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A characterization of Lorentz boosts

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Summary. Suppose that X is a real inner product space of (finite or infinite) dimension at least 2. The following result will be proved in this note. A bijection $\lambda \neq \mathrm{id}$ of the space-time $Z = X \oplus \mathbb{R}$ is an orthochronous Lorentz boost if, and only if,

(i) There exists $e \neq 0$ in X and $\tau : X \to \mathbb{R} \setminus \{0\}$ with

$$\lambda\left(x, \sqrt{1+x^2}\right) = \left(x + \tau(x)e, \sqrt{1 + (x+\tau(x)e)^2}\right)$$

for all $x \in X$, and

(ii) $l(v,w) = 0$ implies $l\left(\lambda(v), \lambda(w)\right) = 0$ for all $v, w \in Z$ where $l(z_1, z_2)$ designates the Lorentz–Minkowski distance of $z_1, z_2 \in Z$.

Moreover, we characterize (general) Lorentz boosts by distance invariance and the behavior on certain subspaces of Z.

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1. Introduction

Let X be a (finite- or infinite-dimensional) real inner product space, i.e., a real vector space equipped with an inner product

$$\sigma : X \times X \to \mathbb{R}, \sigma(x,y) =: xy,$$

satisfying $xy = yx$, $x(y+z) = xy + xz$, $\alpha(xy) = (\alpha x)y$ for all $x, y, z \in X$, $\alpha \in \mathbb{R}$, and moreover, $x^2 = xx > 0$ for all $x \neq 0$ in X. We assume that $\dim X \geq 2$. Define the vector space $Z = X \oplus \mathbb{R}$ consisting of all (x, γ) with $x \in X$ and $\gamma \in \mathbb{R}$. If $y = (\overline{y}, y_0)$, $z = (\overline{z}, z_0)$ are elements of Z, put

$$yz := \overline{y}\,\overline{z} - y_0 z_0, \tag{1}$$

and observe $z_1 z_2 = z_2 z_1$, $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$, $\alpha(z_1 z_2) = (\alpha z_1) z_2$ for all $z_1, z_2, z_3 \in Z$ and $\alpha \in \mathbb{R}$. The *Lorentz–Minkowski distance* of $y, z \in Z$ is defined

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by

$$l(y,z) = (y-z)^2 = (\overline{y}-\overline{z})^2 - (y_0 - z_0)^2. \tag{2}$$

The mapping $\lambda : Z \to Z$ is called a *Lorentz transformation* of Z if, and only if,

$$l(y,z) = l\left(\lambda(y), \lambda(z)\right)$$

holds true for all $y, z \in Z$. Special Lorentz transformations are the so-called *Lorentz boosts*. Suppose that $p \in X$ satisfies $p^2 < 1$, and $k \in \mathbb{R}$ the equation $k^2(1-p^2) = 1$. Define $A_p(z) := (z_0 p, \overline{z}p)$ and

$$\begin{aligned} B_{p,k}(z) &= z + kA_p(z) + \frac{k^2}{k+1} A_p^2(z) \\ &= \left(\overline{z} + \left(kz_0 + \frac{k^2}{k+1}\overline{z}p\right)p, k(z_0 + \overline{z}p)\right) \end{aligned} \tag{3}$$

for $k \neq -1$ and $z = (\overline{z}, z_0) \in Z$. Moreover, put $B_{0,-1}(z) := (\overline{z}, -z_0)$. The Lorentz boosts

$$z \mapsto B_{p,k}(z)$$

are bijective Lorentz transformations of Z, they are linear and they satisfy

$$B_{p,k} \cdot B_{-p,k} = \mathrm{id} \tag{4}$$

with $\mathrm{id}\,(z) := z$ for all $z \in Z$. The boost $B_{p,k}$ is said to be *orthochronous* or *proper* provided $k > 0$, i.e. $k \geq 1$, since $k^2(1-p^2) = 1$. If $k < 0$, i.e. $k \leq -1$, $B_{p,k}$ is called *improper*. All Lorentz transformations λ of Z are given by

$$\lambda(z) = B_{p,k}\left(\omega\left(\overline{z}\right), z_0\right) + \lambda(0) \tag{5}$$

for all $z = (\overline{z}, z_0) \in Z$ where $B_{p,k}$ is a Lorentz boost and $\omega : X \to X$ a linear and orthogonal transformation of X. For these and many other informations in our context and our notations, see the book [4].

2. A functional equation

We would like to show that proper Lorentz boosts $\lambda : Z \to Z$, $\lambda \neq \mathrm{id}$, satisfy the following functional equation.

Find all $f : Z \to Z$ such that there exist $e \neq 0$ in X and $\tau : X \to \mathbb{R} \setminus \{0\}$ with

$$f\left(x, \sqrt{1+x^2}\right) = \left(x + \tau(x)e, \sqrt{1 + \left(x+\tau(x)e\right)^2}\right) \tag{6}$$

for all $x \in X$.

In fact! Suppose that $B_{p,k}$ is a Lorentz boost with $k \geq 1$. Observe $p \neq 0$, because otherwise $B_{p,k} = \mathrm{id}$ would hold true, in view of $k = 1$ from $k^2(1-p^2) = 1$. Put $p =: \|p\|e$ and

$$\tau(x) := k\|p\|\sqrt{1+x^2} + (k-1)xe.$$

Hence, by (3) and $k^2p^2 = k^2 - 1$,

$$B_{p,k}\left(x, \sqrt{1+x^2}\right) = \left(x + \tau(x)e, k\sqrt{1+x^2} + k\|p\|xe\right).$$

Observe $\tau(x) \neq 0$ for all $x \in X$, because otherwise

$$k^2p^2(1+x^2) = \left(k\|p\|\sqrt{1+x^2}\right)^2 = ((1-k)xe)^2 \leq (1-k)^2x^2$$

would hold true, in view of the inequality of Cauchy–Schwarz, i.e., by $k^2p^2 = k^2-1$,

$$(k^2-1)(1+x^2) \leq (1-k)^2x^2.$$

But this is a contradiction, on account of $k > 1$. Moreover, applying the inequality of Cauchy–Schwarz again, we get

$$\begin{aligned} A :&= k\sqrt{1+x^2} + k\|p\|xe = k\left(\sqrt{1+x^2} + xp\right) \\ &\geq k\left(\sqrt{1+x^2} - \|p\|\|x\|\right) \geq k\left(\sqrt{1+x^2} - \sqrt{x^2}\right) > 0. \end{aligned}$$

Notice, finally

$$A^2 = 1 + (x + \tau(x)e)^2.$$

Remark. Suppose that $B_{p,k}$, $k > 1$, is a Lorentz boost, and define $f : Z \to Z$ by

$$f(z) := B_{p,k}(z), z = (\overline{z}, z_0),$$

for $z_0 = \sqrt{1+\overline{z}^2}$, and by $f(z) := z$ otherwise. Obviously, f is a bijection of Z, it solves (6), but is not a Lorentz boost. So we need something more than a bijective solution of (6), in order to characterize boosts. The further and, moreover, mild requirement that f preserves distance 0 turns out to be sufficient for this purpose.

3. All bijective solutions preserving distance zero

We now are interested in all bijective solutions λ of the functional equation (6) of Section 2 satisfying

$$l(v,w) = 0 \Rightarrow l\left(\lambda(v), \lambda(w)\right) = 0 \tag{7}$$

for all $v, w \in Z$.

Theorem 1. *A bijection $\lambda \neq \mathrm{id}$ of $Z = X \oplus \mathbb{R}$ is an orthochronous Lorentz boost, if (7) holds true for all $v, w \in Z$, and if there exists $e \neq 0$ in X and $\tau : X \to \mathbb{R}\backslash\{0\}$ satisfying*

$$\lambda\left(x, \sqrt{1+x^2}\right) = \left(x + \tau(x)e, \sqrt{1 + (x+\tau(x)e)^2}\right) \tag{8}$$

for all $x \in X$.

Proof. Because of Theorem 2 in Section 4.1 of the book [4], λ must be of the form

$$\lambda(z) = \sigma \cdot B_{p,k}\left(\omega\left(\overline{z}\right), z_0\right) + d \tag{9}$$

for all $z = (\overline{z}, z_0)$ in Z where $d \in Z$, $p \in X$, $k \in \mathbb{R}$ with $k^2(1-p^2) = 1$, $0 \neq \sigma \in \mathbb{R}$, and where $\omega : X \to X$ is supposed to be linear, orthogonal and bijective (see also [5]). Observe $\dim Z \geq 3$, because of $\dim X \geq 2$. Theorem 2 ([4, Section 4.1]) was proved under the stronger assumptions $\dim Z < \infty$ and that λ and λ^{-1} preserve Lorentz–Minkowski distance 0 by A. D. Alexandrov (see [1, 2, 3]), however not precisely in the form (9), but in the form $\lambda = \sigma\lambda'$ with

$$l(v, w) = l\left(\lambda'(v), \lambda'(w)\right)$$

for all $v, w \in Z$.

1) We will show that it is sufficient to assume $\sigma > 0$ in (9).

With $\widehat{\omega} = \omega \circ (-\mathrm{id}\big|_X)$ we obtain

$$-B_{p,k}\left(\omega(\overline{z}), z_0\right) = B_{p,k}\left(\widehat{\omega}\left(\overline{z}\right), -z_0\right) = B_{p,k}\left(B_{0,-1}\left(\widehat{\omega}(\overline{z}), z_0\right)\right).$$

Thus, by Theorem 1 of Section 4.1 in [4],

$$-B_{p,k}\left(\omega(\overline{z}), z_0\right) = B_{p',k'}\left(\omega'\left(\overline{z}\right), z_0\right) + \widehat{d}$$

for all $z \in Z$, where $p'^2 < 1$, $k'^2(1-p'^2) = 1$, $\widehat{d} \in \mathbb{R}$ and where $\omega' : X \to X$ is a linear and orthogonal bijection. Accordingly, for $\sigma < 0$, we get for all $z \in Z$

$$\begin{aligned}\lambda(z) &= \sigma B_{p,k}\big(\omega\left(\overline{z}\right), z_0\big) + d = |\sigma|\Big(B_{p',k'}\left(\omega'\left(\overline{z}\right), z_0\right) + \widehat{d}\Big) + d \\ &= |\sigma| B_{p',k'}\left(\omega'\left(\overline{z}\right), z_0\right) + d',\end{aligned}$$

where $d' = |\sigma|\widehat{d} + d$.

2) k^2 must be $\neq 1$ in (9), and hence $p \neq 0$ because of $k^2(1-p^2) = 1$.

Assume $k^2 = 1$ in (9). Take arbitrarily $j \in X$ with $j^2 = 1$. Hence, by $B_{p,k} = B_{0,k}$ and (8), (9)

$$\begin{aligned}\left(j + \tau(j)e, \sqrt{1 + \left(j + \tau(j)e\right)^2}\right) &= \lambda(j, \sqrt{2}) \\ &= \sigma \cdot \left(\omega(j), k\sqrt{2}\right) + d,\end{aligned}$$

i.e. $j + \tau(j)e = \sigma\omega(j) + \overline{d}$ with $d = (\overline{d}, d_0)$, and

$$1 + \left(\sigma\omega(j) + \overline{d}\right)^2 = \left(k\sigma\sqrt{2} + d_0\right)^2.$$

This equation also holds true for $-j$ instead of j. Hence $\sigma\omega(j)\overline{d} = 0$ for all $j \in X$, $j^2 = 1$. Since ω is linear and bijective, this implies that $\overline{d}x = 0$ for all $x \in X$ and thus $\overline{d} = 0$. Similarly,

$$\left(\tau(0)e, \sqrt{1 + \left(\tau(0)e\right)^2}\right) = \lambda(0, 1) = \sigma \cdot (0, k) + \left(\overline{d}, d_0\right),$$

i.e. with respect to the first components, $\tau(0)e = \overline{d} = 0$. But $\tau(x) \neq 0$ for all $x \in X$.

3) $d = 0$ and $\sigma = 1$.

Take arbitrary $t \in \mathbb{R}$ and $j \in X$ with $j^2 = 1$. With the abbreviations $s := \sinh t$, $c := \cosh t$ and by (8), (9), we obtain

$$\begin{aligned}\left(A(t,j), \sqrt{1+(A(t,j))^2}\right) &= \lambda\left(s\omega^{-1}(j), c\right)\\ &= \sigma \cdot B_{p,k}(sj, c) + d\end{aligned}$$

where we put

$$A(t,j) := s\omega^{-1}(j) + \tau\left(s\omega^{-1}(j)\right)e.$$

Hence, by (3) and $k^2(1-p^2) = 1$, i.e. $(k^2p^2)/(k+1) = k-1$,

$$\begin{aligned}A(t,j) &= \overline{d} + \sigma sj + \sigma ckp + \sigma s(k-1)\frac{jp}{p^2}p,\\ \sqrt{1+(A(t,j))^2} &= d_0 + \sigma skjp + \sigma ck.\end{aligned} \tag{10}$$

Thus we obtain

$$(d_0 + \sigma skjp + \sigma ck)^2 - 1 = \left(\overline{d} + \sigma sj + \sigma ckp + \sigma s(k-1)\frac{jp}{p^2}p\right)^2 \tag{$*$}$$

for all $t \in \mathbb{R}$ and all $j \in X$ satisfying $j^2 = 1$.

Choose, especially, $j \in p^\perp := \{x \in X \mid xp = 0\}$. Then $(*)$ implies

$$(d_0 + \sigma ck)^2 - 1 = \left(\overline{d} + \sigma sj + \sigma ckp\right)^2, \tag{11}$$

a formula which also holds true, if we replace j by $-j$. Hence, by $j \in p^\perp$,

$$0 = (\overline{d} + \sigma ckp)\sigma sj = \sigma s\overline{d}j$$

for all $t \in \mathbb{R}$. Thus $\overline{d}j = 0$ for all $j \in p^\perp$. Now (11) implies

$$d_0^2 + 2\sigma ckd_0 = 1 + \overline{d}^2 - \sigma^2 + 2\sigma ck\overline{d}p$$

for all $t \in \mathbb{R}$, i.e. for all $c \geq 1$. Hence $d_0 = \overline{d}p$ and $d_0^2 = 1 + \overline{d}^2 - \sigma^2$. Observe

$$w := \overline{d} - \frac{\overline{d}p}{p^2}p \in p^\perp \tag{12}$$

and $wj = 0$ for all $j \in p^\perp$, $j^2 = 1$, since $\overline{d}j = 0$. Hence $w = 0$, since otherwise $wj = 0$ for $j = w/\|w\|$. Thus, by (12),

$$\overline{d} = \alpha p, \alpha := \frac{\overline{d}p}{p^2}, d_0 = \overline{d}p = \alpha p^2, \tag{13}$$

and, moreover, by $d_0^2 = 1 + \overline{d}^2 - \sigma^2$,

$$\alpha^2 = \frac{\sigma^2 - 1}{p^2(1-p^2)}. \tag{14}$$

Looking again at formula $(*)$, but now under the restriction $jp \neq 0$, we get with (13),

$$\begin{aligned}&\big((\alpha p^2+\sigma ck)+\sigma skjp\big)^2-1\\&\quad=\left(\left((\alpha+\sigma ck)+\sigma s(k-1)\frac{jp}{p^2}\right)p+\sigma sj\right)^2.\end{aligned}$$

This formula also holds true, if we replace j by $-j$. This yields

$$(\alpha p^2+\sigma ck)\sigma skjp=(\alpha+\sigma ck)\sigma s(k-1)jp+(\alpha+\sigma ck)\sigma spj,$$

i.e. $\alpha p^2\sigma ks=\alpha\sigma sk$, i.e., by $t\neq 0$,

$$\alpha(1-p^2)=0.$$

Hence $\alpha=0$, i.e. $\overline{d}=\alpha p=0$, $d_0=\alpha p^2=0$. Thus $d=0$, and, by (14), $\sigma^2=1$, i.e. $\sigma=1$ since $\sigma>0$.

4) Up till now, we know that (9) has the form

$$\lambda(z)=B_{p,k}\left(\omega(\overline{z}),z_0\right) \tag{15}$$

with $k^2\neq 1$ and $p\neq 0$ (see 2), 3)). We would like to show that λ must be orthochronous. In order to be sure that λ is orthochronous, we must prove $k\geq 1$ (see [4, Theorem 5, Chapter 4]). If we apply (10) for $j\in p^{\perp}$ and $t=0$, we obtain, by observing $d=0$, i.e. $d_0=0$, and $\sigma=1$,

$$k=\sqrt{1+[A(0,j)]^2}\geq 1.$$

Hence from $k\neq 1$,

$$k>1. \tag{16}$$

5) Put $p=:\|p\|\cdot b$, by observing $p\neq 0$, and $k=:\cosh t$ with $t>0$. Note that $t>0$ is uniquely determined by k. Also here we will apply the earlier notation

$$c:=\cosh t, s:=\sinh t.$$

Observe $k^2p^2=k^2-1=\sinh^2 t=s^2$, i.e.

$$\|p\|=\tanh t, p=b\tanh t.$$

From (15) we get

$$\lambda\left(x,\sqrt{1+x^2}\right)=B_{b\tanh t,\cosh t}\left(\omega(x),\sqrt{1+x^2}\right) \tag{17}$$

for all $x\in X$; i.e. $\lambda\left(x,\sqrt{1+x^2}\right)$ is given by

$$\left(\omega(x)+\left(s\sqrt{1+x^2}+(c-1)\omega(x)b\right)b, c\sqrt{1+x^2}+s\omega(x)b\right),$$

in view of (3). Hence, by (8),

$$\begin{aligned}\omega(x)+\left(\omega(x)b(c-1)+\sqrt{1+x^2}s\right)b&=x+\tau(x)e\\ \omega(x)bs+\sqrt{1+x^2}c&=\sqrt{1+(x+\tau(x)e)^2}.\end{aligned} \tag{18}$$

Without loss of generality we may assume $e^2 = 1$, since otherwise we would work with

$$\tau'(x) := \tau(x) \cdot \|e\|, e' := e/\|e\|$$

instead of $\tau(x)$ and e. For $x = 0$ we obtain from (18) that $sb = \tau(0)e$ and that $c = \sqrt{1 + \tau(0)^2}$, i.e.

$$e = b \text{ for } \tau(0) > 0 \text{ and } e = -b \text{ for } \tau(0) < 0.$$

Again, without loss of generality, we may choose a special situation, namely $e = b$, since otherwise we would work with

$$\tau''(x) := -\tau(x), e'' := -e$$

instead of $\tau(x)$ and e. From (18) we get

$$\omega(x) - x = \mu(x) \cdot e \tag{19}$$

for all $x \in X$ with a suitable function $\mu : X \to \mathbb{R}$. If $\mu(x) = 0$ for all $x \in X$, we obtain the solution $\omega = \text{id}$ from (19).

6) There is exactly one linear, orthogonal, bijective solution $\omega \neq \text{id}$ of (19), namely

$$\omega(x) = x - 2(xe)e. \tag{20}$$

Obviously, this ω is linear and orthogonal. $\omega \neq \text{id}$ follows from $\omega(e) = -e$. Since ω is involutorial, it must be bijective. Assume now that $\omega' \neq \text{id}$ is a linear, bijective and orthogonal solution of (19). Then there exists $r \in X$ with $\omega'(r) \neq r$. From

$$x^2 = \omega'(x)\omega'(x) = x^2 + 2\mu(x)xe + \mu^2(x),$$

we get $\mu(x) = 0$ or $\mu(x) = -2(xe)$. By assumption, $\mu(r) \neq 0$. Thus $0 \neq \mu(r) = -2(re)$. Then, for arbitrary $x \in X$, we have

$$xr = \omega'(x)\omega'(r) = (x + \mu(x)e)(r - 2(re)e)$$

which, by $re \neq 0$ implies $\mu(x) = -2(xe)$, i.e. ω' satisfies (20).

7) If we are able to show that $\omega(x) = x$ for all $x \in X$, i.e. that $\omega = \text{id}$, the proof of the theorem will be finished. So assume that ω of (17) (see also (15)) is given by (20),

$$\omega(x) = x - 2(xe)e.$$

Define $x' := e \sinh(t/2)$. Hence, by (18), $b = e$ (see step 5)), (20) and $\omega(x') = -x'$,

$$\begin{aligned} x' + \tau(x')e &= -x' + \left(-x'e(c-1) + \sqrt{1 + x'^2}\, s\right) e \\ &= (-c \sinh(t/2) + s \cosh(t/2))\, e \\ &= e \sinh(t/2) = x' \end{aligned}$$

holds true, i.e. $\tau(x')e = 0$. But $\tau : X \to \mathbb{R} \setminus \{0\}$. Hence ω of (17) has not the form (20). Thus $\omega = \text{id}$, and from (15) it follows that

$$\lambda(z) = B_{p,k}(z)$$

with $k > 1$, in view of (16). □

4. How to find the form of Lorentz boosts

Lorentz boosts play a crucial role in the description of all isometries of Z (see [4, Theorem 61, Chapter 3] and also the results from the previous sections). So one might wonder about the definition of $B_{p,k}$ in (3) which could appear to be rather far from being obvious. In this section we want to find some "natural" conditions ensuring that an isometry $\lambda : Z \to Z$ is of the form (3).

Let $q := (0,1) \in Z$, let $p \in X$ and let $k \in \mathbb{R}$. Looking at the proof of Theorem 61 in [4] one realizes that the following properties of $\lambda = B_{p,k} : Z \to Z$ are used:

i) $\lambda(j) = j$ for all $j \in p^\perp := \{x \in X \mid xp = 0\}$
ii) $\lambda(\mathbb{R}p + \mathbb{R}q) \subseteq \mathbb{R}p + \mathbb{R}q$
iii) λ satisfies $l(y,z) = l(\lambda(y), \lambda(z))$ for all $y, z \in Z$.
iv) $\lambda(q) = k(p+q)$.

$\lambda = B_{p,k}$ also satisfies

v) $\lambda(q-p) \in \mathbb{R}q$.

It is clear that $Z = p^\perp \oplus \mathbb{R}p \oplus \mathbb{R}q$. Thus every $z \in Z$ may be written uniquely as

$$z = \tilde{z} + \rho p + \sigma q, \;\; \tilde{z} \in p^\perp, \rho, \sigma \in \mathbb{R} \tag{21}$$

if $p \neq 0$. Concerning the first three properties we may state the following theorem.

Theorem 2. *A mapping $\lambda : Z \to Z$ satisfies* i)–iii) *if, and only if, $\lambda = B_{0,\pm 1}$ in the case $p = 0$ or if*

$$\lambda(\tilde{z} + \rho p + \sigma q) = \tilde{z} + (\rho\alpha + \sigma\gamma)p + (\rho\beta + \sigma\delta)q$$

for all $\tilde{z} \in p^\perp$, $\rho, \sigma \in \mathbb{R}$ in the case $p \neq 0$, where with arbitrary $\beta \in \mathbb{R}$ and arbitrary $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ the numbers $\alpha, \gamma, \delta \in \mathbb{R}$ are given by

$$\alpha = \varepsilon_1 \sqrt{1 + \beta^2/p^2}, \gamma = \varepsilon_2 \frac{\beta}{p^2}, \delta = \varepsilon_1 \varepsilon_2 \sqrt{1 + \beta^2/p^2}. \tag{22}$$

Proof. In the case $p = 0$ it is clear that $B_{0,\pm 1}$ satisfy i)–iii). Moreover, if, still for $p = 0$, $\lambda : Z \to Z$ satisfies i)–iii), we get, by $0 \in p^\perp = X$ that $\lambda(0) = 0$ and thus by [4, Theorem 61, Chapter 3] that λ has to be linear. Thus

$$\lambda(\tilde{z} + \sigma q) = \tilde{z} + \sigma\alpha q$$

for all $\tilde{z} \in X$, all $\sigma \in \mathbb{R}$ and for some $\alpha \in \mathbb{R}$. Property iii) for $y = \tilde{z} + \sigma q$ with $\sigma \neq 0$ and $\tilde{z} = 0$ yields $\alpha^2 = 1$, i.e. $\lambda = B_{0,\pm 1}$.

Now, let $p \neq 0$, and assume first, that λ satisfies i)–iii). Then we again have $\lambda(0) = 0$. So λ is linear also in this case. Moreover

$$\lambda(\tilde{z} + \rho p + \sigma q) = \tilde{z} + (\rho\alpha + \sigma\gamma)p + (\rho\beta + \sigma\delta)q \tag{23}$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, all $\tilde{z} \in p^\perp$ and all $\rho, \sigma \in \mathbb{R}$. Property iii) together with $\tilde{z}p = 0$ implies

$$(\rho\alpha + \sigma\gamma)^2 p^2 - (\rho\beta + \sigma\delta)^2 = \rho^2 p^2 - \sigma^2$$

for all real numbers ρ and σ. Thus

$$\alpha^2 p^2 - \beta^2 = p^2, \quad \gamma^2 p^2 - \delta^2 = -1 \quad \text{and} \quad \alpha\gamma p^2 - \beta\delta = 0. \tag{24}$$

For $\alpha' := \alpha\sqrt{p^2}$, $\beta' := \beta$, $\gamma' := \gamma\sqrt{p^2}$ and $\delta' := \delta$ this means

$$\alpha'^2 - \beta'^2 = p^2, \quad \gamma'^2 - \delta'^2 = -1 \quad \text{and} \quad \alpha'\gamma' - \beta'\delta' = 0. \tag{25}$$

The first two equations of (25) imply $\alpha'^2 \geq p^2 > 0$, i.e. $\alpha' \neq 0$, and $\delta'^2 = 1+\gamma'^2 \geq 1$, i.e. $|\delta'| \geq 1$ (and $\delta' \neq 0$).

The third equation in (25) means that the vectors (α', β') and (δ', γ') are linearly dependent. Thus with $\kappa := \delta'/\alpha'$, which is well-defined and $\neq 0$,

$$(\delta', \gamma') = \kappa(\alpha', \beta').$$

This, together with the second equation of (25), implies $\kappa^2(\beta'^2 - \alpha'^2) = -1$. Now, from the first equation, we get

$$p^2 = \alpha'^2 - \beta'^2 = 1/\kappa^2,\ \kappa = \pm\frac{1}{\sqrt{p^2}}.$$

Thus $\alpha', \beta', \gamma', \delta'$ satisfying (25) also satisfy

$$\alpha' = \varepsilon_1\sqrt{p^2 + \beta'^2}, \delta' = \varepsilon_2\frac{1}{\sqrt{p^2}}\alpha', \gamma' = \varepsilon_2\frac{1}{\sqrt{p^2}}\beta' \tag{26}$$

where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$. It is obvious that for arbitrary $\beta' \in \mathbb{R}$ and arbitrary $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ the values given by (26) indeed solve (25). Thus, using the connection between α and α' etc., we see that $\alpha, \beta, \gamma, \delta$ satisfy (24) iff there is some $\beta_0 \in \mathbb{R}$ and there are $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ such that

$$\alpha = \varepsilon_1\sqrt{1 + \beta_0^2/p^2}, \beta = \beta_0, \gamma = \varepsilon_2\frac{\beta_0}{p^2}, \delta = \varepsilon_1\varepsilon_2\sqrt{1 + \beta_0^2/p^2}. \tag{27}$$

Thus (22) is fulfilled.

If, on the other hand and still for $p \neq 0$,

$$\lambda(\tilde{z} + \rho p + \sigma q) = \tilde{z} + (\rho\alpha + \sigma\gamma)p + (\rho\beta + \sigma\delta)q$$

and if (22) is satisfied with some real β and some $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$, then it is obvious that properties i)–iii) hold true. □

The proof of Theorem 61 in [4] already uses the properties of the lorentz boosts $B_{p,k}$. A theorem similar to our Theorem 2 may be proved independently of Theorem 61 of [4] when iii) is replaced by

iii') λ is a *linear* isometry, i.e., λ is linear and satisfies $l(0, z) = l\,(0, \lambda(z))$ for all $z \in Z$.

The same remark applies to the following theorem.

Theorem 3. *Given $p \in X$ and $k \in \mathbb{R}$ a mapping $\lambda : Z \to Z$ satisfies conditions* i)–iv) *if, and only if, $k^2(1-p^2) = 1$ (which implies $p^2 < 1$) and $\lambda = B_{p,k}$ or, if $p \neq 0$, $\lambda = B_{p,k} \circ \omega'$, where $\omega'(\overline{z}, z_0) :== (\overline{z} - 2\frac{\overline{z}p}{p^2}p, z_0)$ for all $z = (\overline{z}, z_0) \in X \oplus \mathbb{R}$. $\lambda = B_{p,k}$ holds true if, and only if,* v) *is satisfied, too.*

Proof. Obviously $\lambda = B_{p,k}$ with $k^2(1-p^2) = 1$ satisfies i)–v). If, on the other hand, $\lambda : Z \to Z$ satisfies conditions i)–iv), we have, by Theorem 2, $\lambda = B_{0,\pm 1}$ if $p = 0$. Otherwise we know, also by Theorem 2, that

$$\lambda(\tilde{z} + \rho p + \sigma q) = \tilde{z} + (\rho\alpha + \sigma\gamma)p + (\rho\beta + \sigma\delta)q$$

where for $\beta \in \mathbb{R}$ and $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$

$$\alpha = \varepsilon_1\sqrt{1 + \beta^2/p^2}, \gamma = \varepsilon_2\frac{\beta}{p^2}, \delta = \varepsilon_1\varepsilon_2\sqrt{1 + \beta^2/p^2}.$$

Condition iv) implies $\gamma = \delta = k$. Thus $\varepsilon_2\frac{\beta}{p^2} = \varepsilon_1\varepsilon_2\sqrt{1 + \beta^2/p^2}$ implying $(1 - p^2)\beta^2 = (p^2)^2 > 0$ and therefor $p^2 < 1$. Moreover $\gamma^2 = k^2 = \beta^2/(p^2)^2 = (1-p^2)^{-1}$ or $k = \varepsilon\frac{1}{\sqrt{1-p^2}}$ with some $\varepsilon \in \{\pm 1\}$. So $k^2(1-p^2) = 1$ holds true. Then $k = \delta = \varepsilon_1\varepsilon_2\sqrt{1 + \frac{\beta^2}{p^2}}$ implies $\varepsilon = \varepsilon_1\varepsilon_2$. Accordingly

$$\alpha = \varepsilon_2 k, \beta = \varepsilon_2 p^2 k, \gamma = \delta = k$$

holds true.

Since $\lambda(q - p) = k(p + q) - (\alpha p + \beta q) = (k - \varepsilon_2 k)p + (k - \varepsilon_2 p^2 k)q$ condition v) implies $\varepsilon_2 = 1$.

If condition v) is *not* satisfied $\varepsilon_2 = -1$ holds true.

Note, finally, that for $\varepsilon_2 = 1$

$$\lambda(\tilde{z}) = B_{p,k}(\tilde{z}) = \tilde{z}$$

for all $\tilde{z} \in p^\perp$ and that, using (3),

$$\lambda(p) = (kp, kp^2) = B_{p,k}(p),\ \lambda(q) = k(p+q) = B_{p,k}(q).$$

If $\varepsilon_2 = -1$

$$\lambda(\tilde{z}) = (B_{p,k} \circ \omega')(\tilde{z}) = \tilde{z}$$

for all $\tilde{z} \in p^\perp$ and

$$\lambda(p) = -(kp, kp^2) = (B_{p,k} \circ \omega')(p),\ \lambda(q) = k(p+q) = (B_{p,k} \circ \omega')(q). \qquad \square$$

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