Sampling theorem

- Impulse invariance
- Sampling of the impulse response of LTI CT systems

$$h[n] = Th_c(nT),$$

$$H(e^{j\omega}) = H_c\left(j\frac{\omega}{T}\right), \qquad |\omega| \le \pi.$$

- Example: LPF

$$\Omega_c = \omega_c/T < \pi/T$$

$$H_c(j\Omega) = \begin{cases} 1, & |\Omega| < \Omega_c, \\ 0, & |\Omega| \le \Omega_c. \end{cases}$$

Sampling theorem

- The impulse response of the CT LPF:

$$h_c(t) = \frac{\sin(\Omega_c t)}{\pi t}$$

- The impulse response of DT system

$$h[n] = Th_c(nT) = T\frac{\sin(\Omega_c nT)}{\pi nT} = \frac{\sin(\omega_c n)}{\pi n}$$

- And,

$$\omega_c = \Omega_c T$$

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

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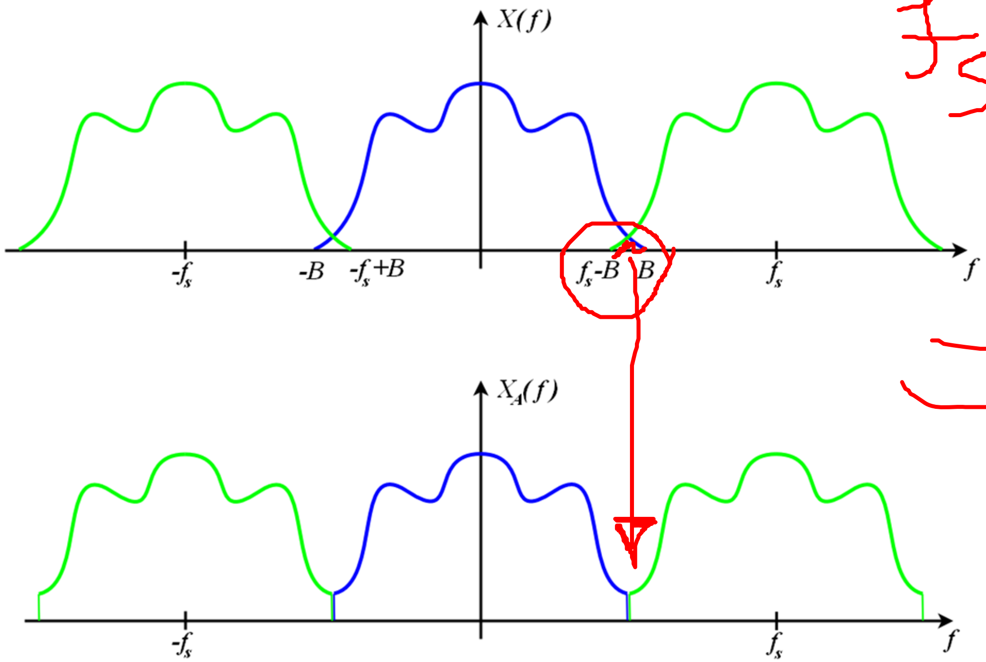
Sampling theorem

- Anti-aliasing filter
  - – Aliased components and Nyquest theorem



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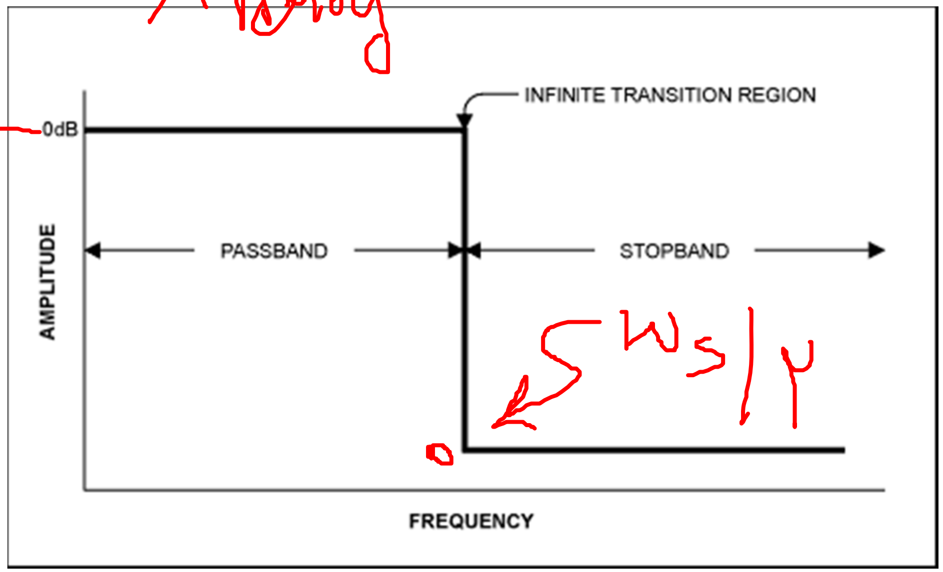
Sampling theorem

- Given a sampling frequency: properly selection of the LPF to ensure “secure” sampling



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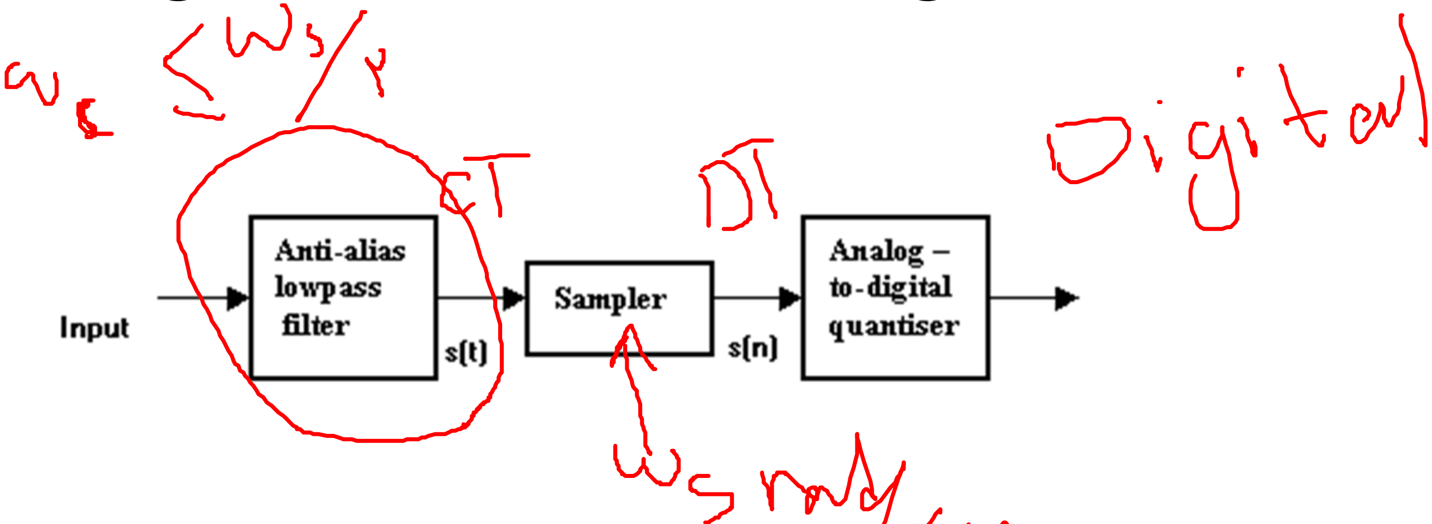
http://www.maximintegrated.com/en/app-notes/index.mvp/id/928

Sampling theorem

- Block diagram with anti-aliasing filter



  – Noise/interference reduction

https://www2.spsc.tugraz.at/www-archive/AdvancedSignalProcessing/SS01-VoiceOverIP/coders/quantization.html

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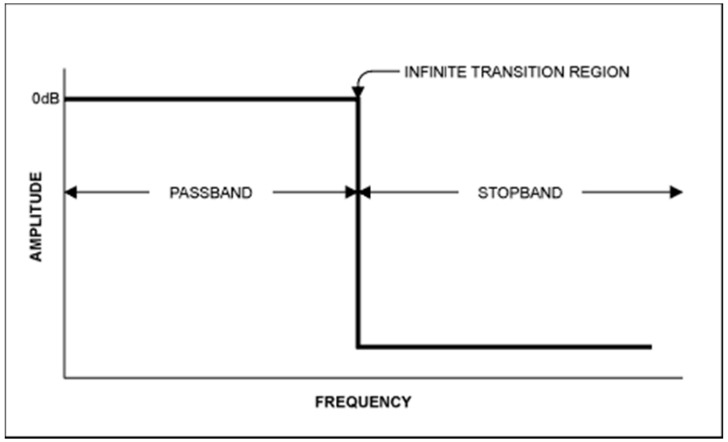
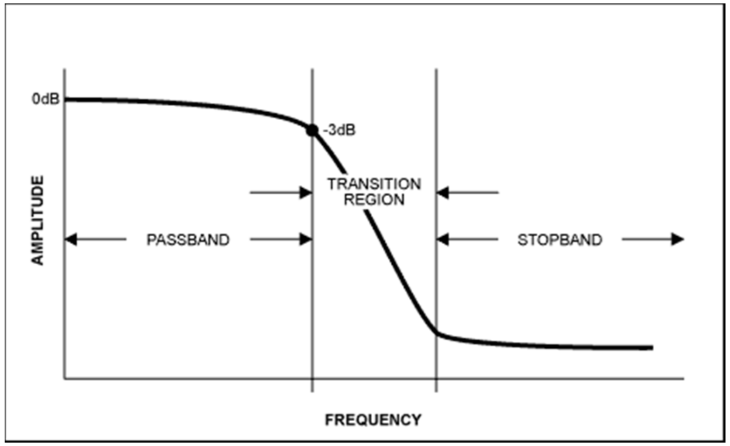
Sampling theorem

- Ideal vs. non-ideal filters
  - Ideal filters are not realizable





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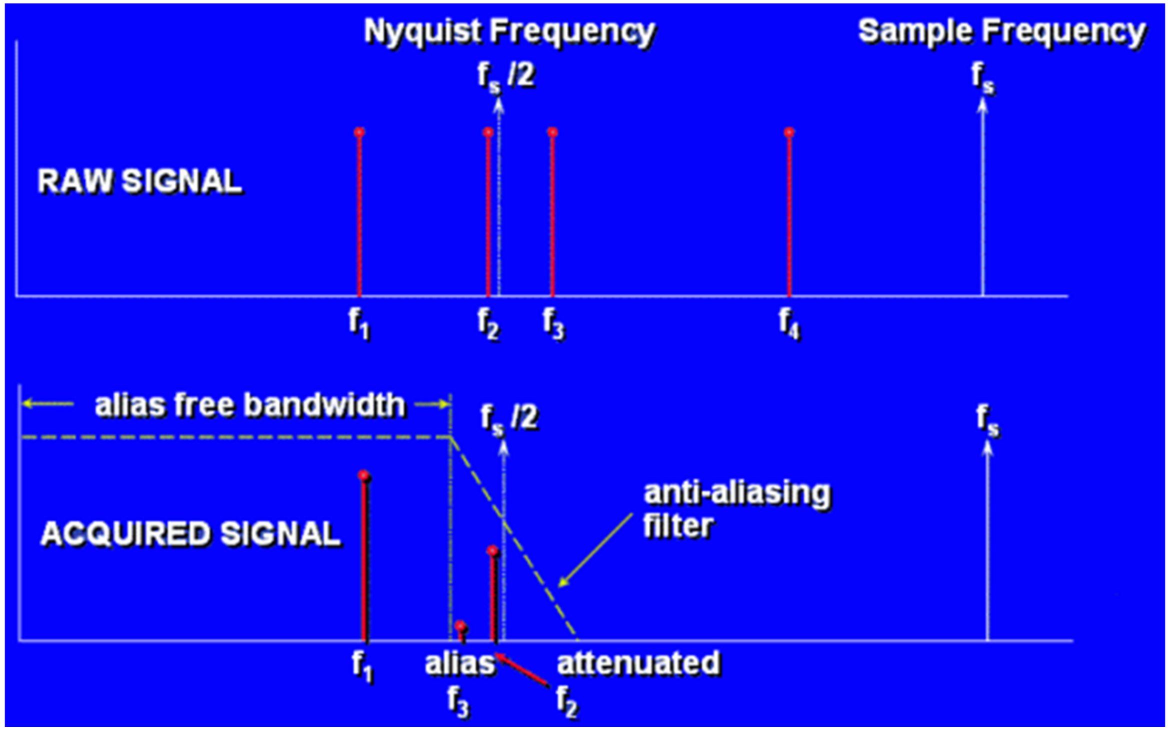
Sampling theorem

- An example of the non-ideal LPF



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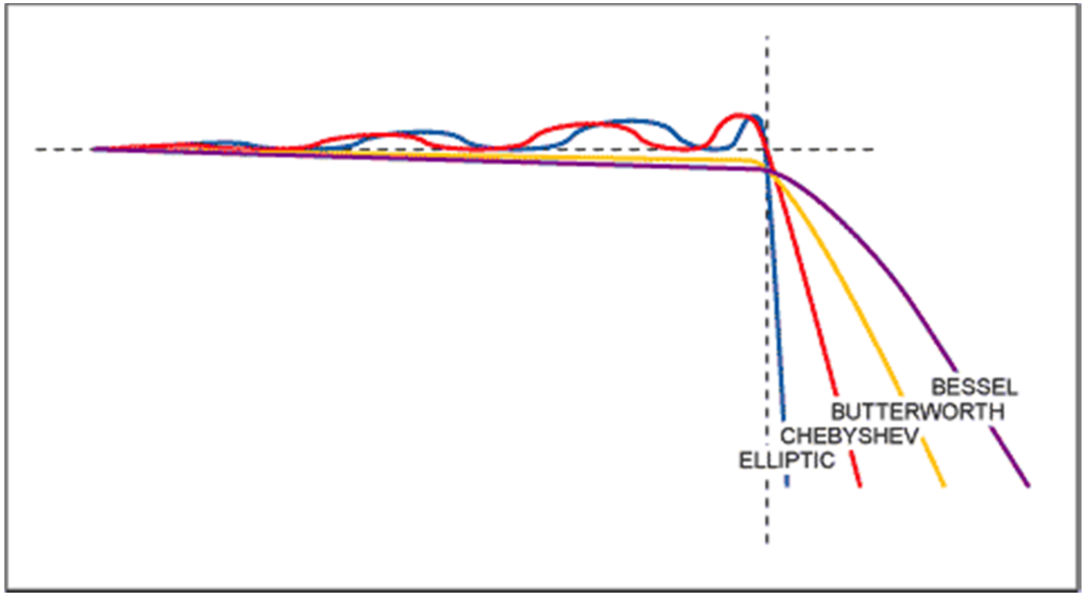
http://digital.ni.com/public.nsf/allkb/68F14E8E26B3D101862569350069E0B9

Sampling theorem

- Addressing filter design
  - Various methods
    - Passband
    - Stopband



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Sampling theorem

- Changing the sampling rate
  - Sampling of continuous-time signals

$$x[n] = x_c(nT)$$

  - Reconstruction: interpolation formula

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h_r(t - nT).$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$

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Sampling theorem

- Changing the sampling rate

$$x[n] = x_c(nT) \Rightarrow x'[n] = x_c(nT')$$

- A suggestion:
  - 1) Reconstruction of the CT signal
  - 2) sampling with desired rate
  - Associated with non-idealities
  - Complexity: ADC, DAC

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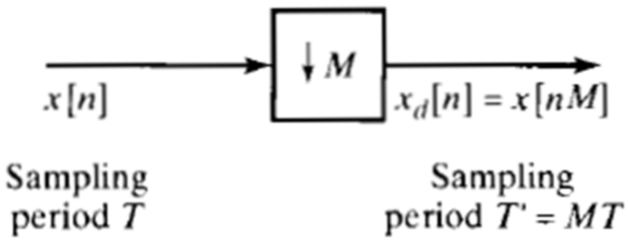
Sampling theorem

- We consider changing the sampling rate in discrete-time domain
- Sampling rate reduction by an integer factor
  - Downsampling/decimation



$$x_d[n] = x[nM] = x_c(nMT)$$

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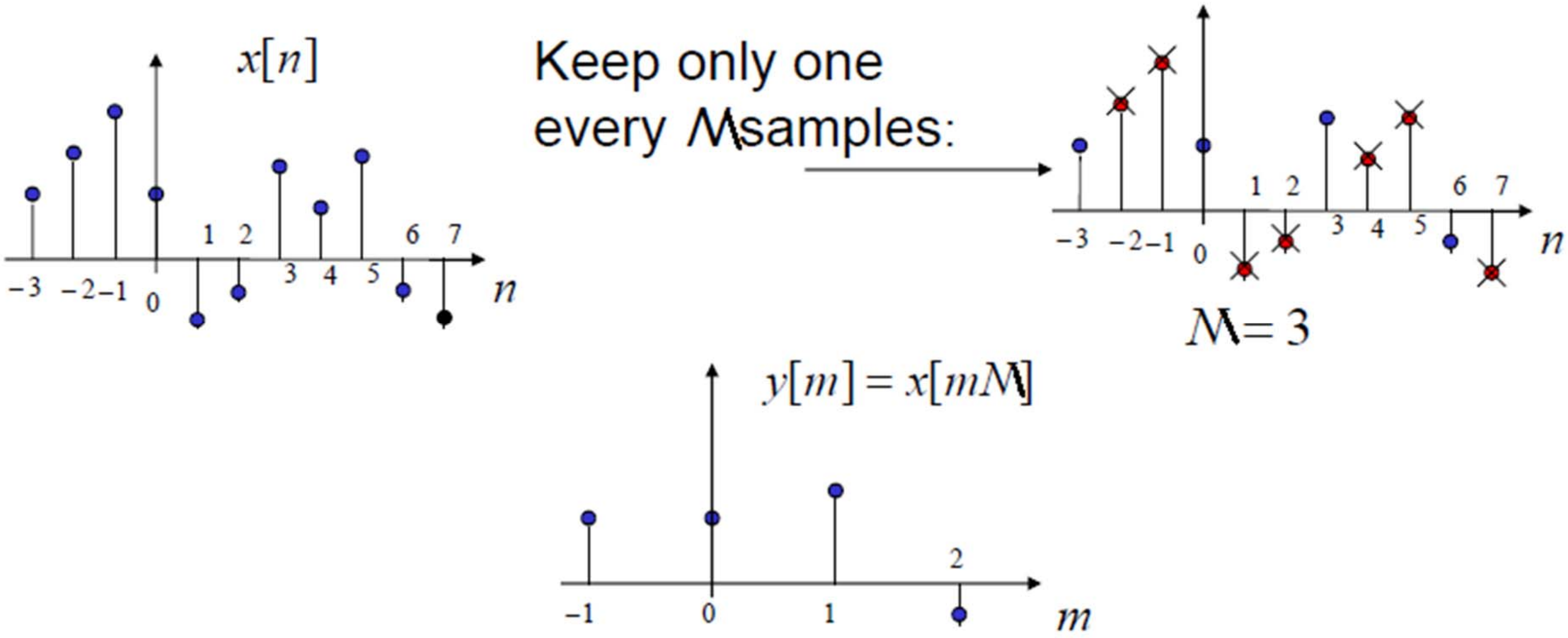
Sampling theorem

- Compressor system: integer *M*
  - Discarding *M-1* samples out of *M*



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http://faculty.nps.edu/.../2_Multirate_Signals.ppt

Sampling theorem

- The compressor is not TI

$$y[n] = x[Mn], \qquad -\infty < n < \infty,$$

  - The response to $x_1[n] = x[n - n_0]$

$$y_1[n] = x_1[Mn] = x[Mn - n_0].$$

  - Compared with

$$y[n - n_0] = x[M(n - n_0)].$$

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Sampling theorem

- Investigation
  - Let $X_c(j\Omega) = 0 \text{ for } |\Omega| \geq \Omega_N$
  - When can we reduce the sampling rate by factor *M*?
  - The original sampling frequency must be at least *M* times of the Nyquest rate
  - More on this later!

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Sampling theorem

- Frequency analysis
  - We had $x[n] = x_c(nT)$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

  - Therefore, for $x_d[n] = x[nM] = x_c(nT')$

$$X_d(e^{j\omega}) = \frac{1}{T'}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T'} - \frac{2\pi r}{T'}\right)\right)$$

  - interpretation

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Sampling theorem

- Next consider that $T' = MT$

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi r}{MT}\right)\right)$$

- Finally, it can be shown that

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\omega/M - 2\pi i/M)})$$

  – M copies!

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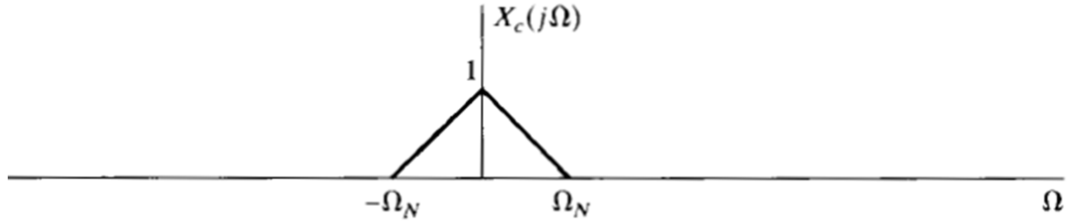
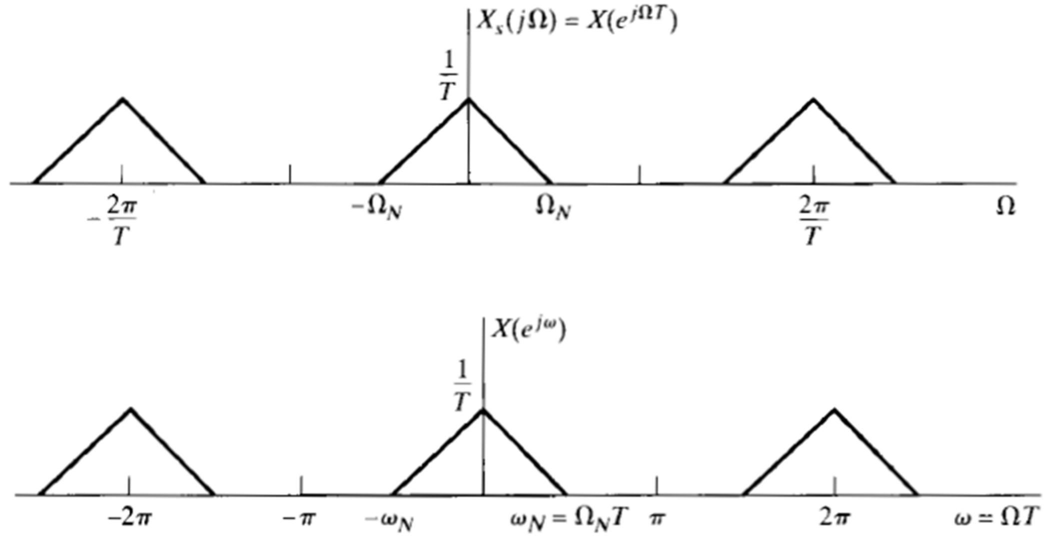
Sampling theorem

- Illustration



$$2\pi/T = 4\Omega_N$$

$$\omega_N = \Omega_N T = \pi/2$$



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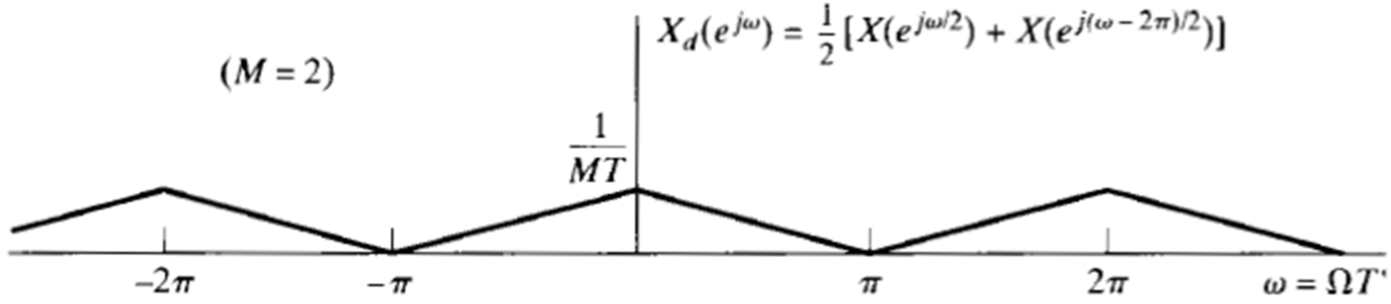
Sampling theorem

- And after, downsampling $2\pi / T = 4\Omega_N$



$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X(e^{j(\omega/M - 2\pi i/M)})$$

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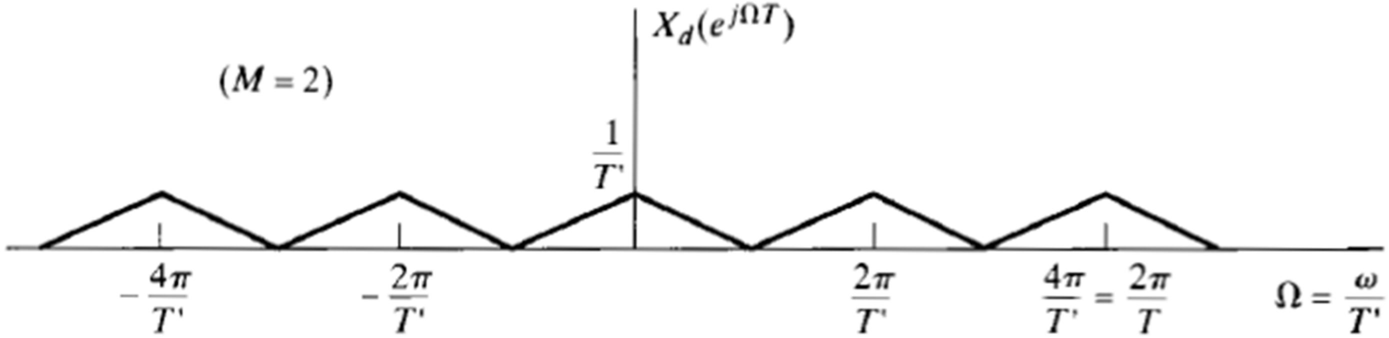
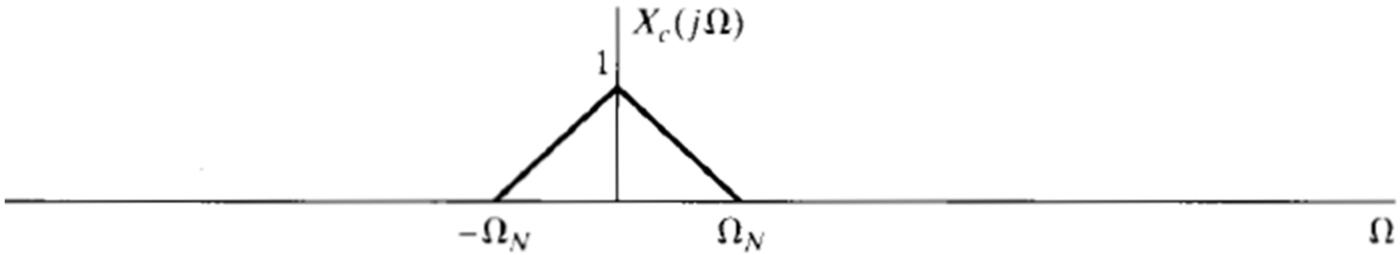
Sampling theorem



- An example of aliasing via downsampling



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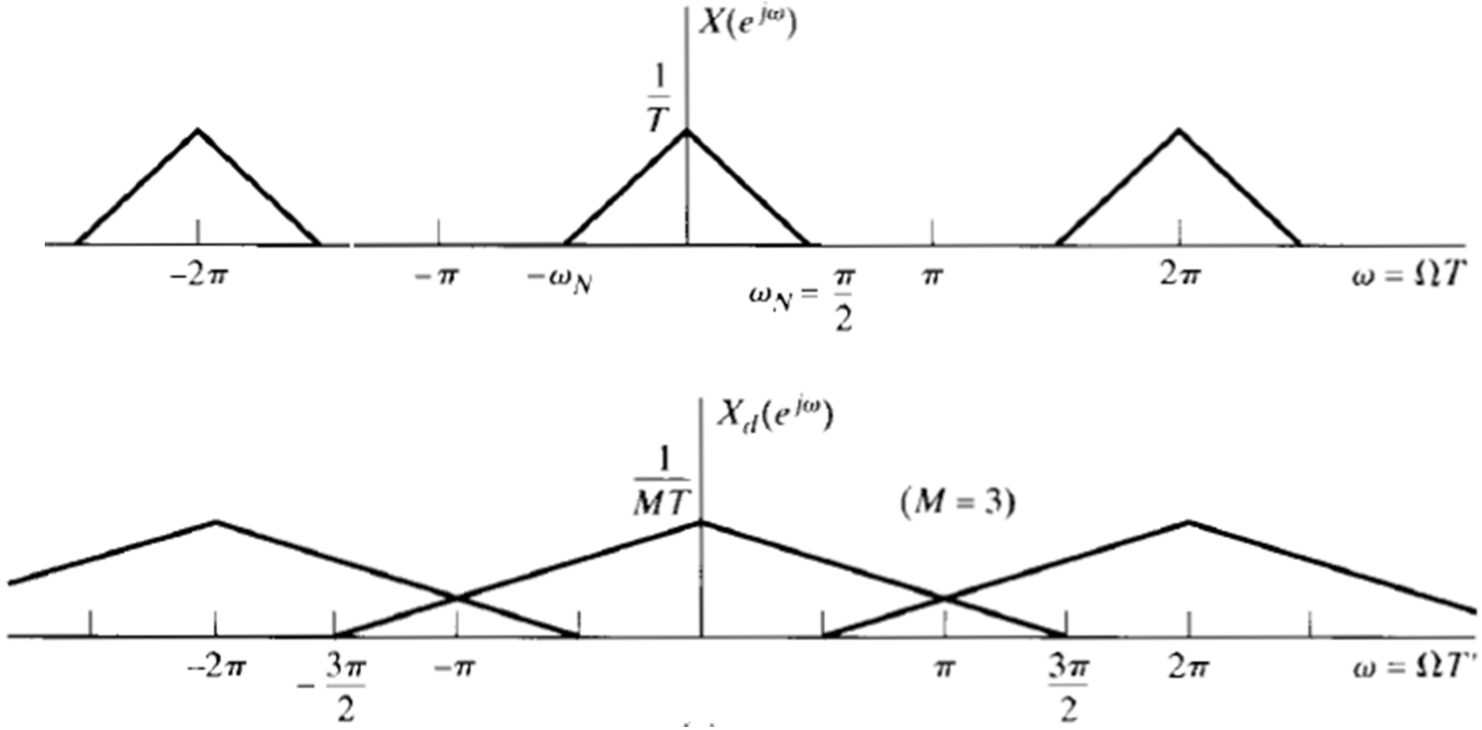
Sampling theorem

- Cont.



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Sampling theorem

- Consequently, for safe downsampling we should have

$$X(e^{j\omega}) = 0, \qquad \omega_N \le |\omega| \le \pi.$$

  – And

$$\omega_N < \pi / M$$

    - Check it!

  – Otherwise, we will have aliasing!!

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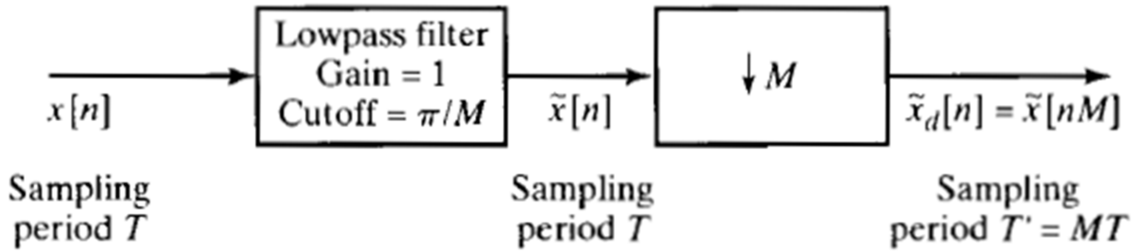
Sampling theorem

- Avoiding aliasing in downsamplin via DT anti-aliasing filter
- The cut-off frequency: $\omega_N < \pi / M$



General system for sampling rate reduction by M.

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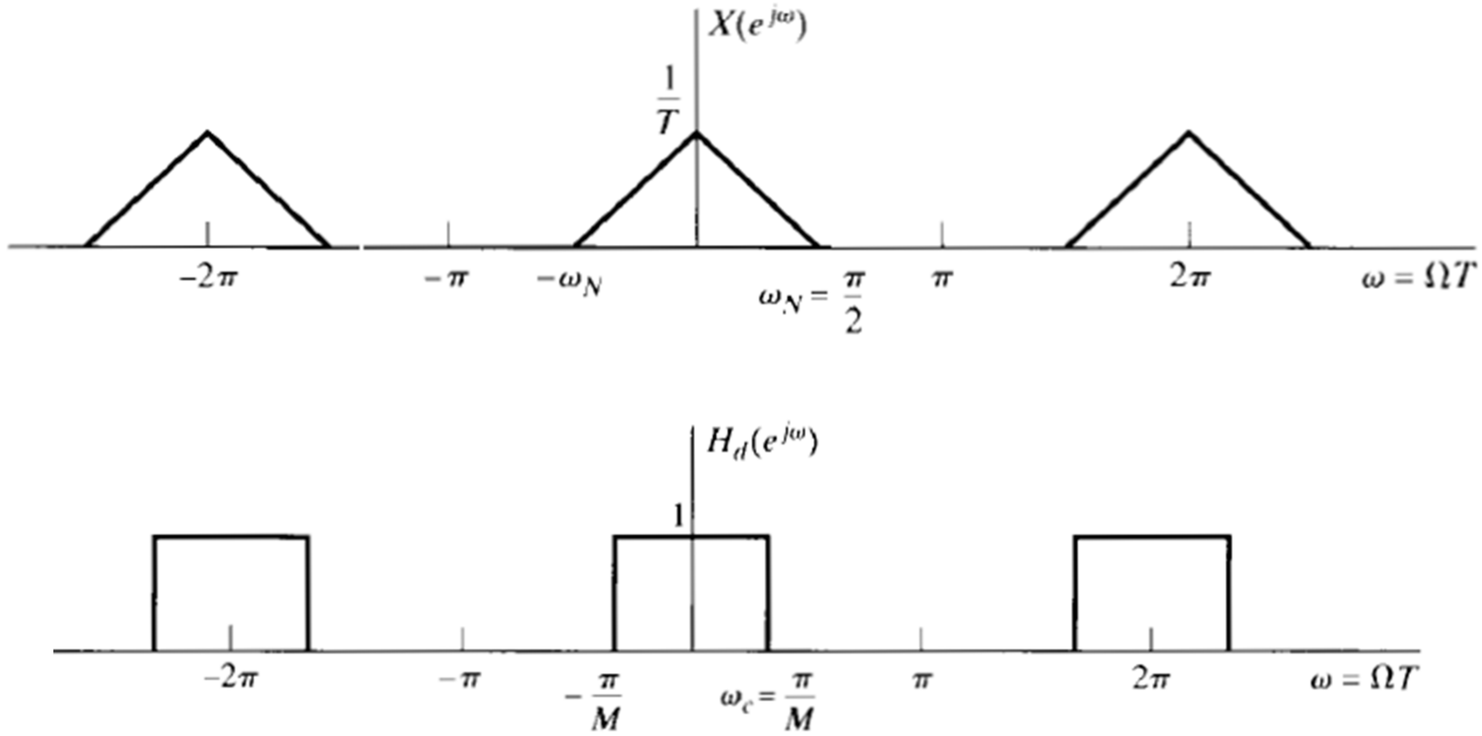
Sampling theorem

- Modified version of the aliased example



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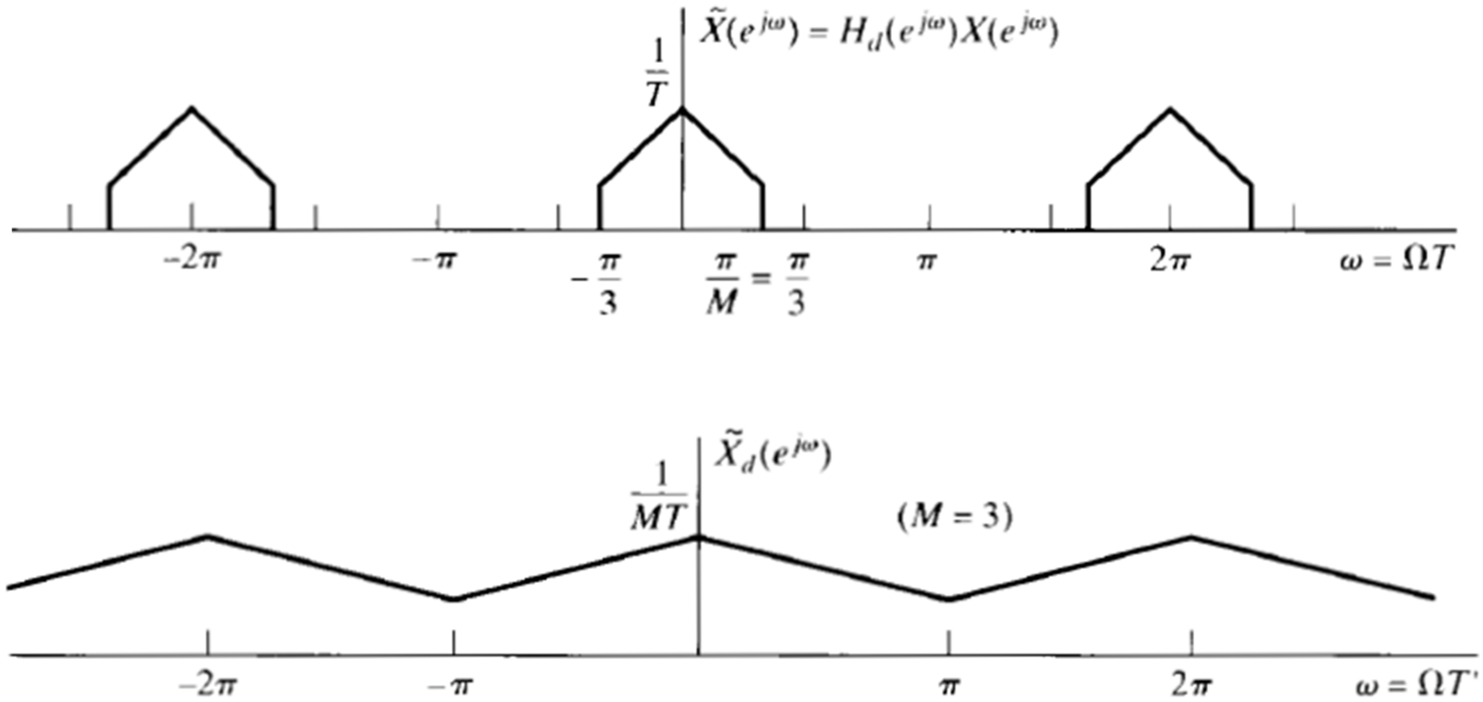
Sampling theorem

- Cont.



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