Digital Signal Processing (DSP)

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DIGITAL SIGNAL PROCESSING (DSP)

Lecture 4
Z-transform and LTI systems

Most of the materials are from the discrete-time signal processing by Oppenheim

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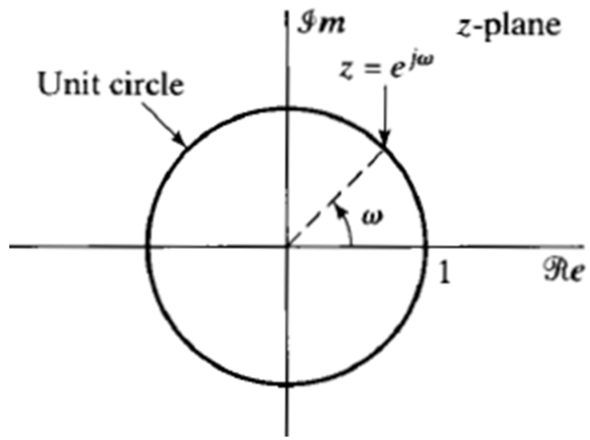
ZT & LTI systems

- The LTI system $Y(z) = H(z)X(z)$

$$x[n] \rightarrow \boxed{\text{LTI } h[n]} \rightarrow y[n] = x[n] * h[n]$$

- Let $x[n] = z^n$
  – Exponential sequence



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– Therefore,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} = H(z)\, z^n$$

– Where $H(z) = \sum_{k=-\infty}^{\infty} z^{-k}h[k]$

– The system function H(z) (eigenvalue)

– Eigenfunction: $x[n] = z^n$ (similar to the DTFT)

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- Note: specifying ROC for H(z); essential
- H(z)+ROC: unique h[n]
- If $z = e^{i\omega}$ be in ROC

$$H(e^{i\omega}) = \sum_{k=-\infty}^{\infty} \left(e^{i\omega}\right)^{-k} h[k] = \sum_{k=-\infty}^{\infty} e^{-i\omega k} h[k]$$

- Properties of the LTI system depends on the ROC

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- Causality of LTI systems
  - The system is causal iff ROC be exterior of a circle including ∞
  - Observe that
  - Right-hand sided

$$H(z) = \sum_{k=-\infty}^{\infty} z^{-k} h[k] = \sum_{k=0}^{\infty} z^{-k} h[k]$$

$$h[n] = 0 \text{ for all } n < 0$$

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- Stability (BIBO)
  - The LTI system is stable iff ROC includes unit circle
  - Assume $|x[n]| \leq B.$
    - Then

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[n-k]h[k] \right| \leq \sum_{k=-\infty}^{\infty} |x[n-k]|\,|h[k]| \leq B \sum_{k=-\infty}^{\infty} |h[k]|$$

    - Depends on absolutely summablity of the impulse response

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

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– Note also that for $|z| = 1$:

$$\sum_{k=-\infty}^{\infty} |z^{-k}h[k]| = \sum_{k=-\infty}^{\infty} |h[k]|$$

– therefore, absolutely summable h[n] means that

an LTI system is stable if and only if the circle $|z| = 1$ is in the ROC

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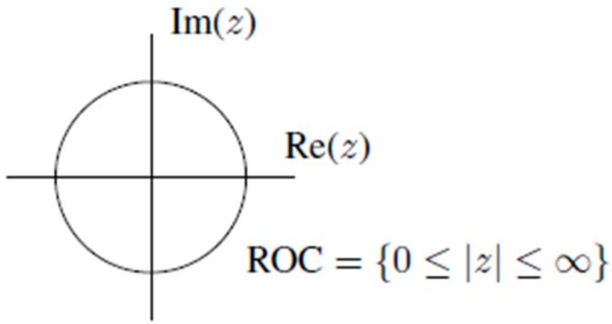
ZT & LTI systems

- Memoryless systems

An LTI system is memoryless iff $h[n] = b_0\,\delta[n]$

– And hence,



Memoryless

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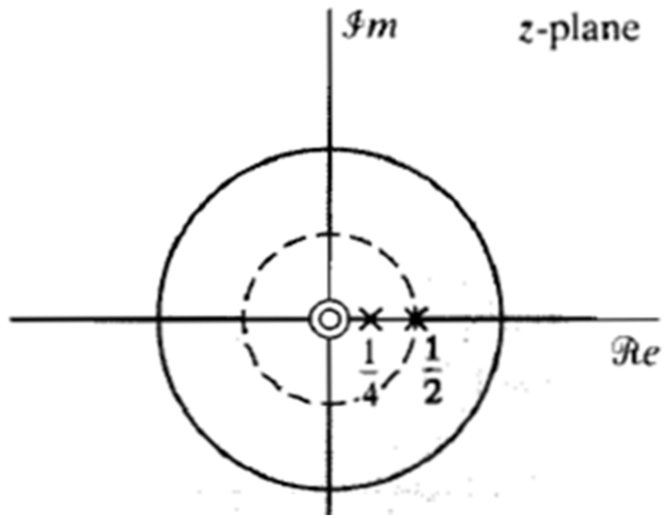
ZT & LTI systems

- The ROC: implicitly stated via properties of LTI systems
- Example:

$$H(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}.$$

  - Causal LTI system
    - We should recover ROC:

$$|z| > \frac{1}{2}$$



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- Another look at LCCDE
  - We had:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

  - Taking ZT of both sides

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z).$$

  - And,

$$\left(\sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z)$$

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– Finally for system function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.$$

– Or,

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}.$$

- Factored form

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- The expression of H(z) directly obtains from LCCDE
- ROC of H(z)?
  - Are obtained considering the properties of LTI systems: causality/stability
  - Remember that: LCCDE with initial rest determines a unique LTI system

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- Example: an LTI system with the following LCCDE

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n]$$

  – Then,

$$H(z) = \frac{1}{1 - \frac{5}{2}z^{-1} + z^{-2}} = \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - 2z^{-1})}$$

  – ROC?

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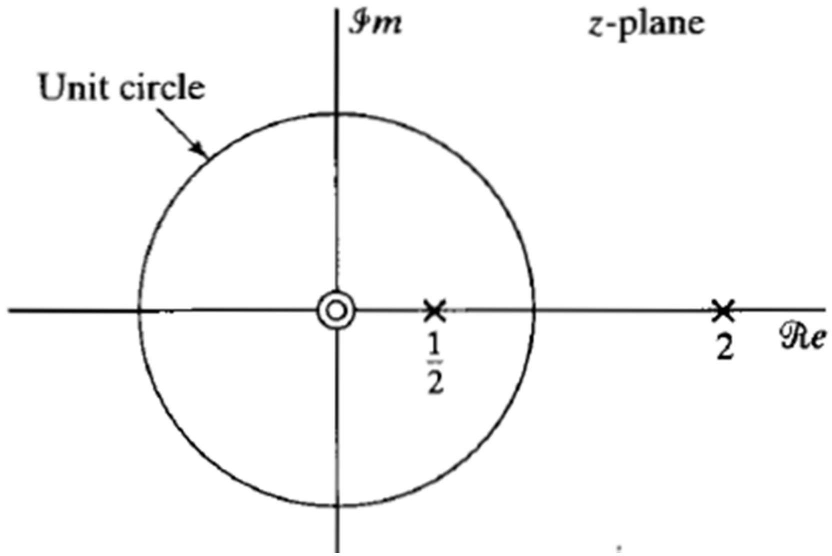
– Three choices for ROC

- Causal system
- Stable system
- Neither causal nor stable



- No causal and stable LTI system!!

ZT & LTI systems

- Note 1: rational systems with real coefficients

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.$$

  - It is checked that H*(z)= H(z*)
  - Therefore,
  - If z is a zero (pole), z* is a zero (pole) as well:
  - Complex zeros (poles) are complex conjugate

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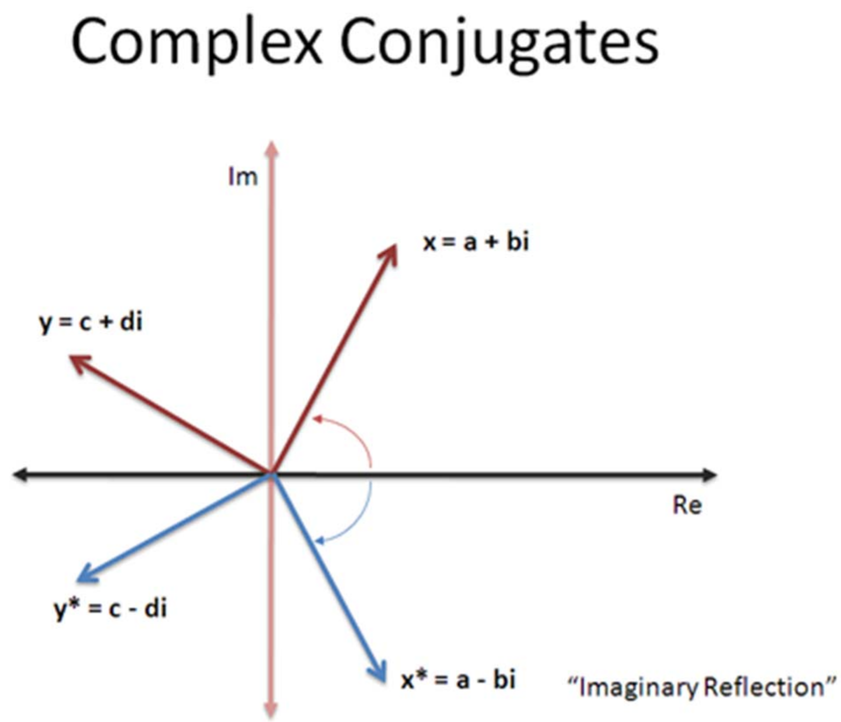
- Complex conjugate pairs



http://betterexplained.com/articles/intuitive-arithmetic-with-complex-numbers/

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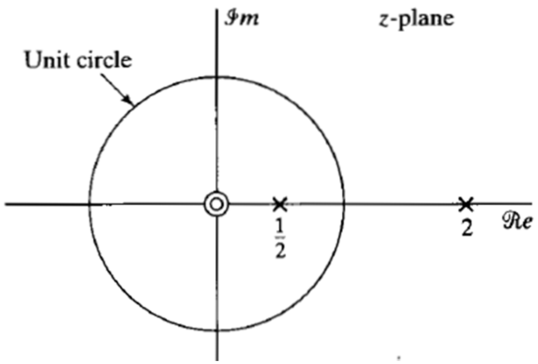
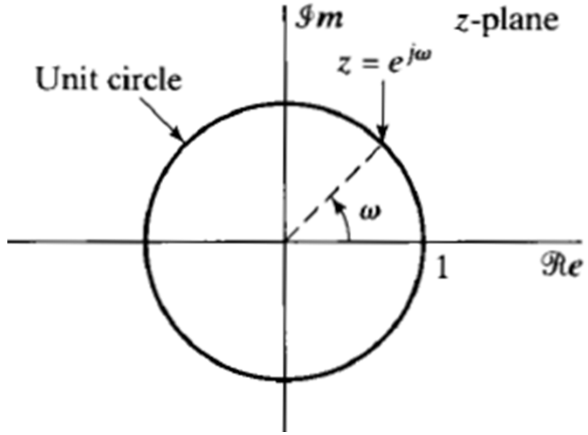
- Note 2: characterizing zero-poles of a causal stable LTI system
- All poles shall be inside the unit-circle





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ZT & LTI systems

- Frequency response
  - Assuming a stable LTI system
  - Evaluating H(z) at $z = e^{i\omega}$
- Example:

$$H(z) = \frac{z}{z-a} = \frac{1}{1-az^{-1}}$$

  - compute magnitude/phase of the frequency response

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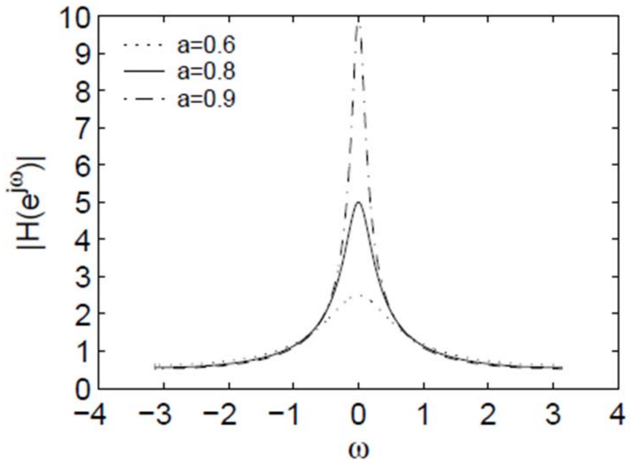
- Magnitude

$$
\begin{aligned}
H\left(e^{j\omega}\right) &= \frac{1}{1-ae^{-j\omega}} \\
\left|H\left(e^{j\omega}\right)\right| &= \frac{1}{\sqrt{(1-ae^{-j\omega})(1-ae^{j\omega})}} \\
&= \frac{1}{\sqrt{1-2a\cos\omega+a^2}} \\
&= \frac{1}{\sqrt{(1-a)^2+2a(1-\cos\omega)}}
\end{aligned}
$$



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- Intuitive insights
  - A pole near the unit-circle leads to peak in the frequency response
  - A zero near the unit-circle leads to null in the frequency response
  - Zeros may be on the unit-circle but poles not

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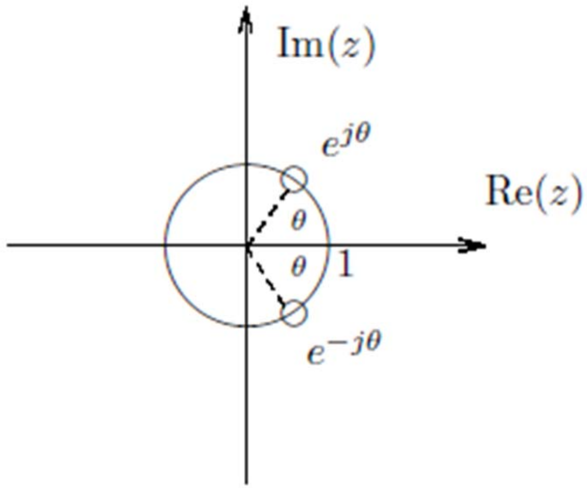
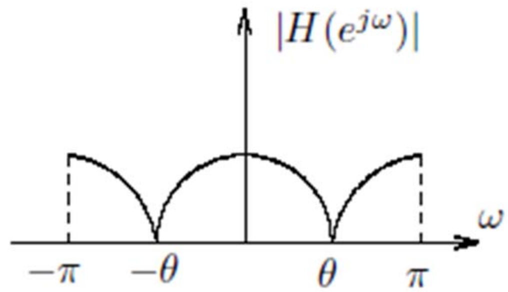
- Illustration



(a) Zeros on the unit circle.



(b) A bandstop filter corresponding to (a).

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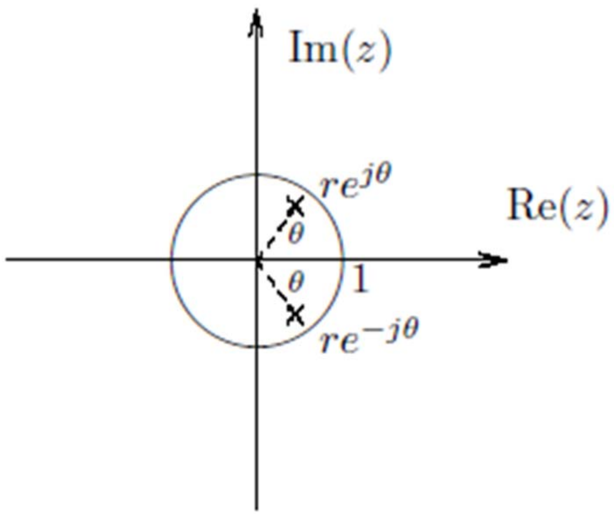
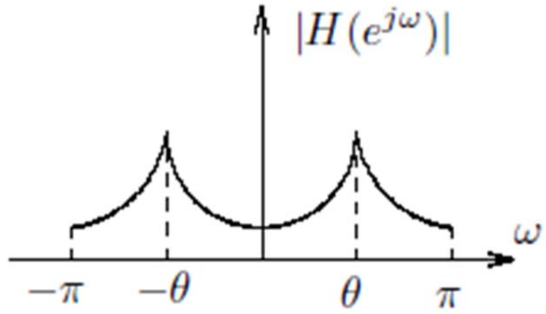
- Illustration



(c) Poles near the unit circle.



(d) A bandpass filter corresponding to (c).

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