CHAPTER 14

BESSEL FUNCTIONS

Bessel functions appear in a wide variety of physical problems. In Section 9.4 we saw that separation of the Helmholtz, or wave, equation in circular cylindrical coordinates led to Bessel's equation in the coordinate describing distance from the axis of the cylindrical system. In that same section, we also identified **spherical Bessel functions** (closely related to Bessel functions of half-integral order) in Helmholtz equations in spherical coordinates. In summarizing the forms of solutions to partial differential equations (PDEs) in these coordinate systems, we not only identified the original and spherical Bessel functions, but also those of imaginary argument (usually expressed as **modified Bessel functions** to avoid the explicit use of imaginary quantities). Since these PDEs can describe many types of problems ranging from stationary problems in quantum mechanics to those of spherical or cylindrical wave propagation, a good familiarity with Bessel functions is important to the practicing physicist.

Often problems in physics involve integrals that can be identified as Bessel functions, even when the original problem did not explicitly involve cylindrical or spherical geometry. Moreover, Bessel and closely related functions form a rich area of mathematical analysis with many representations, many interesting and useful properties, and many interrelations. Some of the major interrelations are developed in the present chapter. In addition to the material presented here, we call attention to further relations in terms of confluent hypergeometric functions; see Section 18.6.

14.1 BESSEL FUNCTIONS OF THE FIRST KIND, $J_\nu(x)$

Bessel functions **of the first kind**, normally labeled J_ν, are those obtained by the Frobenius method for solution of the Bessel ODE,

$$x^2 J_\nu'' + x J_\nu' + (x^2 - \nu^2) J_\nu = 0. \tag{14.1}$$

The term "first kind" reflects the fact that $J_\nu(x)$ includes the functions that, for non-negative integer ν, are regular at $x = 0$. All solutions to the Bessel ordinary differential

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equation (ODE) that are linearly independent of $J_\nu(x)$ are irregular at $x = 0$ for all ν; a specific choice for a second solution is denoted $Y_\nu(x)$ and is called a Bessel function **of the second kind**.[1]

Generating Function for Integral Order

We start our detailed study of Bessel functions by introducing a generating function yielding the J_n for integer n (of either sign). Because the J_n are not polynomials, the generating function cannot be found by the methods of Section 12.1, but we will be able to show that the functions defined by the generating function are indeed the solutions of the Bessel ODE obtained by the Frobenius method.

Our generating function formula, a Laurent series, is

$$g(x,t) = e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n. \tag{14.2}$$

Although the Bessel ODE is homogeneous and its solutions are of arbitrary scale, Eq. (14.2) fixes a specific scale for $J_n(x)$. To relate Eq. (14.2) to the Frobenius solution, Eq. (7.48), we manipulate the exponential as follows:

$$g(x,t) = e^{xt/2} \cdot e^{-x/2t} = \sum_{r=0}^{\infty} \left(\frac{x}{2}\right)^r \frac{t^r}{r!} \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^s \frac{t^{-s}}{s!}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{r+s} \frac{t^{r-s}}{r!s!}.$$

We now change the summation index r to $n = r - s$, yielding

$$g(x,t) = \sum_{n=-\infty}^{\infty} \left[\sum_s \frac{(-1)^s}{(n+s)!s!} \left(\frac{x}{2}\right)^{n+2s} \right] t^n, \tag{14.3}$$

where the s summation starts at $\max(0, -n)$. For $n \geq 0$, the coefficient of t^n is seen to be

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} \left(\frac{x}{2}\right)^{n+2s}. \tag{14.4}$$

Comparing with Eq. (7.48), we confirm that for $n \geq 0$, J_n as given by Eq. (14.4) is the Frobenius solution, at the specific scale given here.

If now we replace n by $-n$, the summation in Eq. (14.3) becomes

$$J_{-n}(x) = \sum_{s=n}^{\infty} \frac{(-1)^s}{s!(s-n)!} \left(\frac{x}{2}\right)^{-n+2s};$$

[1] We use the notation of AMS-55, also used by Watson in his definitive treatise (for both sources, see Additional Readings). The Y_ν are sometimes also called **Neumann functions**; for that reason some workers write them as N_ν. They were denoted N_ν in previous editions of this book.

changing s to $s+n$, we reach

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{s+n}}{s!(s+n)!}\left(\frac{x}{2}\right)^{n+2s} = (-1)^n J_n(x) \quad \text{(integral } n\text{)}, \tag{14.5}$$

confirming both that $J_{-n}(x)$ is a solution to the Bessel ODE and that it is linearly dependent on J_n.

If we now consider J_ν with ν nonintegral, we get no information from the generating function, but the Frobenius method then gives linearly independent solutions for both $+\nu$ and $-\nu$, which are both solutions of the Bessel ODE, Eq. (14.1), for the same value of ν^2. Looking at the details of the development of Eqs. (7.46) to (7.48), we see that the generalization of Eq. (14.4) to noninteger ν is

$$J_\nu(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\Gamma(\nu+s+1)}\left(\frac{x}{2}\right)^{\nu+2s}, \quad (\nu \neq -1, -2, \ldots), \tag{14.6}$$

and that $J_\nu(x)$ as given in Eq. (14.6) is a solution to the Bessel ODE.

For $\nu \geq 0$ the series of Eq. (14.6) is convergent for all x, and for small x is a practical way to evaluate $J_\nu(x)$. Graphs of J_0, J_1, and J_2 are shown in Fig. 14.1. The Bessel functions oscillate but are **not** periodic, except in the limit $x \to \infty$, with the amplitude of the oscillation decreasing asymptotically as $x^{-1/2}$. This behavior is discussed further in Section 14.6.

Recurrence Relations

The Bessel functions $J_n(x)$ satisfy recurrence relations connecting functions of contiguous n, as well as some connecting the derivative J'_n to various J_n. Such recurrence relations may all be obtained by operating on the series, Eq. (14.6), although this requires a bit of clairvoyance (or a lot of trial and error). However, if the recurrence relations are already known, their verification is straightforward; see Exercise 14.1.8. Our approach here will be to obtain them from the generating function $g(x,t)$, using a process similar to that illustrated in Example 12.1.2.

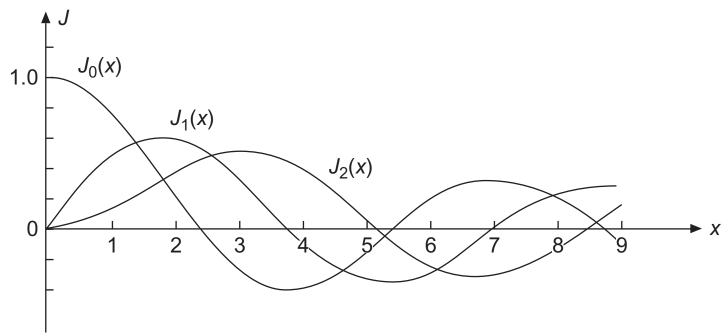

FIGURE 14.1 Bessel functions $J_0(x)$, $J_1(x)$, and $J_2(x)$.

We start by differentiating $g(x,t)$:

$$\frac{\partial}{\partial t}g(x,t) = \frac{x}{2}\left(1+\frac{1}{t^2}\right)e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} nJ_n(x)t^{n-1},$$

$$\frac{\partial}{\partial x}g(x,t) = \frac{1}{2}\left(t-\frac{1}{t}\right)e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n'(x)t^{n}.$$

Inserting the right-hand side of Eq. (14.2) in place of the exponentials and equating the coefficients of equal powers of t (as illustrated in Example 12.1.2), we obtain the two basic Bessel-function recurrence formulas:

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x), \tag{14.7}$$

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x). \tag{14.8}$$

Because Eq. (14.7) is a three-term recurrence relation, its use to generate J_n will require two starting values. For example, given J_0 and J_1, then J_2 (and any other integral order J_n including those for $n<0$) may be computed.

An important special case of Eq. (14.8) is

$$J_0'(x) = -J_1(x). \tag{14.9}$$

Equations (14.7) and (14.8) can also be combined (Exercise 14.1.4) to form the useful additional formulas

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x), \tag{14.10}$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x), \tag{14.11}$$

$$J_n(x) = \pm J_{n\pm1}' + \frac{n\pm1}{x}J_{n\pm1}(x). \tag{14.12}$$

Bessel's Differential Equation

Suppose we consider a set of functions $Z_\nu(x)$ that satisfies the basic recurrence relations, Eqs. (14.7) and (14.8), but with ν not necessarily an integer and Z_ν not necessarily given by the series in Eq. (14.6). It is our objective to show that any functions that satisfy these recurrence relations must also be solutions to Bessel's ODE. We start by forming (1) $x^2Z_\nu''(x)$ from $x^2/2$ times the derivative of Eq. (14.8), (2) $xZ_\nu'(x)$ from Eq. (14.8) multiplied by $x/2$, and (3) $\nu^2Z_\nu(x)$ from Eq. (14.7) multiplied by $\nu x/2$. Putting these together we obtain

$$\begin{aligned} &x^2Z_\nu''(x) + xZ_\nu'(x) - \nu^2 Z_\nu(x) \\ &\quad = \frac{x^2}{2}\left[Z_{\nu-1}'(x) - Z_{\nu+1}'(x) - \frac{\nu-1}{x}Z_{\nu-1}(x) - \frac{\nu+1}{x}Z_{\nu+1}(x)\right]. \end{aligned} \tag{14.13}$$

The terms within square brackets in Eq. (14.13) can now by use of Eq. (14.12) be simplified to $-2Z_\nu(x)$, so Eq. (14.13) can be rewritten

$$x^2 Z_\nu{}''(x) + xZ_\nu'(x) + (x^2 - \nu^2)Z_\nu(x) = 0, \tag{14.14}$$

which is Bessel's ODE. Reiterating, we have shown that any functions $Z_\nu(x)$ that satisfy the basic recurrence formulas, Eqs. (14.7) and (14.8), also satisfy Bessel's equation; that is, the Z_ν are Bessel functions. For later use, we note that if the argument of Z_ν is $k\rho$ rather than x, Eq. (14.14) becomes

$$\rho^2 \frac{d^2}{d\rho^2} Z_\nu(k\rho) + \rho \frac{d}{d\rho} Z_\nu(k\rho) + (k^2\rho^2 - \nu^2)Z_\nu(k\rho) = 0. \tag{14.15}$$

Integral Representation

It is of great value to have integral representations of Bessel functions. Starting from the generating-function formula, we can apply the residue theorem to evaluate the contour integral

$$\oint_C \frac{e^{(x/2)(t+1/t)}}{t^{n+1}} dt = \oint_C \sum_m J_m(x) t^{m-n-1} dt = 2\pi i J_n(x), \tag{14.16}$$

where the contour C encircles the singularity at $t = 0$. The integral on the left-hand side of Eq. (14.16) can now be brought to a convenient form by taking the contour to be the unit circle and changing the integration variable by making the substitution $t = e^{i\theta}$. Then $dt = ie^{i\theta}d\theta$, $e^{(x/2)(t-1/t)} = e^{ix\sin\theta}$, and we have

$$2\pi i J_n(x) = \int_0^{2\pi} \frac{e^{ix\sin\theta}}{e^{(n+1)i\theta}} ie^{i\theta} d\theta = \int_0^{2\pi} e^{i(x\sin\theta - n\theta)} i\, d\theta. \tag{14.17}$$

Assuming x to be real and taking the imaginary parts of both sides of Eq. (14.17), we find

$$J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} \cos(x\sin\theta - n\theta) d\theta = \frac{1}{\pi} \int_0^{\pi} \cos(x\sin\theta - n\theta)\, d\theta, \tag{14.18}$$

where the last equality only holds because we are assuming n to be an integer. Though we will not need it now, the real part of this equation also gives an interesting formula:

$$\int_0^{2\pi} \sin(x\sin\theta - n\theta) d\theta = 0. \tag{14.19}$$

An oft-occurring special case of Eq. (14.18) is

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix\cos\theta} d\theta = \frac{1}{\pi} \int_0^{\pi} \cos(x\sin\theta)\, d\theta. \tag{14.20}$$

Table 14.1 Zeros of the Bessel Functions and Their First Derivatives

Number of zeros	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178
	$J_0'(x)$	$J_1'(x)$	$J_2'(x)$	$J_3'(x)$	$J_4'(x)$	$J_5'(x)$
1	3.8317	1.8412	3.0542	4.2012	5.3176	6.4156
2	7.0156	5.3314	6.7061	8.0152	9.2824	10.5199
3	10.1735	8.5363	9.9695	11.3459	12.6819	13.9872
4	13.3237	11.7060	13.1704	14.5858	15.9641	17.3128
5	16.4706	14.8636	16.3475	17.7887	19.1960	20.5755

Equation (14.18) is only one of many integral representations of J_n, and some of these can be derived (using an appropriately modified contour) for J_ν of a nonintegral order. This topic is explored in the subsection below entitled "Bessel Functions of Nonintegral Order".

Zeros of Bessel Functions

In many physical problems in which phenomena are described by Bessel functions, we are interested in the points where these functions (which have oscillatory character) are zero. For example, in a problem involving standing waves, these zeros identify the positions of the **nodes**. And in boundary value problems, we may need to choose the argument of our Bessel function to put a zero at an appropriate point.

There are no closed formulas for the zeros of Bessel functions; they must be found by numerical methods. Because the need for them arises frequently, tables of the zeros are available, both in compilations such as AMS-55 (see Additional Readings) and at a variety of sources online.[2] Table 14.1 lists the first few zeros of $J_n(x)$ for integer n from $n = 0$ through $n = 5$, giving also the positions of the zeros of J_n'.

Example 14.1.1 FRAUNHOFER DIFFRACTION, CIRCULAR APERTURE

In the theory of diffraction of radiation of wavelength λ, incident normal to a circular aperture of radius a, we encounter the integral

$$\Phi \sim \int_0^a r\,dr \int_0^{2\pi} e^{ibr\cos\theta}\,d\theta, \qquad (14.21)$$

[2] Additional roots of the Bessel functions and those of their first derivatives may be found in C. L. Beattie, Table of first 700 zeros of Bessel functions, *Bell Syst. Tech. J.* **37**, 689 (1958), and Bell Monogr. **3055**. Roots may be also be accessed in Mathematica, Maple, and other symbolic software.

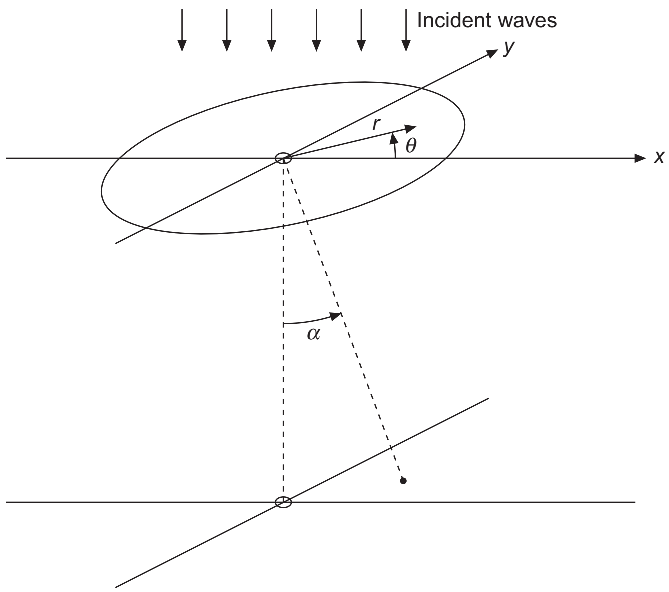

FIGURE 14.2 Geometry for Fraunhofer diffraction, circular aperture.

where Φ is the amplitude of the diffracted wave and (r, θ) identifies points in the aperture. The exponent $br\cos\theta$ is the phase of the radiation through (r, θ) that is diffracted to an angle α from the incident direction, with

$$b = \frac{2\pi}{\lambda}\sin\alpha. \tag{14.22}$$

The geometry is illustrated in Fig. 14.2. **Fraunhofer** diffraction, for which the above are the relevant formulas, applies in the limit that the outgoing radiation is detected at large distances from the aperture.

The behavior of the complex exponential will cause the amplitude to oscillate as α is increased, creating (for each wavelength) a diffraction pattern. To understand the patterns more fully, we need to evaluate the integral in Eq. (14.21). From Eq. (14.20) we may immediately reduce Eq. (14.21) to

$$\Phi \sim 2\pi \int_0^a J_0(br)r\,dr, \tag{14.23}$$

which can be integrated in r using Eq. (14.10):

$$\Phi \sim 2\pi \int_0^a \frac{1}{b^2}\frac{d}{dr}\big[(br)J_1(br)\big]dr = \frac{2\pi}{b^2}\big[br\,J_1(br)\big]_0^a = \frac{2\pi a}{b}J_1(ab), \tag{14.24}$$

where we have used the fact that $J_1(0) = 0$. The intensity of the light in the diffraction

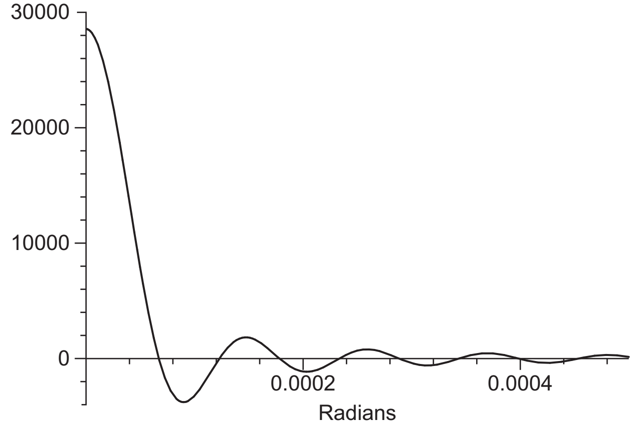

FIGURE 14.3 Amplitude of Fraunhofer diffraction vs. deflection angle (green light, aperture of radius 0.5 cm).

pattern is proportional to Φ^2 and, substituting for b from Eq. (14.22),

$$\Phi^2 \sim \left(\frac{J_1[(2\pi a/\lambda)\sin\alpha]}{\sin\alpha} \right)^2 . \tag{14.25}$$

For visible light and apertures of reasonable size, $2\pi a/\lambda$ is quite small: for green light ($\lambda = 5.5 \times 10^{-5}$ cm) and an aperture with $a = 0.5$ cm, $2\pi a/\lambda = 57120$, and these parameter values lead to the pattern for Φ shown in Fig. 14.3. Note that the figure plots Φ (a plot of Φ^2 would make the oscillations too small to be observable on the same graph as the maximum at $\alpha = 0$). We see that Φ exhibits a central maximum at $\alpha = 0$ of amplitude $\sim$30,000, with subsidiary extrema that by $\alpha = 0.001$ radian have decreased in magnitude to less than 1% of the central maximum. Remembering that the intensity is Φ^2, we see that the diffraction spreading of the incident light is exceedingly small. To make a quantitative analysis of the diffraction pattern, we need to identify the positions of its minima. They correspond to the zeros of J_1; for example, from Table 14.1 we find the first minimum to be where $(2\pi a/\lambda)\sin\alpha = 3.8317$, or $\alpha \approx 14$ seconds of arc. If this analysis had been known in the 17th century, the arguments against the wave theory of light would have collapsed.

In mid-20th century this same diffraction pattern appears in the scattering of nuclear particles by atomic nuclei, a striking demonstration of the wave properties of the nuclear particles. ■

Further examples of the use of Bessel functions and their roots are provided by the following example and by the exercises of this section and Section 14.2.

Example 14.1.2 CYLINDRICAL RESONANT CAVITY

The propagation of electromagnetic waves in hollow metallic cylinders is important in many practical devices. If the cylinder has end surfaces, it is called a **cavity**. Resonant cavities play a crucial role in many particle accelerators.

The resonant frequencies of a cavity are those of the oscillatory solutions to Maxwell's equations that correspond to standing wave patterns. By combining Maxwell's equations, we derived in Example 3.6.2 the vector Laplace equation for the electric field **E** in a region free of electric charges and currents. Taking the z-axis along the axis of the cavity, our concern here is the equation for E_z, which from Eq. (3.71) we found to have the form

$$\nabla^2 E_z = -\frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}, \tag{14.26}$$

which has standing-wave solutions $E_z(x,y,z,t) = E_z(x,y,z)f(t)$, where $f(t)$ has real solutions $\sin\omega t$ and $\cos\omega t$, corresponding to sinusoidal oscillations at angular frequency ω. We are implicitly assuming that our solution has a nonzero component E_z, and we will also set $B_z = 0$, so we intend to obtain solutions that are usually called the TM (for ''transverse magnetic'') modes of oscillation. Additional solutions, with $E_z = 0$ and B_z nonzero, correspond to TE (transverse electric) modes and are the subject of Exercise 14.1.25.

Thus, for the present problem, in which our cavity is that shown in Fig. 14.4, we seek solutions to the spatial PDE:

$$\nabla^2 E_z + k^2 E_z = 0, \quad k = \frac{\omega}{c}. \tag{14.27}$$

The aim of the present example is to find the values of ω for which Eq. (14.27) has solutions consistent with the boundary conditions at the cavity walls. Assuming the metallic walls to be perfect conductors, the boundary conditions are that the tangential components of the electric field vanish there. Taking the cavity to have planar end caps at $z = 0$ and $z = h$, and (in cylindrical coordinates ρ, φ) to be bounded by a curved surface at $\rho = a$, our boundary conditions are $E_x = E_y = 0$ on the end caps, and $E_\varphi = E_z = 0$ on the boundary at $\rho = a$.

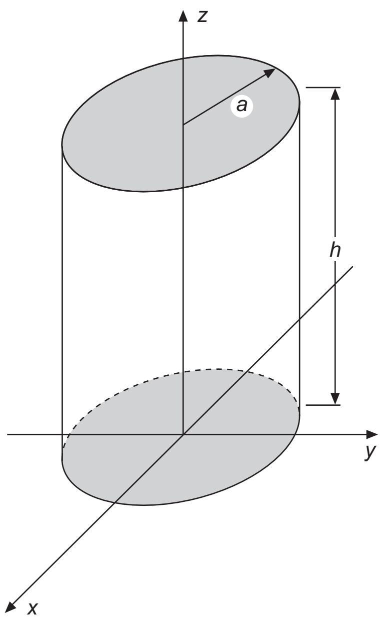

FIGURE 14.4 Resonant cavity.

Once a solution (with $B_z = 0$) has been found for E_z, then the remaining components of **B** and **E** have definite values. For further details, see J. D. Jackson, *Electrodynamics* in Additional Readings.

Equation (14.27) can be solved by the method of separation of variables, with solutions of the form given in Eq. (9.64):

$$E_z(\rho, \theta, z) = P_{lm}(\rho)\Phi_m(\varphi)Z_l(z), \tag{14.28}$$

with $\Phi_m(\theta) = e^{\pm im\varphi}$ or its equivalent in terms of sines and cosines, while $Z_l(z)$ and $P_{lm}(\rho)$ are solutions of the ODEs

$$\frac{d^2 Z_l}{dz^2} = -l^2 Z_l, \tag{14.29}$$

$$\rho\frac{d}{d\rho}\left(\rho\frac{dP_{lm}}{d\rho}\right) + \left((k^2 - l^2)\rho^2 - m^2\right)P_{lm} = 0. \tag{14.30}$$

Equation (14.29) corresponds to Eq. (9.58), but with a different choice of the sign for the separation constant in anticipation of the fact that Z_l will turn out to be oscillatory. This change causes n^2 in Eq. (9.60) to become $k^2 - l^2$, and Eq. (14.30) is then seen to correspond exactly with Eq. (9.63).

Recognizing now Eq. (14.30) as Bessel's ODE and Eq. (14.29) as the ODE for a classical harmonic oscillator, we find, before imposing boundary conditions,

$$E_z = J_m(n\rho)e^{\pm im\varphi}\big[A\sin lz + B\cos lz\big], \tag{14.31}$$

and the general solution will be an arbitrary linear combination of the above for different values of n, m, and l. We have chosen the solution to Bessel's ODE to be of the first kind to maintain regularity at $\rho = 0$, since this ρ value is inside the cavity. We have written the φ dependence of the solution as a complex exponential for notational convenience. The physically relevant solutions will be arbitrary mixtures of the corresponding real quantities, $\sin m\varphi$ and $\cos m\varphi$. Continuity and single-valuedness in φ dictate that m have integer values.

The condition that $E_z = 0$ on the curved boundary translates into the requirement $J_m(na) = 0$. Letting α_{mj} stand for the jth positive zero of J_m, we find that

$$na = \alpha_{mj}, \quad \text{or} \quad k^2 - l^2 = \left(\frac{\alpha_{mj}}{a}\right)^2. \tag{14.32}$$

To complete the solution we need to identify the boundary condition on Z. Because $\partial E_x/\partial x = \partial E_y/\partial y = 0$ on the end caps, we have from the Maxwell equation for $\nabla \cdot \mathbf{E}$:

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \quad \longrightarrow \quad \frac{\partial E_z}{\partial z} = 0, \tag{14.33}$$

so we have the requirement $Z'(0) = Z'(h) = 0$, and we must choose

$$Z = B\cos lz, \quad \text{with} \quad l = \frac{p\pi}{h}, \quad p = 0, 1, 2, \ldots. \tag{14.34}$$

Combining Eqs. (14.32) and (14.34), we find

$$k^2 = \left(\frac{\alpha_{mj}}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2 = \frac{\omega^2}{c^2}, \tag{14.35}$$

thereby providing an equation for the resonant frequencies:

$$\omega_{mjp} = c\sqrt{\frac{\alpha_{mj}^2}{a^2} + \frac{p^2\pi^2}{h^2}}, \qquad \begin{cases} m = 0, 1, 2, \dots, \\ j = 1, 2, 3, \dots, \\ p = 0, 1, 2, \dots. \end{cases} \tag{14.36}$$

Recapitulating, the functions we have found, labeled by the indices m, j, and p, are the spatial parts of standing-wave solutions of TM character whose time dependence and overall amplitude are of the form $Ce^{\pm i\omega_{mjp}t}$. ■

Bessel Functions of Nonintegral Order

While J_ν of noninteger ν are not produced from a generating-function approach, they are readily identified from the Taylor series expansion, and they are conventionally given a scale consistent with that of the J_n of integer n. They then satisfy the same recurrence relations as those derived from the generating function.

If ν is not an integer, there is actually an important simplification. The functions J_ν and $J_{-\nu}$ are then independent solutions of the same ODE, and a relation of the form of Eq. (14.5) does not exist. On the other hand, for $\nu = n$, an integer, we need another solution. The development of this second solution and an investigation of its properties form the subject of Section 14.3.

Schlaefli Integral

It is useful to modify the integral representation, Eq. (14.16), so that it can be applied for Bessel functions of nonintegral order. Our first step in doing so is to deform the circular contour by stretching it to infinity on the negative real axis and opening the contour there, as shown in Fig. 14.5. Our integral, written

$$F_\nu(x) = \frac{1}{2\pi i}\int_C \frac{e^{(x/2)(t-1/t)}}{t^{\nu+1}}dt, \tag{14.37}$$

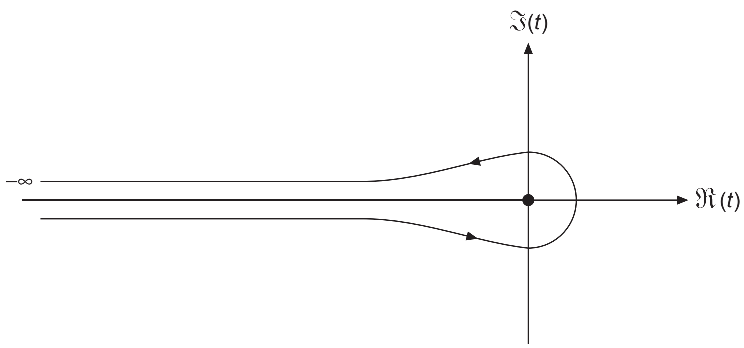

FIGURE 14.5 Contour, Schlaefli integral for J_ν.

now has a branch point at $t = 0$, and because we have opened the contour we can place the branch cut along the negative real axis. We might anticipate that this procedure will not affect our integral representation, as the integrand vanishes at $t = -\infty$ on both sides of the cut. However, that remains to be proved.

Our first step toward a proof that F_ν is actually J_ν is to verify that F_ν still satisfies Bessel's ODE. If we substitute F_ν and its x derivatives into the ODE, we can, after some manipulation, reach the expression

$$\frac{1}{2\pi i}\int_C \frac{d}{dt}\left\{\frac{e^{(x/2)(t-1/t)}}{t^\nu}\left[\nu + \frac{x}{2}\left(t + \frac{1}{t}\right)\right]\right\}dt, \tag{14.38}$$

and because the integration is within a region of analyticity of the integrand, the integral reduces to

$$\left\{\frac{e^{(x/2)(t-1/t)}}{t^\nu}\left[\nu + \frac{x}{2}\left(t + \frac{1}{t}\right)\right]\right\}_{\text{end}} - \left\{\frac{e^{(x/2)(t-1/t)}}{t^\nu}\left[\nu + \frac{x}{2}\left(t + \frac{1}{t}\right)\right]\right\}_{\text{start}}.$$

We therefore conclude that the ODE is satisfied if the above expression vanishes; in our present situation each of the quantities in braces is zero for large negative t and positive x, confirming that F_ν satisfies Bessel's ODE.

We still need to show that F_ν is the solution designated J_ν; to accomplish this we consider its value for small $x > 0$. Deforming the contour to a large open circle and making a change of variable to $u = e^{i\pi}xt/2$, we get (to lowest order in x)

$$F_\nu(x) \approx \frac{1}{2\pi i}\left(\frac{x}{2}\right)^\nu e^{i\nu\pi}\int_{C'} \frac{e^{-u}}{u^{\nu+1}}du. \tag{14.39}$$

Because of the change of variable, the contour C' becomes that which we introduced when developing a Schlaefli integral representation of the gamma function, and, using Eq. (13.31), we reduce Eq. (14.39) to

$$F_\nu(x) \approx \left(\frac{x}{2}\right)^\nu \frac{\sin[(\nu+1)\pi]\Gamma(-\nu)}{\pi} = \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu, \tag{14.40}$$

where the last step used the reflection formula for the gamma function, Eq. (13.23). Since this is the leading term of the expansion for J_ν, our proof is complete.

Exercises

14.1.1 From the product of the generating functions $g(x,t)g(x,-t)$, show that

$$1 = [J_0(x)]^2 + 2[J_1(x)]^2 + 2[J_2(x)]^2 + \cdots$$

and therefore that $|J_0(x)| \le 1$ and $|J_n(x)| \le 1/\sqrt{2}$, $n = 1, 2, 3, \ldots$.

Hint. Use uniqueness of power series, (Section 1.2).

14.1.2 Using a generating function $g(x,t) = g(u+v,t) = g(u,t)g(v,t)$, show that

(a) $J_n(u+v) = \sum_{s=-\infty}^{\infty} J_s(u)J_{n-s}(v)$,

(b) $J_0(u+v) = J_0(u)J_0(v) + 2\sum_{s=1}^{\infty} J_s(u)J_{-s}(v)$.

These are addition theorems for the Bessel functions.

14.1.3 Using only the generating function

$$e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n$$

and not the explicit series form of $J_n(x)$, show that $J_n(x)$ has odd or even parity according to whether n is odd or even, that is,

$$J_n(x) = (-1)^n J_n(-x).$$

14.1.4 Use the basic recurrence formulas, Eqs. (14.7) and (14.8), to prove the following formulas:

(a) $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$,

(b) $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$,

(c) $J_n(x) = J'_{n+1} + \frac{n+1}{x} J_{n+1}(x)$.

14.1.5 Derive the Jacobi-Anger expansion

$$e^{i\rho\cos\varphi} = \sum_{m=-\infty}^{\infty} i^m J_m(\rho)e^{im\varphi}.$$

This is an expansion of a plane wave in a series of cylindrical waves.

14.1.6 Show that

(a) $\cos x = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$,

(b) $\sin x = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(x)$.

14.1.7 To help remove the generating function from the realm of magic, show that it can be **derived** from the recurrence relation, Eq. (14.7).

Hint. (a) Assume a generating function of the form

$$g(x,t) = \sum_{m=-\infty}^{\infty} J_m(x)t^m.$$

(b) Multiply Eq. (14.7) by t^n and sum over n.

(c) Rewrite the preceding result as

$$\left(t + \frac{1}{t}\right) g(x,t) = \frac{2t}{x}\frac{\partial g(x,t)}{\partial t}.$$

(d) Integrate and adjust the "constant" of integration (a function of x) so that the coefficient of the zeroth power, t^0, is $J_0(x)$ as given by Eq. (14.6).

14.1.8 Show, by direct differentiation, that

$$J_\nu(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\Gamma(s+\nu+1)} \left(\frac{x}{2}\right)^{\nu+2s}$$

satisfies the two recurrence relations

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x),$$
$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_\nu(x),$$

and Bessel's differential equation

$$x^2 J''_\nu(x) + x J'_\nu(x) + (x^2 - \nu^2) J_\nu(x) = 0.$$

14.1.9 Prove that

$$\frac{\sin x}{x} = \int_0^{\pi/2} J_0(x\cos\theta)\cos\theta\, d\theta, \qquad \frac{1-\cos x}{x} = \int_0^{\pi/2} J_1(x\cos\theta)\, d\theta.$$

Hint. The definite integral

$$\int_0^{\pi/2} \cos^{2s+1}\theta\, d\theta = \frac{2\cdot 4\cdot 6\cdots(2s)}{1\cdot 3\cdot 5\cdots(2s+1)}$$

may be useful.

14.1.10 Derive

$$J_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n J_0(x).$$

Hint. Try mathematical induction (Section 1.4).

14.1.11 Show that between any two consecutive zeros of $J_n(x)$ there is one and only one zero of $J_{n+1}(x)$.

Hint. Equations (14.10) and (14.11) may be useful.

14.1.12 An analysis of antenna radiation patterns for a system with a circular aperture involves the equation

$$g(u) = \int_0^1 f(r) J_0(ur) r dr.$$

If $f(r) = 1 - r^2$, show that

$$g(u) = \frac{2}{u^2} J_2(u).$$

14.1.13 The differential cross section in a nuclear scattering experiment is given by $d\sigma/d\Omega = |f(\theta)|^2$. An approximate treatment leads to

$$f(\theta) = \frac{-ik}{2\pi} \int_0^{2\pi} \int_0^R \exp[ik\rho \sin\theta \sin\varphi]\rho\, d\rho\, d\varphi.$$

Here θ is an angle through which the scattered particle is scattered. R is the nuclear radius. Show that

$$\frac{d\sigma}{d\Omega} = (\pi R^2)\frac{1}{\pi}\left[\frac{J_1(kR\sin\theta)}{\sin\theta}\right]^2.$$

14.1.14 A set of functions $C_n(x)$ satisfies the recurrence relations

$$C_{n-1}(x) - C_{n+1}(x) = \frac{2n}{x} C_n(x),$$

$$C_{n-1}(x) + C_{n+1}(x) = 2C_n'(x).$$

(a) What linear second-order ODE does the $C_n(x)$ satisfy?

(b) By a change of variable transform your ODE into Bessel's equation. This suggests that $C_n(x)$ may be expressed in terms of Bessel functions of transformed argument.

14.1.15 (a) Show by direct differentiation and substitution that

$$J_\nu(x) = \frac{1}{2\pi i} \int_C e^{(x/2)(t-1/t)} t^{-\nu-1} dt$$

(this is the Schlaefli integral representation of J_ν), and that the equivalent equation,

$$J_\nu(x) = \frac{1}{2\pi i}\left(\frac{x}{2}\right)^\nu \int_C e^{s-x^2/4s} s^{-\nu-1} ds,$$

both satisfy Bessel's equation. C is the contour shown in Fig. 14.5. The negative real axis is the cut line.

Hint. This exercise is aimed at providing details of the discussion that starts at Eq. (14.38).

(b) Show that the first integral (with n an integer) may be transformed into

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(x\sin\theta - n\theta)} d\theta = \frac{i^{-n}}{2\pi}\int_0^{2\pi} e^{i(x\cos\theta + n\theta)} d\theta.$$

14.1.16 The contour C in Exercise 14.1.15 is deformed to the path $-\infty$ to -1, unit circle $e^{-i\pi}$ to $e^{i\pi}$, and finally -1 to $-\infty$. Show that

$$J_\nu(x) = \frac{1}{\pi}\int_0^\pi \cos(\nu\theta - x\sin\theta) d\theta - \frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-\nu\theta - x\sinh\theta} d\theta.$$

This is Bessel's integral.

Hint. The negative values of the variable of integration u must be represented in a manner consistent with the presence of the branch cut, for example, by writing $u = te^{\pm ix}$.

14.1.17 (a) Show that

$$J_\nu(x) = \frac{2}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})}\left(\frac{x}{2}\right)^\nu \int_0^{\pi/2} \cos(x\sin\theta)\cos^{2\nu}\theta\, d\theta,$$

where $\nu > -\frac{1}{2}$.

Hint. Here is a chance to use series expansion and term-by-term integration. The formulas of Section 13.3 will prove useful.

(b) Transform the integral in part (a) into

$$J_\nu(x) = \frac{1}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})}\left(\frac{x}{2}\right)^\nu \int_0^{\pi} \cos(x\cos\theta)\sin^{2\nu}\theta\, d\theta$$

$$= \frac{1}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})}\left(\frac{x}{2}\right)^\nu \int_0^{\pi} e^{\pm ix\cos\theta}\sin^{2\nu}\theta\, d\theta$$

$$= \frac{1}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})}\left(\frac{x}{2}\right)^\nu \int_{-1}^{1} e^{\pm ipx}(1-p^2)^{\nu-1/2}\, dp.$$

These are alternate integral representations of $J_\nu(x)$.

14.1.18 Given that C is the contour in Fig. 14.5,

(a) From

$$J_\nu(x) = \frac{1}{2\pi i}\left(\frac{x}{2}\right)^\nu \int_C t^{-\nu-1} e^{t-x^2/4t}\, dt$$

derive the recurrence relation

$$J_\nu'(x) = \frac{\nu}{x}J_\nu(x) - J_{\nu+1}(x).$$

(b) From

$$J_\nu(x) = \frac{1}{2\pi i}\int_C t^{-\nu-1} e^{(x/2)(t-1/t)}\, dt$$

derive the recurrence relation

$$J_\nu'(x) = \frac{1}{2}\left[J_{\nu-1}(x) - J_{\nu+1}(x)\right].$$

14.1.19 Show that the recurrence relation

$$J_n'(x) = \frac{1}{2}\big[J_{n-1}(x) - J_{n+1}(x)\big]$$

follows directly from differentiation of

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta.$$

14.1.20 Evaluate

$$\int_0^\infty e^{-ax} J_0(bx)dx, \quad a, b > 0.$$

Actually the results hold for $a \ge 0$, $-\infty < b < \infty$. This is a Laplace transform of J_0.

Hint. Either an integral representation of J_0 or a series expansion will be helpful.

14.1.21 Using the symmetries of the trigonometric functions, confirm that for integer n,

$$\frac{1}{2\pi}\int_0^{2\pi} \cos(x\sin\theta - n\theta)d\theta = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta - n\theta)d\theta.$$

14.1.22 (a) Plot the intensity, Φ^2 of Eq. (14.25), as a function of $(\sin\alpha/\lambda)$ along a diameter of the circular diffraction pattern. Locate the first two minima.

(b) Estimate the fraction of the total light intensity that falls within the central maximum.

Hint. $[J_1(x)]^2/x$ may be written as a derivative and the area integral of the intensity integrated by inspection.

14.1.23 The fraction of light incident on a circular aperture (normal incidence) that is transmitted is given by

$$T = 2\int_0^{2ka} J_2(x)\frac{dx}{x} - \frac{1}{2ka}\int_0^{2ka} J_2(x)dx.$$

Here a is the radius of the aperture and k is the wave number, $2\pi/\lambda$. Show that

(a) $T = 1 - \dfrac{1}{ka}\displaystyle\sum_{n=0}^{\infty} J_{2n+1}(2ka)$, (b) $T = 1 - \dfrac{1}{2ka}\displaystyle\int_0^{2ka} J_0(x)dx$.

14.1.24 The amplitude $U(\rho, \varphi, t)$ of a vibrating circular membrane of radius a satisfies the wave equation

$$\nabla^2 U \equiv \frac{\partial^2 U}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial\varphi^2} = \frac{1}{v^2}\frac{\partial^2 U}{\partial t^2}.$$

Here v is the phase velocity of the wave, determined by the properties of the membrane.

(a) Show that a physically relevant solution is

$$U(\rho, \varphi, t) = J_m(k\rho)\left(c_1 e^{im\varphi} + c_2 e^{-im\varphi}\right)\left(b_1 e^{i\omega t} + b_2 e^{-i\omega t}\right).$$

(b) From the Dirichlet boundary condition $J_m(ka) = 0$, find the allowable values of k.

14.1.25 Example 14.1.2 describes the TM modes of electromagnetic cavity oscillation. To obtain the transverse electric (TE) modes, we set $E_z = 0$ and work from the z component of the magnetic induction $\mathbf{B}$:

$$\nabla^2 B_z + \alpha^2 B_z = 0$$

with boundary conditions

$$B_z(0) = B_z(l) = 0 \quad \text{and} \quad \left.\frac{\partial B_z}{\partial \rho}\right|_{\rho=a} = 0.$$

Show that the TE resonant frequencies are given by

$$\omega_{mnp} = c\sqrt{\frac{\beta_{mn}^2}{a^2} + \frac{p^2\pi^2}{l^2}}, \qquad p = 1, 2, 3, \ldots,$$

and identify the quantities β_{mn}.

14.1.26 A conducting cylinder can accommodate traveling electromagnetic waves; when used for this purpose it is called a wave guide. The equations describing traveling waves are the same as those of Example 14.1.2, but there is no boundary condition on E_z at $z = 0$ or $z = h$ other than that its z dependence be oscillatory. For each TM mode (values of m and j of Example 14.1.2), there is a minimum frequency that can be transmitted through a wave guide of radius a. Explain why this is so, and give a formula for the cutoff frequencies.

14.1.27 Plot the three lowest TM and the three lowest TE angular resonant frequencies, ω_{mnp}, as a function of the ratio radius/length (a/l) for $0 \le a/l \le 1.5$.

Hint. Try plotting ω^2 (in units of c^2/a^2) vs. $(a/l)^2$. Why this choice?

14.1.28 Show that the integral

$$\int_0^a x^m J_n(x)dx, \qquad m \ge n \ge 0,$$

(a) is integrable for $m + n$ odd in terms of Bessel functions and powers of x, i.e., is expressible as linear combinations of $a^p J_q(a)$;

(b) may be reduced for $m + n$ even to integrated terms plus $\int_0^a J_0(x)dx$.

14.1.29 Show that

$$\int_0^{\alpha_{0n}} \left(1 - \frac{y}{\alpha_{0n}}\right) J_0(y)y\,dy = \frac{1}{\alpha_{0n}} \int_0^{\alpha_{0n}} J_0(y)dy.$$

Here α_{0n} is the nth zero of $J_0(y)$. This relation is useful (see Exercise 14.2.9): The expression on the right is easier and quicker to evaluate, and is much more accurate. Taking the difference of two terms in the expression on the left leads to a large relative error.

14.2 ORTHOGONALITY

To identify the orthogonality properties of Bessel functions, it is convenient to start by writing Bessel's ODE in a form that we can recognize as a Sturm-Liouville eigenvalue problem, the general properties of which were discussed in detail starting from Eq. (8.15). If we divide Eq. (14.15) through by ρ^2 and rearrange slightly, we have

$$-\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{\nu^2}{\rho^2}\right)Z_\nu(k\rho)=k^2 Z_\nu(k\rho), \tag{14.41}$$

showing that $Z_\nu(k\rho)$ is an eigenfunction of the operator

$$\mathcal{L}=-\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{\nu^2}{\rho^2}\right) \tag{14.42}$$

with eigenvalue k^2. Since we are most often interested in problems whose solutions in cylindrical coordinates (ρ,φ,z) separate into products $P(\rho)\Phi(\varphi)Z(z)$ and which are for the region within a cylindrical boundary at some $\rho=a$, we usually have $\Phi(\varphi)=e^{im\varphi}$ with m an integer (thereby causing $\nu^2\to m^2$), and find that $P(\rho)=J_m(k\rho)$. We choose P to be a Bessel function of the first kind because $\rho=0$ is interior to our region and we want a solution that is nonsingular there.

From Sturm-Liouville theory, we find that the weight factor needed to make $\mathcal{L}$ of Eq. (14.42) self-adjoint (as an ODE) is $w(\rho)=\rho$, and the orthogonality integral for the two eigenfunctions $J_\nu(k\rho)$ and $J_\nu(k'\rho)$, a case of Eq. (8.20), is (whether or not ν is an integer)

$$\frac{a\big[k' J_\nu(ka)J'_\nu(k'a)-kJ'_\nu(ka)J_\nu(k'a)\big]}{k^2-k'^2}=\int_0^a \rho J_\nu(k\rho)J_\nu(k'\rho)d\rho. \tag{14.43}$$

In writing Eq. (14.43) we have used the fact that the presence of a factor ρ in the boundary terms causes there to be no contribution from the lower limit $\rho=0$.[3]

Equation (14.43) shows us that the $J_\nu(k)$ of different k will be orthogonal (with weight factor ρ) if we can cause the left-hand side of that equation to vanish. We may do so by choosing k and k' in such a way that $J_\nu(ka)=J_\nu(k'a)=0$. In other words, we can require that k and k' be such that ka and $k'a$ are zeros of J_ν, and our Bessel functions will then satisfy Dirichlet boundary conditions.

If now we let $\alpha_{\nu i}$ denote the ith zero of J_ν, the above analysis corresponds to the following orthogonality formula for the interval $[0,a]$:

$$\int_0^a \rho J_\nu\left(\alpha_{\nu i}\frac{\rho}{a}\right)J_\nu\left(\alpha_{\nu j}\frac{\rho}{a}\right)d\rho=0, \quad i\neq j. \tag{14.44}$$

[3]This will be true for all $\nu\geq -1$, as will become more evident when we discuss Bessel functions of the second kind.

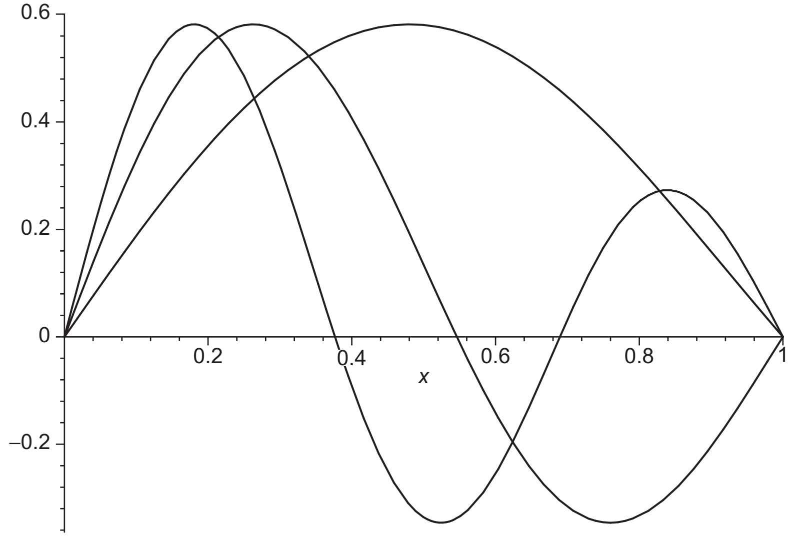

FIGURE 14.6 Bessel functions $J_1(\alpha_{1n}\rho)$, $n = 1, 2, 3$ on range $0 \le \rho \le 1$.

Note that all members of our orthogonal set of Bessel functions have the same value of the index ν, differing only in the scale of the argument of J_ν. Successive members of the orthogonal set will have increasing numbers of oscillations in the interval $(0, a)$. Note also that the weight factor, ρ, is just that which corresponds to **unweighted** orthogonality over the region within a circle of radius a. We show in Fig. 14.6 the first three Bessel functions of order $\nu = 1$ that are orthogonal within the unit circle.

An alternative to the foregoing analysis would be to ensure the vanishing of the boundary term of Eq. (14.43) at $\rho = a$ by choosing values of k corresponding to the Neumann boundary condition $J'_\nu(ka) = 0$. The functions obtained in this way would also form an orthogonal set.

Normalization

Our orthogonal sets of Bessel functions are not normalized, and to use them in expansions we need their normalization integrals. These integrals may be developed by returning to Eq. (14.43), which is valid for all k and k', whether or not the boundary terms vanish. We take the limits of both sides of that equation as $k' \to k$, evaluating the limit on the left-hand side using l'Hôpital's rule, which here corresponds to taking the derivatives of numerator and denominator with respect to k':

$$\int_0^a \rho\,[J_\nu(k\rho)]^2\,d\rho = \lim_{k'\to k} \frac{a\left[J_\nu(ka)\dfrac{d}{dk'}\Big(k'J'_\nu(k'a)\Big) - kJ'_\nu(ka)\dfrac{d}{dk'}\Big(J_\nu(k'a)\Big)\right]}{\dfrac{d}{dk'}(k^2 - k'^2)}.$$

We now simplify this equation for the case that $ka = \alpha_{\nu i}$, so we set $J_\nu(ka) = 0$ and reach

$$\int_0^a \rho \left[J_\nu\left(\alpha_{\nu i}\frac{\rho}{a}\right)\right]^2 d\rho = \frac{-a^2 k\left[J'_\nu(ka)\right]^2}{-2k} = \frac{a^2}{2}\left[J'_\nu(\alpha_{\nu i})\right]^2. \tag{14.45}$$

Now, because $\alpha_{\nu i}$ is a zero of J_ν, Eq. (14.12) permits us to recognize that $J'_\nu(\alpha_{\nu i}) = -J_{\nu+1}(\alpha_{\nu i})$. We then obtain from Eq. (14.45) the desired result,

$$\int_0^a \rho \left[J_\nu\left(\alpha_{\nu i}\frac{\rho}{a}\right)\right]^2 d\rho = \frac{a^2}{2}\left[J_{\nu+1}(\alpha_{\nu i})\right]^2. \tag{14.46}$$

Bessel Series

If we assume that the set of Bessel functions $J_\nu(\alpha_{\nu j}\rho/a)$ for fixed ν and for $j = 1, 2, 3, \ldots$ is complete, then any well-behaved but otherwise arbitrary function $f(\rho)$ may be expanded in a Bessel series

$$f(\rho) = \sum_{j=1}^{\infty} c_{\nu j} J_\nu\left(\alpha_{\nu j}\frac{\rho}{a}\right), \quad 0 \le \rho \le a, \quad \nu > -1. \tag{14.47}$$

The coefficients $c_{\nu j}$ are determined by the usual rules for orthogonal expansions. With the aid of Eq. (14.46) we have

$$c_{\nu j} = \frac{2}{a^2[J_{\nu+1}(\alpha_{\nu j})]^2}\int_0^a f(\rho) J_\nu\left(\alpha_{\nu j}\frac{\rho}{a}\right)\rho d\rho. \tag{14.48}$$

As pointed out earlier, it is also possible to obtain an orthogonal set of Bessel functions of given order ν by imposing the Neumann boundary condition $J'_\nu(k\rho) = 0$ at $\rho = a$, corresponding to $k = \beta_{\nu j}/a$, where $\beta_{\nu j}$ is the jth zero of J'_ν. These functions can also be used for orthogonal expansions. This approach is explored in Exercises 14.2.2 and 14.2.5.

The following example illustrates the usefulness of Bessel series.

Example 14.2.1 Electrostatic Potential in a Hollow Cylinder

We consider a hollow cylinder, which in cylindrical coordinates (ρ, φ, z) is bounded by a curved surface at $\rho = a$ and end caps at $z = 0$ and $z = h$. The base ($z = 0$) and curved surface are assumed to be grounded, and therefore at potential $\psi = 0$, while the end cap at $z = h$ has a known potential distribution $V(\rho, \varphi, h)$. Our problem is to determine the potential $V(\rho, \varphi, z)$ throughout the interior of the cylinder.

We proceed by finding separated-variable solutions to the Laplace equation in cylindrical coordinates, along the lines discussed in Section 9.4. Our first step is to identify product solutions, which, as in Eq. (9.64), must take the form[4]

$$\psi_{lm}(\rho, \varphi, z) = P_{lm}(\rho)\Phi_m(\varphi)Z_l(z), \tag{14.49}$$

with $\Phi_m = e^{\pm im\varphi}$, and

$$\frac{d^2}{dz^2} Z_l(z) = l^2 Z_l(z), \tag{14.50}$$

$$\rho^2 \frac{d^2}{d\rho^2} P_{lm} + \rho \frac{d}{d\rho} P_{lm} + (l^2\rho^2 - m^2) P_{lm} = 0. \tag{14.51}$$

The equation for P_{lm} is Bessel's ODE, with solutions of relevance here $J_m(l\rho)$. To satisfy the boundary condition at $\rho = a$ we need to choose $l = \alpha_{mj}/a$, where j can be any positive integer and α_{mj} is the jth zero of J_m.

The equation for Z_l has solutions $e^{\pm lz}$; to satisfy the boundary condition at $z = 0$ we need to take the linear combination of these solutions that is equivalent to $\sinh lz$. Combining these observations, we see that possible solutions to the Laplace equation that satisfy all the boundary conditions other than that at $z = h$ can be written

$$\psi_{mj} = c_{mj} J_m\left(\alpha_{mj}\frac{\rho}{a}\right) e^{im\varphi} \sinh\left(\alpha_{mj}\frac{z}{a}\right). \tag{14.52}$$

Since Laplace's equation is homogeneous, any linear combination of the ψ_{mj} with arbitrary values of the c_{mj} will be a solution, and our remaining task is to find the linear combination of such solutions that satisfies the boundary condition at $z = h$. Therefore,

$$V(\rho, \varphi, z) = \sum_{m=-\infty}^{\infty} \sum_{j=1}^{\infty} \psi_{mj}, \tag{14.53}$$

with the boundary condition at $z = h$ expressed as

$$\sum_{m=-\infty}^{\infty} \sum_{j=1}^{\infty} c_{mj} J_m\left(\alpha_{mj}\frac{\rho}{a}\right) e^{im\varphi} \sinh\left(\alpha_{mj}\frac{h}{a}\right) = V(\rho, \varphi, h). \tag{14.54}$$

Our solution is both a trigonometric series and a Bessel series, each with orthogonality properties that can be used to determine the coefficients. From Eq. (14.48) and the formula

$$\int_0^{2\pi} e^{-im\varphi} e^{im'\varphi} = 2\pi\delta_{mm'}, \tag{14.55}$$

we find

$$c_{mj} = \left[\pi a^2 \sinh\left(\alpha_{mj}\frac{h}{a}\right) J_{m+1}^2(\alpha_{mj})\right]^{-1}$$

$$\int_0^{2\pi} d\varphi \int_0^a V(\rho, \varphi, h) J_m\left(\alpha_{mj}\frac{\rho}{a}\right) e^{-im\varphi} \rho d\rho. \tag{14.56}$$

[4] Note that here Z_l is a function of z arising from the separation of variables; the notation is not intended to identify it as a Bessel function.

These are definite integrals, that is, numbers. Substituting back into Eq. (14.52), the series in Eq. (14.53) is specified and the potential $V(\rho, \varphi, z)$ is determined. ■

Exercises

14.2.1 Show that

$$(k^2 - k'^2)\int_0^a J_\nu(kx)J_\nu(k'x)x\,dx = a[k' J_\nu(ka)J_\nu'(k'a) - kJ_\nu'(ka)J_\nu(k'a)],$$

where $J_\nu'(ka) = \dfrac{d}{d(kx)} J_\nu(kx)\,|_{x=a}$, and that

$$\int_0^a [J_\nu(kx)]^2 x\,dx = \frac{a^2}{2}\left\{[J_\nu'(ka)]^2 + \left(1 - \frac{\nu^2}{k^2a^2}\right)[J_\nu(ka)]^2\right\}, \quad \nu > -1.$$

These two integrals are usually called the **first and second Lommel integrals**.

14.2.2 (a) If $\beta_{\nu m}$ is the mth zero of $(d/d\rho)J_\nu(\beta_{\nu m}\rho/a)$, show that the Bessel functions are orthogonal over the interval $[0, a]$ with an orthogonality integral

$$\int_0^a J_\nu\left(\beta_{\nu m}\frac{\rho}{a}\right) J_\nu\left(\beta_{\nu n}\frac{\rho}{a}\right)\rho\,d\rho = 0, \quad m \neq n, \quad \nu > -1.$$

(b) Derive the corresponding normalization integral $(m = n)$.

$$ANS. \quad (b)\ \frac{a^2}{2}\left(1 - \frac{\nu^2}{\beta_{\nu m}^2}\right)[J_\nu(\beta_{\nu m})]^2, \quad \nu > -1.$$

14.2.3 Verify that the orthogonality equation, Eq. (14.44), and the normalization equation, Eq. (14.46), hold for $\nu > -1$.

Hint. Using power-series expansions, examine the behavior of Eq. (14.43) as $\rho \to 0$.

14.2.4 From Eq. (11.49), develop a proof that $J_\nu(z)$, $\nu > -1$ has no complex roots (with a nonzero imaginary part).

Hint. (a) Use the series form of $J_\nu(z)$ to exclude pure imaginary roots.

(b) Assume $\alpha_{\nu m}$ to be complex and take $\alpha_{\nu n}$ to be $\alpha_{\nu m}^*$.

14.2.5 (a) In the series expansion

$$f(\rho) = \sum_{m=1}^{\infty} c_{\nu m} J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right), \quad 0 \le \rho \le a, \quad \nu > -1,$$

with $J_\nu(\alpha_{\nu m}) = 0$, show that the coefficients are given by

$$c_{\nu m} = \frac{2}{a^2[J_{\nu+1}(\alpha_{\nu m})]^2}\int_0^a f(\rho)J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\rho\,d\rho.$$

(b) In the series expansion

$$f(\rho) = \sum_{m=1}^{\infty} d_{\nu m} J_\nu\left(\beta_{\nu m}\frac{\rho}{a}\right), \quad 0 \le \rho \le a, \quad \nu > -1,$$

with $(d/d\rho) J_\nu(\beta_{\nu m}\rho/a)\,|_{\rho=a} = 0$, show that the coefficients are given by

$$d_{\nu m} = \frac{2}{a^2(1-\nu^2/\beta_{\nu m}^2)[J_\nu(\beta_{\nu m})]^2}\int_0^a f(\rho) J_\nu\left(\beta_{\nu m}\frac{\rho}{a}\right)\rho\, d\rho.$$

14.2.6 A right circular cylinder has an electrostatic potential of $\psi(\rho,\varphi)$ on both ends. The potential on the curved cylindrical surface is zero. Find the potential at all interior points.

Hint. Choose your coordinate system and adjust your z dependence to exploit the symmetry of your potential.

14.2.7 A function $f(x)$ is expressed as a Bessel series:

$$f(x) = \sum_{n=1}^{\infty} a_n J_m(\alpha_{mn} x),$$

with α_{mn} the nth root of J_m. Prove the Parseval relation,

$$\int_0^1 [f(x)]^2 x\, dx = \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 [J_{m+1}(\alpha_{mn})]^2.$$

14.2.8 Prove that

$$\sum_{n=1}^{\infty} (\alpha_{mn})^{-2} = \frac{1}{4(m+1)}.$$

Hint. Expand x^m in a Bessel series and apply the Parseval relation.

14.2.9 A right circular cylinder of length l and radius a has on its end caps a potential

$$\psi\left(z = \pm\frac{l}{2}\right) = 100\left(1 - \frac{\rho}{a}\right).$$

The potential on the curved surface (the side) is zero. Using the Bessel series from Exercise 14.2.6, calculate the electrostatic potential for $\rho/a = 0.0(0.2)1.0$ and $z/l = 0.0(0.1)0.5$. Take $a/l = 0.5$.

Hint. From Exercise 14.1.29 you have

$$\int_0^{\alpha_{0n}} \left(1 - \frac{y}{\alpha_{0n}}\right) J_0(y)\, y\, dy.$$

Show that this equals

$$\frac{1}{\alpha_{0n}} \int_0^{\alpha_{0n}} J_0(y)\, dy.$$

Numerical evaluation of this latter form rather than the former is both faster and more accurate.

Note. For $\rho/a = 0.0$ and $z/l = 0.5$ the convergence is slow, 20 terms giving only 98.4 rather than 100.

Check value. For $\rho/a = 0.4$ and $z/l = 0.3$, $\psi = 24.558$.

14.3 Neumann Functions, Bessel Functions of the Second Kind

From the theory of ODEs, it is known that Bessel's equation has two independent solutions. Indeed, for nonintegral order ν we have already found two solutions and labeled them $J_\nu(x)$ and $J_{-\nu}(x)$ using the infinite series, Eq. (14.6). The trouble is that when ν is integral, Eq. (14.5) holds and we have but one independent solution. A second solution may be developed by the methods of Section 7.6. This yields a perfectly good second solution of Bessel's equation. However, that solution is not the standard form, which is called a **Bessel function of the second kind** or alternatively, a **Neumann function**.

Definition and Series Form

The standard definition of the Neumann functions is the following linear combination of $J_\nu(x)$ and $J_{-\nu}(x)$:

$$Y_\nu(x) = \frac{\cos \nu\pi\, J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}. \tag{14.57}$$

For nonintegral ν, $Y_\nu(x)$ clearly satisfies Bessel's equation, for it is a linear combination of known solutions, $J_\nu(x)$ and $J_{-\nu}(x)$. The behavior of $Y_\nu(x)$ for small x (and nonintegral ν) can be determined from the power-series expansion of $J_{-\nu}$, Eq. (14.6); we may write, calling upon Eq. (13.23),

$$Y_\nu(x) = -\frac{1}{\sin\nu\pi}\left[\frac{1}{\Gamma(1-\nu)}\left(\frac{x}{2}\right)^{-\nu} - \cdots\right]$$
$$= -\frac{\Gamma(\nu)\Gamma(1-\nu)}{\pi}\left[\frac{1}{\Gamma(1-\nu)}\left(\frac{x}{2}\right)^{-\nu} - \cdots\right]$$
$$= -\frac{\Gamma(\nu)}{\pi}\left(\frac{x}{2}\right)^{-\nu} + \cdots. \tag{14.58}$$

However, for integral ν, Eq. (14.57) becomes indeterminate; in fact, $Y_n(x)$ for integral n is defined as

$$Y_n(x) = \lim_{\nu\to n} Y_\nu(x). \tag{14.59}$$

To determine that the limit represented by Eq. (14.59) exists and is not identically zero (so that $Y_n(x)$ has a meaningful definition), we apply l'Hôpital's rule to Eq. (14.57), obtaining initially

$$Y_n(x) = \frac{1}{\pi}\left[\frac{dJ_\nu}{d\nu} - (-1)^n \frac{dJ_{-\nu}}{d\nu}\right]_{\nu=n}. \tag{14.60}$$

Inserting the expansions of J_ν and $J_{-\nu}$ from Eq. (14.6), the differentiations of $(x/2)^{2s\pm\nu}$ combine to yield $(2/\pi)J_n(x)\ln(x/2)$, while the derivatives of $1/\Gamma(s\pm n+1)$ yield terms containing $\psi(s\pm n+1)/\Gamma(s\pm n+1)$, where ψ is the digamma function (Section 13.2). The final result, whose verification is the topic of Exercise 14.3.8, is

$$Y_n(x) = \frac{2}{\pi}J_n(x)\ln\left(\frac{x}{2}\right) - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n}$$
$$-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(n+k)!}\left[\psi(k+1)+\psi(n+k+1)\right]\left(\frac{x}{2}\right)^{2k+n}, \tag{14.61}$$

An explicit form for $\psi(n)$ for integer n is given in Eq. (13.40).

Equation (14.61) shows that for $n > 0$, the most divergent term for small x is in agreement with the result for noninteger n given in Eq. (14.58). We also see that all solutions for integer n contain a logarithmic term with the regular function J_n multiplying the logarithm. In our earlier study of ODEs, we found that a second solution will usually have a contribution of this type when the indicial equation causes the exponents of the power-series expansion to be integers. We may also conclude from Eq. (14.61) that Y_n is linearly independent of J_n, confirming that we indeed have a second solution to Bessel's ODE.

It is of some interest to obtain the expansion of $Y_0(x)$ in a more explicit form. Returning to Eq. (14.61), we note that its first summation is vacant, and we have the relatively simple expansion

$$Y_0(x) = \frac{2}{\pi}J_0(x)\ln\left(\frac{x}{2}\right) - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!k!}\left[-\gamma + H_k\right]\left(\frac{x}{2}\right)^{2k}$$
$$= \frac{2}{\pi}J_0(x)\left[\gamma + \ln\left(\frac{x}{2}\right)\right] - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k!k!}H_k\left(\frac{x}{2}\right)^{2k}, \tag{14.62}$$

where H_k is the harmonic number $\sum_{m=1}^{k} m^{-1}$ and γ is the Euler-Mascheroni constant.

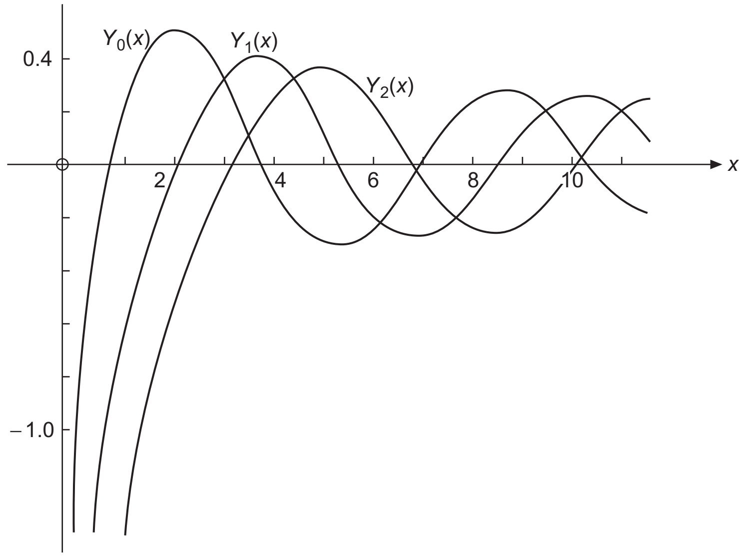

FIGURE 14.7 Neumann functions $Y_0(x)$, $Y_1(x)$, and $Y_2(x)$.

The Neumann functions $Y_n(x)$ are irregular at $x = 0$, but with increasing x become oscillatory, as may be seen from the graphs of Y_0, Y_1, and Y_2 in Fig. 14.7. The definition of Eq. (14.57) was specifically chosen to cause the oscillatory behavior to be at the same scale as that of J_n and displaced asymptotically in phase by $\pi/2$, similarly to the relative behavior of the sine and cosine. However, unlike the sine and cosine, J_n and Y_n only exhibit exact periodicity in the asymptotic limit. This point is covered in detail in Section 14.6. Figure 14.8 compares $J_0(x)$ and $Y_0(x)$ over a large range of x.

Integral Representations

As with all the other Bessel functions, $Y_\nu(x)$ has integral representations. For $Y_0(x)$ we have

$$Y_0(x) = -\frac{2}{\pi}\int_0^\infty \cos(x\cosh t)dt = -\frac{2}{\pi}\int_1^\infty \frac{\cos(xt)}{(t^2-1)^{1/2}}dt, \quad x > 0. \tag{14.63}$$

See Exercise 14.3.7, which shows that the above integral is a solution to Bessel's ODE that is linearly independent of $J_0(x)$. Specific identification as Y_0 is the topic of Exercise 14.4.8.

Recurrence Relations

Substituting Eq. (14.57) for $Y_\nu(x)$ (nonintegral ν) into the recurrence relations for $J_n(x)$, Eqs. (14.7) and (14.8), we see immediately that $Y_\nu(x)$ satisfies these same recurrence relations. This actually constitutes a proof that Y_ν is a solution to the Bessel ODE. Note that the converse is not necessarily true. All solutions need not satisfy the same recurrence relations, as the relations depend on the scales assigned to the solutions of different ν. An example of this sort of trouble appears in Section 14.5.

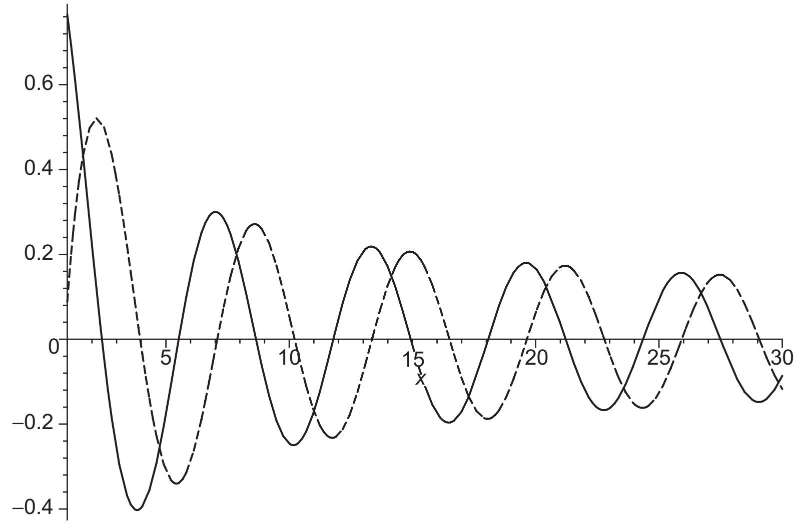

FIGURE 14.8 Oscillatory behavior of $J_0(x)$ (solid line) and $Y_0(x)$ (dashed line) for $1 \le x \le 30$.

Wronskian Formulas

An ODE $p(x)y'' + q(x)y' + r(x)y = 0$ in self-adjoint form (so $q = p'$) was found in Exercise 7.6.1 to have the following Wronskian formula connecting its solutions u and v:

$$u(x)v'(x) - u'(x)v(x) = \frac{A}{p(x)}. \tag{14.64}$$

To bring Bessel's equation to self-adjoint form, we need to write it as $xy'' + y' + (x - \nu^2/x)y = 0$, thereby showing that for our present purposes $p(x) = x$, and we therefore have for each noninteger ν

$$J_\nu J'_{-\nu} - J'_\nu J_{-\nu} = \frac{A_\nu}{x}. \tag{14.65}$$

Since A_ν is a constant but can be expected to depend on ν, it may be identified for each ν at any convenient point, such as $x = 0$. From the power-series expansion, Eq. (14.6), we obtain the following limiting behaviors for small x:

$$\begin{aligned} J_\nu &\to \frac{1}{\Gamma(1+\nu)}\left(\frac{x}{2}\right)^\nu, \qquad & J'_\nu &\to \frac{\nu}{2\Gamma(1+\nu)}\left(\frac{x}{2}\right)^{\nu-1}, \\ J_{-\nu} &\to \frac{1}{\Gamma(1-\nu)}\left(\frac{x}{2}\right)^{-\nu}, & J'_{-\nu} &\to \frac{-\nu}{2\Gamma(1-\nu)}\left(\frac{x}{2}\right)^{-\nu-1}. \end{aligned} \tag{14.66}$$

Substitution into Eq. (14.65) yields

$$J_\nu(x)J'_{-\nu}(x) - J'_\nu(x)J_{-\nu}(x) = \frac{-2\nu}{x\Gamma(1+\nu)\Gamma(1-\nu)} = -\frac{2\sin\nu\pi}{\pi x}, \tag{14.67}$$

using Eq. (13.23). Although Eq. (14.67) was obtained for $x \to 0$, comparison with Eq. (14.65) shows that it must be true for all x, and that $A_\nu = -(2/\pi)\sin\nu\pi$. Note that A_ν vanishes for integral ν, showing that the Wronskian of J_n and J_{-n} vanishes and that these Bessel functions are linearly dependent.

Using our recurrence relations, we may readily develop a large number of alternate forms, among which are

$$J_\nu J_{-\nu+1} + J_{-\nu} J_{\nu-1} = \frac{2\sin\nu\pi}{\pi x}, \tag{14.68}$$

$$J_\nu J_{-\nu-1} + J_{-\nu} J_{\nu+1} = -\frac{2\sin\nu\pi}{\pi x}, \tag{14.69}$$

$$J_\nu Y'_\nu - J'_\nu Y_\nu = \frac{2}{\pi x}, \tag{14.70}$$

$$J_\nu Y_{\nu+1} - J_{\nu+1} Y_\nu = -\frac{2}{\pi x}. \tag{14.71}$$

Many more will be found in the Additional Readings.

You will recall that in Chapter 7, Wronskians were of great value in two respects: (1) in establishing the linear independence or linear dependence of solutions of differential equations, and (2) in developing an integral form of a second solution. Here the specific forms of the Wronskians and Wronskian-derived combinations of Bessel functions are useful primarily in development of the general behavior of the various Bessel functions. Wronskians are also of great use in checking tables of Bessel functions.

Uses of Neumann Functions

The Neumann functions $Y_\nu(x)$ are of importance for a number of reasons:

1. They are second, independent solutions of Bessel's equation, thereby completing the general solution.
2. They are needed for physical problems in which they are not excluded by a requirement of regularity at $x = 0$. Specific examples include electromagnetic waves in coaxial cables and quantum mechanical scattering theory.
3. They lead directly to the two Hankel functions, whose definition and use, particularly in studies of wave propagation, are discussed in Section 14.4.

We close with one example in which Neumann functions play a vital role.

Example 14.3.1 Coaxial Wave Guides

We are interested in an electromagnetic wave confined between the concentric, conducting cylindrical surfaces $\rho = a$ and $\rho = b$. The equations governing the wave propagation are the same as those discussed in Example 14.1.2, but the boundary conditions are now different, and our interest is in solutions that are traveling waves (compare Exercise 14.1.26).

For wave propagation problems, it is convenient to write the solution in terms of complex exponentials, with the actual physical quantities involved ultimately identified as their real (or imaginary) parts. Thus, in place of Eq. (14.31) (the solution for standing waves in a cylindrical cavity), we now have for E_z solutions in which the ρ dependence must involve both J_m and Y_m (as the latter is not ruled out by a requirement for regularity at $\rho = 0$). Including the time dependence, we have for the TM (transverse magnetic) solutions the separated-variable forms

$$E_z = \left[c_{mn} J_m(\gamma_{mn}\rho) + d_{mn} Y_m(\gamma_{mn}\rho) \right] e^{\pm im\varphi} e^{i(lz-\omega t)}, \tag{14.72}$$

with l now permitted to have any real value (there is no boundary condition on z). The index n identifies different possible values of γ_{mn}. As in Eq. (14.30), the relation between γ_{mn}, l, and ω is

$$\frac{\omega^2}{c^2} = \gamma_{mn}^2 + l^2. \tag{14.73}$$

The most general TM traveling-wave solution will be an arbitrary linear combination of all functions of the form given by Eq. (14.72) with γ_{mn}, c_{mn}, and d_{mn} chosen so that E_z will vanish at $\rho = a$ and $\rho = b$. A main difference between this problem and that of Example 14.1.2 is that the condition on E_z is not given by the zeros of the Bessel functions J_m, but by zeros of linear combinations of J_m and Y_m. Specifically, we require that

$$c_{mn} J_m(\gamma_{mn}a) + d_{mn} Y_m(\gamma_{mn}a) = 0, \tag{14.74}$$

$$c_{mn} J_m(\gamma_{mn}b) + d_{mn} Y_m(\gamma_{mn}b) = 0. \tag{14.75}$$

These transcendental equations may be solved, for each relevant m, to yield an infinite set of solutions (indexed by n) for γ_{mn} and the ratio d_{mn}/c_{mn}. An example of this process is in Exercise 14.3.10.

Returning now to the equation for ω, we observe that the smallest value it can attain for the solution indexed by m and n is $c\gamma_{mn}$, showing that TM waves can only propagate if the angular frequency ω of the electromagnetic radiation is equal to or larger than this **cutoff**. In general, larger values of γ_{mn} correspond to higher degrees of transverse oscillation, and modes with greater transverse oscillation will therefore have higher cutoff frequencies.

As for the circular wave guide (the subject of Exercise 14.1.26, there will also be TE modes of propagation, also with mode-dependent cutoffs. However, the coaxial guide can also support traveling waves in TEM (transverse electric and magnetic) modes. These modes, not possible for a circular waveguide, do not exhibit a cutoff, are the confined equivalent of plane waves, and correspond to the flow of current (in opposite directions) on the coaxial conductors. ■

Exercises

14.3.1 Prove that the Neumann functions Y_n (with n an integer) satisfy the recurrence relations

$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x),$$

$$Y_{n-1}(x) - Y_{n+1}(x) = 2Y_n'(x).$$

Hint. These relations may be proved by differentiating the recurrence relations for J_ν or by using the limit form of Y_ν but **not** dividing everything by zero.

14.3.2 Show that for integer n

$$Y_{-n}(x) = (-1)^n Y_n(x).$$

14.3.3 Show that

$$Y_0'(x) = -Y_1(x).$$

14.3.4 If X and Z are any two solutions of Bessel's equation, show that

$$X_\nu(x) Z_\nu'(x) - X_\nu'(x) Z_\nu(x) = \frac{A_\nu}{x},$$

in which A_ν may depend on ν but is independent of x. This is a special case of Exercise 7.6.11.

14.3.5 Verify the Wronskian formulas

$$J_\nu(x) J_{-\nu+1}(x) + J_{-\nu}(x) J_{\nu-1}(x) = \frac{2 \sin \nu\pi}{\pi x},$$

$$J_\nu(x) Y_\nu'(x) - J_\nu'(x) Y_\nu(x) = \frac{2}{\pi x}.$$

14.3.6 As an alternative to letting x approach zero in the evaluation of the Wronskian constant, we may invoke the uniqueness of power-series expansions. The coefficient of x^{-1} in the series expansion of $u_\nu(x) v_\nu'(x) - u_\nu'(x) v_\nu(x)$ is then A_ν. Show by series expansion that the coefficients of x^0 and x^1 of $J_\nu(x) J_{-\nu}'(x) - J_\nu'(x) J_{-\nu}(x)$ are each zero.

14.3.7 (a) By differentiating and substituting into Bessel's ODE for $\nu = 0$, show that $\int_0^\infty \cos(x \cosh t) dt$ is a solution.

Hint. Rearrange the final integral to $\int_0^\infty \frac{d}{dt}\big[x \sin(x \cosh t) \sinh t\big] dt$.

(b) Show that $Y_0(x) = -\frac{2}{\pi} \int_0^\infty \cos(x \cosh t) dt$ is linearly independent of $J_0(x)$.

14.3.8 Verify the expansion formula for $Y_n(x)$ given in Eq. (14.61).

Hint. Start from Eq. (14.60) and perform the indicated differentiations on the power-series expansions of J_ν and $J_{-\nu}$. The digamma functions ψ arise from the differentiation of the gamma function. You will need the identity (not derived in this book) $\lim_{z \to -n} \psi(z)/\Gamma(z) = (-1)^{n-1} n!$, where n is a positive integer.

14.3.9 If Bessel's ODE (with solution J_ν) is differentiated with respect to ν, one obtains

$$x^2 \frac{d^2}{dx^2}\left(\frac{\partial J_\nu}{\partial \nu}\right) + x \frac{d}{dx}\left(\frac{\partial J_\nu}{\partial \nu}\right) + (x^2 - \nu^2) \frac{\partial J_\nu}{\partial \nu} = 2\nu J_\nu.$$

Use the above equation to show that $Y_n(x)$ is a solution to Bessel's ODE.

Hint. Equation (14.60) will be useful.

14.3.10 For the case $m = 0$, $a = 1$, and $b = 2$, the coaxial wave-guide TM boundary conditions become $f(\lambda) = 0$, with

$$f(x) = \frac{J_0(2x)}{Y_0(2x)} - \frac{J_0(x)}{Y_0(x)}.$$

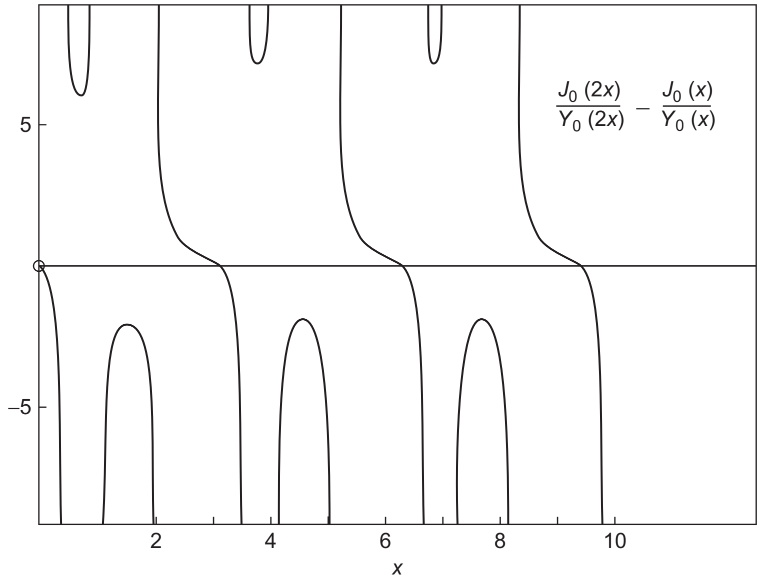

FIGURE 14.9 The function $f(x)$ of Exercise 14.3.10.

This function is plotted in Fig. 14.9.

(a) Calculate $f(x)$ for $x = 0.0(0.1)10.0$ and plot $f(x)$ vs. x to find the approximate location of the roots.

(b) Call a root-finding program to determine the first three roots to higher precision.

ANS. (b) 3.1230, 6.2734, 9.4182.

Note. The higher roots can be expected to appear at intervals whose length approaches π. Why? AMS-55 (see Additional Readings) gives an approximate formula for the roots. The function $g(x) = J_0(x)Y_0(2x) - J_0(2x)Y_0(x)$ is much better behaved than the $f(x)$ previously discussed.

14.4 HANKEL FUNCTIONS

Hankel functions are solutions of Bessel's ODE with asymptotic properties that make them particularly useful in problems involving the propagation of spherical or cylindrical waves. Since the functions J_ν and Y_ν form the complete solution of this ODE, the Hankel functions cannot be anything completely new; they must be linear combinations of the solutions we have already found. We introduce them here via straightforward algebraic definitions; later in this section we identify integral representations that some authors have used as a starting point.

Definitions

Starting from the Bessel functions of the first and second kinds, namely $J_\nu(x)$ and $Y_\nu(x)$, we define the two Hankel functions $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ (sometimes, but nowadays infrequently referred to as **Bessel functions of the third kind**) as follows:

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x), \tag{14.76}$$

$$H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x). \tag{14.77}$$

This is exactly analogous to taking

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta. \tag{14.78}$$

For real arguments, $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are complex conjugates. The extent of the analogy will be seen even better when their asymptotic forms are considered. Indeed, it is their asymptotic behavior that makes the Hankel functions useful. This behavior is discussed in Section 14.6, and in that section we provide an illustrative example in which the asymptotic properties play a key role.

Series expansion of $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ may be obtained by combining Eqs. (14.6) and (14.62). Often only the first term is of interest; it is given by

$$H_0^{(1)}(x) \approx i\frac{2}{\pi}\ln x + 1 + i\frac{2}{\pi}(\gamma - \ln 2) + \cdots, \tag{14.79}$$

$$H_\nu^{(1)}(x) \approx -i\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu + \cdots, \quad \nu > 0, \tag{14.80}$$

$$H_0^{(2)}(x) \approx -i\frac{2}{\pi}\ln x + 1 - i\frac{2}{\pi}(\gamma - \ln 2) + \cdots, \tag{14.81}$$

$$H_\nu^{(2)}(x) \approx i\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu + \cdots, \quad \nu > 0. \tag{14.82}$$

In these equations γ is the Euler-Mascheroni constant, defined in Eq. (1.13).

Since the Hankel functions are linear combinations (with constant coefficients) of J_ν and Y_ν, they satisfy the same recurrence relations, Eqs. (14.7) and (14.8). For both $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$,

$$H_{\nu-1}(x) + H_{\nu+1}(x) = \frac{2\nu}{x}H_\nu(x), \tag{14.83}$$

$$H_{\nu-1}(x) - H_{\nu+1}(x) = 2H_\nu'(x). \tag{14.84}$$

A variety of Wronskian formulas can be developed, including:

$$H_\nu^{(2)}H_{\nu+1}^{(1)} - H_\nu^{(1)}H_{\nu+1}^{(2)} = \frac{4}{i\pi x}, \tag{14.85}$$

$$J_{\nu-1}H_\nu^{(1)} - J_\nu H_{\nu-1}^{(1)} = \frac{2}{i\pi x}, \tag{14.86}$$

$$J_{\nu-1}H_\nu^{(2)} - J_\nu H_{\nu-1}^{(2)} = -\frac{2}{i\pi x}. \tag{14.87}$$

Contour Integral Representation of the Hankel Functions

The integral representation (Schlaefli integral) for $J_\nu(x)$ was introduced in Section 14.1, where we established that

$$J_\nu(x) = \frac{1}{2\pi i} \int\limits_C e^{(x/2)(t-1/t)} \frac{dt}{t^{\nu+1}}, \tag{14.88}$$

with C the contour shown in Fig. 14.5. Recall that when ν is nonintegral, the integrand has a branch point at $t = 0$ and the contour had to avoid a cut line that was drawn along the negative real axis. In developing the Schlaefli integral for general ν, we began by showing that Bessel's ODE was satisfied for any open contour for which an expression of the form

$$\frac{e^{(x/2)(t-1/t)}}{t^\nu} \left[\nu + \frac{x}{2}\left(t + \frac{1}{t}\right) \right] \tag{14.89}$$

vanished at both endpoints of the contour.

We now make further use of those observations by noting that the expression in Eq. (14.89) not only vanishes at $t = -\infty$ on the real axis both below and above the cut, but that it also vanishes at $t = 0$ when that point is approached from positive t.

We therefore consider the contour shown in Fig. 14.10, calling attention to the fact that the upper half of the contour (from $t = 0+$ to $t = \infty e^{\pi i}$), labeled C_1, meets the conditions necessary to yield a solution to Bessel's ODE, and that the remaining (lower) half of the contour, labeled C_2, also yields a solution. What remains to be determined is the identification of these solutions: We will show that they are the Hankel functions. For $x > 0$, we assert that

$$H_\nu^{(1)}(x) = \frac{1}{\pi i} \int\limits_{C_1} e^{(x/2)(t-1/t)} \frac{dt}{t^{\nu+1}}, \tag{14.90}$$

$$H_\nu^{(2)}(x) = \frac{1}{\pi i} \int\limits_{C_2} e^{(x/2)(t-1/t)} \frac{dt}{t^{\nu+1}}. \tag{14.91}$$

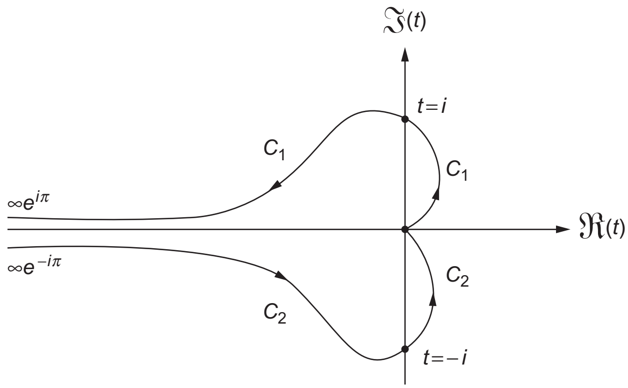

FIGURE 14.10 Hankel function contours.

These expressions are particularly convenient because they may be handled by the method of steepest descents (Section 12.7). $H_\nu^{(1)}(x)$ has a saddle point at $t = +i$, whereas $H_\nu^{(2)}(x)$ has a saddle point at $t = -i$.

There remains the problem of relating Eqs. (14.90) and (14.91) to our earlier definition of the Hankel functions, Eqs. (14.76) and (14.77). Since the contours of Eqs. (14.90) and (14.91) combine to produce a contour yielding J_ν, Eq. (14.88), we have, from the integral representations,

$$J_\nu(x) = \frac{1}{2}\left[H_\nu^{(1)}(x) + H_\nu^{(2)}(x)\right]. \tag{14.92}$$

If we can show (also from the integral representations) that

$$Y_\nu(x) = \frac{1}{2i}\left[H_\nu^{(1)}(x) - H_\nu^{(2)}(x)\right], \tag{14.93}$$

we will be able to recover the original definitions of the $H_\nu^{(i)}$.

We therefore rewrite Eq. (14.90) by replacing the integration variable t by $e^{i\pi}/s$, so the integrand of that equation becomes $-e^{(x/2)(s-1/s)}e^{-i\nu\pi}s^{\nu-1}$. After the substitution the contour (in s) is found to be the same as C_1, but traversed in the opposite direction (thereby compensating the initial minus sign in the transformed integrand). The result, with details left as Exercise 14.4.3, is that the contour integral representation of $H^{(1)}$ is consistent with the identification

$$H_\nu^{(1)}(x) = e^{-i\nu\pi} H_{-\nu}^{(1)}(x). \tag{14.94}$$

Similar processing of Eq. (14.91), with $t = e^{-i\pi}/s$, leads to

$$H_\nu^{(2)}(x) = e^{i\nu\pi} H_{-\nu}^{(2)}(x). \tag{14.95}$$

We now combine Eqs. (14.94) and (14.95) to reach

$$J_{-\nu}(x) = \frac{1}{2}\left[e^{i\nu\pi} H_\nu^{(1)}(x) + e^{-i\nu\pi} H_\nu^{(2)}(x)\right], \tag{14.96}$$

where again the $H_\nu^{(i)}$ refer to the contour integral representations. Substituting Eqs. (14.92) and (14.96) into the defining equation for Y_ν, Eq. (14.57), we confirm that Y_ν is described properly when the $H_\nu^{(i)}$ stand for their contour integral representations. This completes the proof that Eqs. (14.90) and (14.91) are consistent with the original definitions of the Hankel functions.

The reader may wonder why so much stress is placed on the development of integral representations. There are several reasons. The first is simply aesthetic appeal. Second, the integral representations facilitate manipulations, analysis, and the development of relations among the various special functions. We have already seen an example of this in the development of Eqs. (14.94) to (14.96). And, probably most important of all, integral representations are extremely useful in developing asymptotic expansions. Such expansions can often be obtained using the method of steepest descents (Section 12.7), or by methods involving expansion in negative powers of the expansion variable, as in Section 12.6.

In conclusion, the Hankel functions are introduced here for the following reasons:

- As analogs of $e^{\pm ix}$ they are useful for describing traveling waves. These applications are best studied when the asymptotic properties of the functions are in hand, and therefore are postponed to Section 14.6.
- They offer an alternate (contour integral) and rather elegant definition of Bessel functions.
- We will see in Section 14.5 that they offer a route to the definition of the quantities known as **modified Bessel functions**, and that in Section 14.6 they are useful for the development of the asymptotic properties of Bessel functions.

Exercises

14.4.1 Verify the Wronskian formulas

(a) $J_\nu(x)H_\nu^{(1)\prime}(x) - J'_\nu(x)H_\nu^{(1)}(x) = \frac{2i}{\pi x}$,

(b) $J_\nu(x)H_\nu^{(2)\prime}(x) - J'_\nu(x)H_\nu^{(2)}(x) = -\frac{2i}{\pi x}$,

(c) $Y_\nu(x)H_\nu^{(1)\prime}(x) - Y'_\nu(x)H_\nu^{(1)}(x) = -\frac{2}{\pi x}$,

(d) $Y_\nu(x)H_\nu^{(2)\prime}(x) - Y'_\nu(x)H_\nu^{(2)}(x) = -\frac{2}{\pi x}$,

(e) $H_\nu^{(1)}(x)H_\nu^{(2)\prime}(x) - H_\nu^{(1)\prime}(x)H_\nu^{(2)}(x) = -\frac{4i}{\pi x}$,

(f) $H_\nu^{(2)}(x)H_{\nu+1}^{(1)}(x) - H_\nu^{(1)}(x)H_{\nu+1}^{(2)}(x) = \frac{4}{i\pi x}$,

(g) $J_{\nu-1}(x)H_\nu^{(1)}(x) - J_\nu(x)H_{\nu-1}^{(1)}(x) = \frac{2}{i\pi x}$.

14.4.2 Show that the integral forms

(a) $$\frac{1}{i\pi}\int_{0C_1}^{\infty e^{i\pi}} e^{(x/2)(t-1/t)}\frac{dt}{t^{\nu+1}} = H_\nu^{(1)}(x),$$

(b) $$\frac{1}{i\pi}\int_{\infty e^{-i\pi}C_2}^{0} e^{(x/2)(t-1/t)}\frac{dt}{t^{\nu+1}} = H_\nu^{(2)}(x)$$

satisfy Bessel's ODE. The contours C_1 and C_2 are shown in Fig. 14.10.

14.4.3 Show that the substitution $t = e^{i\pi}/s$ into Eq. (14.90) for $H_\nu^{(1)}(x)$ not only produces the integrand for the similar integral representation of $H_{-\nu}^{(1)}(x)$ but that the contour in s is identical to the original contour in t.

14.4.4 Using the integrals and contours given in Exercise 14.4.2, show that

$$\frac{1}{2i}[H_\nu^{(1)}(x) - H_\nu^{(2)}(x)] = Y_\nu(x).$$

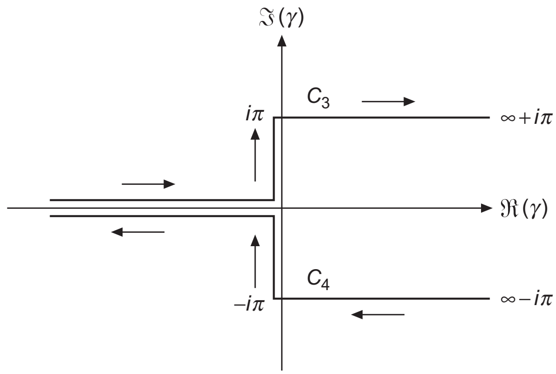

FIGURE 14.11 Hankel function contours for Exercise 14.4.5.

14.4.5 Show that the integrals in Exercise 14.4.2 may be transformed to yield

(a) $H_\nu^{(1)}(x) = \frac{1}{\pi i}\int_{C_3} e^{x\sinh\gamma - \nu\gamma}\,d\gamma$,

(b) $H_\nu^{(2)}(x) = \frac{1}{\pi i}\int_{C_4} e^{x\sinh\gamma - \nu\gamma}\,d\gamma$,

where C_3 and C_4 are the contours in Fig. 14.11.

14.4.6 (a) Transform $H_0^{(1)}(x)$, Eq. (14.90), into

$$H_0^{(1)}(x) = \frac{1}{i\pi}\int_C e^{ix\cosh s}\,ds,$$

where the contour C runs from $-\infty - i\pi/2$ through the origin of the s-plane to $\infty + i\pi/2$.

(b) Justify rewriting $H_0^{(1)}(x)$ as

$$H_0^{(1)}(x) = \frac{2}{i\pi}\int_0^{\infty + i\pi/2} e^{ix\cosh s}\,ds.$$

(c) Verify that this integral representation actually satisfies Bessel's differential equation. (The $i\pi/2$ in the upper limit is not essential. It serves as a convergence factor. We can replace it by $ia\pi/2$ and take the limit $a \to 0$.)

14.4.7 From

$$H_0^{(1)}(x) = \frac{2}{i\pi}\int_0^\infty e^{ix\cosh s}\,ds$$

show that

(a) $J_0(x) = \frac{2}{\pi}\int_0^\infty \sin(x\cosh s)\,ds$, (b) $J_0(x) = \frac{2}{\pi}\int_1^\infty \frac{\sin(xt)}{\sqrt{t^2-1}}\,dt$.

This last result is a Fourier sine transform.

14.4.8 From $H_0^{(1)}(x) = \frac{2}{i\pi}\int_0^\infty e^{ix\cosh s}ds$ (see Exercises 14.4.5 and 14.4.6), show that

(a) $Y_0(x) = -\frac{2}{\pi}\int_0^\infty \cos(x\cosh s)ds,$

(b) $Y_0(x) = -\frac{2}{\pi}\int_1^\infty \frac{\cos(xt)}{\sqrt{t^2-1)}}dt.$

These are the integral representations in Eq. (14.63). This last result is a Fourier cosine transform.

14.5 Modified Bessel Functions, $I_\nu(x)$ and $K_\nu(x)$

The Laplace and Helmholtz equations, when separated in circular cylindrical coordinates, may lead to Bessel's ODE in the coordinate ρ that describes distance from the cylindrical axis. When that is the case, the behavior of the solutions as a function of ρ is inherently oscillatory; as we have already seen, the Bessel functions $J_\nu(k\rho)$, and also $Y_\nu(k\rho)$, have for any value of ν an infinite number of zeros, and this property may be useful in causing satisfaction of boundary conditions. However, as already shown in Section 9.4, the connection constants arising when the variables are separated may have a sign opposite to that required to yield Bessel's ODE, and the equation in the ρ coordinate then assumes the form

$$\rho^2\frac{d^2}{d\rho^2}P_\nu(k\rho) + \rho\frac{d}{d\rho}P_\nu(k\rho) - (k^2\rho^2+\nu^2)P_\nu(k\rho) = 0. \tag{14.97}$$

Equation (14.97), known as the **modified Bessel equation**, differs from the Bessel ODE only in the sign of the quantity $k^2\rho^2$, but this small change is sufficient to alter the nature of the solutions. As we shall shortly discuss in more detail, the solutions to Eq. (14.97), called **modified Bessel functions**, are **not** oscillatory and have behavior that is exponential (rather than trigonometric) in character.

Fortunately, the knowledge we have developed regarding the Bessel ODE can be put to good use for the modified Bessel equation, since the substitution $k \to ik$ converts the conventional Bessel ODE to its modified form, and shows that if $P_\nu(k\rho)$ is a solution to the Bessel ODE, then $P_\nu(ik\rho)$ must be a solution to the modified Bessel equation. One way of stating this fact is to note that the solutions of Eq. (14.97) are Bessel functions of imaginary argument.

Series Solution

Since any solution of Bessel's ODE can be converted into a solution of the modified ODE by insertion of i into its argument, let's start by looking at the series expansion

$$J_\nu(ix) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\Gamma(s+\nu+1)} \left(\frac{ix}{2}\right)^{\nu+2s} = i^\nu \sum_{s=0}^{\infty} \frac{1}{s!\Gamma(s+\nu+1)} \left(\frac{x}{2}\right)^{\nu+2s}. \tag{14.98}$$

Since all the terms of the summation have the same sign, it is evident that $J_\nu(ix)$ cannot exhibit oscillatory behavior. It is convenient to choose the solutions of the modified Bessel equation in a way that causes them to be real, and we accordingly defined the **modified Bessel functions of the first kind**, denoted $I_\nu(x)$, as

$$I_\nu(x) = i^{-\nu} J_\nu(ix) = e^{-i\nu\pi/2} J_\nu(xe^{i\pi/2}) = \sum_{s=0}^{\infty} \frac{1}{s!\Gamma(s+\nu+1)} \left(\frac{x}{2}\right)^{\nu+2s}. \tag{14.99}$$

Like J_ν for $\nu \geq 0$, I_ν is finite at the origin, with a power-series expansion that is convergent for all x. At small x, its limiting behavior will be of the form

$$I_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} + \cdots . \tag{14.100}$$

From the relation between J_ν and $J_{-\nu}$, we may also conclude that I_ν and $I_{-\nu}$ are linearly independent unless ν is an integer n; taking cognizance of the factor i^{-n} in the definition of I_n, the linear dependence takes the form

$$I_n(x) = I_{-n}(x). \tag{14.101}$$

Graphs of I_0 and I_1 are shown in Fig. 14.12.

Recurrence Relations for I_ν

The recurrence relations satisfied by $I_\nu(x)$ may be developed from the series expansions, but it is perhaps easier to work from the existing recurrence relations for $J_\nu(x)$. Our starting point is Eq. (14.7), written for ix:

$$J_{\nu-1}(ix) + J_{\nu+1}(ix) = \frac{2n}{ix} J_n(ix). \tag{14.102}$$

We change J to I, related according to Eq. (14.99) by

$$J_\nu(ix) = i^\nu I_\nu(x), \tag{14.103}$$

thereby obtaining

$$i^{\nu-1} I_{\nu-1}(x) + i^{\nu+1} I_{\nu+1}(x) = \frac{2\nu}{ix} i^\nu I_\nu(x),$$

which simplifies to

$$I_{\nu-1}(x) - I_{\nu+1}(x) = \frac{2\nu}{x} I_\nu(x). \tag{14.104}$$

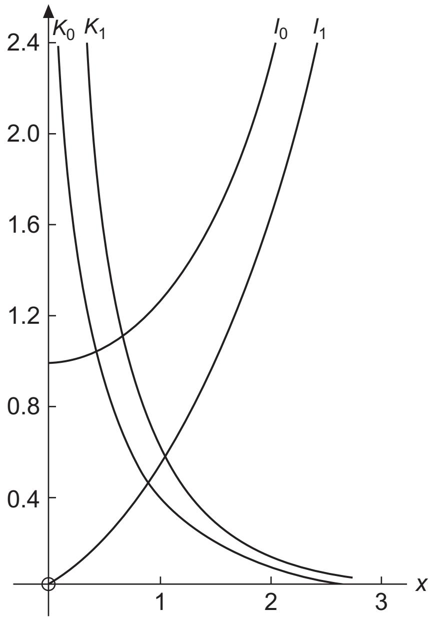

FIGURE 14.12 Modified Bessel functions.

In a similar fashion, Eq. (14.8) transforms into

$$I_{\nu-1}(x) + I_{\nu+1}(x) = 2I'_\nu(x). \tag{14.105}$$

The above analysis is also the topic of Exercise 14.1.14.

Second Solution K_ν

As already pointed out we have but one independent solution when ν is an integer, exactly as for the Bessel functions J_ν. The choice of a second, independent solution of Eq. (14.97) is essentially a matter of convenience. The second solution given here is selected on the basis of its asymptotic behavior, which we examine in the next section. The confusion of choice and notation for this solution is perhaps greater than anywhere else in this field.[5] There is also no universal nomenclature; the K_ν are sometimes referred to as Whittaker functions. Following AMS-55 (see Additional Readings for reference), we here define a second solution in terms of the Hankel function $H_\nu^{(1)}(x)$ as

$$K_\nu(x) \equiv \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) = \frac{\pi}{2} i^{\nu+1}\left[J_\nu(ix) + iY_\nu(ix)\right]. \tag{14.106}$$

[5]Discussion and comparison of notations will be found in *Math. Tables Aids Comput.* **1**: 207–308 (1944) and in AMS-55 (see Additional Readings).

The factor $i^{\nu+1}$ makes $K_\nu(x)$ real when x is real.[6] Using Eqs. (14.57) and (14.99), we may transform Eq. (14.106) to[7]

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi}, \tag{14.107}$$

somewhat analogous to Eq. (14.57) for $Y_\nu(x)$. The choice of Eq. (14.106) as a definition is somewhat unfortunate in that the function $K_\nu(x)$ does not satisfy the same recurrence relations as $I_\nu(x)$. The recurrence formulas for the K_ν are

$$K_{\nu-1}(x) - K_{\nu+1}(x) = -\frac{2\nu}{x}K_\nu(x), \tag{14.108}$$

$$K_{\nu-1}(x) + K_{\nu+1}(x) = -2K'_\nu(x). \tag{14.109}$$

To avoid this discrepancy in the recurrence relations, some authors[8] have included an additional factor of $\cos \nu\pi$ in the definition of K_ν. This would permit K_ν to satisfy the same recurrence relations as I_ν (see Exercise 14.5.8), but it has the disadvantage of making $K_\nu = 0$ for $\nu = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$.

The series expansion of $K_\nu(x)$ follows directly from the series form of $H_\nu^{(1)}(ix)$, providing that we choose the branch of $\ln ix$ appropriately (see Exercise 14.5.9). Using Eqs. (14.79) and (14.80), the lowest-order terms are then found to be

$$K_0(x) = -\ln x - \gamma + \ln 2 + \cdots, \tag{14.110}$$

$$K_\nu(x) = 2^{\nu-1}\Gamma(\nu)x^{-\nu} + \cdots. \tag{14.111}$$

Because the modified Bessel function I_ν is related to the Bessel function J_ν, much as sinh is related to sine, the modified Bessel functions I_ν and K_ν are sometimes referred to as **hyperbolic Bessel functions**. K_0 and K_1 are shown in Fig. 14.12.

Integral Representations

$I_0(x)$ and $K_0(x)$ have the integral representations

$$I_0(x) = \frac{1}{\pi}\int_0^\pi \cosh(x\cos\theta)d\theta, \tag{14.112}$$

$$K_0(x) = \int_0^\infty \cos(x \sinh t)dt = \int_0^\infty \frac{\cos(xt)dt}{(t^2+1)^{1/2}}, \quad x > 0. \tag{14.113}$$

Equation (14.112) may be derived from Eq. (14.20) for $J_0(x)$ or may be taken as a special case of Exercise 14.5.14. The integral representation of K_0, Eq. (14.113), is derived in Section 14.6. A variety of other forms of integral representations (including $\nu \neq 0$) appear

[6]If ν is not an integer, $K_\nu(z)$ has a branch point at $z = 0$ due to the presence of a fractional power; if $\nu = n$, an integer, $K_n(z)$ has a branch point at $z = 0$ due to the term $\ln z$. We normally identify $K_n(z)$ as the branch that is real for real z.

[7]For integral index n we take the limit as $\nu \to n$.

[8]For example, Whittaker and Watson (see Additional Readings).

in the exercises. These integral representations are useful in developing asymptotic forms (Section 14.6) and in connection with Fourier transforms (Chapter 19).

Example 14.5.1 A Green's Function

We wish to develop an expansion for the fundamental Green's function for the Laplace equation in cylindrical coordinates (ρ, φ, z). The defining equation is

$$\left[\frac{\partial^2}{\partial \rho_1^2} + \frac{1}{\rho_1}\frac{\partial}{\partial \rho_1} + \frac{1}{\rho_1^2}\frac{\partial^2}{\partial \varphi_1^2} + \frac{\partial^2}{\partial z_1^2}\right] G(\mathbf{r}_1, \mathbf{r}_2) = \delta(\rho_1 - \rho_2)\frac{1}{\rho_1^2}\delta(\varphi_1 - \varphi_2)\delta(z_1 - z_2). \tag{14.114}$$

We now write the Dirac delta function for the φ coordinate in the form corresponding to Eq. (5.27):

$$\delta(\varphi_1 - \varphi_2) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\varphi_1 - \varphi_2)}.$$

For the z coordinate, we use the continuum limit of the above formula, or, equivalently, the large-n limit of Eq. (1.155),

$$\delta(z_1 - z_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(z_1 - z_2)} dk = \frac{1}{\pi}\int_0^{\infty} \cos k(z_1 - z_2) dk.$$

We use the last form of the above equation so that k will never be negative.

We now expand $G(\mathbf{r}_1, \mathbf{r}_2)$ as

$$G(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2\pi^2}\sum_m \int_0^{\infty} dk g_m(k, \rho_1, \rho_2) e^{im(\varphi_1 - \varphi_2)} \cos k(z_1 - z_2). \tag{14.115}$$

For φ_1 and φ_2, this is simply an expansion in orthogonal functions; the dependence on z_1, z_2, and k is actually an integral transform that will be more completely justified in Chapter 20. For our present purposes, what is significant is that we can apply the orthogonality properties of the expansion to find that Eq. (14.114) will be satisfied if (for all relevant values of k and m)

$$\left[\frac{\partial^2}{\partial \rho_1^2} + \frac{1}{\rho_1}\frac{\partial}{\partial \rho_1} - \frac{m^2}{\rho_1^2} - k^2\right] g_m(k, \rho_1, \rho_2) = \delta(\rho_1 - \rho_2). \tag{14.116}$$

We now have a one-dimensional (1-D) Green's function problem for which the homogeneous equation can be identified as the modified Bessel equation, with solutions $I_m(k\rho)$ and $K_m(k\rho)$. Keeping in mind that I_m is regular at the origin, that K_m is regular at infinity, and that the Green's function we seek must be regular at both these limits, we write our 1-D **axial Green's function** in the more explicit form

$$g_m(k\rho_1, k\rho_2) = -I_m(k\rho_<)K_m(k\rho_>), \tag{14.117}$$

where $\rho_<$ and $\rho_>$ are, respectively, the smaller and larger of ρ_1 and ρ_2. The coefficient in the above equation, -1, is evaluated according to Eq. (10.19), from

$$\Big(p(k\rho)\big[K_m'(k\rho) I_m(k\rho) - I_m'(k\rho) K_m(k\rho) \big] \Big)^{-1}.$$

The coefficient p is from the differential equation, and has here the value $k\rho$; the form involving modified Bessel functions is their Wronskian, and has the value $-1/k\rho$; that is the topic of Exercise 14.5.11.

Given our explicit formula for g_m, Eq. (14.115) assumes the final form

$$G(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2\pi^2} \sum_m \int_0^\infty dk g_m(k\rho_1, k\rho_2) e^{im(\varphi_1 - \varphi_2)} \cos k(z_1 - z_2). \qquad (14.118)$$

This is the form quoted in Section 10.2. ■

Summary

To put the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$ in proper perspective, note that we have introduced them here because:

- These functions are solutions of the frequently encountered modified Bessel equation, which arises in a variety of physically important problems,
- $K_\nu(x)$ will be found useful in determining the asymptotic behavior of all the Bessel and modified Bessel functions (Section 14.6), and
- $I_\nu(x)$ and $K_\nu(x)$ arise in our discussion of Green's functions (Example 14.5.1).

Exercises

14.5.1 Show that $e^{(x/2)(t+1/t)} = \sum_{n=-\infty}^{\infty} I_n(x) t^n$, thus generating modified Bessel functions, $I_n(x)$.

14.5.2 Verify the following identities

(a) $1 = I_0(x) + 2\sum_{n=1}^{\infty} (-1)^n I_{2n}(x)$,

(b) $e^x = I_0(x) + 2\sum_{n=1}^{\infty} I_n(x)$,

(c) $e^{-x} = I_0(x) + 2\sum_{n=1}^{\infty} (-1)^n I_n(x)$,

(d) $\cosh x = I_0(x) + 2\sum_{n=1}^{\infty} I_{2n}(x)$,

(e) $\sinh x = 2\sum_{n=1}^{\infty} I_{2n-1}(x)$.

14.5.3 (a) From the generating function of Exercise 14.5.1 show that

$$I_n(x) = \frac{1}{2\pi i} \oint e^{(x/2)(t+1/t)} \frac{dt}{t^{n+1}}.$$

(b) For $n = \nu$, not an integer, show that the preceding integral representation may be generalized to

$$I_\nu(x) = \frac{1}{2\pi i} \int_C e^{(x/2)(t+1/t)} \frac{dt}{t^{\nu+1}}.$$

The contour C is the same as that for $J_\nu(x)$ (Fig. 14.5).

14.5.4 For $\nu > -\frac{1}{2}$ show that $I_\nu(z)$ may be represented by

$$\begin{aligned} I_\nu(z) &= \frac{1}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})} \left(\frac{z}{2}\right)^\nu \int_0^\pi e^{\pm z\cos\theta} \sin^{2\nu}\theta \, d\theta \\ &= \frac{1}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})} \left(\frac{z}{2}\right)^\nu \int_{-1}^{1} e^{\pm zp}(1-p^2)^{\nu-1/2}\, dp \\ &= \frac{2}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})} \left(\frac{z}{2}\right)^\nu \int_0^{\pi/2} \cosh(z\cos\theta)\sin^{2\nu}\theta\, d\theta. \end{aligned}$$

14.5.5 The cylindrical cavity depicted in Fig. 14.4 has radius a and height h. For this exercise, the end caps $z = 0$ and h are at zero potential, while the cylindrical wall $\rho = a$ has a potential of functional form $V = V(\varphi, z)$.

(a) Show that the electrostatic potential $\Phi(\rho, \varphi, z)$ has the functional form

$$\Phi(\rho,\varphi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} I_m(k_n\rho)\,(a_{mn}\sin m\varphi + b_{mn}\cos m\varphi)\sin k_n z,$$

where $k_n = n\pi/h$.

(b) Show that the coefficients a_{mn} and b_{mn} are given by

$$\left.\begin{matrix} a_{mn} \\ b_{mn} \end{matrix}\right\} = \frac{2-\delta_{m0}}{\pi l I_m(k_n a)} \int_0^{2\pi}\int_0^{l} V(\varphi,z) \left\{\begin{matrix} \sin m\varphi \\ \cos m\varphi \end{matrix}\right\} \sin k_n z dz d\varphi.$$

Hint. Expand $V(\varphi, z)$ as a double series and use the orthogonality of the trigonometric functions.

14.5.6 Verify that $K_\nu(x)$ as defined in Eq. (14.106) is equivalent to

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin\nu\pi}$$

and from this show that

$$K_\nu(x) = K_{-\nu}(x).$$

14.5.7 Show that $K_\nu(x)$ satisfies the following recurrence relations:

$$K_{\nu-1}(x) - K_{\nu+1}(x) = -\frac{2\nu}{x} K_\nu(x),$$

$$K_{\nu-1}(x) + K_{\nu+1}(x) = -2K'_\nu(x).$$

Note. These differ from the recurrence relations for I_ν.

14.5.8 If $\mathcal{K}_\nu = e^{\nu\pi i} K_\nu$, show that $\mathcal{K}_\nu$ satisfies the same recurrence relations as I_ν.

14.5.9 Show that when K_0 is evaluated from its series expansion about $x = 0$, the formula given as Eq. (14.110) only follows if a specific branch of its logarithmic term is chosen.

14.5.10 For $\nu > -\frac{1}{2}$ show that $K_\nu(z)$ may be represented by

$$K_\nu(z) = \frac{\pi^{1/2}}{\Gamma(\nu + \frac{1}{2})} \left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-z\cosh t} \sinh^{2\nu} t \, dt, \qquad -\frac{\pi}{2} < \arg z < \frac{\pi}{2}$$

$$= \frac{\pi^{1/2}}{\Gamma(\nu + \frac{1}{2})} \left(\frac{z}{2}\right)^\nu \int_1^\infty e^{-zp} (p^2 - 1)^{\nu - 1/2} dp.$$

14.5.11 Show that $I_\nu(x)$ and $K_\nu(x)$ satisfy the Wronskian relation

$$I_\nu(x) K'_\nu(x) - I'_\nu(x) K_\nu(x) = -\frac{1}{x}.$$

14.5.12 Verify that the coefficient in the axial Green's function of Eq. (14.117) is -1.

14.5.13 If $r = (x^2 + y^2)^{1/2}$, prove that

$$\frac{1}{r} = \frac{2}{\pi} \int_0^\infty \cos(xt) K_0(yt) dt.$$

This is a Fourier cosine transform of K_0.

14.5.14 Derive the integral representation

$$I_n(x) = \frac{1}{\pi} \int_0^\pi e^{x\cos\theta} \cos(n\theta) d\theta.$$

Hint. Start with the corresponding integral representation of $J_n(x)$. Equation (14.112) is a special case of this representation.

14.5.15 Show that

$$K_0(z) = \int_0^\infty e^{-z\cosh t} dt$$

satisfies the modified Bessel equation. How can you establish that this form is linearly independent of $I_0(z)$?

14.5.16 The cylindrical cavity of Exercise 14.5.5 has along the cylinder walls the potential walls:

$$V(z)=\begin{cases} 100\,\dfrac{z}{h}, & 0\le \dfrac{z}{h}\le 1/2,\\ 100\Big(1-\dfrac{z}{h}\Big), & 1/2\le \dfrac{z}{h}\le 1. \end{cases}$$

With the radius-height ratio $a/h = 0.5$, calculate the potential for $z/h = 0.1(0.1)0.5$ and $\rho/a = 0.0(0.2)1.0$.

Check value. For $z/h = 0.3$ and $\rho/a = 0.8$, $V = 26.396$.

14.6 Asymptotic Expansions

Frequently in physical problems there is a need to know how a given Bessel or modified Bessel function behaves for large values of the argument, that is, its asymptotic behavior. This is one occasion when computers are not very helpful. One possible approach is to develop a power-series solution of the differential equation, but now using negative powers. This is Stokes' method, illustrated in Exercise 14.6.10. The limitation is that starting from some positive value of the argument (for convergence of the series), we do not know what mixture of solutions or multiple of a given solution we have. The problem is to relate the asymptotic series (useful for large values of the variable) to the power-series or related definition (useful for small values of the variable). This relationship can be established is various ways, one of which is to introduce a suitable **integral representation** whose asymptotic behavior can be studied by application of the method of steepest descents, Section 12.7.

We start this process with a study of the Hankel functions, for which a contour integral representation was introduced in Section 14.4.

Asymptotic Forms of Hankel Functions

In Section 14.4 it was shown that the Hankel functions, which satisfy Bessel's equation, may be defined by the contour integrals

$$H_\nu^{(1)}(t)=\frac{1}{\pi i}\int_{C_1} e^{(t/2)(z-1/z)}\,\frac{dz}{z^{\nu+1}}, \tag{14.119}$$

$$H_\nu^{(2)}(t)=\frac{1}{\pi i}\int_{C_2} e^{(t/2)(z-1/z)}\,\frac{dz}{z^{\nu+1}}, \tag{14.120}$$

where C_1 and C_2 are the contours shown in Fig. 14.10. We desire formulas based on these representations for the asymptotic behavior of the Hankel functions at large positive t.

The direct and exact evaluation of these integrals appears to be nearly impossible, but the situation does have features permitting us to use the method of steepest descents to make an asymptotic evaluation. Referring to the exposition of that method in Section 12.7, we have the approximate evaluation

$$\int\limits_C g(z,t)e^{w(z,t)}dz \approx g(z_0,t)e^{w(z_0,t)}e^{i\theta}\sqrt{\frac{2\pi}{|w''(z_0,t)|}}, \tag{14.121}$$

where the contour C passes through a saddle point at $z = z_0$ and

$$\theta = -\frac{\arg(w''(z_0,t))}{2} + \left(\frac{\pi}{2} \text{ or } \frac{3\pi}{2}\right) \tag{14.122}$$

is a phase arising from the direction of passage through the saddle point.

We regard the common integrand of Eqs. (14.119) and (14.120) as possessing a slowly varying factor $g(z) = z^{-\nu-1}$ and an exponential e^w with $w = (t/2)(z - z^{-1})$, and seek saddle points by finding the zeros of

$$w' = \frac{t}{2}\left(1 + \frac{1}{z^2}\right). \tag{14.123}$$

Solving the above equation, we identify the two saddle points $z_0 = +i$ and $z_0 = -i$.

Limiting attention to $H_\nu^{(1)}(t)$, we see that we can deform the contour C_1 so that it passes through the saddle point at $z_0 = i$; there is neither the need nor the possibility to deform this contour to pass through $z_0 = -i$. Thus, at the saddle point, we have

$$w(+i) = it, \quad w''(+i) = -\frac{t}{z_0^3}\bigg|_{z_0=i} = -it. \tag{14.124}$$

The argument of $w''(z_0)$ is $-\pi/2$, so the possible values of the phase θ (the direction of descent from the saddle point) are $3\pi/4$ and $7\pi/4$. We must choose $\theta = 3\pi/4$ since we cannot get into position to cross the saddle point in the direction $\theta = 7\pi/4 = -\pi/4$ without first crossing a region where the integrand is larger in absolute value than its value at the saddle point.

We now have all the information needed to use Eq. (14.121) to estimate the integral. The result is

$$\begin{aligned} H_\nu^{(1)}(t) &\approx \frac{1}{\pi i}e^{(i\pi/2)(-\nu-1)}e^{3i\pi/4}e^{it}\sqrt{\frac{2\pi}{t}} \\ &\approx \sqrt{\frac{2}{\pi t}}e^{i(t-\nu\pi/2-\pi/4)}. \end{aligned} \tag{14.125}$$

This is the leading term of the asymptotic expansion of the Hankel function $H_\nu^{(1)}(t)$ for large t. The other Hankel function can be treated similarly, but using the saddle point at $z = -i$, with result

$$H_\nu^{(2)}(t) \approx \sqrt{\frac{2}{\pi t}}e^{-i(t-\nu\pi/2-\pi/4)}. \tag{14.126}$$

Equations (14.125) and (14.126) permit us to obtain the leading terms in the asymptotic behavior of all the Bessel and modified Bessel functions. In particular, inserting the asymptotic form for $H^{(1)}(ix)$ into Eq. (14.106), which defines $K_\nu(x)$, we find

$$K_\nu(x) \sim \frac{\pi}{2} i^{\nu+1} \sqrt{\frac{2}{i\pi x}} e^{-x} e^{t-\nu\pi/2-\pi/4)},$$
$$\sim \sqrt{\frac{\pi}{2x}} e^{-x}. \tag{14.127}$$

Another solution to the modified Bessel equation can be obtained from $H^{(2)}(ix)$; its asymptotic behavior will be proportional to e^{+x}. Combining the present observations with Eqs. (14.100), (14.110), and (14.111), we can conclude that:

1. The modified Bessel function $K_\nu(x)$ will be irregular at $x = 0$ as given by Eqs. (14.110) or (14.111), and will decay exponentially at large x;
2. The modified Bessel function $I_\nu(x)$ will (for $\nu \geq 0$) be finite at the origin, as given by Eq. (14.100), and will increase exponentially at large x.

Rather than developing additional asymptotic forms from Eq. (14.127), we find it more interesting to obtain more complete asymptotic expansions by use of a particular integral representation of K_ν.

Expansion of an Integral Representation for K_ν

Here we start from the integral representation

$$K_\nu(z) = \frac{\pi^{1/2}}{\Gamma(\nu+\frac{1}{2})} \left(\frac{z}{2}\right)^\nu \int_1^\infty e^{-zx}(x^2-1)^{\nu-1/2} dx, \quad \nu > -\frac{1}{2}. \tag{14.128}$$

For the present let us take z to be real, although Eq. (14.128) may be established for $-\pi/2 < \arg z < \pi/2$ (i.e., for $\Re e(z) > 0$).

Before using Eq. (14.128) we need to verify that (1) the form claimed to be $K_\nu(z)$ satisfies the modified Bessel equation, (2) that it has the small-z behavior required for K_ν, and (3) that it has the required exponentially decaying asymptotic value. These three features suffice to establish the validity of Eq. (14.128).

The fact that Eq. (14.128) is a solution of the modified Bessel equation may be verified by direct substitution into Eq. (14.97). After some manipulation, we obtain

$$z^{\nu+1} \int_1^\infty \frac{d}{dx}\left[e^{-zx}(x^2-1)^{\nu+1/2}\right] dx = 0,$$

which transforms the combined integrand into the derivative of a function that vanishes at both endpoints.

We next consider how Eq. (14.128) behaves for small z. We proceed by substituting $x = 1 + t/z$:

$$\frac{\pi^{1/2}}{\Gamma(\nu+\frac{1}{2})}\left(\frac{z}{2}\right)^{\nu}\int_1^{\infty} e^{-zx}(x^2-1)^{\nu-1/2}dx$$

$$= \frac{\pi^{1/2}}{\Gamma(\nu+\frac{1}{2})}\left(\frac{z}{2}\right)^{\nu} e^{-z}\int_0^{\infty} e^{-t}\left(\frac{t^2}{z^2}+\frac{2t}{z}\right)^{\nu-1/2}\frac{dt}{z}$$

$$= \frac{\pi^{1/2}}{\Gamma(\nu+\frac{1}{2})}\frac{e^{-z}}{2^{\nu}z^{\nu}}\int_0^{\infty} e^{-t}t^{2\nu-1}\left(1+\frac{2z}{t}\right)^{\nu-1/2}dt. \tag{14.129}$$

This substitution has changed the limits of integration to a more convenient range and has isolated the negative exponential dependence e^{-z}. The integral in Eq. (14.129) may now (for $\nu > 0$) be evaluated for $z = 0$ to yield $\Gamma(2\nu)$. Then, using the duplication formula, Eq. (13.27), we have

$$\lim_{z\to 0} K_{\nu}(z) = \frac{\Gamma(\nu)2^{\nu-1}}{z^{\nu}}, \quad \nu > 0. \tag{14.130}$$

Equation (14.130) agrees with Eq. (14.111), showing that Eq. (14.128) has the proper small-z behavior to represent K_{ν}. Note that for $\nu = 0$, Eq. (14.128) diverges logarithmically at $z = 0$ and the verification of its scale requires a different approach, which is the topic of Exercise 14.6.4.

Finally, to complete the identification of Eq. (14.128) with K_{ν}, we need to verify that it decays exponentially at large z. That feature will be a by-product of our main interest here, which is to develop an asymptotic series for $K_{\nu}(z)$. We do so by rewriting Eq. (14.129) as

$$K_{\nu}(z) = \sqrt{\frac{\pi}{2z}}\frac{e^{-z}}{\Gamma(\nu+\frac{1}{2})}\int_0^{\infty} e^{-t}t^{\nu-1/2}\left(1+\frac{t}{2z}\right)^{\nu-1/2}dt. \tag{14.131}$$

We next expand $(1+t/2z)^{\nu-1/2}$ by the binomial theorem and interchange the summation and integration (valid for the asymptotic series we plan to obtain), reaching

$$K_{\nu}(z) = \sqrt{\frac{\pi}{2z}}\frac{e^{-z}}{\Gamma(\nu+\frac{1}{2})}\sum_{r=0}^{\infty}\binom{\nu-\frac{1}{2}}{r}(2z)^{-r}\int_0^{\infty} e^{-t}t^{\nu+r-1/2}dt$$

$$= \sqrt{\frac{\pi}{2z}}e^{-z}\sum_{r=0}^{\infty}\frac{\Gamma(\nu+r+\frac{1}{2})}{r!\,\Gamma(\nu-r+\frac{1}{2})}(2z)^{-r}. \tag{14.132}$$

Equation (14.132) can now be rearranged to

$$K_{\nu}(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}\left[1+\frac{(4\nu^2-1^2)}{1!8z}+\frac{(4\nu^2-1^2)(4\nu^2-3^2)}{2!(8z)^2}+\cdots\right]. \tag{14.133}$$

Equation (14.133) yields the anticipated exponential dependence, confirming that Eq. (14.128) actually represents K_ν.

Although the integral of Eq. (14.128), integrating along the real axis, was convergent only for $-\pi/2 < \arg z < \pi/2$, Eq. (14.133) may be extended to $-3\pi/2 < \arg z < 3\pi/2$. Considered as an infinite series, Eq. (14.133) is actually divergent. However, this series is asymptotic, in the sense that for large enough z, $K_\nu(z)$ may be approximated to any fixed degree of accuracy with a small number of terms. Compare Section 12.6 for a definition and discussion of asymptotic series. The asymptotic character arises because our binomial expansion was valid only for $t < 2z$ but we integrated t out to infinity. The exponential decrease of the integrand has prevented a disaster, but the series is only asymptotic and not convergent. By Table 7.1, $z = \infty$ is an essential singularity of the Bessel (and modified Bessel) equations. Fuchs' theorem does not guarantee a convergent series and we did not get one.

It is convenient to rewrite Eq. (14.133) as

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}\big[P_\nu(iz) + iQ_\nu(iz)\big], \tag{14.134}$$

where

$$P_\nu(z) \sim 1 - \frac{(\mu-1)(\mu-9)}{2!(8z)^2} + \frac{(\mu-1)(\mu-9)(\mu-25)(\mu-49)}{4!(8z)^4} - \cdots, \tag{14.135}$$

$$Q_\nu(z) \sim \frac{\mu-1}{1!(8z)} - \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3} + \cdots, \tag{14.136}$$

and $\mu = 4\nu^2$. It should be noted that although $P_\nu(z)$ of Eq. (14.135) and $Q_\nu(z)$ of Eq. (14.136) have alternating signs, the series for $P_\nu(iz)$ and $Q_\nu(iz)$ in Eq. (14.134) have all positive signs. Finally, note that for z large, P_ν dominates.

Additional Asymptotic Forms

We started our detailed study of asymptotic behavior with K_ν because, with its properties in hand, we can deduce the asymptotic expansions of the other members of the family of Bessel-related functions.

1. Rearranging the definition of K_ν to

$$H_\nu^{(1)}(x) = \frac{2}{\pi} e^{-(i\pi/2)(\nu+1)} K_\nu(-ix), \tag{14.137}$$

we have

$$H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}} \exp\left\{i\left[z - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right]\right\}\big[P_\nu(z) + iQ_\nu(z)\big], \tag{14.138}$$

which although originally derived for real values of $-ix$, can be analytically continued into the larger range $-\pi < \arg z < 2\pi$.

2. The second Hankel function is just (for real arguments) the complex conjugate of the first, and therefore

$$H_\nu^{(2)}(z) = \sqrt{\frac{2}{\pi z}} \exp\left\{-i\left[z - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right]\right\}\big[P_\nu(z) - iQ_\nu(z)\big], \tag{14.139}$$

valid for $-2\pi < \arg z < \pi$.

3. Since $J_\nu(z)$ is the real part of $H_\nu^{(1)}(z)$ for real z,

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \left\{ P_\nu(z) \cos\left[z - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right] - Q_\nu(z) \sin\left[z - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right]\right\}, \tag{14.140}$$

valid for $-\pi < \arg z < \pi$.

4. The Neumann function is the imaginary part of $H_\nu^{(1)}(z)$ for real z, or

$$Y_\nu(z) = \sqrt{\frac{2}{\pi z}} \left\{ P_\nu(z) \sin\left[z - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right] + Q_\nu(z) \cos\left[z - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right]\right\}, \tag{14.141}$$

also valid for $-\pi < \arg z < \pi$.

5. Finally, the modified Bessel function $I_\nu(z)$ is given by

$$I_\nu(z) = i^{-\nu} J_\nu(iz), \tag{14.142}$$

so

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}} \left[P_\nu(iz) - i Q_\nu(iz)\right], \tag{14.143}$$

valid for $-\pi/2 < \arg z < \pi/2$.

Properties of the Asymptotic Forms

Having derived the asymptotic forms of the various Bessel functions, it is opportune to note their essential characteristics. Remembering that in the limit of large z, P_ν approaches unity while $Q_\nu \sim 1/z$, we see that at large z, all the Bessel functions have leading terms with a $1/z^{1/2}$ dependence, multiplied by either a real or complex exponential. The modified functions K_ν and I_ν, respectively, contain decreasing and increasing exponentials, while the ordinary Bessel functions J_ν and Y_ν have leading terms with sinusoidal oscillation (damped by the $z^{-1/2}$ factor). When multiplied by a time factor $e^{\pm i\omega t}$, the Hankel functions can describe incoming and outgoing traveling waves.

Looking at the oscillatory functions J_ν, Y_ν, $H_\nu^{(i)}$ in more detail, we see that exact sinusoidal behavior is only reached in the limit of large z, as for finite z the terms involving Q_ν will to some extent alter the periodicity. The reader may wish to compare the positions of the zeros of J_n in Table 14.1 with those predicted by its leading term, namely the zeros of

$$\cos\left[z - \left(n + \frac{1}{2}\right)\frac{\pi}{2}\right].$$

We see that J_n behaves asymptotically like a phase-shifted cosine function, with the phase shift a function of n. The asymptotic form of Y_n will be that of a sine function, with (for the same n) the same phase shift. This causes the zeros of J_n and Y_n for large z to alternate, as we saw for J_0 and Y_0 in Fig. 14.8.

The asymptotic behavior of the two solutions to a problem described by ordinary or modified Bessel functions may be sufficient to eliminate immediately one of these functions as a solution for a physical problem. This observation may enable us to use the behavior at $z = \infty$ as well as that at $z = 0$ to restrict the functional forms we need to consider.

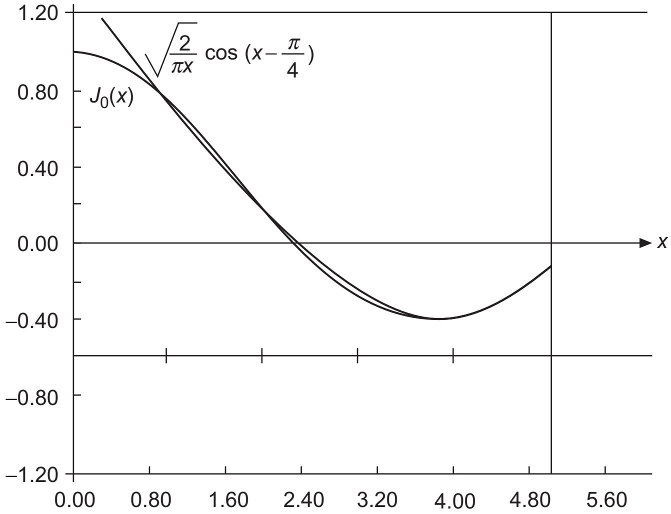

FIGURE 14.13 Asymptotic approximation of $J_0(x)$.

Finally, we note that the asymptotic series $P_\nu(z)$ and $Q_\nu(z)$, Eqs. (14.135) and (14.136), terminate for $\nu = \pm 1/2, \pm 3/2, \ldots$ and become polynomials (in negative powers of z). For these special values of ν the asymptotic approximations become exact solutions.

It is of some interest to consider the accuracy of the asymptotic forms, taking for example just the first term

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left[x - \left(n + \frac{1}{2}\right)\left(\frac{\pi}{2}\right)\right]. \tag{14.144}$$

Clearly, the condition for Eq. (14.144) to be accurate is that the sine term of Eq. (14.140) be negligible; that is,

$$8x \gg 4n^2 - 1. \tag{14.145}$$

In Fig. 14.13 we plot $J_0(x)$ and the leading term of its asymptotic approximation. The agreement is nearly quantitative for $x > 5$. However, for n or $\nu > 1$ the asymptotic region may be far out.

Another use of the asymptotic formulas is to establish the constants in Wronskian formulas, where we know the Wronskian of any two Bessel functions of argument x has a $1/x$ functional dependence but with a premultiplying constant that depends on the Bessel functions involved.

Example 14.6.1 CYLINDRICAL TRAVELING WAVES

As an illustration of a problem in which we have chosen a specific Bessel function because of its asymptotic properties, consider a two-dimensional (2-D) wave problem similar to the vibrating circular membrane of Exercise 14.1.24. Now imagine that the waves are generated at $r = 0$ and move outward to infinity. We replace our standing waves by traveling ones. The differential equation remains the same, but the boundary conditions change. We now demand that for large r the wave behave like

$$U \sim e^{i(kr-\omega t)}, \tag{14.146}$$

to describe an outgoing wave with wavelength $2\pi/k$. We assume, for simplicity, that there is no azimuthal dependence, so we have circular symmetry, implying $m = 0$. The Bessel function of order zero with this asymptotic dependence is $H_0^{(1)}(kr)$, as can be seen from Eq. (14.138). This boundary condition at infinity then determines our wave solution as

$$U(r,t) = H_0^{(1)}(kr)e^{-i\omega t}. \tag{14.147}$$

This solution diverges as $r \to 0$, which is the behavior to be expected with a source at the origin. ■

Exercises

14.6.1 Determine the asymptotic dependence of the modified Bessel functions $I_\nu(x)$, given

$$I_\nu(x) = \frac{1}{2\pi i}\int_C e^{(x/2)(t+1/t)}\frac{dt}{t^{\nu+1}}.$$

The contour starts and ends at $t = -\infty$, encircling the origin in a positive sense. There are two saddle points. Only the one at $z = +1$ contributes significantly to the asymptotic form.

14.6.2 Determine the asymptotic dependence of the modified Bessel function of the second kind, $K_\nu(x)$, by using

$$K_\nu(x) = \frac{1}{2}\int_0^\infty e^{(-x/2)(s+1/s)}\frac{ds}{s^{1-\nu}}.$$

14.6.3 Verify that the integral representations

$$I_n(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos t}\cos(nt)dt,$$

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh(\nu t)dt, \quad \Re e(z) > 0,$$

satisfy the modified Bessel equation by direct substitution into that equation. How can you check the normalization?

14.6.4 (a) Show that when K_ν is defined by Eq. (14.128),

$$\frac{dK_0(z)}{dz} = -K_1(z).$$

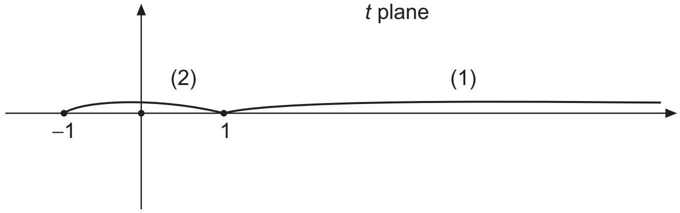

FIGURE 14.14 Modified Bessel function contours.

(b) Show that the indefinite integral of $-K_1(x)$ as defined by Eq. (14.128) has in the limit of small z the value $-\ln z + C$, and therefore, by comparison with Eq. (14.110), that K_0 as defined by Eq. (14.128) has the correct normalization.

14.6.5 Verify that Eq. (14.132) can be rearranged to the form given as Eq. (14.133).

14.6.6 (a) Show that

$$y(z) = z^{\nu} \int e^{-zt}(t^2 - 1)^{\nu - 1/2} dt$$

satisfies the modified Bessel equation, provided the contour is chosen so that

$$e^{-zt}(t^2 - 1)^{\nu + 1/2}$$

has the same value at the initial and final points of the contour.

(b) Verify that the contours shown in Fig. 14.14 are suitable for this problem.

14.6.7 Use the asymptotic expansions to verify the following Wronskian formulas:

(a) $J_\nu(x)J_{-\nu-1}(x) + J_{-\nu}(x)J_{\nu+1}(x) = -2\sin\nu\pi/\pi x$,

(b) $J_\nu(x)N_{\nu+1}(x) - J_{\nu+1}(x)N_\nu(x) = -2/\pi x$,

(c) $J_\nu(x)H^{(2)}_{\nu-1}(x) - J_{\nu-1}(x)H^{(2)}_\nu(x) = 2/i\pi x$,

(d) $I_\nu(x)K'_\nu(x) - I'_\nu(x)K_\nu(x) = -1/x$,

(e) $I_\nu(x)K_{\nu+1}(x) + I_{\nu+1}(x)K_\nu(x) = 1/x$.

14.6.8 Verify that the Green's function for the 2-D Helmholtz equation (operator $\nabla^2 + k^2$) with outgoing-wave boundary conditions is

$$G(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \frac{i}{4} H_0^{(1)}(k|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|).$$

Hint. $H_0^{(1)}(k\rho)$ is known to be an outgoing-wave solution to the homogeneous Helmholtz equation.

14.6.9 From the asymptotic form of $K_\nu(z)$, Eq. (14.134), derive the asymptotic form of $H_\nu^{(1)}(z)$, Eq. (14.138). Note particularly the phase, $(\nu + \frac{1}{2})\pi/2$.

14.6.10 Apply Stokes' method for obtaining an asymptotic expansion for the Hankel function $H_\nu^{(1)}$ as follows:

(a) Replace the Bessel function in Bessel's equation by $x^{-1/2}y(x)$ and show that $y(x)$ satisfies

$$y''(x) + \left(1 - \frac{\nu^2 - \frac{1}{4}}{x^2}\right) y(x) = 0.$$

(b) Develop a power-series solution with negative powers of x starting with the assumed form

$$y(x) = e^{ix} \sum_{n=0}^{\infty} a_n x^{-n}.$$

Obtain the recurrence relation giving a_{n+1} in terms of a_n. Check your result against the asymptotic series, Eq. (14.138).

(c) From Eq. (14.125), determine the initial coefficient, a_0.

14.6.11 Using the method of steepest descents, evaluate the second Hankel function given by

$$H_\nu^{(2)}(t) = \frac{1}{\pi i} \int_{C_2} e^{(t/2)(z-1/z)} \frac{dz}{z^{\nu+1}},$$

with contour C_2 as shown in Fig. 14.10.

ANS. $H_\nu^{(2)}(t) \approx \sqrt{\frac{2}{\pi t}}\, e^{-i(t-\pi/4-\nu\pi/2)}.$

14.6.12 (a) In applying the method of steepest descents to the Hankel function $H_\nu^{(1)}(t)$, show that $w(z,t)$, which appears in Eq. (14.121), satisfies

$$\Re e[w(z,t)] < \Re e[w(z_0,t)] = 0$$

for z on the contour C_1 (Fig. 14.10) but away from the point $z = z_0 = i$.

(b) For general values of $z = re^{i\theta}$, show that

$$\Re e[w(z,t)] > 0 \quad \text{for} \quad 0 < r < 1, \quad \begin{cases} \dfrac{\pi}{2} < \theta \le \pi \\ -\pi \le \theta < \dfrac{\pi}{2} \end{cases}$$

and

$$\Re e[w(z,t)] < 0 \quad \text{for} \quad r > 1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Your demonstration verifies that the distribution of the sign of w is as shown schematically in Fig. 14.15.

(c) Explain why the contour C_1 (Fig. 14.10) cannot be deformed to go through both saddle points, and why it may not go through the saddle point at $-i$ if it is to end at $z = -\infty$ with argument $+\pi$.

14.6.13 Calculate the first 15 partial sums of $P_0(x)$ and $Q_0(x)$, Eqs. (14.135) and (14.136). Let x vary from 4 to 10 in unit steps. Determine the number of terms to be retained for maximum accuracy and the accuracy achieved as a function of x. Specifically, how small may x be without raising the error above 3×10^{-6}?

ANS. $x_{\min} = 6.$

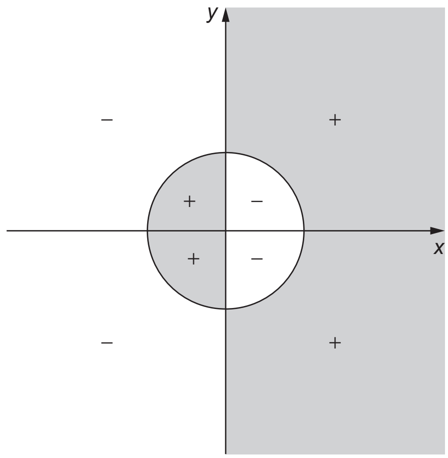

FIGURE 14.15 Sign of $w(z,t)$, occurring in Eq. (14.121), for integral representation of Hankel functions.

14.7 SPHERICAL BESSEL FUNCTIONS

In Section 9.4 we discussed the separation of the Helmholtz equation in spherical coordinates. We showed there that in the oft-occurring case that the boundary conditions of the problem have spherical symmetry, the radial equation has the form given in Eq. (9.80), namely,

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}+\left[k^2r^2-l(l+1)\right]R=0. \tag{14.148}$$

We remind the reader that the parameter k is that from the original Helmholtz equation, while $l(l+1)$ is the separation constant associated with solutions of the angular equations identified by the index l (which is required by the boundary conditions to be an integer).

In Section 9.4 we went on to discuss the fact that the substitution

$$R(kr)=\frac{Z(kr)}{(kr)^{1/2}} \tag{14.149}$$

permits us to rewrite Eq. (14.148) as

$$r^2\frac{d^2Z}{dr^2}+r\frac{dZ}{dr}+\left[k^2r^2-\left(l+\frac{1}{2}\right)^2\right]Z=0, \tag{14.150}$$

which we identified in Eq. (9.84) as Bessel's equation of order $l+\frac{1}{2}$.

We can now identify the general solution $Z(kr)$ as a linear combination of $J_{l+1/2}(kr)$ and $Y_{l+1/2}(kr)$, which in turn means that we can write $R(kr)$ in terms of these Bessel functions of half-integral order, illustrated (for $J_{l+1/2}$) by

$$R(kr)=\frac{C}{\sqrt{kr}}J_{l+1/2}(kr).$$

Since the $R(kr)$ describe radial functions in spherical coordinates, they are termed **spherical Bessel functions**. Note also that since Eq. (14.148) is homogeneous, we are free to define our spherical Bessel functions at any scale; the scale ordinarily used is that introduced in the next subsection.

Definitions

We define our spherical Bessel functions by the following equations. It is not ordinarily useful to introduce spherical Bessel functions with indices that are not integers, so we assume the index n to be integral (but not necessarily nonnegative).

$$\begin{aligned} j_n(x) &= \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x), \\ y_n(x) &= \sqrt{\frac{\pi}{2x}} Y_{n+1/2}(x), \\ h_n^{(1)}(x) &= \sqrt{\frac{\pi}{2x}} H_{n+1/2}^{(1)}(x) = j_n(x) + i y_n(x), \\ h_n^{(2)}(x) &= \sqrt{\frac{\pi}{2x}} H_{n+1/2}^{(2)}(x) = j_n(x) - i y_n(x). \end{aligned} \tag{14.151}$$

Referring to the definition of $Y_{n+1/2}$, we see that

$$Y_{n+1/2}(x) = \frac{\cos(n+\frac{1}{2})\pi\, J_{n+1/2}(x) - J_{-n-1/2}(x)}{\sin(n+\frac{1}{2})\pi} = (-1)^{n+1} J_{-n-\frac{1}{2}}(x),$$

which means that

$$y_n(x) = (-1)^{n+1} j_{-n-1}(x). \tag{14.152}$$

These spherical Bessel functions (Figs. 14.16 and 14.17) can be expressed in series form. Using Eq. (14.6), we have initially

$$j_n(x) = \sqrt{\frac{\pi}{2x}} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(s+n+\frac{3}{2})} \left(\frac{x}{2}\right)^{2s+n+1/2}. \tag{14.153}$$

Writing

$$\Gamma(s+n+\tfrac{3}{2}) = \Gamma(n+\tfrac{3}{2})(n+\tfrac{3}{2})_s, \tag{14.154}$$

where $(..)_s$ is a Pochhammer symbol, defined in Eq. (1.72), we can bring Eq. (14.153) to the form

$$\begin{aligned} j_n(x) &= \sqrt{\frac{\pi}{2x}} \left(\frac{x}{2}\right)^{n+1/2} \frac{1}{\Gamma(n+\frac{3}{2})} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+\frac{3}{2})_s} \left(\frac{x}{2}\right)^{2s} \\ &= \frac{x^n}{(2n+1)!!} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+\frac{3}{2})_s} \left(\frac{x}{2}\right)^{2s}. \end{aligned} \tag{14.155}$$

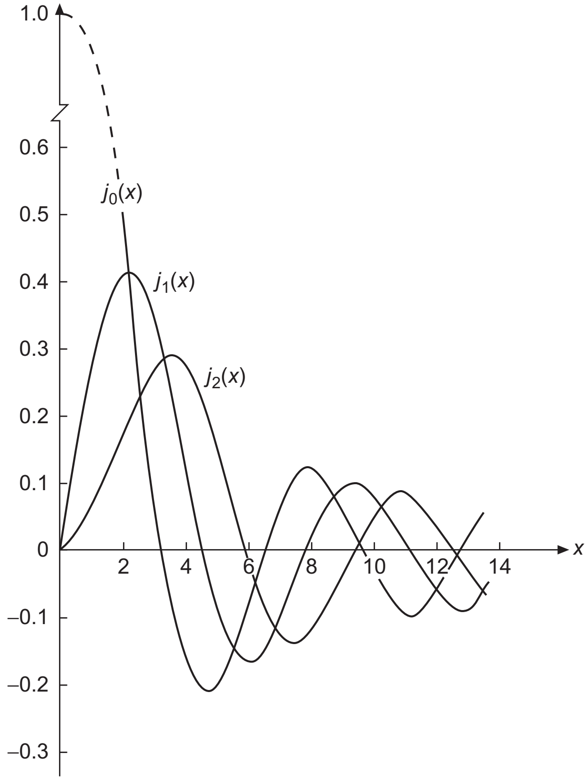

FIGURE 14.16 Spherical Bessel functions.

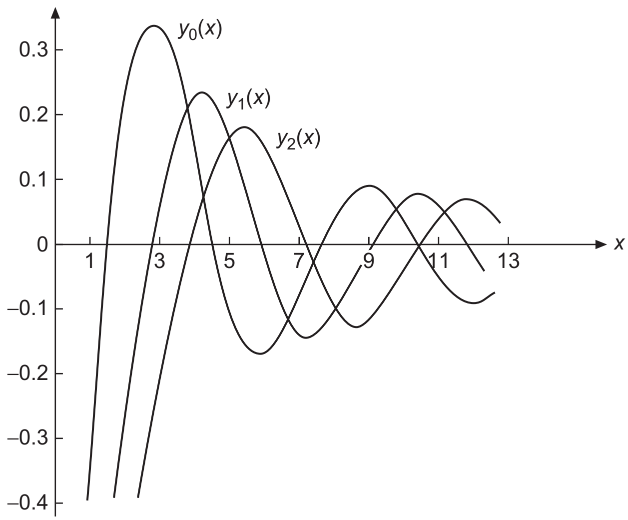

FIGURE 14.17 Spherical Neumann functions.

We reached the last line of Eq. (14.155) by writing $\Gamma(n+\frac{3}{2})$ using the double factorial notation (compare with Exercise 13.1.14).

If we now develop a series expansion for $y_n(x)$ by the same method that was used for $j_n(x)$, but starting from Eq. (14.152), we get

$$y_n(x) = -\frac{(2n-1)!!}{x^{n+1}} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(\frac{1}{2}-n)_s} \left(\frac{x}{2}\right)^{2s}. \tag{14.156}$$

The spherical Bessel functions are oscillatory, as can be seen from the graphs in Figs. 14.16 and 14.17. Note that $j_n(x)$ are regular at $x=0$, with limiting behavior there proportional to x^n. The y_n are all irregular at $x=0$, approaching that point as x^{-n-1}.

The infinite series in Eqs. (14.155) and (14.156) can be evaluated in closed form (but with increasing difficulty as n increases). For the special case $n=0$, we can substitute into Eq. (14.155) $s! = 2^{-s}(2s)!!$ and $(3/2)_s = 2^{-s}(2s+1)!!$, reaching

$$\begin{aligned} j_0(x) &= \sum_{s=0}^{\infty} \frac{(-1)^s 2^{2s}}{(2s)!!(2s+1)!!} \left(\frac{x}{2}\right)^{2s} = \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} x^{2s} \\ &= \frac{\sin x}{x}. \end{aligned} \tag{14.157}$$

A similar treatment of the expansion for y_0 yields

$$y_0(x) = -\frac{\cos x}{x}. \tag{14.158}$$

From the definition of the spherical Hankel functions, Eq. (14.151), we also have

$$h_0^{(1)}(x) = \frac{1}{x}(\sin x - i\cos x) = -\frac{i}{x}e^{ix}, \tag{14.159}$$

$$h_0^{(2)}(x) = \frac{1}{x}(\sin x + i\cos x) = \frac{i}{x}e^{-ix}. \tag{14.160}$$

Since we anticipate the availability of recurrence formulas for the spherical Bessel functions, and since y_0 is just $-j_{-1}$, we expect all the j_n and y_n to be linear combinations of sines and cosines. In fact, the recurrence formulas are good ways of getting these functions for small n. However, we identify here an alternate approach, which depends on the fact, noted in Section 14.6, that the asymptotic expansion for the Hankel functions actually terminates when the order is a half-integer, thereby yielding exact, closed expressions. We start from

$$\begin{aligned} h_n^{(1)}(x) &= \sqrt{\frac{\pi}{2x}} H_{n+1/2}^{(1)}(x) \\ &= (-i)^{n+1} \frac{e^{ix}}{x} \left[P_{n+1/2}(x) + i Q_{n+1/2}(x) \right], \end{aligned} \tag{14.161}$$

where P_ν and Q_ν are given by Eqs. (14.135) and (14.136). Now, $P_{n+1/2}$ and $Q_{n+1/2}$ are **polynomials**, and we can bring Eq. (14.161) to the form

$$\begin{aligned} h_n^{(1)}(x) &= (-i)^{n+1}\frac{e^{ix}}{x}\sum_{s=0}^{n}\frac{i^s}{s!(8x)^s}\frac{(2n+2s)!!}{(2n-2s)!!} \\ &= (-i)^{n+1}\frac{e^{ix}}{x}\sum_{s=0}^{n}\frac{i^s}{s!(2x)^s}\frac{(n+s)!}{(n-s)!}. \end{aligned} \tag{14.162}$$

For real x, $j_n(x)$ is the real part of this, $y_n(x)$ the imaginary part, and $h_n^{(2)}(x)$ the complex conjugate. Specifically,

$$h_1^{(1)}(x) = e^{ix}\left(-\frac{1}{x}-\frac{i}{x^2}\right), \tag{14.163}$$

$$h_2^{(1)}(x) = e^{ix}\left(\frac{i}{x}-\frac{3}{x^2}-\frac{3i}{x^3}\right), \tag{14.164}$$

$$j_1(x) = \frac{\sin x}{x^2}-\frac{\cos x}{x}, \tag{14.165}$$

$$j_2(x) = \left(\frac{3}{x^3}-\frac{1}{x}\right)\sin x-\frac{3}{x^2}\cos x, \tag{14.166}$$

$$y_1(x) = -\frac{\cos x}{x^2}-\frac{\sin x}{x}, \tag{14.167}$$

$$y_2(x) = -\left(\frac{3}{x^3}-\frac{1}{x}\right)\cos x-\frac{3}{x^2}\sin x. \tag{14.168}$$

Recurrence Relations

The recurrence relations to which we now turn provide a convenient way of developing the higher-order spherical Bessel functions. These recurrence relations may be derived from the power-series expansions, but it is easier to substitute into the known recurrence relations, Eqs. (14.8) and (14.9). This gives

$$f_{n-1}(x)+f_{n+1}(x) = \frac{2n+1}{x}f_n(x), \tag{14.169}$$

$$nf_{n-1}(x)-(n+1)f_{n+1}(x) = (2n+1)f_n'(x). \tag{14.170}$$

Rearranging these relations, or substituting into Eqs. (14.10) and (14.11), we obtain

$$\frac{d}{dx}[x^{n+1}f_n(x)] = x^{n+1}f_{n-1}(x), \tag{14.171}$$

$$\frac{d}{dx}[x^{-n}f_n(x)] = -x^{-n}f_{n+1}(x). \tag{14.172}$$

In these equations f_n may represent j_n, y_n, $h_n^{(1)}$, or $h_n^{(2)}$.

By mathematical induction (Section 1.4) we may establish the **Rayleigh formulas**:

$$j_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right), \tag{14.173}$$

$$y_n(x) = -(-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\cos x}{x}\right), \tag{14.174}$$

$$h_n^{(1)}(x) = -i(-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{e^{ix}}{x}\right), \tag{14.175}$$

$$h_n^{(2)}(x) = i(-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{e^{-ix}}{x}\right). \tag{14.176}$$

Limiting Values

For $x \ll 1$,[9] Eqs. (14.155) and (14.156) yield

$$j_n(x) \approx \frac{x^n}{(2n+1)!!}, \tag{14.177}$$

$$y_n(x) \approx -\frac{(2n-1)!!}{x^{n+1}}. \tag{14.178}$$

The limiting values of the spherical Hankel functions for small x go as $\pm i y_n(x)$.

The asymptotic values of j_n, y_n, $h_n^{(1)}$, and $h_n^{(2)}$ may be obtained from the asymptotic forms of the corresponding Bessel functions, as given in Section 14.6. We find

$$j_n(x) \sim \frac{1}{x}\sin\left(x - \frac{n\pi}{2}\right), \tag{14.179}$$

$$y_n(x) \sim -\frac{1}{x}\cos\left(x - \frac{n\pi}{2}\right), \tag{14.180}$$

$$h_n^{(1)}(x) \sim (-i)^{n+1}\frac{e^{ix}}{x} = -i\,\frac{e^{i(x-n\pi/2)}}{x}, \tag{14.181}$$

$$h_n^{(2)}(x) \sim i^{n+1}\frac{e^{-ix}}{x} = i\,\frac{e^{-i(x-n\pi/2)}}{x}. \tag{14.182}$$

The condition for these spherical Bessel forms is that $x \gg n(n+1)/2$. From these asymptotic values we see that $j_n(x)$ and $y_n(x)$ are appropriate for a description of **standing spherical waves**; $h_n^{(1)}(x)$ and $h_n^{(2)}(x)$ correspond to **traveling spherical waves**. If the time dependence for the traveling waves is taken to be $e^{-i\omega t}$, then $h_n^{(1)}(x)$ yields an outgoing traveling spherical wave, and $h_n^{(2)}(x)$ an incoming wave. Radiation theory in electromagnetism and scattering theory in quantum mechanics provide many applications.

[9] The condition that the second term in the series be negligible compared to the first is actually $x \ll 2[(2n+2)(2n+3)/(n+1)]^{1/2}$ for $j_n(x)$.

Orthogonality and Zeros

We may take the orthogonality integral for the ordinary Bessel functions, Eqs. (11.49) and (11.50),

$$\int_0^a J_\nu\left(\alpha_{\nu p}\frac{\rho}{a}\right) J_\nu\left(\alpha_{\nu q}\frac{\rho}{a}\right)\rho\, d\rho = \frac{a^2}{2}\left[J_{\nu+1}(\alpha_{\nu p})\right]^2\delta_{pq},$$

and rewrite it in terms of j_n to obtain

$$\int_0^a j_n\left(\alpha_{np}\frac{r}{a}\right) j_n\left(\alpha_{nq}\frac{r}{a}\right) r^2\, dr = \frac{a^3}{2}\left[j_{n+1}(\alpha_{np})\right]^2\delta_{pq}. \tag{14.183}$$

Here α_{np} is the p-th positive zero of j_n.

Note that in contrast to the formula for the orthogonality of the J_ν, Eq. (14.183) has the weight factor r^2, not r. This of course comes from the factors $x^{-1/2}$ in the definition of $j_n(x)$, but also has the effect that if the integration is construed as being over a **spherical volume** rather than a linear interval, it is the factor corresponding to uniform weight of all volume elements. (Remember that the weight ρ for the J_ν integral produces uniform weight if we construe the integration in that case as over the area within a circle.)

As for the ordinary Bessel functions, the functions that are orthogonal on $(0, a)$ all satisfy a Dirichlet boundary condition, with zeros at $r = a$. We therefore find it useful to know the values of the zeros of the j_n. The first few zeros for small n, and also the locations of the zeros of j_n', are listed in Table 14.2.

The following example illustrates a problem in which the zeros of the j_n play an essential role.

Table 14.2 Zeros of the Spherical Bessel Functions and Their First Derivatives

Number of zero	$j_0(x)$	$j_1(x)$	$j_2(x)$	$j_3(x)$	$j_4(x)$	$j_5(x)$
1	3.1416	4.4934	5.7635	6.9879	8.1826	9.3558
2	6.2832	7.7253	9.0950	10.4171	11.7049	12.9665
3	9.4248	10.9041	12.3229	13.6980	15.0397	16.3547
4	12.5664	14.0662	15.5146	16.9236	18.3013	19.6532
5	15.7080	17.2208	18.6890	20.1218	21.5254	22.9046
	$j_0'(x)$	$j_1'(x)$	$j_2'(x)$	$j_3'(x)$	$j_4'(x)$	$j_5'(x)$
1	4.4934	2.0816	3.3421	4.5141	5.6467	6.7565
2	7.7253	5.9404	7.2899	8.5838	9.8404	11.0702
3	10.9041	9.2058	10.6139	11.9727	13.2956	14.5906
4	14.0662	12.4044	13.8461	15.2445	16.6093	17.9472
5	17.2208	15.5792	17.0429	18.4681	19.8624	21.2311

Example 14.7.1 Particle in a Sphere

An illustration of the use of the spherical Bessel functions is provided by the problem of a quantum mechanical particle of mass m in a sphere of radius a. Quantum theory requires that the wave function ψ, describing our particle, satisfy the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi, \tag{14.184}$$

subject to the conditions that (1) $\psi(r)$ is finite for all $0 \le r \le a$, and (2) $\psi(a) = 0$. This corresponds to a square-well potential $V = 0$ for $r \le a$, $V = \infty$ for $r > a$. Here $\hbar$ is Planck's constant divided by 2π. Equation (14.184) with its boundary conditions is an eigenvalue equation; its eigenvalues E are the possible values of the particle's energy.

Let us determine the **minimum** value of the energy for which our wave equation has an acceptable solution. Equation (14.184) is Helmholtz's equation, which after separation of variables leads to the radial equation previously presented as Eq. (14.148):

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[k^2 - \frac{l(l+1)}{r^2}\right]R = 0, \tag{14.185}$$

with

$$k^2 = 2mE/\hbar^2 \tag{14.186}$$

and l (determined from the angular equation) a nonnegative integer. Comparing with Eq. (14.150) and the definitions of the spherical Bessel functions, Eq. (14.151), we see that the general solution to Eq. (14.185) is

$$R = Aj_l(kr) + By_l(kr). \tag{14.187}$$

To satisfy the boundary conditions of the present problem, we must reject the solution y_l because it is singular at $r = 0$, and we must choose k such that $j_l(ka) = 0$. This boundary condition at $r = a$ can be satisfied if

$$k \equiv k_{li} = \frac{\alpha_{li}}{a}, \tag{14.188}$$

where α_{li} is the ith positive zero of j_l. From Eq. (14.186) we see that the smallest E will correspond to the smallest acceptable k, which in turn corresponds to the smallest α_{li}. Thus, scanning Table 14.2, we identify the smallest α_{li} as the first zero of j_0, a result which we would expect after we have learned that the value $l = 0$ is associated with an angular function with no kinetic energy.

We conclude this example by solving Eq. (14.186) for E with k assigned the value $\alpha_{01}/a = \pi/a$[10]:

$$E_{\min} = \frac{\pi^2\hbar^2}{2ma^2} = \frac{h^2}{8ma^2}. \tag{14.189}$$

[10]Most of the entries in Table 14.2 are only accessible numerically, but the zeros of j_0 are readily identified due to their simple form, $\alpha_{0m} = m\pi$.

This example illustrates several features common to bound-state problems in quantum mechanics. First, we see that for any finite sphere the particle will have a positive minimum or zero-point energy. Second, we note that the particle cannot have a continuous range of energy values; the energy is restricted to discrete values corresponding to the eigenvalues of the Schrödinger equation. Third, the possible energies in this spherically symmetric problem depend on l; as is evident from the table of zeros of j_l, the minimum energy for a given l increases with l. Finally, note that the orthogonality of the j_l under the conditions of this problem shows us that the eigenfunctions corresponding to the same l but different i are orthogonal (with the weight factor corresponding to spherical polar coordinates). ■

We close this subsection with the observation that, in addition to orthogonality with respect to the scaling (to bring zeros to a specified r value), the spherical Bessel functions also possess orthogonality with respect to the indices:

$$\int_{-\infty}^{\infty} j_m(x) j_n(x) dx = 0, \quad m \neq n, \; m, \; n \geq 0. \tag{14.190}$$

The proof is left as Exercise 14.7.12. If $m = n$ (compare Exercise 14.7.13), we have

$$\int_{-\infty}^{\infty} [j_n(x)]^2 \, dx = \frac{\pi}{2n+1}. \tag{14.191}$$

The spherical Bessel functions will enter again in connection with spherical waves, but further consideration is postponed until the corresponding angular functions, the Legendre functions, have been more thoroughly discussed.

Modifed Spherical Bessel Functions

Problems involving the radial equation

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} - \left[k^2 r^2 + l(l+1) \right] R = 0, \tag{14.192}$$

which differs from Eq. (14.148) only in the sign of k^2, also arise frequently in physics. The solutions to this equation are spherical Bessel functions with imaginary arguments, leading us to define **modified spherical Bessel functions** (Fig. 14.18) as follows:

$$i_n(x) = \sqrt{\frac{\pi}{2x}} I_{n+1/2}(x), \tag{14.193}$$

$$k_n(x) = \sqrt{\frac{2}{\pi x}} K_{n+1/2}(x). \tag{14.194}$$

Note that the scale factor in the definition of k_n differs from that of the other spherical Bessel functions.

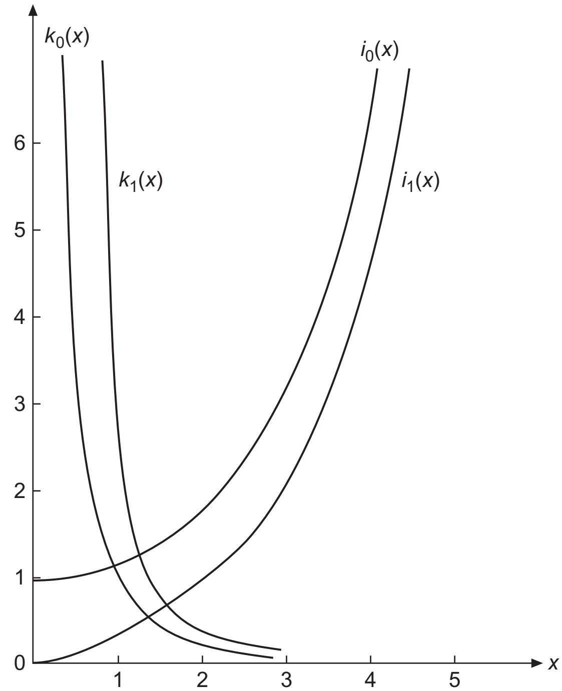

FIGURE 14.18 Modified spherical Bessel functions.

With the above definitions, these functions have the following recurrence relations:

$$\begin{aligned} i_{n-1}(x) - i_{n+1}(x) &= \frac{2n+1}{x} i_n(x), \\ n i_{n-1}(x) + (n+1) i_{n+1}(x) &= (2n+1) i_n'(x), \\ k_{n-1}(x) - k_{n+1}(x) &= -\frac{2n+1}{x} k_n(x), \\ n k_{n-1}(x) + (n+1) k_{n+1}(x) &= -(2n+1) k_n'(x). \end{aligned} \tag{14.195}$$

The first few of these functions are

$$\begin{aligned} i_0(x) &= \frac{\sinh x}{x}, & k_0(x) &= \frac{e^{-x}}{x}, \\ i_1(x) &= \frac{\cosh x}{x} - \frac{\sinh x}{x^2}, & k_1(x) &= e^{-x}\left(\frac{1}{x} + \frac{1}{x^2}\right), \\ i_2(x) &= \sinh x\left(\frac{1}{x} + \frac{3}{x^3}\right) - \frac{3\cosh x}{x^2}, & k_2(x) &= e^{-x}\left(\frac{1}{x} + \frac{3}{x^2} + \frac{3}{x^3}\right). \end{aligned} \tag{14.196}$$

Limiting values of the modified spherical Bessel functions are, for small x,

$$i_n(x) \approx \frac{x^n}{(2n+1)!!}, \quad k_n(x) \approx \frac{(2n-1)!!}{x^{n+1}}. \tag{14.197}$$

For large z, the asymptotic behavior of these functions is

$$i_n(x) \sim \frac{e^x}{2x}, \quad k_n(x) \sim \frac{e^{-x}}{x}. \tag{14.198}$$

Example 14.7.2 Particle in a Finite Spherical Well

As a final example, we return to the problem of a particle trapped in a spherical potential well of radius a (Example 14.7.1), but instead of confining the particle by a wall at potential $V = \infty$ (equivalent to requiring that its wave function ψ vanish at $r = a$), we now consider a well of finite depth, corresponding to

$$V(r) = \begin{cases} V_0 < 0, & 0 \le r \le a, \\ 0, & r > a. \end{cases}$$

If the particle can have an energy $E < 0$, it will be localized in and near the potential well, with a wave function that decays to zero as r increases to values greater than a. A simple case of this problem was one of our examples of an eigenvalue problem (Example 8.3.3), but in that case we did not proceed with enough generality to identify its solutions as Bessel functions.

This problem is governed by the Schrödinger equation, which now has the form

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(r)\psi = E\psi.$$

This is an eigenvalue equation, to be solved for ψ and E over the full three-dimensional space, subject to the condition that ψ be continuous and differentiable for all r, and that it be normalizable (thus approaching zero asymptotically at large r). Here m is the mass of the particle and $\hbar$ is Planck's constant divided by 2π.

While this problem is more difficult than that of Example 14.7.1, it becomes manageable if we realize that it is equivalent to two separate problems for the respective regions $0 \le r \le a$ and $r > a$, within each of which the potential has a constant value, but constrained to (1) have the same eigenvalue E, and (2) connect smoothly (so the r derivative will exist) at $r = a$.

When our Schrödinger equation is processed by the method of separation of variables, we obtain as its radial component

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left(\frac{2m}{\hbar^2}\left[E - V(r)\right] - \frac{l(l+1)}{r^2}\right)R = 0,$$

which is either the spherical Bessel equation, Eq. (14.150), or the modified spherical Bessel equation, Eq. (14.192), depending on the sign of $E - V(r)$. We see that if $V_0 < E < 0$, then for $r \le a$ we will have $E - V(r) > 0$, yielding a Bessel ODE with an acceptable solution involving j_l, while for $r > a$ we have $E - V(r) < 0$, leading to a modified Bessel ODE for which we can choose the k_l solution to obtain the necessary asymptotic behavior.

Summarizing the above, we have, for the two regions:

$$R_{\text{in}}(r) = A j_l(kr), \quad k^2 = \frac{2m}{\hbar^2}(E - V_0) \quad r \le a,$$

$$R_{\text{out}}(r) = B k_l(k'r), \quad k'^2 = -\frac{2m}{\hbar^2}E \quad r > a.$$

Smooth connection at $r = a$ then corresponds to the equations

$$R_{\text{in}}(a) = R_{\text{out}}(a) \quad \longrightarrow \quad A j_l(ka) = B k_l(k'r), \tag{14.199}$$

$$\left.\frac{dR_{\text{in}}}{dr}\right|_{r=a} = \left.\frac{dR_{\text{out}}}{dr}\right|_{r=a} \quad \longrightarrow \quad kAj_l'(ka) = k'Bk_l'(k'a). \tag{14.200}$$

For $l = 0$ this problem reduces to that considered in Example 8.3.3, where we indicate a numerical procedure of solving it, but we are now in a position to obtain solutions for all l. ■

Exercises

14.7.1 Show how one can obtain Eq. (14.162) starting from Eq. (14.161).

14.7.2 Show that if

$$y_n(x) = \sqrt{\frac{\pi}{2x}}\, Y_{n+1/2}(x),$$

it automatically equals

$$(-1)^{n+1}\sqrt{\frac{\pi}{2x}}\, J_{-n-1/2}(x).$$

14.7.3 Derive the trigonometric-polynomial forms of $j_n(z)$ and $y_n(z)$[11]:

$$j_n(z) = \frac{1}{z}\sin\left(z - \frac{n\pi}{2}\right)\sum_{s=0}^{[n/2]}\frac{(-1)^s(n+2s)!}{(2s)!(2z)^{2s}(n-2s)!}$$

$$+ \frac{1}{z}\cos\left(z - \frac{n\pi}{2}\right)\sum_{s=0}^{[(n-1)/2]}\frac{(-1)^s(n+2s+1)!}{(2s+1)!(2z)^{2s}(n-2s-1)!},$$

[11] The upper summation limit $[n/2]$ means the largest **integer** that does not exceed $n/2$.

$$y_n(z) = \frac{(-1)^{n+1}}{z} \cos\left(z + \frac{n\pi}{2}\right) \sum_{s=0}^{[n/2]} \frac{(-1)^s (n+2s)!}{(2s)!(2z)^{2s}(n-2s)!}$$

$$+ \frac{(-1)^{n+1}}{z} \sin\left(z + \frac{n\pi}{2}\right) \sum_{s=0}^{[(n-1)/2]} \frac{(-1)^s (n+2s+1)!}{(2s+1)!(2z)^{2s+1}(n-2s-1)!}.$$

14.7.4 Use the integral representation of $J_\nu(x)$,

$$J_\nu(x) = \frac{1}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})} \left(\frac{x}{2}\right)^\nu \int_{-1}^{1} e^{\pm ixp}(1-p^2)^{\nu-1/2} dp,$$

to show that the spherical Bessel functions $j_n(x)$ are expressible in terms of trigonometric functions; that is, for example,

$$j_0(x) = \frac{\sin x}{x}, \quad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}.$$

14.7.5 (a) Derive the recurrence relations

$$f_{n-1}(x) + f_{n+1}(x) = \frac{2n+1}{x} f_n(x),$$

$$n f_{n-1}(x) - (n+1) f_{n+1}(x) = (2n+1) f_n'(x)$$

satisfied by the spherical Bessel functions $j_n(x)$, $y_n(x)$, $h_n^{(1)}(x)$, and $h_n^{(2)}(x)$.

(b) Show, from these two recurrence relations, that the spherical Bessel function $f_n(x)$ satisfies the differential equation

$$x^2 f_n''(x) + 2x f_n'(x) + \left[x^2 - n(n+1)\right] f_n(x) = 0.$$

14.7.6 Prove by mathematical induction (Section 1.4) that

$$j_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right)$$

for n, an arbitrary nonnegative integer.

14.7.7 From the discussion of orthogonality of the spherical Bessel functions, show that a Wronskian relation for $j_n(x)$ and $n_n(x)$ is

$$j_n(x) y_n'(x) - j_n'(x) y_n(x) = \frac{1}{x^2}.$$

14.7.8 Verify

$$h_n^{(1)}(x) h_n^{(2)\prime}(x) - h_n^{(1)\prime}(x) h_n^{(2)}(x) = -\frac{2i}{x^2}.$$

14.7.9 Verify Poisson's integral representation of the spherical Bessel function,

$$j_n(z) = \frac{z^n}{2^{n+1} n!} \int_0^\pi \cos(z\cos\theta) \sin^{2n+1}\theta \, d\theta.$$

14.7.10 A well-known integral representation for $K_\nu(x)$ has the form

$$K_\nu(x) = \frac{2^\nu \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}\, x^\nu} \int_0^\infty \frac{\cos xt}{(t^2+1)^{\nu+1/2}} dt.$$

Starting from this formula, show that

$$k_n(x) = \frac{2^{n+2}(n+1)!}{\pi x^{n+1}} \int_0^\infty \frac{k^2 j_0(kx)}{(k^2+1)^{n+2}} dk.$$

14.7.11 Show that $\displaystyle\int_0^\infty J_\mu(x) J_\nu(x) \frac{dx}{x} = \frac{2}{\pi} \frac{\sin[(\mu-\nu)\pi/2]}{\mu^2-\nu^2}, \quad \mu+\nu > 0.$

14.7.12 Derive Eq. (14.190): $\displaystyle\int_{-\infty}^\infty j_m(x) j_n(x) dx = 0, \quad \begin{cases} m \neq n, \\ m, n \geq 0. \end{cases}$

14.7.13 Derive Eq. (14.191): $\displaystyle\int_{-\infty}^\infty \left[j_n(x) \right]^2 dx = \frac{\pi}{2n+1}.$

14.7.14 The Fresnel integrals (Fig. 14.19 and Exercise 12.7.2) occurring in diffraction theory are given by

$$x(t) = \sqrt{\frac{\pi}{2}}\, C\left(\sqrt{\frac{\pi}{2}}\, t\right) = \int_0^t \cos(v^2) dv,$$

$$y(t) = \sqrt{\frac{\pi}{2}}\, s\left(\sqrt{\frac{\pi}{2}}\, t\right) = \int_0^t \sin(v^2) dv.$$

Show that these integrals may be expanded in series of spherical Bessel functions as follows:

$$x(s) = \frac{1}{2} \int_0^s j_{-1}(u) u^{1/2} du = s^{1/2} \sum_{n=0}^\infty j_{2n}(s),$$

$$y(s) = \frac{1}{2} \int_0^s j_0(u) u^{1/2} x du = s^{1/2} \sum_{n=0}^\infty j_{2n+1}(s).$$

Hint. To establish the equality of the integral and the sum, you may wish to work with their derivatives. The spherical Bessel analogs of Eqs. (14.8) and (14.12) may be helpful.

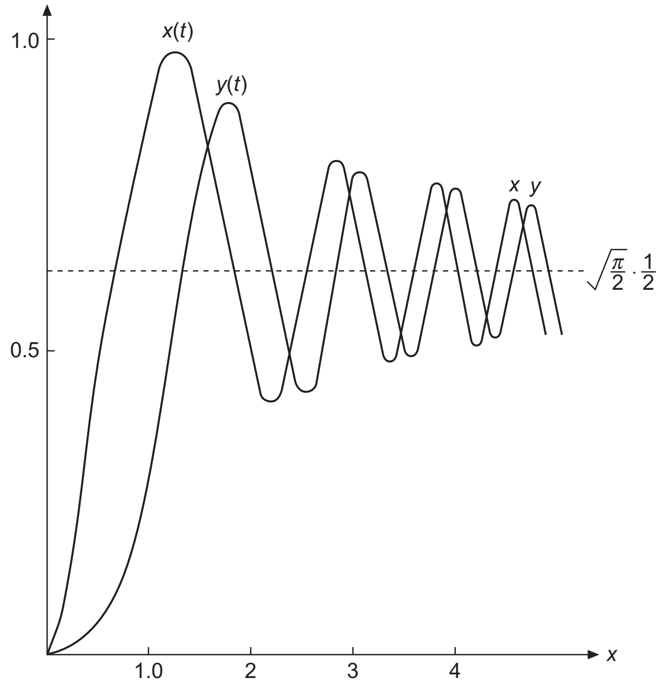

FIGURE 14.19 Fresnel integrals.

14.7.15 A hollow sphere of radius a (Helmholtz resonator) contains standing sound waves. Find the minimum frequency of oscillation in terms of the radius a and the velocity of sound v. The sound waves satisfy the wave equation

$$\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$$

and the boundary condition $\dfrac{\partial \psi}{\partial r} = 0, \quad r = a.$

The spatial part of this PDE is the same as the PDE discussed in Example 14.7.1, but here we have a Neumann boundary condition, in contrast to the Dirichlet boundary condition of that example.

ANS. $\nu_{\min} = 0.3313 v/a, \quad \lambda_{\max} = 3.018a.$

14.7.16 (a) Show that the parity of $i_n(x)$ (the behavior under $x \to -x$) is $(-1)^n$.

(b) Show that $k_n(x)$ has no definite parity.

14.7.17 Show that the Wronskian of the spherical modified Bessel functions is given by

$$i_n(x)k_n'(x) - i_n'(x)k_n(x) = -\frac{1}{x^2}.$$

Additional Readings

Abramowitz, M., and I. A. Stegun, eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (AMS-55). Washington, DC: National Bureau of Standards (1972), reprinted, Dover (1974).

Jackson, J. D., *Classical Electrodynamics*, 3rd ed. New York: Wiley (1999).

Morse, P. M., and H. Feshbach, *Methods of Theoretical Physics*, 2 vols. New York: McGraw-Hill (1953). This work presents the mathematics of much of theoretical physics in detail but at a rather advanced level.

Watson, G. N., *A Treatise on the Theory of Bessel Functions*, 1st ed. Cambridge: Cambridge University Press (1922).

Watson, G. N., *A Treatise on the Theory of Bessel Functions*, 2nd ed. Cambridge: Cambridge University Press (1952). This is the definitive text on Bessel functions and their properties. Although difficult reading, it is invaluable as the ultimate reference.

Whittaker, E. T., and G. N. Watson, *A Course of Modern Analysis*, 4th ed. Cambridge: Cambridge University Press (1962), paperback.