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Characterizations of homomorphisms of skew fields

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Summary. The functional equations

$$f\left(x\,(x+y)^{-1}\right)\big(f\,(x)+f\,(y)\big)=f\,(x)$$

and

$$f\big((x+y)x^{-1}\big)\,f\,(x)=f\,(x)+f\,(y)$$

are solved for skew fields.

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1. The functional equation

$$f\left(\frac{x+y}{x-y}\right)=\frac{f\,(x)+f\,(y)}{f\,(x)-f\,(y)} \tag{1}$$

where $f:\mathbb{R}\to\mathbb{R}$ is supposed to be injective was solved by S. Reich (American Math. Monthly *78* (1971), 675). Replacing $\mathbb{R}$ by a prime field or by certain Galois extensions of $\mathbb{Q}$, the solutions of (1) were found by K. S. Sarkaria in [5]. T. M. K. Davison, [3], posed the problem to solve (1) for arbitrary fields F. It was possible, [1], to find the solution, even for skew fields, replacing (1) by (2) or (3) (see below) in order to avoid the injectivity assumption. In [1] we proved the following theorem. *If F is an arbitrary skew field with $F\neq F_5$ and* $\operatorname{char} F\neq 2$, *then every $f:F\to F$ satisfying equation* (2), *equation* (3), *respectively,*

$$f\left((x+y)(x-y)^{-1}\right)\big(f\,(x)-f\,(y)\big)\;=\;f\,(x)+f\,(y), \tag{2}$$

$$\big(f\,(x)-f\,(y)\big)\Big(f\left((x+y)(x-y)^{-1}\right)\Big)\;=\;f\,(x)+f\,(y), \tag{3}$$

for all elements $x\neq y$ of F must be a homomorphism, an anti-homomorphism, respectively, of F. In the present note we would like to find a characterization of homomorphisms including the case $\operatorname{char} F=2$. The following equations which are of a similar type as equation (2), will be considered.

$$f\left(x\,(x+y)^{-1}\right)\big(f\,(x)+f\,(y)\big) \;=\; f\,(x), \tag{4}$$

$$f\big((x+y)x^{-1}\big)\,f\,(x) \;=\; f\,(x)+f\,(y). \tag{5}$$

As a matter of fact, it is not difficult to verify

Proposition 1. *If F is an arbitrary skew field and if $f : F \to F$ satisfies* (5) *for all $x, y \in F$ with $x \neq 0$, then f is $\equiv 2$ or a homomorphism of F.*

With respect to equation (4) the following result will be proved in this note.

Theorem 2. *Let F be an arbitrary skew field and $f : F \to F$ be a mapping satisfying* (4) *for all $x, y \in F$ with $x + y \neq 0$. If $f\,(1) = 1$, then f is a monomorphism of F. If $f\,(1) \neq 1$, then $f \equiv 0$, or $f\,(0) = 1$ and $f\,(x) = 0$ for all $x \neq 0$, or $2f \equiv 1$ for $2 \neq 0$.*

As a consequence of this theorem we get

Corollary 3. *If F is a skew field and $f : F \to F$ a mapping satisfying $2\,[f\,(1)]^2 \neq f\,(1)$ and* (4) *for all $x, y \in F$ with $y \neq -x$, then f is a monomorphism of F.*

In our context we also would like to refer to results of F. Halter-Koch and L. Reich [4].

Concerning the algebraic notions in this paper see P. M. Cohn [2]. Note that in the terminology of P. M. Cohn [2] fields are special skew fields, namely skew fields satisfying the commutative law of multiplication.

2. In this section the first part of Theorem 2 will be proved. So let $f : F \to F$ satisfy $f\,(1) = 1$ and (4) for all $x, y \in F$ with $y \neq -x$.

2.1. $f\,(0) = 0$ *and* $f\,(z) \neq 0$ *for all* $z \neq 0$. *Moreover,*

$$\forall_{y \neq -1}\; f\left((1+y)^{-1}\right)\big(1+f\,(y)\big) = 1. \tag{6}$$

Proof. Apply (4) for $x = 1$ and $y = 0$. Hence $1 + f\,(0) = 1$. For $x = 1$ and $y \neq -1$ we get (6) from (4). If $z \neq 0$, then $y := \frac{1}{z} - 1 \neq -1$. Hence, by (6), $f\,(z) \neq 0$.

2.2. *For all $a, b \in F$ with $a \neq -1$*

$$f\,(ab) = f\,(a)\,f\,(b) \tag{7}$$

holds true.

Proof. We may assume $ab \neq 0$. Hence (6) and (4) yield

$$f\left((1+a)^{-1}\right)\left(1+f(a)\right) = 1, \tag{8}$$

$$f\left(b\cdot(b+ab)^{-1}\right)\left(f(b)+f(ab)\right) = f(b). \tag{9}$$

Observe $(1+a)^{-1} = b\cdot(b+ab)^{-1}$ and $f(b) \neq 0$. Hence

$$1+f(a) = \left[f(b)+f(ab)\right]\left[f(b)\right]^{-1},$$

i.e. (7).

2.3. *If* char $F=2$*, then, obviously,* (7) *holds true for all* $a, b \in F$.

2.4. $f(a)+f(1-a) = 1$ for all $a \in F$.

Proof. We may assume $a \neq 0$. Hence, by 2.1, $f(a) \neq 0$. Apply (4) for $x = a$ and $y = 1-a$. Then $f(a)\left(f(a)+f(1-a)\right) = f(a)$, i.e. 2.4.

2.5. $f(2)+f(-1) = 1$ *and* $f(2) = f(-2)\,f(-1)$ *hold true.*

Proof. The first equation follows from 2.4, the second one from 2.2 since $-2 \neq -1$.

2.6. $f(-1)\left(1+f(-1-a)\right) = f(a)$ for all $a \neq 0$.

Proof. We may assume $a \neq -1$. Hence, by 2.2,

$$f(-1) = f\left(a\cdot\frac{1}{-a}\right) = f(a)\cdot f\left(\frac{1}{-a}\right).$$

If we put $y := -1-a$ in (6), then

$$f\left(\frac{1}{-a}\right)\left(1+f(-1-a)\right) = 1.$$

2.7. $f(-a)\left(f(a)+f(-1-a)\right) = f(a)$ *holds true for all* $a \in F$.

Proof. Put $x = a$ and $y = -1-a$ in (4) and observe $x+y \neq 0$.

2.8. $\dfrac{1}{f(a)} + \dfrac{1}{f(-a)} = 1 + \dfrac{1}{f(-1)}$ for all $a \neq 0$.

Proof. Observe $f(-1) \neq 0$, in view of $-1 \neq 0$. Put $k := f(-1)$ and $r := f(-1-a)$. Now 2.6, 2.7 imply

$$1+r = k^{-1}f(a) \text{ and } f(a)+r = \left[f(-a)\right]^{-1}f(a), \text{ i.e.}$$

$$1-f(a) = \left(\frac{1}{k}-\frac{1}{f(-a)}\right)f(a).$$

2.9. *If* char $F = 2$, *then* $f(-1) = -1$, *and, if* char $F \neq 2$, *then* $f(-1) \in \{-1, \frac{1}{2}\}$. *The element* $f(-1)$ *is hence in the center of* F.

Proof. The first statement is trivial, so assume $2 \neq 0$. With $k := f(-1)$ we get $f(2) = 1 - k$ and $f(-2) = f(2)k^{-1}$ from 2.5. Hence, by 2.8 with $a = 2$,

$$\frac{1}{1-k} + \left[(1-k)\,k^{-1}\right]^{-1} = 1 + k^{-1}. \tag{10}$$

Multiplying (10) from the right by $1 - k$, we get $(k+1)(k^{-1} - 2) = 0$, i.e. $k = -1$ or $k^{-1} = 2$.

2.10. *If* $a, b \in F$ *are not both equal to* -1, *then* (7) *holds true.*

Proof. Because of 2.3 we only need to consider the case $2 \neq 0$. In view of 2.2 the only case left is $a = -1$ and $b \neq -1$. Here we have

$$f(ab) = f(-b) = f\big(b \cdot (-1)\big) = f(b) \cdot f(-1) = f(-1)\,f(b),$$

on account of 2.2 and the fact that $f(-1)$ is in the center of F (see 2.9).

2.11. *The equation* (7) *holds true for all* $a, b \in F$. *Moreover,* $f(-1) = -1$.

Proof. Because of 2.3 we may assume $2 \neq 0$. If $3 = 0$, then $\frac{1}{2} = -1$ and thus

$$f(-1)\,f(-1) = 1 = f(1).$$

Suppose now that $2 \cdot 3 \neq 0$. There hence exists $\alpha \in F \backslash \{0, 1, -1\}$. By observing $(-1)\,\alpha \neq -1$, $\alpha \neq -1$, $\alpha^{-1} \neq -1$ we get, by 2.10,

$$\begin{aligned} 1 &= f\big((-1)(-1)\big) = f\big((-1)\,\alpha \cdot \alpha^{-1}(-1)\big) = f\big((-1)\,\alpha\big) \cdot f\big(\alpha^{-1}(-1)\big) \\ &= f(-1)\,f(\alpha) \cdot f(\alpha^{-1})\,f(-1) = f(-1)\,f(\alpha\alpha^{-1})\,f(-1) = f(-1)\,f(-1). \end{aligned}$$

By 2.9, $f(-1) = -1$ for $2 = 0$. If $3 = 0$, then $\frac{1}{2} = -1$. If $2 \cdot 3 \neq 0$, then $\left[f(-1)\right]^2 = 1$ implies $f(-1) \neq \frac{1}{2}$. So $f(-1) = -1$ by 2.9.

2.12. $f(-a) = -f(a)$ *for all* $a \in F$.

Proof. This follows from 2.11.

2.13. $f(1 + a) = 1 + f(a)$ for all $a \in F$.

Proof. $f(1 + a) = 1 - f(-a) = 1 + f(a)$ by 2.4 and 2.12.

2.14. $f(a + b) = f(a) + f(b)$ for all $a, b \in F$.

Proof. We may assume $a \neq 0$. Then

$$\begin{aligned} f(a+b) &= f\big(a(1+a^{-1}b)\big) = f(a)\, f(1+a^{-1}b) \\ &= f(a)\big(1+f(a^{-1}b)\big) = f(a)+f(b) \end{aligned}$$

by 2.11 and 2.13.

2.11 and 2.14 finally prove Theorem 2 in the case $f(1) = 1$.

3. In this section we will solve the functional equation (4) in the case $f(1) \neq 1$. Apply (4) for $x \neq 0$ and $y = 0$. Then we get

$$\big(1 - f(1)\big)\, f(x) = f(1)\, f(0) \quad \forall_{x \neq 0}. \tag{11}$$

Hence $f(x) = \text{const}$ for all $x \neq 0$, i.e.

$$f(x) = f(1) \quad \forall_{x \neq 0} \tag{12}$$

since $1 \neq 0$.

If $f(0) = 0$, then (11) implies $f(x) = 0$ also in the case $x \neq 0$. This leads to the solution $f \equiv 0$ of (4). If $f(0) \neq 0$, then

$$f(0) + f(y) = 1 \quad \forall_{y \neq 0} \tag{13}$$

holds true by applying (4) for $x = 0$ and $y \neq 0$. In the case $f(0) \neq 0$ we will consider the two subcases a) $f(1) = 0$ and b) $f(1) \neq 0$. If $f(1) = 0$ holds true, (13) implies $f(0) = 1$. This leads to another solution of (4), namely to

$$f(x) = \begin{cases} 0 & x \neq 0 \\ & \text{for} \\ 1 & x = 0 \end{cases}. \tag{14}$$

Assume now that $f(1) \neq 0$ holds true. Since also $f(1) \neq 1$ we get $F \neq F_2$. Suppose that $\alpha \in F \backslash \{0, 1\}$. Put $x = 1$ and

$$y = \begin{cases} \alpha & 2 = 0 \\ & \text{for} \\ 1 & 2 \neq 0 \end{cases}.$$

Then (4) and (12) imply

$$f(1)\big(f(1) + f(1)\big) = f(1),$$

i.e. $2f(1) = 1$. This is impossible for $2 = 0$. In the case $2 \neq 0$, we hence get $f(x) = \frac{1}{2}$ for all $x \in F$, in view of (12) and (13). This $f \equiv \frac{1}{2}$ solves (4) for $2 \neq 0$.

References

[1] W. Benz, *A characterization of monomorphisms of skew fields*, Aequationes Math. *60* (2000), 142–147.
[2] P. M. Cohn, *Algebra I, II*, John Wiley & Sons. London–New York, 1974, 1977.
[3] T. M. K. Davison, *Problem 7*, Aequationes Math. *60* (2000), 189.
[4] F. Halter-Koch and L. Reich, *Additive functions commuting with Möbius transformations and field homomorphisms*, Aequationes Math. *58* (1999), 176–182.
[5] K. S. Sarkaria, *On some questions concerning a functional equation involving Möbius transformations*, Aequationes Math. *60* (2000), 137–141.

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