Graphs and Algorithms

Problem set #2
Due date: 17 Khordad 99 (via lms.iut.ac.ir)

Problem 1. Show that a bipartite graph $G[X, Y]$ has an f-factor with $f(x) = 1$ for all $x \in X$ and $f(y) \leq k$ for all $y \in Y$ if and only if $|N(S)| \geq |S|/k$ for all $S \subseteq X$.

Problem 2. An *edge cover* of a graph is a subset F of edges such that for every vertex $v \in V$ there exists an edge of F adjacent to v. Consider a (not necessarily bipartite) connected graph $G = (V, E)$ with $n = |V|$ vertices, and assume that the maximum matching has size $M(G)$. Prove that the minimum cardinality of an edge cover is precisely $n - M(G)$. Using this give an algorithm for detecting the minimum edge cover of the graph. Compute the runtime of the algorithm.

Problem 3. Let M be a perfect matching in a graph G and S a subset of V.

(a) Show that $|M \cap \partial(S)| \equiv |S| \pmod 2$.

(b) Deduce that if M is a perfect matching of the Petersen graph, and C is the edge set of one of its 5-cycles, then $|M \cap C|$ is even.

Problem 4. Consider the following game. Two players alternately pick distinct vertices $v_0, v_1, v_2, \ldots$ of a graph G, where, for $i \geq 0$, v_{i+1} must be adjacent to v_i. The last player able to choose a vertex wins the game. Show that the first player has a winning strategy if and only if the graph G has no perfect matching.

Problem 5. Consider a $m \times n$ checkerboard where m is even, and cells are alternatively colored black and white. Show that if we remove arbitrarily one black cell and one white cell, the resulting $mn - 2$ cells can be covered by dominoes.

Problem 6. Prove that a regular bipartite graph has a perfect matching. (Hint: use Hall's theorem.)

Theorem 7. (a) Let T be an APS-tree rooted at u returned by Edmond's Algorithm (after shrinking the blossoms). Show that every M-alternating path from u in G that terminates in a blue vertex is of odd length.

(b) Deduce that the APS-tree rooted at u returned Edmond's Algorithm contains all vertices of G that can be reached by M-alternating paths from u.

Good Luck.