution when $\mu = 1$; the solution for $\mu \neq 1$ follows by substituting μx for x. Consequently, we seek solutions of the form

$$y(x) = \sum_{k=0}^{\infty} B_k x^{2k+s}, \tag{6.5.3}$$

---

[18] Bessel, F. W., 1824: Untersuchung des Teils der planetarischen Störungen, welcher aus der Bewegung der Sonne entsteht. *Abh. d. K. Akad. Wiss. Berlin*, 1–52. See Dutka, J., 1995: On the early history of Bessel functions. *Arch. Hist. Exact Sci.*, **49**, 105–134. The classic reference on Bessel functions is Watson, G. N., 1966: *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge.

[19] This case is much simpler than for arbitrary n. See Hildebrand, F. B., 1962: *Advanced Calculus for Applications.* Prentice-Hall, Englewood Cliffs, NJ, Section 4.8.

Figure 6.5.1: It was Friedrich William Bessel's (1784–1846) apprenticeship to the famous mercantile firm of Kulenkamp that ignited his interest in mathematics and astronomy. As the founder of the German school of practical astronomy, Bessel discovered his functions while studying the problem of planetary motion. Bessel functions arose as coefficients in one of the series that described the gravitational interaction between the sun and two other planets in elliptic orbit. (Portrait courtesy of Photo AKG, London.)

$$y'(x) = \sum_{k=0}^{\infty}(2k+s)B_k x^{2k+s-1} \tag{6.5.4}$$

and

$$y''(x) = \sum_{k=0}^{\infty}(2k+s)(2k+s-1)B_k x^{2k+s-2}, \tag{6.5.5}$$

where we formally assume that we can interchange the order of differentiation and summation. The substitution of (6.5.3)–(6.5.5) into (6.5.1) with $\mu = 1$ yields

$$\sum_{k=0}^{\infty}(2k+s)(2k+s-1)B_k x^{2k+s} + \sum_{k=0}^{\infty}(2k+s)B_k x^{2k+s} + \sum_{k=0}^{\infty} B_k x^{2k+s+2} - n^2 \sum_{k=0}^{\infty} B_k x^{2k+s} = 0 \tag{6.5.6}$$

or

$$\sum_{k=0}^{\infty}[(2k+s)^2 - n^2]B_k x^{2k} + \sum_{k=0}^{\infty} B_k x^{2k+2} = 0. \tag{6.5.7}$$

If we explicitly separate the $k = 0$ term from the other terms in the first summation in (6.5.7),

$$(s^2 - n^2)B_0 + \sum_{m=1}^{\infty}[(2m+s)^2 - n^2]B_m x^{2m} + \sum_{k=0}^{\infty} B_k x^{2k+2} = 0. \tag{6.5.8}$$

We now change the dummy integer in the first summation of (6.5.8) by letting $m = k + 1$ so that

$$(s^2 - n^2)B_0 + \sum_{k=0}^{\infty}\{[(2k+s+2)^2 - n^2]B_{k+1} + B_k\}x^{2k+2} = 0. \tag{6.5.9}$$

Because (6.5.9) must be true for all x, each power of x must vanish identically. This yields $s = \pm n$ and

$$[(2k+s+2)^2 - n^2]B_{k+1} + B_k = 0. \tag{6.5.10}$$

Since the difference of the larger indicial root from the lower root equals the integer $2n$, we are only guaranteed a power series solution of the form (6.5.3) for $s = n$. If we use this indicial root and the recurrence formula (6.5.10), this solution, known as the Bessel function of the first kind of order n and denoted by $J_n(x)$, is

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!(n+k)!}. \tag{6.5.11}$$

To find the second general solution to Bessel's equation, the one corresponding to $s = -n$, the most economical method[20] is to express it in terms of partial derivatives of $J_n(x)$ with respect to its order n:

$$Y_n(x) = \left[\frac{\partial J_\nu(x)}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}(x)}{\partial \nu}\right]_{\nu=n}. \tag{6.5.12}$$

---

[20] See Watson, G. N., 1966: *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, Section 3.5 for the derivation.

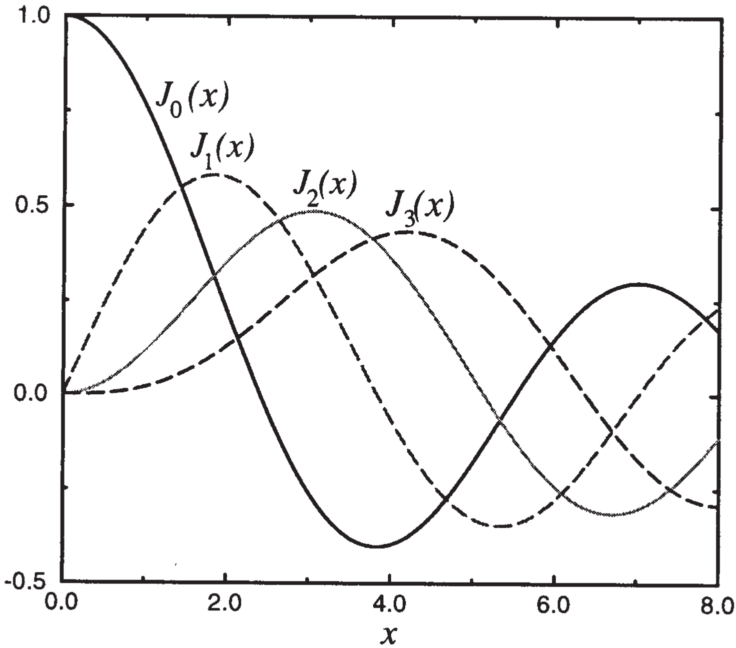


Figure 6.5.2: The first four Bessel functions of the first kind over $0 \leq x \leq 8$.

Upon substituting the power series representation (6.5.11) into (6.5.12),

$$Y_n(x) = \frac{2}{\pi} J_n(x) \ln(x/2) - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-n}$$
$$- \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!(n+k)!} [\psi(k+1) + \psi(k+n+1)], \qquad \textbf{(6.5.13)}$$

where

$$\psi(m+1) = -\gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{m}, \qquad \textbf{(6.5.14)}$$

$\psi(1) = -\gamma$ and γ is Euler's constant (0.5772157). In the case $n = 0$, the first sum in (6.5.13) disappears. This function $Y_n(x)$ is Neumann's Bessel function of the second kind of order n. Consequently, the general solution to (6.5.1) is

$$y(x) = AJ_n(\mu x) + BY_n(\mu x). \qquad \textbf{(6.5.15)}$$

Figure 6.5.2 illustrates the functions $J_0(x)$, $J_1(x)$, $J_2(x)$, and $J_3(x)$ while Figure 6.5.3 gives $Y_0(x)$, $Y_1(x)$, $Y_2(x)$, and $Y_3(x)$.

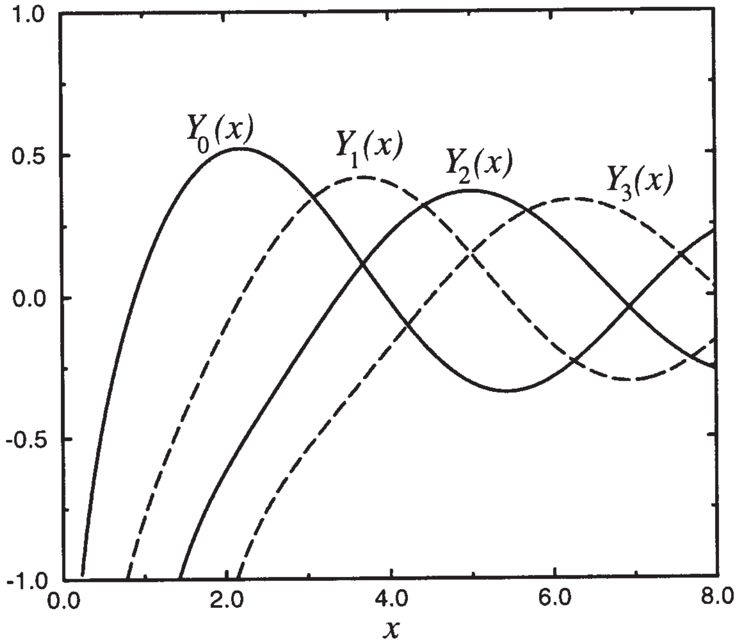


Figure 6.5.3: The first four Bessel functions of the second kind over $0 \leq x \leq 8$.

An equation which is very similar to (6.5.1) is

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (n^2 + x^2)y = 0. \tag{6.5.16}$$

It arises in the solution of partial differential equations in cylindrical coordinates. If we substitute $ix = t$ (where $i = \sqrt{-1}$) into (6.5.16), it becomes Bessel's equation:

$$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - n^2)y = 0. \tag{6.5.17}$$

Consequently, we may immediately write the solution to (6.5.16) as

$$y(x) = c_1 J_n(ix) + c_2 Y_n(ix), \tag{6.5.18}$$

if n is an integer. Traditionally the solution to (6.5.16) has been written

$$y(x) = c_1 I_n(x) + c_2 K_n(x) \tag{6.5.19}$$

rather than in terms of $J_n(ix)$ and $Y_n(ix)$, where

$$I_n(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+n}}{k!(k+n)!} \tag{6.5.20}$$

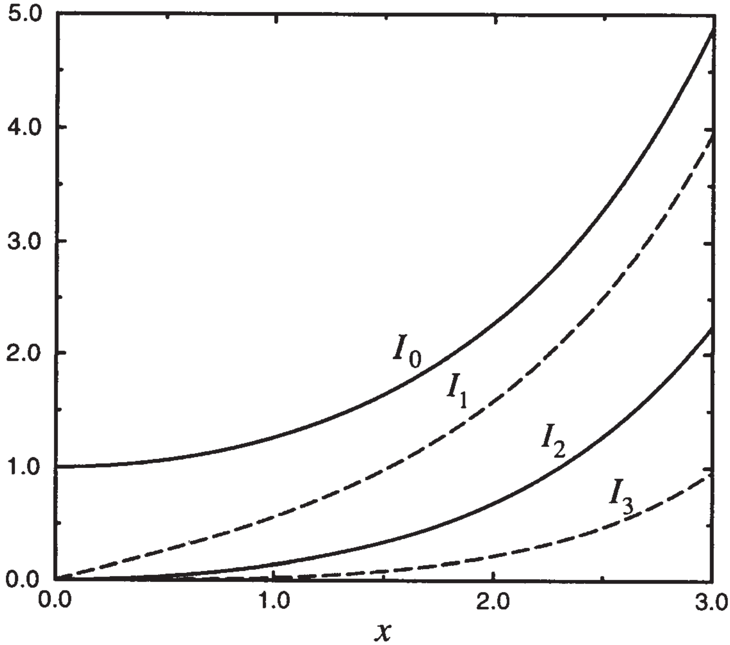


Figure 6.5.4: The first four modified Bessel functions of the first kind over $0 \leq x \leq 3$.

and

$$K_n(x) = \frac{\pi}{2} i^{n+1} \left[J_n(ix) + iY_n(ix) \right]. \qquad \textbf{(6.5.21)}$$

The function $I_n(x)$ is the modified Bessel function of the first kind, of order n, while $K_n(x)$ is the modified Bessel function of the second kind, of order n. Figure 6.5.4 illustrates $I_0(x)$, $I_1(x)$, $I_2(x)$, and $I_3(x)$ while in Figure 6.5.5 $K_0(x)$, $K_1(x)$, $K_2(x)$, and $K_3(x)$ have been graphed. Note that $K_n(x)$ has no real zeros while $I_n(x)$ equals zero only at $x = 0$ for $n \geq 1$.

As our derivation suggests, modified Bessel functions are related to ordinary Bessel functions via complex variables. In particular, $J_n(iz) = i^n I_n(z)$ and $I_n(iz) = i^n J_n(z)$ for z complex.

Although we have found solutions to Bessel's equation (6.5.1), as well as (6.5.16), can we use any of them in an eigenfunction expansion? From Figures 6.5.2–6.5.5 we see that $J_n(x)$ and $I_n(x)$ remain finite at $x = 0$ while $Y_n(x)$ and $K_n(x)$ do not. Furthermore, the products $xJ_n'(x)$ and $xI_n'(x)$ tend to zero at $x = 0$. Thus, both $J_n(x)$ and $I_n(x)$ satisfy the first requirement of an eigenfunction for a Fourier-Bessel expansion.

What about the second condition that the eigenfunction must satisfy the homogeneous boundary condition (6.1.2) at $x = L$? From Figure 6.5.4 we see that $I_n(x)$ can never satisfy this condition while from Fig-

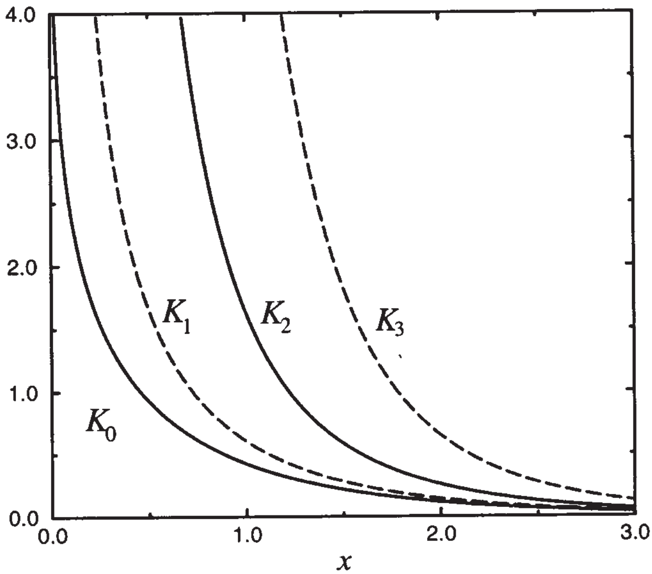


Figure 6.5.5: The first four modified Bessel functions of the second kind over $0 \le x \le 3$.

ure 6.5.2 $J_n(x)$ can. For that reason, we discard $I_n(x)$ from further consideration and continue our analysis only with $J_n(x)$.

Before we can derive the expressions for a Fourier-Bessel expansion, we need to find how $J_n(x)$ is related to $J_{n+1}(x)$ and $J_{n-1}(x)$. Assuming that n is a positive integer, we multiply the series (6.5.11) by x^n and then differentiate with respect to x. This gives

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{k=0}^{\infty} \frac{(-1)^k (2n+2k) x^{2n+2k-1}}{2^{n+2k} k!(n+k)!} \tag{6.5.22}$$

$$= x^n \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n-1+2k}}{k!(n-1+k)!} \tag{6.5.23}$$

$$= x^n J_{n-1}(x) \tag{6.5.24}$$

or

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x) \tag{6.5.25}$$

for $n = 1, 2, 3, \ldots$. Similarly, multiplying (6.5.11) by x^{-n}, we find that

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x) \tag{6.5.26}$$

for $n = 0, 1, 2, 3, \ldots$. If we now carry out the differentiation on (6.5.25) and (6.5.26) and divide by the factors $x^{\pm n}$, we have that

$$J_n'(x) + \frac{n}{x} J_n(x) = J_{n-1}(x) \tag{6.5.27}$$

and

$$J_n'(x) - \frac{n}{x} J_n(x) = -J_{n+1}(x). \tag{6.5.28}$$

Equations (6.3.27)–(6.3.28) immediately yield the *recurrence relationships*

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \tag{6.5.29}$$

and

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x) \tag{6.5.30}$$

for $n = 1, 2, 3, \ldots$ For $n = 0$, we replace (6.5.30) by $J_0'(x) = -J_1(x)$.

Let us now construct a Fourier-Bessel series. The exact form of the expansion depends upon the boundary condition at $x = L$. There are three possible cases. One of them is the requirement that $y(L) = 0$ and results in the condition that $J_n(\mu_k L) = 0$. Another condition is $y'(L) = 0$ and gives $J_n'(\mu_k L) = 0$. Finally, if $hy(L) + y'(L) = 0$, then $hJ_n(\mu_k L) + \mu_k J_n'(\mu_k L) = 0$. In all of these cases, the eigenfunction expansion is the same, namely

$$f(x) = \sum_{k=1}^{\infty} A_k J_n(\mu_k x), \tag{6.5.31}$$

where μ_k is the kth positive solution of either $J_n(\mu_k L) = 0$, $J_n'(\mu_k L) = 0$ or $hJ_n(\mu_k L) + \mu_k J_n'(\mu_k L) = 0$.

We now need a mechanism for computing A_k. We begin by multiplying (6.5.31) by $xJ_n(\mu_m x)\,dx$ and integrate from 0 to L. This yields

$$\sum_{k=1}^{\infty} A_k \int_0^L xJ_n(\mu_k x)J(\mu_m x)\,dx = \int_0^L xf(x)J_n(\mu_m x)\,dx. \tag{6.5.32}$$

From the general orthogonality condition (6.2.1),

$$\int_0^L xJ_n(\mu_k x)J_n(\mu_m x)\,dx = 0 \tag{6.5.33}$$

if $k \neq m$. Equation (6.5.32) then simplifies to

$$A_m \int_0^L xJ_n^2(\mu_m x)\,dx = \int_0^L xf(x)J_n(\mu_m x)\,dx \tag{6.5.34}$$

or

$$\boxed{A_k = \frac{1}{C_k}\int_0^L xf(x)J_n(\mu_k x)\,dx,} \tag{6.5.35}$$

where

$$C_k = \int_0^L xJ_n^2(\mu_k x)\,dx \tag{6.5.36}$$

and k has replaced m in (6.5.34).

The factor C_k depends upon the nature of the boundary conditions at $x = L$. In all cases we start from Bessel's equation

$$[xJ_n'(\mu_k x)]' + \left(\mu_k^2 x - \frac{n^2}{x}\right) J_n(\mu_k x) = 0. \tag{6.5.37}$$

If we multiply both sides of (6.5.37) by $2xJ_n'(\mu_k x)$, the resulting equation is

$$\left(\mu_k^2 x^2 - n^2\right)\left[J_n^2(\mu_k x)\right]' = -\frac{d}{dx}\left[xJ_n'(\mu_k x)\right]^2. \tag{6.5.38}$$

An integration of (6.5.38) from 0 to L, followed by the subsequent use of integration by parts, results in

$$(\mu_k^2 x^2 - n^2)J_n^2(\mu_k x)\Big|_0^L - 2\mu_k^2 \int_0^L xJ_n^2(\mu_k x)\,dx = -\left[xJ_n'(\mu_k x)\right]^2\Big|_0^L. \tag{6.5.39}$$

Because $J_n(0) = 0$ for $n > 0$, $J_0(0) = 1$ and $xJ_n'(x) = 0$ at $x = 0$, the contribution from the lower limits vanishes. Thus,

$$C_k = \int_0^L xJ_n^2(\mu_k x)\,dx \tag{6.5.40}$$

$$= \frac{1}{2\mu_k^2}\left[(\mu_k^2 L^2 - n^2)J_n^2(\mu_k L) + L^2 J_n'^2(\mu_k L)\right]. \tag{6.5.41}$$

Because

$$J_n'(\mu_k x) = \frac{n}{x}J_n(\mu_k x) - \mu_k J_{n+1}(\mu_k x) \tag{6.5.42}$$

from (6.5.28), C_k becomes

$$\boxed{C_k = \tfrac{1}{2}L^2 J_{n+1}^2(\mu_k L),} \tag{6.5.43}$$

if $J_n(\mu_k L) = 0$. Otherwise, if $J_n'(\mu_k L) = 0$, then

$$\boxed{C_k = \frac{\mu_k^2 L^2 - n^2}{2\mu_k^2} J_n^2(\mu_k L).} \tag{6.5.44}$$

Finally,

$$\boxed{C_k = \frac{\mu_k^2 L^2 - n^2 + h^2 L^2}{2\mu_k^2} J_n^2(\mu_k L),} \tag{6.5.45}$$

if $\mu_k J_n'(\mu_k L) = -hJ_n(\mu_k L)$.

All of the preceding results must be slightly modified when $n = 0$ and the boundary condition is $J_0'(\mu_k L) = 0$ or $\mu_k J_1(\mu_k L) = 0$. This modification results from the additional eigenvalue $\mu_0 = 0$ being present and we must add the extra term A_0 to the expansion. For this case the series reads

$$f(x) = A_0 + \sum_{k=1}^{\infty} A_k J_0(\mu_k x), \tag{6.5.46}$$

where the equation for finding A_0 is

$$A_0 = \frac{2}{L^2}\int_0^L f(x)\, x\, dx \tag{6.5.47}$$

and (6.5.35) and (6.5.44) with $n = 0$ give the remaining coefficients.

• Example 6.5.1

Starting with Bessel's equation, we want to show that the solution to

$$y'' + \frac{1-2a}{x}y' + \left(b^2c^2x^{2c-2} + \frac{a^2 - n^2c^2}{x^2}\right)y = 0 \tag{6.5.48}$$

is

$$y(x) = Ax^a J_n(bx^c) + Bx^a Y_n(bx^c), \tag{6.5.49}$$

provided that $bx^c > 0$ so that $Y_n(bx^c)$ exists.

The general solution to

$$\xi^2\frac{d^2\eta}{d\xi^2} + \xi\frac{d\eta}{d\xi} + (\xi^2 - n^2)\eta = 0 \tag{6.5.50}$$

is

$$\eta = AJ_n(\xi) + BY_n(\xi). \tag{6.5.51}$$

If we now let $\eta = y(x)/x^a$ and $\xi = bx^c$, then

$$\frac{d}{d\xi} = \frac{dx}{d\xi}\frac{d}{dx} = \frac{x^{1-c}}{bc}\frac{d}{dx}, \tag{6.5.52}$$

$$\frac{d^2}{d\xi^2} = \frac{x^{2-2c}}{b^2c^2}\frac{d^2}{dx^2} - \frac{(c-1)x^{1-2c}}{b^2c^2}\frac{d}{dx}, \tag{6.5.53}$$

$$\frac{d}{dx}\left(\frac{y}{x^a}\right) = \frac{1}{x^a}\frac{dy}{dx} - \frac{a}{x^{a+1}}y \tag{6.5.54}$$

and

$$\frac{d^2}{dx^2}\left(\frac{y}{x^a}\right) = \frac{1}{x^a}\frac{d^2y}{dx^2} - \frac{2a}{x^{a+1}}\frac{dy}{dx} + \frac{a(1+a)}{x^{a+2}}y. \tag{6.5.55}$$

Substituting (6.5.52)–(6.5.55) into (6.5.51) and simplifying, yields the desired result.

• Example 6.5.2

We want to show that

$$x^2J_n''(x) = (n^2 - n - x^2)J_n(x) + xJ_{n+1}(x). \tag{6.5.56}$$

From (6.5.28),

$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x), \tag{6.5.57}$$

$$J_n''(x) = -\frac{n}{x^2} J_n(x) + \frac{n}{x} J_n'(x) - J_{n+1}'(x) \tag{6.5.58}$$

and

$$\begin{aligned} J_n''(x) = &-\frac{n}{x^2} J_n(x) + \frac{n}{x}\left[\frac{n}{x} J_n(x) - J_{n+1}(x)\right] \\ &- \left[J_n(x) - \frac{n+1}{x} J_{n+1}(x)\right] \end{aligned} \tag{6.5.59}$$

after using (6.5.27) and (6.5.28). Simplifying,

$$J_n''(x) = \left(\frac{n^2 - n}{x^2} - 1\right) J_n(x) + \frac{J_{n+1}(x)}{x}. \tag{6.5.60}$$

After multiplying (6.5.60) by x^2, we obtain (6.5.56).

• **Example 6.5.3**

Show that

$$\int_0^a x^5 J_2(x)\, dx = a^5 J_3(a) - 2a^4 J_4(a). \tag{6.5.61}$$

We begin by integrating (6.5.61) by parts. If $u = x^2$ and $dv = x^3 J_2(x)\, dx$, then

$$\int_0^a x^5 J_2(x)\, dx = x^5 J_3(x)\Big|_0^a - 2\int_0^a x^4 J_3(x)\, dx, \tag{6.5.62}$$

because $d[x^3 J_3(x)]/dx = x^2 J_2(x)$ by (6.5.25). Finally, since $x^4 J_3(x) = d[x^4 J_4(x)]/dx$ by (6.5.25),

$$\int_0^a x^5 J_2(x)\, dx = a^5 J_3(a) - 2x^4 J_4(x)\Big|_0^a = a^5 J_3(a) - 2a^4 J_4(a). \tag{6.5.63}$$

• **Example 6.5.4**

Let us expand $f(x) = x$, $0 < x < 1$, in the series

$$f(x) = \sum_{k=1}^{\infty} A_k J_1(\mu_k x), \tag{6.5.64}$$

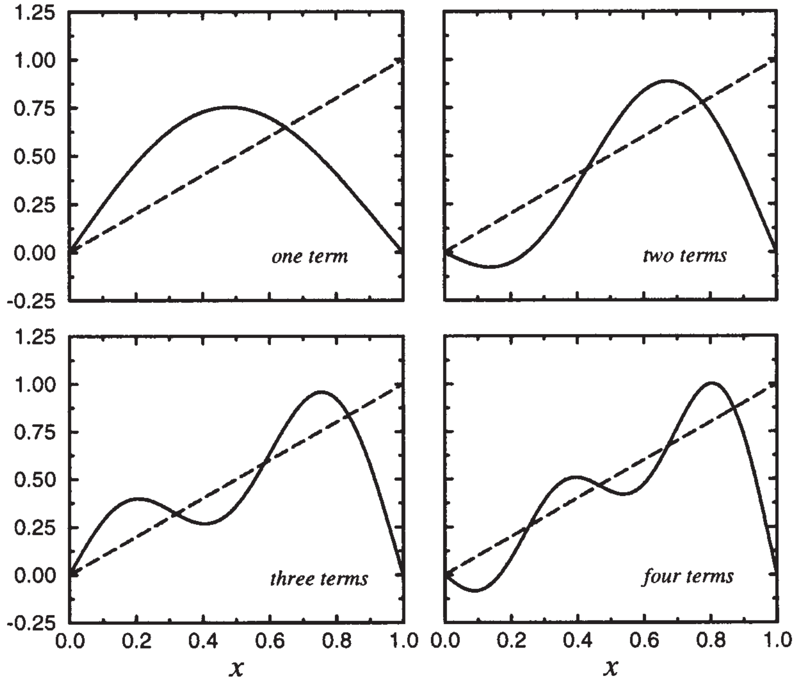


Figure 6.5.6: The Fourier-Bessel series representation (6.5.68) for $f(x) = x$, $0 < x < 1$, when we truncate the series so that it includes only the first, first two, first three, and first four terms.

where μ_k denotes the kth zero of $J_1(\mu)$. From (6.5.35) and (6.5.43),

$$A_k = \frac{2}{J_2^2(\mu_k)} \int_0^1 x^2 J_1(\mu_k x)\, dx. \tag{6.5.65}$$

However, from (6.5.25),

$$\frac{d}{dx}\left[x^2 J_2(x)\right] = x^2 J_1(x), \tag{6.5.66}$$

if $n = 2$. Therefore, (6.5.65) becomes

$$A_k = \left. \frac{2x^2 J_2(x)}{\mu_k^3 J_2^2(\mu_k)} \right|_0^{\mu_k} = \frac{2}{\mu_k J_2(\mu_k)} \tag{6.5.67}$$

and the resulting expansion is

$$x = 2 \sum_{k=1}^{\infty} \frac{J_1(\mu_k x)}{\mu_k J_2(\mu_k)}, \qquad 0 < x < 1. \tag{6.5.68}$$

Figure 6.5.6 shows the Fourier-Bessel expansion of $f(x) = x$ in truncated form when we only include one, two, three, and four terms.

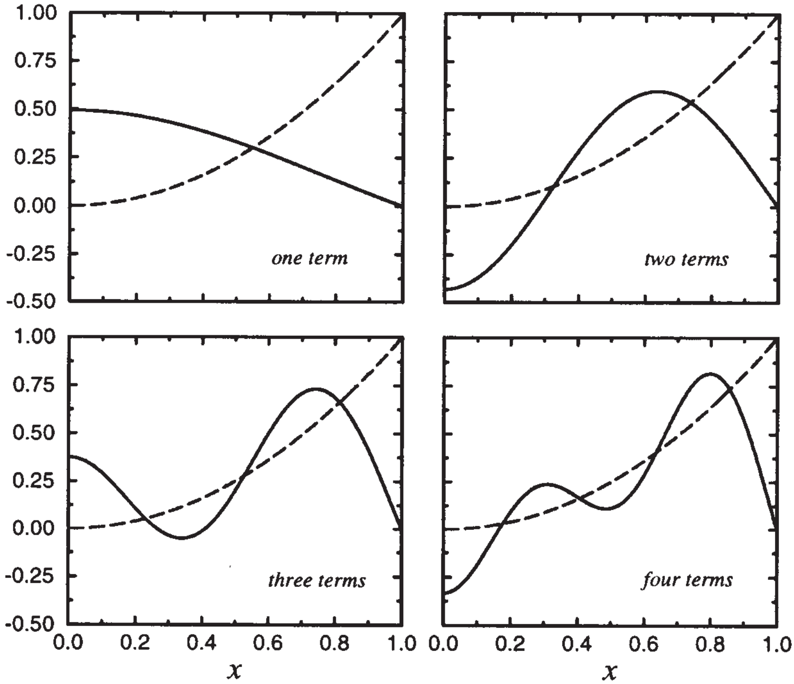


Figure 6.5.7: The Fourier-Bessel series representation (6.5.79) for $f(x) = x^2$, $0 < x < 1$, when we truncate the series so that it includes only the first, first two, first three, and first four terms.

• Example 6.5.5

Let us expand the function $f(x) = x^2$, $0 < x < 1$, in the series

$$f(x) = \sum_{k=1}^{\infty} A_k J_0(\mu_k x), \tag{6.5.69}$$

where μ_k denotes the kth positive zero of $J_0(\mu)$. From (6.5.35) and (6.5.43),

$$A_k = \frac{2}{J_1^2(\mu_k)} \int_0^1 x^3 J_0(\mu_k x)\, dx. \tag{6.5.70}$$

If we let $t = \mu_k x$, the integration (6.5.70) becomes

$$A_k = \frac{2}{\mu_k^4 J_1^2(\mu_k)} \int_0^{\mu_k} t^3 J_0(t)\, dt. \tag{6.5.71}$$

We now let $u = t^2$ and $dv = tJ_0(t)\, dt$ so that integration by parts results in

$$A_k = \frac{2}{\mu_k^4 J_1^2(\mu_k)} \left[t^3 J_1(t) \Big|_0^{\mu_k} - 2 \int_0^{\mu_k} t^2 J_1(t)\, dt \right] \tag{6.5.72}$$

$$= \frac{2}{\mu_k^4 J_1^2(\mu_k)} \left[\mu_k^3 J_1(\mu_k) - 2 \int_0^{\mu_k} t^2 J_1(t)\, dt \right], \tag{6.5.73}$$

because $v = tJ_1(t)$ from (6.5.25). If we integrate by parts once more, we find that

$$A_k = \frac{2}{\mu_k^4 J_1^2(\mu_k)} \left[\mu_k^3 J_1(\mu_k) - 2\mu_k^2 J_2(\mu_k) \right] \tag{6.5.74}$$

$$= \frac{2}{J_1^2(\mu_k)} \left[\frac{J_1(\mu_k)}{\mu_k} - \frac{2J_2(\mu_k)}{\mu_k^2} \right]. \tag{6.5.75}$$

However, from (6.5.29) with $n = 1$,

$$J_1(\mu_k) = \tfrac{1}{2}\mu_k \left[J_2(\mu_k) + J_0(\mu_k) \right] \tag{6.5.76}$$

or

$$J_2(\mu_k) = \frac{2J_1(\mu_k)}{\mu_k}, \tag{6.5.77}$$

because $J_0(\mu_k) = 0$. Therefore,

$$A_k = \frac{2(\mu_k^2 - 4)J_1(\mu_k)}{\mu_k^3 J_1^2(\mu_k)} \tag{6.5.78}$$

and

$$x^2 = 2 \sum_{k=1}^{\infty} \frac{(\mu_k^2 - 4) J_0(\mu_k x)}{\mu_k^3 J_1(\mu_k)}, \qquad 0 < x < 1. \tag{6.5.79}$$

Figure 6.5.7 shows the representation of x^2 by the Fourier-Bessel series (6.5.79) when we truncate it so that it includes only one, two, three, or four terms. As we add each additional term in the orthogonal expansion, the expansion fits $f(x)$ better in the "least squares" sense of (6.3.5).

Problems

1. Show from the series solution that

$$\frac{d}{dx}\left[J_0(kx) \right] = -kJ_1(kx).$$

From the recurrence formulas, show these following relations:

2.
$$2J_0''(x) = J_2(x) - J_0(x)$$

3.
$$J_2(x) = J_0''(x) - J_0'(x)/x$$

4.
$$J_0'''(x) = \frac{J_0(x)}{x} + \left(\frac{2}{x^2} - 1\right) J_0'(x)$$

5.
$$\frac{J_2(x)}{J_1(x)} = \frac{1}{x} - \frac{J_0''(x)}{J_0'(x)} = \frac{2}{x} - \frac{J_0(x)}{J_1(x)} = \frac{2}{x} + \frac{J_0(x)}{J_0'(x)}$$

6.
$$J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) - \left(\frac{24}{x^2} - 1\right) J_0(x)$$

7.
$$J_{n+2}(x) = \left[2n + 1 - \frac{2n(n^2-1)}{x^2}\right] J_n(x) + 2(n+1)J_n''(x)$$

8.
$$J_3(x) = \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$$

9.
$$4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$$

10. Show that the maximum and minimum values of $J_n(x)$ occur when

$$x = \frac{nJ_n(x)}{J_{n+1}(x)}, \quad x = \frac{nJ_n(x)}{J_{n-1}(x)}, \quad \text{and} \quad J_{n-1}(x) = J_{n+1}(x).$$

Show that

11.
$$\frac{d}{dx}\left[x^2 J_3(2x)\right] = -xJ_3(2x) + 2x^2 J_2(2x)$$

12.
$$\frac{d}{dx}\left[xJ_0(x^2)\right] = J_0(x^2) - 2x^2 J_1(x^2)$$