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Characterization of 16-dimensional Hughes planes

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Abstract. The well-known finite Hughes planes have compact analoga with 16-dimensional point space. The automorphism group of such a plane is a 36-dimensional Lie group. Theorem: *Assume that the compact projective plane $\mathscr{P}$ is not isomorphic to the classical Moufang plane over the octonions. Let Δ be a closed subgroup of* $\operatorname{Aut}\mathscr{P}$. *If* $\dim\Delta \geqq 31$ *and if Δ has a normal torus subgroup, then $\mathscr{P}$ is a Hughes plane,* $\Delta = \operatorname{Aut}\mathscr{P}$, *and* $\Delta' \cong \mathrm{PSL}_3\mathbb{H}$.

A finite Hughes plane $\mathscr{F}$ is a projective plane of order n^2 having a Desarguesian subplane $\mathscr{E}$ of order n such that each linear collineation of $\mathscr{E}$ is induced by an automorphism of $\mathscr{F}$, compare Lüneburg [3]. Similarly, a compact 16-dimensional topological projective plane $\mathscr{H}$ with automorphism group Σ is called a Hughes plane if $\mathscr{H}$ has a Σ-invariant subplane $\mathscr{E}$ isomorphic to the classical Desarguesian quaternion plane $\mathscr{P}_2\mathbb{H}$ such that Σ induces on $\mathscr{E}$ the full automorphism group $\mathrm{PSL}_3\mathbb{H}$. There exist infinitely many non-isomorphic 16-dimensional Hughes planes, see [8, § 86]. These and their 8-dimensional analoga play a prominent role in the classification of compact, connected planes with an automorphism group of sufficiently large dimension, compare [8, Chap. 8, Introduction] and Theorem S below.

In the following, $\mathscr{P} = (P, \mathfrak{L})$ will always denote a topological projective plane with compact, 16-dimensional point space P. Taken with the compact-open topology, the automorphism group $\Sigma = \operatorname{Aut}\mathscr{P}$ is a locally compact transformation group of P, and Σ has a countable basis [8, 44.3, p. 237]. Let Δ be a connected closed (hence locally compact) subgroup of Σ. If the topological dimension $\dim\Delta \geqq 27$, then Δ is even a Lie group (Priwitzer-Salzmann [6]).

Theorem S. *Assume that $\mathscr{P}$ is not the classical Moufang plane and that Δ is semi-simple. If* $28 < \dim\Delta < 36$, *then* $\Delta \cong \mathrm{SL}_3\mathbb{H}$, *and $\mathscr{P}$ is a Hughes plane.*

Proof. Priwitzer [5] and Hähl [2].

Here, a related characterization will be given:

Theorem T. *Let $\mathscr{P}$ be as above, and assume that Δ has a normal torus subgroup $\Theta \cong \mathbb{T}$. If* $\dim\Delta > 28$ *and if the involution $\iota \in \Theta$ is not a reflection, or if* $\dim\Delta > 30$, *then Θ fixes a Baer subplane,* $\Delta' \cong \mathrm{SL}_3\mathbb{H}$ *and $\mathscr{P}$ is a Hughes plane.*

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Proof of the first part. (a) Since $\varDelta$ is connected and Aut $\varTheta$ is finite, the normal subgroup $\varTheta$ is contained in the center Z of $\varDelta$, compare [8, 93.19]. In particular, $\iota \in Z$.

(b) By assumption, ι is not a reflection, and [8, 55.29] shows that the fixed elements of ι form a Baer subplane $\mathscr{E}$.

(c) $\iota \in Z$ implies $\mathscr{E}^{\varDelta} = \mathscr{E}$. Let $\varDelta^* = \varDelta|_{\mathscr{E}} \cong \varDelta/\varPhi$ be the effective action of $\varDelta$ on $\mathscr{E}$, and consider its kernel $\varPhi = \varDelta_{[\mathscr{E}]}$. By the result mentioned above, $\varDelta$ and $\varPhi$ are Lie groups. From [8, 83.22] it follows that $\varPhi$ is compact and that $\dim \varPhi \leqq 3$. Consequently, $\dim \varDelta^* > 25$, and $\mathscr{E} \cong \mathscr{P}_2\mathbb{H}$ by [8, 84.27]. Moreover, the center of $\varDelta^*$ is trivial [8, 84.10], and $\varTheta$ is contained in the connected component $\varPhi^1$ of $\varPhi$. This excludes the possibility $\varPhi^1 \cong \mathrm{Spin}_3\mathbb{R}$ and shows that $\varTheta = \varPhi^1$, see [8, 83.22]. Hence $\dim \varDelta^* > 27$, and $\varDelta^*$ fixes no element of $\mathscr{E}$. The second part of [8, 84.27] now gives $\varDelta^* \cong \mathrm{PSL}_3\mathbb{H}$.

(d) According to [8, 94.27], the group $\varDelta$ has a subgroup $\varGamma$ locally isomorphic to the simple group $\varDelta^*$, and Theorem S may be applied to $\varGamma$. This shows that $\varGamma \cong \mathrm{SL}_3\mathbb{H}$ and that $\mathscr{P}$ is a Hughes plane. Finally, $\varDelta = \varGamma\varTheta$ implies that $\varGamma = \varDelta'$ is the commutator group of $\varDelta$. □

Proof of the second part. Assume that $\dim \varDelta > 30$ and that the involution $\iota \in \varTheta$ is a reflection with axis W and center $a \notin W$. Again $\iota \in Z$ and hence $W^{\varDelta} = W$. Moreover, $\varDelta$ is a Lie group. The dimension formula

$$\dim \varDelta = \dim x^{\varDelta} + \dim \varDelta_x$$

will be used repeatedly, see [8, 96.10].

The following theorem of Bödi [1] plays an essential role:

(0) *If the fixed elements of a connected Lie group $\varLambda$ form a connected subplane $\mathscr{F}_\varLambda$, then $\varLambda$ is isomorphic to the compact* 14-*dimensional group* G_2, *or* $\varLambda \cong \mathrm{SU}_3\mathbb{C}$, *or* $\dim \varLambda < 8$.

(1) *$\varTheta$ acts trivially on W and consists of homologies with center a.*

Otherwise, $z^{\varTheta} \neq z$ for some $z \in W$. Let $x \in az \setminus \{a, z\}$, and consider the connected component $\varLambda$ of the stabilizer $\varDelta_x$. Because $\varTheta \leqq Z$, the fixed subplane $\mathscr{F}_\varLambda$ contains the connected orbit $z^{\varTheta}$ and hence is itself connected. The dimension formula together with (0) gives $\dim \varDelta \leqq 16 + 14$, a contradiction. □

By combining (0) and (1) we get

(2) *If $\varLambda$ fixes any quadrangle, then* $\dim \varLambda \leqq 8$.

We may assume, in fact, that $\varLambda$ is connected. If $\varLambda \cong \mathrm{G}_2$, then $\mathscr{F}_\varLambda$ would be a 2-dimensional subplane [8, 83.24], but such a plane does not admit a torus group of homologies.

(2') *No subgroup of $\varDelta$ is isomorphic to* G_2.

Proof. Let $\mathrm{G}_2 \cong \Upsilon < \varDelta$. All involutions in Υ are conjugate [8, 11.31(d)], and there are commuting involutions α and β in Υ. These are either reflections or Baer involutions [8, 55.29]. In the first case, one of α, β, or $\alpha\beta$ would have axis W and would coincide with ι, see [8, 55. 35 and 32 (ii)], but $\iota \notin \Upsilon$ because G_2 is simple [8, 11.32]. Hence every involution in Υ is planar, and from [8, 55.39 and Note 6] it follows that $W \approx \mathbb{S}_8$. Repeated application of

[8, 96.35] shows that Υ fixes a quadrangle and, in fact, a 2-dimensional subplane. This is impossible by (2). □

Four cases will be treated separately: (i) Δ is transitive on W, (ii) Δ has a fixed point $v \in W$, (iii) Δ is doubly transitive on some orbit $V \subset W$, or (iv) Δ has none of these properties. The last case will turn out to be the most difficult one.

(3) *The group* $\Omega = \Delta_{[a,W]}$ *of homologies with axis* W *has dimension* $\dim \Omega \leqq 2$*, and* Δ *induces on* W *a group* Δ/Ω *of dimension at least* 29.

Proof. Let Ψ be a maximal compact subgroup of the connected component Ω^1. Then $\Omega^1 = \Psi$ or $\Omega^1 \cong \Psi \times \mathbb{R}$ by [8, 61.2]. The compact Lie group Ψ does not contain commuting involutions and hence has torus rank at most 1, see [8, 55.32(ii) or 35]. From (1) follows $\Theta \trianglelefteq \Psi$ and $\Psi \ncong \mathrm{Spin}_3$. This leaves only the possibility $\Psi = \Theta$. □

(4) If Δ is transitive on W, then $W \approx \mathbb{S}_8$, see [8, 52.3 and 96.14]. A maximal compact subgroup Φ of Δ is also transitive on W by [8, 96.19], and $\Phi|_W \cong \mathrm{SO}_9$, compare [8, 96.22]. According to [8, 94.27], the group Δ contains a covering group H of SO_9, but then $\Phi \geqq H\Theta$ would have torus rank > 4. This contradicts [8, 55.37], and case (i) is impossible.

(5) **Lemma.** *Assume that G is a locally compact, connected transitive transformation group of* $S \approx \mathbb{S}_8 \setminus \{a, b\}$. *Consider a maximal compact subgroup K of G and the stabilizer* $H = G_c$ *of some point* $c \in S$. *If* $H \cong \mathrm{SU}_3\mathbb{C}$, *then* $K \cong \mathrm{SU}_4\mathbb{C}$.

The proof depends on the exact homotopy sequence

$$\ldots \to \pi_{q+1}S \to \pi_q H \to \pi_q G \to \pi_q S \to \pi_{q-1}H \to \ldots$$

for the action of G on S, see [8, 96.12]. Note that G is a Lie group by [8, 96.14], and that there are homotopy equivalences $S \simeq \mathbb{S}_7$ and $G \simeq K$; the second one follows from the Mal'cev-Iwasawa theorem [8, 93.10]. Up to $q = 8$ (and beyond), the homotopy groups of S and of all compact simple Lie groups are known, compare the remarks preceding 94.36 in [8]. We have $\pi_q S = 0$ for $q < 7$ and $\pi_7 S \cong \mathbb{Z}$. The homotopy sequence gives $\pi_q K \cong \pi_q H$ for $q \leqq 5$. In particular, $\pi_1 K = 0$ and, therefore, K is semi-simple [8, 94.31(c)]. Whenever C is compact and almost simple, then $\pi_3 C \cong \mathbb{Z}$, see [8, 94.36]. Hence $\pi_3 K \cong \mathbb{Z}$, and K is even almost simple. The dimension formula shows that $8 \leqq \dim K \leqq 16$. Because of (2'), only the groups $\pi_q C$ with $C \cong \mathrm{SU}_3$, SU_4, or $\mathrm{U}_2\mathbb{H}$ are actually needed; these can be found in Mimura [4, § 3.2]. Generally, $\pi_5 C \cong \mathbb{Z}$ if and only if C is locally isomorphic to a group $\mathrm{SU}_n\mathbb{C}$ with $n > 2$. Moreover, $\pi_6 H \cong \mathbb{Z}_6$ and $\pi_7 H = 0$. The exact sequence

$$\pi_7 H \to \pi_7 K \to \pi_7 \mathbb{S} \to \pi_6 H$$

shows that $\pi_7 K \cong \mathbb{Z}$, and $K \cong \mathrm{SU}_4\mathbb{C}$. □

We are now able to deal with case (ii).

(6) *If* $v^\Delta = v \in W$, *then* Δ *is transitive on* $W \setminus \{v\}$.

Proof. Let $v \neq z \in W$. Together with (2), the dimension formula implies first $z^\Delta \neq z$ and then $31 - 2 \cdot \dim z^\Delta \leqq 8 + 8$. Hence $\dim z^\Delta = 8$, and z^Δ is open in W by [8, 96.11]. Because $W \setminus \{v\}$ is connected, the assertion follows. □

(7) *If* $v^{\Delta} = v \in W$, *then* Δ *is even doubly transitive on* $W \setminus \{v\}$.

Proof. Let $\nabla = \Delta_u$ for some $u \in W$, $u \neq v$, and note that $23 \leqq \dim \nabla \leqq 24$. If ∇ is not transitive on $W \setminus \{u, v\}$, then, by similar arguments as in (6), there is a 7-dimensional orbit $z^{\nabla} \subset W$, and ∇_z is transitive on $S = av \setminus \{a, v\}$. From (0), (2), and (5) we conclude that a maximal subgroup Φ of ∇_z must be isomorphic to $\mathrm{SU}_4\mathbb{C}$, but $\Theta \lhd \Phi$, a contradiction. □

(8) Remarks. All locally compact doubly transitive transformation groups (Γ, M) have been determined by Tits [9]. Either Γ is simple and M is a projective space or a sphere, or $M \approx \mathbb{R}^k$ and Γ is an extension of $\mathbb{R}^k$ by a transitive subgroup $G \leqq \mathrm{GL}_k\mathbb{R}$, compare [8, 96. 15–23]. A convenient description of the possibilities for G can be found in Völklein [10]. The group G has an almost simple normal subgroup H which is transitive on the $(k-1)$-sphere S consisting of the rays in $\mathbb{R}^k$, and a maximal compact subgoup K of H is also transitive on S, see [8, 96.19]. It is now easy to detect the possible groups H among the irreducible representations of almost simple Lie groups [8, 95.10], and G is contained in the product of H and its centralizer. In particular, $\dim G/H \leqq 4$, even $\leqq 2$ if $\mathrm{Cs}H \ncong \mathbb{H}$.

(9) *If* $u^{\Delta} = W \setminus \{v\}$ *and* $\nabla = \Delta_u$, *then* ∇ *is an almost direct product of the solvable radical* $\sqrt{\nabla}$ *and a group* $\Psi \cong \mathrm{Sp}_4\mathbb{C}$.

Proof. By (2) and (3), the effective group $\Upsilon = \nabla|_W \cong \nabla/\Omega$ satisfies $21 \leqq \dim \Upsilon \leqq 23$, and Υ has no subgroup locally isomorphic to Spin_7 by (2'). With the remarks (8) it follows that the commutator subgoup Υ' is isomorphic to the simply connected group $\mathrm{Sp}_4\mathbb{C}$. The center Z of Υ is contained in $\mathbb{C}^{\times}$, and $\Upsilon = \Upsilon' Z$. The group Υ' is covered by a normal subgoup Ψ of ∇. □

A maximal compact subgroup of $\mathrm{Sp}_4\mathbb{C}$ is isomorphic to $\mathrm{U}_2\mathbb{H}$ and does not contain $\mathrm{SU}_3\mathbb{C}$. Hence (0) and (9) imply

(10) **Corollary.** *If* $u^{\Delta} = W \setminus \{v\}$, *and if* Λ *fixes a quadrangle, then* $\dim \Lambda \leqq 7$.

(11) *If* $u^{\Delta} = W \setminus \{v\}$ *and* $c \in av \setminus \{a, v\}$, *then* Δ_c *is doubly transitive on* $W \setminus \{v\}$ *and* $\Gamma = \Delta_{c,u} \cong \mathrm{SL}_2\mathbb{H}$.

Proof. Let $u \neq z \in u^{\Delta}$. By (10) and the dimension formula, we have

$$15 \leqq \dim \Gamma = \dim \Gamma_z + \dim z^{\Gamma} \leqq 7 + 8,$$

and $\dim z^{\Gamma} = 8$. Hence each orbit z^{Γ} is open in W and Γ is transitive on $W \setminus \{u, v\}$ by the arguments of (6). The last assertion follows with the remarks (8). □

Because of Levi's Theorem [8, 94.28], we conclude from (9) and (11) that $\mathrm{SL}_2\mathbb{H}$ must be a subgroup of $\mathrm{Sp}_4\mathbb{C}$. There are several ways to show that this is impossible. A simple reason is the following: both groups have $\mathrm{U}_2\mathbb{H}$ as maximal compact subgroups, but these are even maximal among all subgroups [8, 94.34]. More generally, Tits [9, Th. IV B.3.3] has determined all large maximal subgroups of the classical simple Lie groups. Thus, case (ii) has finally led to a contradiction.

All actions of Δ on W having only fixed points and 8-dimensional orbits are covered by (i) and (ii). Hence we may assume in case (iii) that Δ is doubly transitive on some orbit $V \subset W$ with $0 < \dim V = k < 8$. Let $u, v, w \in V$, and denote the connected component of the stabilizer $\Delta_{u,v,w}$ by Ξ. From (2) and the dimension formula we obtain $\dim \Xi \leqq 16$ and then $31 \leqq \dim \Delta \leqq 3k + 16$. Consequently, $\dim V \geqq 5$. If $V \approx \mathbb{R}^k$, then $\Delta_{u,v}$ fixes a 1-dimensional subspace of $\mathbb{R}^k$, and we get even $2k \geqq 31 - 16$, a contradiction. Thus, V is compact and $\Delta|_V$ is simple by (8). If V is a projective space, then $\Delta_{u,v}$ fixes the (real or complex) line through u and v, and $\dim \Delta \leqq 2k + 2 + \dim \Xi$. This implies $V \approx P_7\mathbb{R}$ and $\Delta|_V \cong PSL_8\mathbb{R}$, see [8, 96.17]. But then $\dim \Delta > 63$ would be to large. By (8) or [8, 96.17] we have

(12) *If Δ is doubly transitive on $V \subset W$, then V is homeomorphic to a sphere $\mathbb{S}_k$ with* $5 \leqq k \leqq 7$.

Because $k > 4$, the kernel Φ of the action of Δ on $v^\Delta = V$ acts freely on $av \setminus \{a, v\}$, and $\dim \Phi \leqq 8$, $\dim \Delta|_V \geqq 23$. By [8, 96.19 and 23], each transitive group on $\mathbb{S}_6$ contains G_2, and (2') shows that $k \neq 6$. Therefore, only one possibility of the list [8, 96.17(b)] remains:

(13) *If Δ is doubly transitive on $V \subset W$, then $V \approx \mathbb{S}_7$ and* $\Delta|_V \cong PSU_5(\mathbb{C}, 1)$.

If Δ is as in (13), then Δ contains an almost simple subgroup Ψ which is locally isomorphic to $SU_5(\mathbb{C}, 1)$, see [8, 94.27]. The kernel $\Phi = \Delta_{[V]}$ has dimension 7 or 8, and each representation of Ψ on the Lie algebra of Φ is trivial [8, 95.10]. Hence $\Phi \leqq Cs_\Delta \Psi$. The group Ψ has torus rank $rk\,\Psi \geqq 3$. By [8, 55. 29 and 35], there exist involutions $\alpha, \omega \in \Psi$ such that ω is planar, α is not the reflection with axis W, and $\alpha\omega = \omega\alpha$. Then α induces on the fixed plane $\mathcal{F}_\omega$ either a reflection or a Baer involution. The common fixed point set $C = F_{\alpha,\omega}$ is 4-dimensional, and Φ acts freely on some orbit $c^\Phi \subset C$, but $\dim \Phi \geqq 7$. This contradiction finally excludes case (iii). □

The general case (iv). Again, there is an orbit $v^\Delta = V \subset W$ with $0 < \dim V < 8$. Let $v \neq u \in V$, and consider the connected components Γ of Δ_v and ∇ of Γ_u.

(14) *The orbit $u^\Gamma = U$ is a 6-dimensional connected manifold.*

Proof. $u^\Gamma \approx \Gamma/\Gamma_u$ is a connected manifold [8, 94.3(a)]. Assume that $\dim U = m < 6$. Choose $w \in U \setminus \{u\}$ and $c \in av \setminus \{a, v\}$, and denote the connected component of $\nabla_{c,w}$ by Λ. The dimension formula gives $\dim \Lambda \geqq 31 - 7 - 8 - 2m \geqq 6$, and (2) implies $m \geqq 4$. By [8, 83.22] and because Θ is a torus group of homologies, the fixed elements of Λ form a 4-dimensional subplane $\mathcal{F}_\Lambda = \mathcal{F}$. Choose $z \in U \setminus \mathcal{F}$. Then $\Lambda_z \neq \mathbb{1}$ and $\mathcal{F}_{\Lambda_z} = \langle \mathcal{F}, z \rangle$ is a Baer subplane. From [8, 83.9] it follows that Λ is compact. In fact, $\Lambda \cong SU_3$ or $\Lambda \cong SO_4$, see Salzmann [7, (2.1)]. In the second case, Λ contains a central involution η, and Λ induces a group Λ/K on the Baer subplane $\mathcal{F}_\eta$. Now $\dim \Lambda/K \leqq 1$ by [8, 83.11], and $\dim K \leqq 3$ by [8, 83.22]. This contradiction shows that $\Lambda \cong SU_3$. For a point z as above, [8, 83.22] implies $\Lambda_z \cong SU_2$ and $z^\Lambda \approx \mathbb{S}_5$. Hence $m = 5$ and z^Λ is open and closed in U, compare [8, 92.14 or 96.11(a)]. Because U is connected, Λ must be transitive on U, but $z^\Lambda \subseteq U \setminus \mathcal{F} \neq U$. □

(15) *If $c \in S = av \setminus \{a, v\}$, then* $\dim \nabla_c \leqq 10$.

Proof. Note that Γ_c acts effectively on U and that $\dim \Gamma_c \leqq 20$ by (14) and (2). If Γ_c is doubly transitive on U, then the remarks (8) and [8, 96.16 and 17] show that $U \approx \mathbb{R}^6$. Moreover, a maximal semi-simple subgroup of Γ_c is isomorphic to SU_3, and $\dim \nabla_c \leqq 10$. Assume now that $\dim \nabla_c > 10$, and let Π denote the connected component of ∇_c. Then Γ_c is not doubly transitive on U, and there is some $w \in U \setminus \{u\}$ such that $\dim w^\Pi < 6$. The connected component Λ of Π_w satisfies $6 \leqq \dim \Lambda \leqq 8$, and $\dim w^\Pi > 2$. As is the proof of step (14) it follows that $\mathcal{F}_\Lambda$ is 4-dimensional, that $\Lambda_z \neq \mathbb{1}$ for $z \in w^\Pi \setminus \mathcal{F}_\Lambda$, and that Λ is compact. As before, $\Lambda \cong \mathrm{SU}_3$ and $z^\Lambda \approx \mathbb{S}_5$. Again Λ would be transitive on the connected manifold w^Π, an obvious contradiction. □

(16) **Corollary.** $\dim \Delta = 31$ *and* ∇ *is transitive on* S.

The same technique as in the proof of Lemma (5) can now be applied. However, only the dimension of the stabilizer $\nabla_c = \Pi$ is known, but neither the structure nor the topology of Π, and there are several distinct possibilities. Consider maximal compact subgroups Ψ of Π and Φ of ∇ with $\Psi \leqq \Phi$ and the respective semi-simple commutator subgroups Ψ' and Φ'. Because ∇ is not compact and Θ is normal in Φ, we have $\dim \Phi' \leqq 16$. The group Ψ' is a product of 3-dimensional factors, or Ψ' is locally isomorphic to SU_3, or $\Psi' = \Pi \cong \mathrm{U}_2\mathbb{H} \cong \mathrm{Spin}_5$. The exact homotopy sequence for the action of ∇ on S becomes

$$\ldots \to \pi_{q+1} S \to \pi_q \Psi \to \pi_q \Phi \to \pi_q S \to \ldots \to \pi_1 S = 0.$$

If C is any compact, connected Lie group and $q > 1$, then $\pi_q C' \cong \pi_q C$ by [8, 94.31(c)]. Moreover, $\pi_1 \Psi \cong \pi_1 \Phi$ is infinite (because Θ is a factor of Φ), and $\Psi' < \Psi$. This excludes the possibility $\Psi' = \Pi$. All the relevant homotopy groups of small compact simple Lie groups C can be found in Mimura [4, § 3.2]. In particular, $\pi_5 \mathrm{SU}_n \cong \mathbb{Z}$ for $n \geqq 3$, all other groups $\pi_5 C$ and all groups $\pi_6 C$ are finite.

(17) $\mathbb{1} < \Psi' < \Phi'$ *and* $\dim \Phi' \leqq \dim \Psi' + 7$.

Proof. If $\Psi' = \Phi'$, then $\pi_q \Psi \cong \pi_q \Phi$ in the exact homotopy sequence, and $\pi_q S \to \pi_{q-1} \Psi$ is injective, but $\pi_7 S \cong \mathbb{Z}$ and $\pi_6 \Psi$ is finite. Hence $\Psi' < \Phi'$. If $\Psi' = \mathbb{1}$, then $\pi_3 \Phi' = 0$ and $\Phi' = \mathbb{1}$ by [8, 94.36]. This contradicts the first step of the proof. From $\pi_1 \Psi \cong \pi_1 \Phi$ and [8, 94.31(c)] it follows that the torus factors of Φ and of Ψ have the same dimension. Because Φ is compact and Ψ is the connected component of Φ_c, we obtain $\dim \Phi'/\Psi' = \dim \Phi/\Psi = \dim c^\Phi < 8$. □

The remaining possibilities will be discussed separately. We will need the following lemma:

(18) *If Φ contains a reflection σ with center u or axis au, then the elation group E with center v (and axis av) is sharply transitive on U, and E is a* 6*-dimensional Lie group.*

Proof. Assume that σ has center u. Choose $\rho = \sigma^\eta$ with $\eta \in \Gamma$ and $u^\eta \neq u$. Then $\sigma\rho$ is the elation with axis av mapping u to u^ρ. Thus, σ is unique and $(\gamma \mapsto \sigma\sigma^\gamma)$ maps the coset space Γ/Γ_u continuously and injectively into E. Hence $\dim E = \dim U = 6$. By [8, 96.11(a)], each E-orbit in U is open, and $u^E = U$ because U is connected. □

(19) $\dim \Psi' \neq 3$.

Proof. If Ψ' is locally isomorphic to SU_2, then $\pi_3 \Phi' \cong \pi_3 \Psi' \cong \mathbb{Z}$, and Φ' is almost simple by [8, 94.36]. The last statement of (17) implies $\dim \Phi' \leqq 10$. Since $\pi_5 \Phi' \cong \pi_5 \Psi' \cong \pi_5 \mathbb{S}_3 \cong \mathbb{Z}_2$ is finite, the group Φ' is not locally isomorphic to SU_3 by the remarks preceding (17). Consequently, $\dim \Phi' = 10$. Because the group SO_5 cannot act on any plane [8, 55.40], it follows that $\Phi' \cong \mathrm{Spin}_5 \cong \mathrm{U}_2\mathbb{H}$ is the simply connected covering group of SO_5. Again by [8, 55.40], the central involution $\sigma \in \Phi'$ cannot be planar, and σ is a reflection. If the axis of σ is different from W, then (18) implies that the elation group E with center v is a 6-dimensional connected Lie group. The group E is not known to be commutative, but σ inverts each element of E. Therefore, Φ' induces a faithful representation on the Lie algebra $\mathfrak{l}E$ of E. The list of irreducible representations given in [8, 95.10] shows that $\dim E = 8$, a contradiction. Hence σ has axis W, and $\sigma \in \Theta$.

Consider now the involution $\beta \in \Phi'$ corresponding to the element $\operatorname{diag}(1, -1) \in U_2\mathbb{H}$. The centralizer $\Phi' \cap \mathrm{Cs}\beta$ is a direct product $A \times B$, where $A \cong B \cong SU_2$ and $\beta \in B$. The properties of $U_2\mathbb{H}$ show that $\alpha = \beta\sigma$ is the central involution in A, and that α and β are conjugate in Φ'. If β is a reflection, then α and β have centers u and v and cannot be conjugate within ∇. Hence β is a Baer involution, its fixed elements form an 8-dimensional subplane $\mathcal{F}_\beta = \mathcal{B}$. Either B induces the identity on $\mathcal{B}$, or $B|_{\mathcal{B}} \cong SO_3$ (note that $\beta \in B$). In the latter case, the fixed elements of B would form a 2-dimensional subplane $\mathcal{E}$, and Θ would act as a group of homologies on $\mathcal{E}$, but this is impossible by [8, 32.17 or 61.26]. Therefore, $B|_{\mathcal{B}} = \mathbb{1}$ and, analogously, $A|_{\mathcal{F}_\alpha} = \mathbb{1}$. Because α and β commute, it follows from [8, 55.32] that $\alpha|_{\mathcal{B}} \neq \mathbb{1}$ and, hence, that A acts faithfully on $\mathcal{B}$. Consequently, $A\Theta \cong U_2\mathbb{C}$ would induce a 4-dimensional compact group of homologies on $\mathcal{B}$. This contradicts [8, 61.26]. □

The next case can be treated in the same way:

(20) $\dim \Psi' \neq 6$.

Proof. If Ψ' is locally isomorphic to SO_4, then Φ' has two almost simple factors by [8, 94.36]. With (17) we obtain $\dim \Phi' = 13$, and Φ' has a factor $\Xi \cong \mathrm{Spin}_5$. As in the last step, the existence of such a group leads to a contradiction. □

(21) $\dim \Psi' \neq 9$.

Proof. If Ψ' is a product of 3 almost simple factors, then so is Φ', again by [8, 94.36]. Because $\Psi' < \Phi'$, one of the factors of Φ' must have torus rank at least 2. This implies that $\mathrm{rk}\Phi' \geqq 4$ and then $\mathrm{rk}\,\Theta\Phi' > 4$. According to [8, 55.37], however, the torus rank can never exceed 4. □

(22) $\dim \Psi' \neq 8$.

Proof. We argue as in step (19). If Ψ' is locally isomorphic to SU_3, then $\pi_3\Phi' \cong \mathbb{Z}$ and Φ' is almost simple. From $\pi_5\Phi' \cong \mathbb{Z}$ and $8 < \dim \Phi' \leqq 15$ we infer that Φ' is locally isomorphic to $SU_4\mathbb{C} \cong \mathrm{Spin}_6$. Because SO_5 cannot act on a plane, Φ' is even isomorphic to SU_4, and its central involution σ is a reflection. In fact, σ has the axis W, or else Φ' would act effectively on the elation group E, see (18). The involution β corresponding to $\operatorname{diag}(1, 1, -1, -1) \in SU_4$ fixes a Baer subplane $\mathcal{B}$ because it commutes with 5 conjugates, see [8, 55.35]. The centralizer of β contains a direct product $A \times B$, where $A \cong B \cong SU_2$ and $\beta \in B$. Exactly as in (19), it follows that ΘA induces on $\mathcal{B}$ a compact, 4-dimensional group of homologies with axis $W \cap \mathcal{B}$. This final contradiction completes the proof of Theorem T.

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