6.5 ANOTHER SINGULAR STURM-LIOUVILLE PROBLEM: BESSEL'S EQUATION

In the previous section we discussed the solutions to Legendre's equation, especially with regard to their use in orthogonal expansions. In the section we consider another classic equation, Bessel's equation[18]

$$x^2y'' + xy' + (\mu^2x^2 - n^2)y = 0 \tag{6.5.1}$$

or

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \left(\mu^2 x - \frac{n^2}{x}\right)y = 0. \tag{6.5.2}$$

Once again, our ultimate goal is the use of its solutions in orthogonal expansions. These orthogonal expansions, in turn, are used in the solution of partial differential equations in cylindrical coordinates.

A quick check of Bessel's equation shows that it conforms to the canonical form of the Sturm-Liouville problem: $p(x) = x$, $q(x) = -n^2/x$, $r(x) = x$, and $\lambda = \mu^2$. Restricting our attention to the interval $[0, L]$, the Sturm-Liouville problem involving (6.5.2) is singular because $p(0) = 0$. From (6.4.1) in the previous section, the eigenfunctions to a singular Sturm-Liouville problem will still be orthogonal over the interval $[0, L]$ if (1) $y(x)$ is finite and $xy'(x)$ is zero at $x = 0$, and (2) $y(x)$ satisfies the homogeneous boundary condition (6.1.2) at $x = L$. Consequently, we will only seek solutions that satisfy these conditions.

We cannot write down the solution to Bessel's equation in a simple closed form; as in the case with Legendre's equation, we must find the solution by power series. Because we intend to make the expansion about $x = 0$ and this point is a regular singular point, we must use the method of Frobenius, where n is an integer.[19] Moreover, because the quantity n^2 appears in (6.5.2), we may take n to be nonnegative without any loss of generality.

To simplify matters, we first find the solution when $\mu = 1$; the solution for $\mu \neq 1$ follows by substituting μx for x. Consequently, we seek solutions of the form

$$y(x) = \sum_{k=0}^{\infty} B_k x^{2k+s}, \tag{6.5.3}$$

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[18] Bessel, F. W., 1824: Untersuchung des Teils der planetarischen Störungen, welcher aus der Bewegung der Sonne entsteht. *Abh. d. K. Akad. Wiss. Berlin*, 1–52. See Dutka, J., 1995: On the early history of Bessel functions. *Arch. Hist. Exact Sci.*, **49**, 105–134. The classic reference on Bessel functions is Watson, G. N., 1966: *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge.

[19] This case is much simpler than for arbitrary n. See Hildebrand, F. B., 1962: *Advanced Calculus for Applications*. Prentice-Hall, Englewood Cliffs, NJ, Section 4.8.

Figure 6.5.1: It was Friedrich William Bessel's (1784–1846) apprenticeship to the famous mercantile firm of Kulenkamp that ignited his interest in mathematics and astronomy. As the founder of the German school of practical astronomy, Bessel discovered his functions while studying the problem of planetary motion. Bessel functions arose as coefficients in one of the series that described the gravitational interaction between the sun and two other planets in elliptic orbit. (Portrait courtesy of Photo AKG, London.)

$$y'(x) = \sum_{k=0}^{\infty} (2k+s) B_k x^{2k+s-1} \tag{6.5.4}$$

and

$$y''(x) = \sum_{k=0}^{\infty} (2k+s)(2k+s-1) B_k x^{2k+s-2}, \tag{6.5.5}$$

where we formally assume that we can interchange the order of differentiation and summation. The substitution of (6.5.3)–(6.5.5) into (6.5.1) with $\mu = 1$ yields

$$\sum_{k=0}^{\infty}(2k+s)(2k+s-1)B_k x^{2k+s} + \sum_{k=0}^{\infty}(2k+s)B_k x^{2k+s} + \sum_{k=0}^{\infty} B_k x^{2k+s+2} - n^2 \sum_{k=0}^{\infty} B_k x^{2k+s} = 0 \qquad (6.5.6)$$

or

$$\sum_{k=0}^{\infty}[(2k+s)^2 - n^2]B_k x^{2k} + \sum_{k=0}^{\infty} B_k x^{2k+2} = 0. \qquad (6.5.7)$$

If we explicitly separate the $k = 0$ term from the other terms in the first summation in (6.5.7),

$$(s^2 - n^2)B_0 + \sum_{m=1}^{\infty}[(2m+s)^2 - n^2]B_m x^{2m} + \sum_{k=0}^{\infty} B_k x^{2k+2} = 0. \quad (6.5.8)$$

We now change the dummy integer in the first summation of (6.5.8) by letting $m = k + 1$ so that

$$(s^2 - n^2)B_0 + \sum_{k=0}^{\infty}\{[(2k+s+2)^2 - n^2]B_{k+1} + B_k\}x^{2k+2} = 0. \quad (6.5.9)$$

Because (6.5.9) must be true for all x, each power of x must vanish identically. This yields $s = \pm n$ and

$$[(2k+s+2)^2 - n^2]B_{k+1} + B_k = 0. \qquad (6.5.10)$$

Since the difference of the larger indicial root from the lower root equals the integer $2n$, we are only guaranteed a power series solution of the form (6.5.3) for $s = n$. If we use this indicial root and the recurrence formula (6.5.10), this solution, known as the Bessel function of the first kind of order n and denoted by $J_n(x)$, is

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!(n+k)!}. \qquad (6.5.11)$$

To find the second general solution to Bessel's equation, the one corresponding to $s = -n$, the most economical method[20] is to express it in terms of partial derivatives of $J_n(x)$ with respect to its order n:

$$Y_n(x) = \left[\frac{\partial J_\nu(x)}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}(x)}{\partial \nu}\right]_{\nu=n}. \qquad (6.5.12)$$

[20] See Watson, G. N., 1966: *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, Section 3.5 for the derivation.

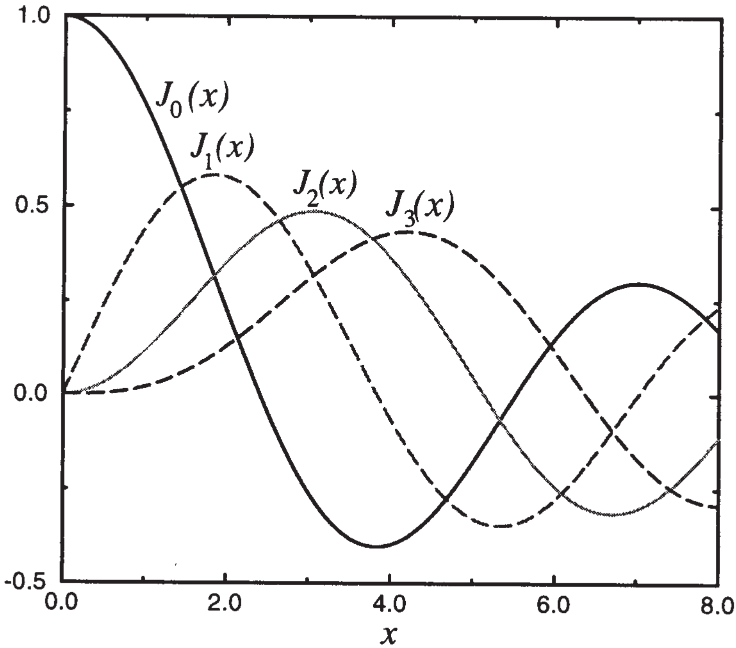

Figure 6.5.2: The first four Bessel functions of the first kind over $0 \le x \le 8$.

Upon substituting the power series representation (6.5.11) into (6.5.12),

$$Y_n(x) = \frac{2}{\pi} J_n(x)\ln(x/2) - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n}$$
$$-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k (x/2)^{n+2k}}{k!(n+k)!}[\psi(k+1)+\psi(k+n+1)], \quad \textbf{(6.5.13)}$$

where

$$\psi(m+1) = -\gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{m}, \quad \textbf{(6.5.14)}$$

$\psi(1) = -\gamma$ and γ is Euler's constant (0.5772157). In the case $n = 0$, the first sum in (6.5.13) disappears. This function $Y_n(x)$ is Neumann's Bessel function of the second kind of order n. Consequently, the general solution to (6.5.1) is

$$y(x) = AJ_n(\mu x) + BY_n(\mu x). \quad \textbf{(6.5.15)}$$

Figure 6.5.2 illustrates the functions $J_0(x)$, $J_1(x)$, $J_2(x)$, and $J_3(x)$ while Figure 6.5.3 gives $Y_0(x)$, $Y_1(x)$, $Y_2(x)$, and $Y_3(x)$.

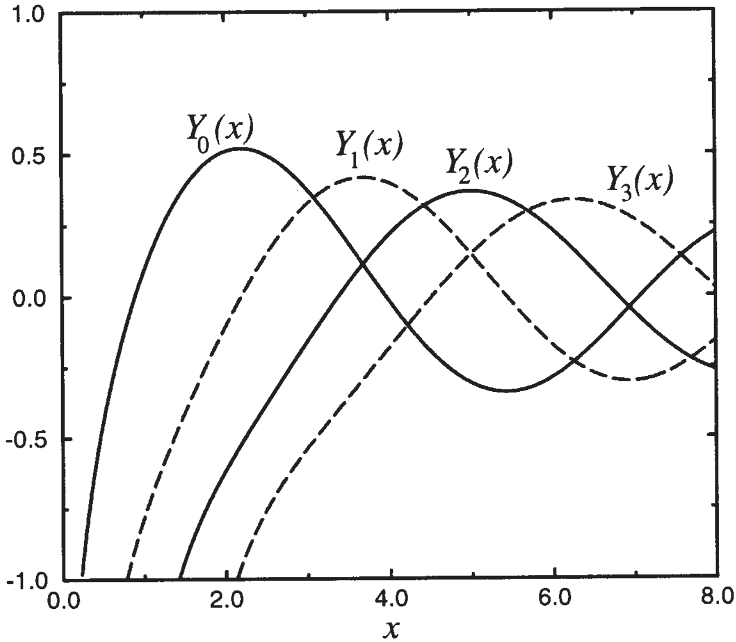

Figure 6.5.3: The first four Bessel functions of the second kind over $0 \leq x \leq 8$.

An equation which is very similar to (6.5.1) is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (n^2 + x^2)y = 0. \qquad (6.5.16)$$

It arises in the solution of partial differential equations in cylindrical coordinates. If we substitute $ix = t$ (where $i = \sqrt{-1}$) into (6.5.16), it becomes Bessel's equation:

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - n^2)y = 0. \qquad (6.5.17)$$

Consequently, we may immediately write the solution to (6.5.16) as

$$y(x) = c_1 J_n(ix) + c_2 Y_n(ix), \qquad (6.5.18)$$

if n is an integer. Traditionally the solution to (6.5.16) has been written

$$y(x) = c_1 I_n(x) + c_2 K_n(x) \qquad (6.5.19)$$

rather than in terms of $J_n(ix)$ and $Y_n(ix)$, where

$$I_n(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+n}}{k!(k+n)!} \qquad (6.5.20)$$

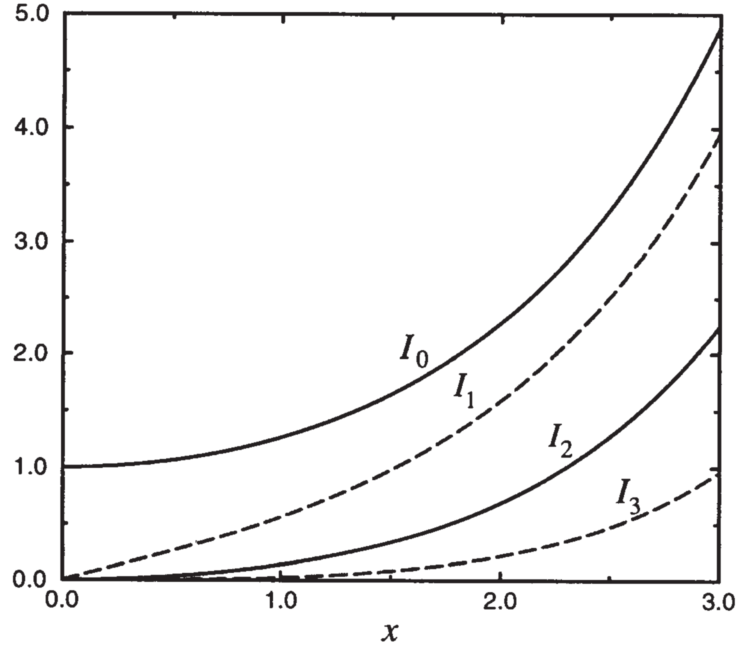

Figure 6.5.4: The first four modified Bessel functions of the first kind over $0 \le x \le 3$.

and

$$K_n(x) = \frac{\pi}{2} i^{n+1} \left[J_n(ix) + iY_n(ix)\right]. \tag{6.5.21}$$

The function $I_n(x)$ is the modified Bessel function of the first kind, of order n, while $K_n(x)$ is the modified Bessel function of the second kind, of order n. Figure 6.5.4 illustrates $I_0(x)$, $I_1(x)$, $I_2(x)$, and $I_3(x)$ while in Figure 6.5.5 $K_0(x)$, $K_1(x)$, $K_2(x)$, and $K_3(x)$ have been graphed. Note that $K_n(x)$ has no real zeros while $I_n(x)$ equals zero only at $x = 0$ for $n \ge 1$.

As our derivation suggests, modified Bessel functions are related to ordinary Bessel functions via complex variables. In particular, $J_n(iz) = i^n I_n(z)$ and $I_n(iz) = i^n J_n(z)$ for z complex.

Although we have found solutions to Bessel's equation (6.5.1), as well as (6.5.16), can we use any of them in an eigenfunction expansion? From Figures 6.5.2–6.5.5 we see that $J_n(x)$ and $I_n(x)$ remain finite at $x = 0$ while $Y_n(x)$ and $K_n(x)$ do not. Furthermore, the products $xJ_n'(x)$ and $xI_n'(x)$ tend to zero at $x = 0$. Thus, both $J_n(x)$ and $I_n(x)$ satisfy the first requirement of an eigenfunction for a Fourier-Bessel expansion.

What about the second condition that the eigenfunction must satisfy the homogeneous boundary condition (6.1.2) at $x = L$? From Figure 6.5.4 we see that $I_n(x)$ can never satisfy this condition while from Fig-

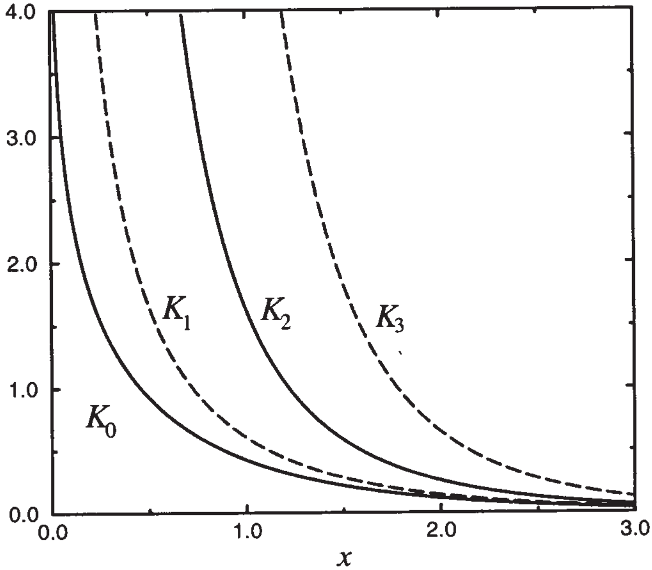

Figure 6.5.5: The first four modified Bessel functions of the second kind over $0 \leq x \leq 3$.

ure 6.5.2 $J_n(x)$ can. For that reason, we discard $I_n(x)$ from further consideration and continue our analysis only with $J_n(x)$.

Before we can derive the expressions for a Fourier-Bessel expansion, we need to find how $J_n(x)$ is related to $J_{n+1}(x)$ and $J_{n-1}(x)$. Assuming that n is a positive integer, we multiply the series (6.5.11) by x^n and then differentiate with respect to x. This gives

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{k=0}^{\infty} \frac{(-1)^k (2n+2k) x^{2n+2k-1}}{2^{n+2k} k!(n+k)!} \quad (6.5.22)$$

$$= x^n \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n-1+2k}}{k!(n-1+k)!} \quad (6.5.23)$$

$$= x^n J_{n-1}(x) \quad (6.5.24)$$

or

$$\boxed{\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)} \quad (6.5.25)$$

for $n = 1, 2, 3, \ldots$. Similarly, multiplying (6.5.11) by x^{-n}, we find that

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x) \qquad \textbf{(6.5.26)}$$

for $n = 0, 1, 2, 3, \ldots$. If we now carry out the differentiation on (6.5.25) and (6.5.26) and divide by the factors $x^{\pm n}$, we have that

$$J_n'(x) + \frac{n}{x} J_n(x) = J_{n-1}(x) \qquad \textbf{(6.5.27)}$$

and

$$J_n'(x) - \frac{n}{x} J_n(x) = -J_{n+1}(x). \qquad \textbf{(6.5.28)}$$

Equations (6.3.27)–(6.3.28) immediately yield the *recurrence relationships*

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \qquad \textbf{(6.5.29)}$$

and

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x) \qquad \textbf{(6.5.30)}$$

for $n = 1, 2, 3, \ldots$ For $n = 0$, we replace (6.5.30) by $J_0'(x) = -J_1(x)$.

Let us now construct a Fourier-Bessel series. The exact form of the expansion depends upon the boundary condition at $x = L$. There are three possible cases. One of them is the requirement that $y(L) = 0$ and results in the condition that $J_n(\mu_k L) = 0$. Another condition is $y'(L) = 0$ and gives $J_n'(\mu_k L) = 0$. Finally, if $hy(L) + y'(L) = 0$, then $hJ_n(\mu_k L) + \mu_k J_n'(\mu_k L) = 0$. In all of these cases, the eigenfunction expansion is the same, namely

$$f(x) = \sum_{k=1}^{\infty} A_k J_n(\mu_k x), \qquad \textbf{(6.5.31)}$$

where μ_k is the kth positive solution of either $J_n(\mu_k L) = 0$, $J_n'(\mu_k L) = 0$ or $hJ_n(\mu_k L) + \mu_k J_n'(\mu_k L) = 0$.

We now need a mechanism for computing A_k. We begin by multiplying (6.5.31) by $xJ_n(\mu_m x)\,dx$ and integrate from 0 to L. This yields

$$\sum_{k=1}^{\infty} A_k \int_0^L xJ_n(\mu_k x)J(\mu_m x)\,dx = \int_0^L xf(x)J_n(\mu_m x)\,dx. \qquad (6.5.32)$$

From the general orthogonality condition (6.2.1),

$$\int_0^L xJ_n(\mu_k x)J_n(\mu_m x)\,dx = 0 \qquad (6.5.33)$$

if $k \neq m$. Equation (6.5.32) then simplifies to

$$A_m \int_0^L xJ_n^2(\mu_m x)\,dx = \int_0^L xf(x)J_n(\mu_m x)\,dx \qquad (6.5.34)$$

or

$$\boxed{A_k = \frac{1}{C_k}\int_0^L xf(x)J_n(\mu_k x)\,dx,} \qquad (6.5.35)$$

where

$$C_k = \int_0^L xJ_n^2(\mu_k x)\,dx \qquad (6.5.36)$$

and k has replaced m in (6.5.34).

The factor C_k depends upon the nature of the boundary conditions at $x = L$. In all cases we start from Bessel's equation

$$[xJ_n'(\mu_k x)]' + \left(\mu_k^2 x - \frac{n^2}{x}\right) J_n(\mu_k x) = 0. \qquad (6.5.37)$$

If we multiply both sides of (6.5.37) by $2xJ_n'(\mu_k x)$, the resulting equation is

$$\left(\mu_k^2 x^2 - n^2\right)\left[J_n^2(\mu_k x)\right]' = -\frac{d}{dx}\left[xJ_n'(\mu_k x)\right]^2. \qquad (6.5.38)$$

An integration of (6.5.38) from 0 to L, followed by the subsequent use of integration by parts, results in

$$(\mu_k^2 x^2 - n^2)J_n^2(\mu_k x)\Big|_0^L - 2\mu_k^2 \int_0^L xJ_n^2(\mu_k x)\,dx = -\left[xJ_n'(\mu_k x)\right]^2\Big|_0^L. \qquad (6.5.39)$$

Because $J_n(0) = 0$ for $n > 0$, $J_0(0) = 1$ and $xJ_n'(x) = 0$ at $x = 0$, the contribution from the lower limits vanishes. Thus,

$$C_k = \int_0^L xJ_n^2(\mu_k x)\,dx \tag{6.5.40}$$

$$= \frac{1}{2\mu_k^2}\left[(\mu_k^2 L^2 - n^2)J_n^2(\mu_k L) + L^2 J_n'^{\,2}(\mu_k L)\right]. \tag{6.5.41}$$

Because

$$J_n'(\mu_k x) = \frac{n}{x}J_n(\mu_k x) - \mu_k J_{n+1}(\mu_k x) \tag{6.5.42}$$

from (6.5.28), C_k becomes

$$C_k = \tfrac{1}{2}L^2 J_{n+1}^2(\mu_k L), \tag{6.5.43}$$

if $J_n(\mu_k L) = 0$. Otherwise, if $J_n'(\mu_k L) = 0$, then

$$C_k = \frac{\mu_k^2 L^2 - n^2}{2\mu_k^2} J_n^2(\mu_k L). \tag{6.5.44}$$

Finally,

$$C_k = \frac{\mu_k^2 L^2 - n^2 + h^2 L^2}{2\mu_k^2} J_n^2(\mu_k L), \tag{6.5.45}$$

if $\mu_k J_n'(\mu_k L) = -hJ_n(\mu_k L)$.

All of the preceding results must be slightly modified when $n = 0$ and the boundary condition is $J_0'(\mu_k L) = 0$ or $\mu_k J_1(\mu_k L) = 0$. This modification results from the additional eigenvalue $\mu_0 = 0$ being present and we must add the extra term A_0 to the expansion. For this case the series reads

$$f(x) = A_0 + \sum_{k=1}^{\infty} A_k J_0(\mu_k x), \tag{6.5.46}$$

where the equation for finding A_0 is

$$A_0 = \frac{2}{L^2} \int_0^L f(x)\, x\, dx \tag{6.5.47}$$

and (6.5.35) and (6.5.44) with $n = 0$ give the remaining coefficients.

• Example 6.5.1

Starting with Bessel's equation, we want to show that the solution to

$$y'' + \frac{1-2a}{x} y' + \left(b^2 c^2 x^{2c-2} + \frac{a^2 - n^2 c^2}{x^2} \right) y = 0 \tag{6.5.48}$$

is

$$y(x) = A x^a J_n(bx^c) + B x^a Y_n(bx^c), \tag{6.5.49}$$

provided that $bx^c > 0$ so that $Y_n(bx^c)$ exists.

The general solution to

$$\xi^2 \frac{d^2\eta}{d\xi^2} + \xi \frac{d\eta}{d\xi} + (\xi^2 - n^2)\eta = 0 \tag{6.5.50}$$

is

$$\eta = A J_n(\xi) + B Y_n(\xi). \tag{6.5.51}$$

If we now let $\eta = y(x)/x^a$ and $\xi = bx^c$, then

$$\frac{d}{d\xi} = \frac{dx}{d\xi}\frac{d}{dx} = \frac{x^{1-c}}{bc}\frac{d}{dx}, \tag{6.5.52}$$

$$\frac{d^2}{d\xi^2} = \frac{x^{2-2c}}{b^2c^2}\frac{d^2}{dx^2} - \frac{(c-1)x^{1-2c}}{b^2c^2}\frac{d}{dx}, \tag{6.5.53}$$

$$\frac{d}{dx}\left(\frac{y}{x^a}\right) = \frac{1}{x^a}\frac{dy}{dx} - \frac{a}{x^{a+1}} y \tag{6.5.54}$$

and

$$\frac{d^2}{dx^2}\left(\frac{y}{x^a}\right) = \frac{1}{x^a}\frac{d^2y}{dx^2} - \frac{2a}{x^{a+1}}\frac{dy}{dx} + \frac{a(1+a)}{x^{a+2}} y. \tag{6.5.55}$$

Substituting (6.5.52)–(6.5.55) into (6.5.51) and simplifying, yields the desired result.

• Example 6.5.2

We want to show that

$$x^2 J_n''(x) = (n^2 - n - x^2) J_n(x) + x J_{n+1}(x). \tag{6.5.56}$$

From (6.5.28),

$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x), \tag{6.5.57}$$

$$J_n''(x) = -\frac{n}{x^2} J_n(x) + \frac{n}{x} J_n'(x) - J_{n+1}'(x) \tag{6.5.58}$$

and

$$J_n''(x) = -\frac{n}{x^2} J_n(x) + \frac{n}{x}\left[\frac{n}{x} J_n(x) - J_{n+1}(x)\right] - \left[J_n(x) - \frac{n+1}{x} J_{n+1}(x)\right] \tag{6.5.59}$$

after using (6.5.27) and (6.5.28). Simplifying,

$$J_n''(x) = \left(\frac{n^2 - n}{x^2} - 1\right) J_n(x) + \frac{J_{n+1}(x)}{x}. \tag{6.5.60}$$

After multiplying (6.5.60) by x^2, we obtain (6.5.56).

• **Example 6.5.3**

Show that

$$\int_0^a x^5 J_2(x)\, dx = a^5 J_3(a) - 2a^4 J_4(a). \tag{6.5.61}$$

We begin by integrating (6.5.61) by parts. If $u = x^2$ and $dv = x^3 J_2(x)\, dx$, then

$$\int_0^a x^5 J_2(x)\, dx = x^5 J_3(x)\Big|_0^a - 2\int_0^a x^4 J_3(x)\, dx, \tag{6.5.62}$$

because $d[x^3 J_3(x)]/dx = x^2 J_2(x)$ by (6.5.25). Finally, since $x^4 J_3(x) = d[x^4 J_4(x)]/dx$ by (6.5.25),

$$\int_0^a x^5 J_2(x)\, dx = a^5 J_3(a) - 2x^4 J_4(x)\Big|_0^a = a^5 J_3(a) - 2a^4 J_4(a). \tag{6.5.63}$$

• **Example 6.5.4**

Let us expand $f(x) = x$, $0 < x < 1$, in the series

$$f(x) = \sum_{k=1}^{\infty} A_k J_1(\mu_k x), \tag{6.5.64}$$

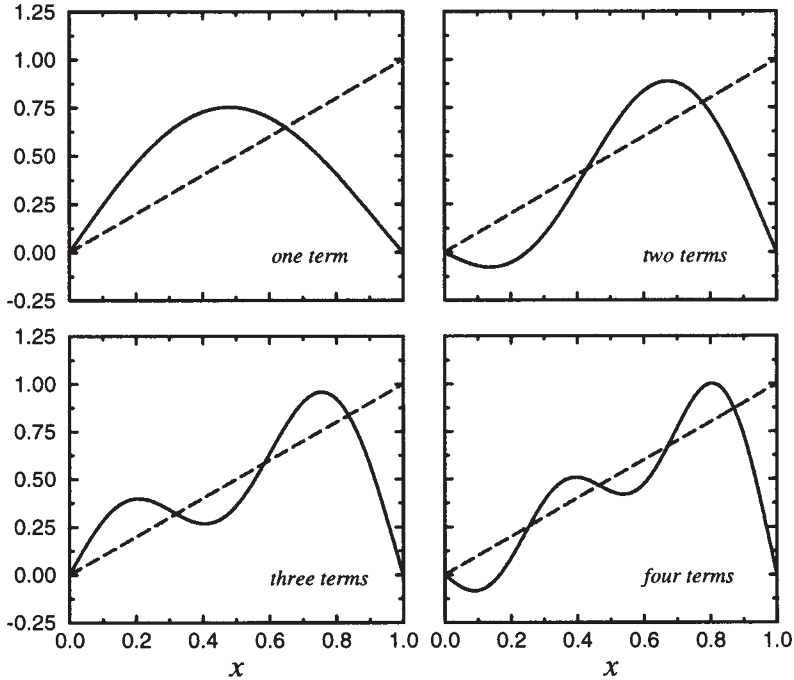

Figure 6.5.6: The Fourier-Bessel series representation (6.5.68) for $f(x) = x$, $0 < x < 1$, when we truncate the series so that it includes only the first, first two, first three, and first four terms.

where μ_k denotes the kth zero of $J_1(\mu)$. From (6.5.35) and (6.5.43),

$$A_k = \frac{2}{J_2^2(\mu_k)} \int_0^1 x^2 J_1(\mu_k x)\, dx. \tag{6.5.65}$$

However, from (6.5.25),

$$\frac{d}{dx}\left[x^2 J_2(x)\right] = x^2 J_1(x), \tag{6.5.66}$$

if $n = 2$. Therefore, (6.5.65) becomes

$$A_k = \left. \frac{2x^2 J_2(x)}{\mu_k^3 J_2^2(\mu_k)} \right|_0^{\mu_k} = \frac{2}{\mu_k J_2(\mu_k)} \tag{6.5.67}$$

and the resulting expansion is

$$x = 2 \sum_{k=1}^{\infty} \frac{J_1(\mu_k x)}{\mu_k J_2(\mu_k)}, \qquad 0 < x < 1. \tag{6.5.68}$$

Figure 6.5.6 shows the Fourier-Bessel expansion of $f(x) = x$ in truncated form when we only include one, two, three, and four terms.

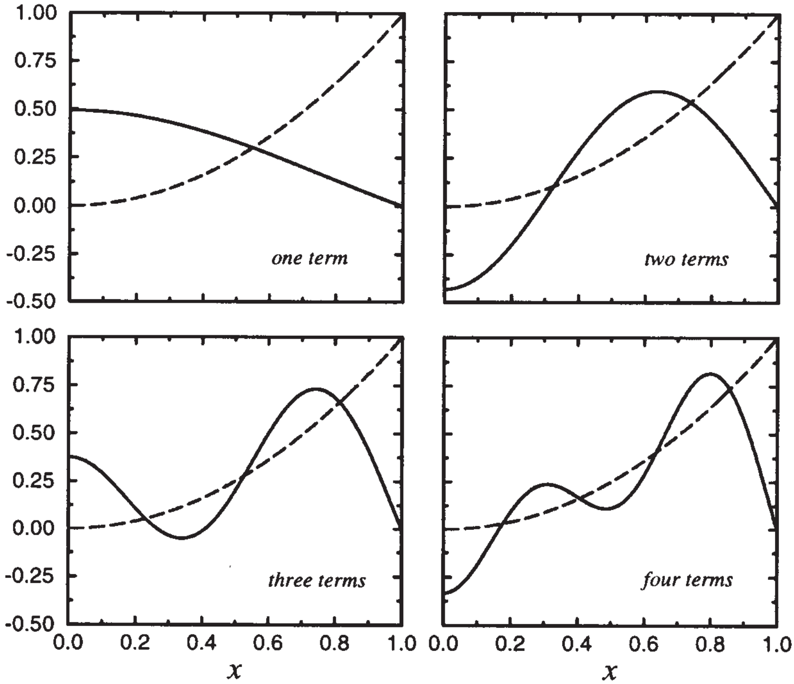

Figure 6.5.7: The Fourier-Bessel series representation (6.5.79) for $f(x) = x^2$, $0 < x < 1$, when we truncate the series so that it includes only the first, first two, first three, and first four terms.

• **Example 6.5.5**

Let us expand the function $f(x) = x^2$, $0 < x < 1$, in the series

$$f(x) = \sum_{k=1}^{\infty} A_k J_0(\mu_k x), \tag{6.5.69}$$

where μ_k denotes the kth positive zero of $J_0(\mu)$. From (6.5.35) and (6.5.43),

$$A_k = \frac{2}{J_1^2(\mu_k)} \int_0^1 x^3 J_0(\mu_k x)\, dx. \tag{6.5.70}$$

If we let $t = \mu_k x$, the integration (6.5.70) becomes

$$A_k = \frac{2}{\mu_k^4 J_1^2(\mu_k)} \int_0^{\mu_k} t^3 J_0(t)\, dt. \tag{6.5.71}$$

We now let $u = t^2$ and $dv = tJ_0(t)\, dt$ so that integration by parts results in

$$A_k = \frac{2}{\mu_k^4 J_1^2(\mu_k)} \left[t^3 J_1(t) \Big|_0^{\mu_k} - 2 \int_0^{\mu_k} t^2 J_1(t)\, dt \right] \tag{6.5.72}$$

$$= \frac{2}{\mu_k^4 J_1^2(\mu_k)} \left[\mu_k^3 J_1(\mu_k) - 2 \int_0^{\mu_k} t^2 J_1(t)\, dt \right], \tag{6.5.73}$$

because $v = tJ_1(t)$ from (6.5.25). If we integrate by parts once more, we find that

$$A_k = \frac{2}{\mu_k^4 J_1^2(\mu_k)} \left[\mu_k^3 J_1(\mu_k) - 2\mu_k^2 J_2(\mu_k) \right] \tag{6.5.74}$$

$$= \frac{2}{J_1^2(\mu_k)} \left[\frac{J_1(\mu_k)}{\mu_k} - \frac{2J_2(\mu_k)}{\mu_k^2} \right]. \tag{6.5.75}$$

However, from (6.5.29) with $n = 1$,

$$J_1(\mu_k) = \tfrac{1}{2}\mu_k \left[J_2(\mu_k) + J_0(\mu_k) \right] \tag{6.5.76}$$

or

$$J_2(\mu_k) = \frac{2J_1(\mu_k)}{\mu_k}, \tag{6.5.77}$$

because $J_0(\mu_k) = 0$. Therefore,

$$A_k = \frac{2(\mu_k^2 - 4) J_1(\mu_k)}{\mu_k^3 J_1^2(\mu_k)} \tag{6.5.78}$$

and

$$x^2 = 2 \sum_{k=1}^{\infty} \frac{(\mu_k^2 - 4) J_0(\mu_k x)}{\mu_k^3 J_1(\mu_k)}, \qquad 0 < x < 1. \tag{6.5.79}$$

Figure 6.5.7 shows the representation of x^2 by the Fourier-Bessel series (6.5.79) when we truncate it so that it includes only one, two, three, or four terms. As we add each additional term in the orthogonal expansion, the expansion fits $f(x)$ better in the "least squares" sense of (6.3.5).

Problems

1. Show from the series solution that

$$\frac{d}{dx}\left[J_0(kx) \right] = -kJ_1(kx).$$

From the recurrence formulas, show these following relations:

2.

$$2J_0''(x) = J_2(x) - J_0(x)$$

3.

$$J_2(x) = J_0''(x) - J_0'(x)/x$$

4.

$$J_0'''(x) = \frac{J_0(x)}{x} + \left(\frac{2}{x^2} - 1\right) J_0'(x)$$

5.

$$\frac{J_2(x)}{J_1(x)} = \frac{1}{x} - \frac{J_0''(x)}{J_0'(x)} = \frac{2}{x} - \frac{J_0(x)}{J_1(x)} = \frac{2}{x} + \frac{J_0(x)}{J_0'(x)}$$

6.

$$J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) - \left(\frac{24}{x^2} - 1\right) J_0(x)$$

7.

$$J_{n+2}(x) = \left[2n + 1 - \frac{2n(n^2 - 1)}{x^2}\right] J_n(x) + 2(n+1)J_n''(x)$$

8.

$$J_3(x) = \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$$

9.

$$4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$$

10. Show that the maximum and minimum values of $J_n(x)$ occur when

$$x = \frac{nJ_n(x)}{J_{n+1}(x)}, \quad x = \frac{nJ_n(x)}{J_{n-1}(x)}, \quad \text{and} \quad J_{n-1}(x) = J_{n+1}(x).$$

Show that

11.

$$\frac{d}{dx}\left[x^2 J_3(2x)\right] = -xJ_3(2x) + 2x^2 J_2(2x)$$

12.

$$\frac{d}{dx}\left[xJ_0(x^2)\right] = J_0(x^2) - 2x^2 J_1(x^2)$$