ZT

– Applying time-reversal property:

$$X(z) = \frac{1}{1 - az^{-1}} \qquad |z| > |a|$$

– This yields

$$X(z) = \frac{1}{1 - az} = \frac{-a^{-1}z^{-1}}{1 - a^{-1}z^{-1}}, \qquad |z| < |a^{-1}|.$$

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- Convolution in time-domain

$$x_1[n] * x_2[n] \overset{z}{\longleftrightarrow} X_1(z)X_2(z), \qquad \text{ROC contains } R_{x_1} \cap R_{x_2}$$

- Pole-zero cancellation
- An important role in LTI system analysis

$$Y(z) = H(z)X(z),$$

$$x[n] \rightarrow \boxed{\text{LTI } h[n]} \rightarrow y[n] = x[n] * h[n]$$

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- Example: $x_1[n] = a^n u[n]$ and $x_2[n] = u[n]$

  – We compute ZT:

$$X_1(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|,$$

$$X_2(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}}, \qquad |z| > 1.$$

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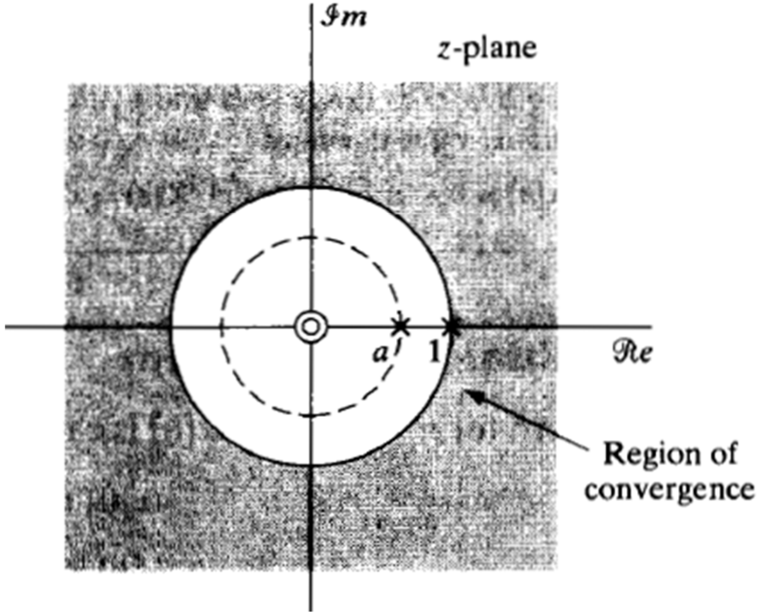
- And if $|a| < 1$,

$$Y(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})} = \frac{z^2}{(z-a)(z-1)}, \qquad |z| > 1$$



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- How to obtain y[n]?
  - **The inverse ZT!**
- Using partial fractional expansion

$$Y(z) = \frac{1}{1-a}\left(\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right), \qquad |z| > 1$$

- therefore,

$$y[n] = \frac{1}{1-a}(u[n] - a^{n+1}u[n]).$$

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- Initial value theorem

If $x[n]$ is zero for $n < 0$ (i.e., if $x[n]$ is causal), then

$$x[0] = \lim_{z \to \infty} X(z).$$

  - Simple proof

- Be careful of ROC when applying properties

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- Table of some properties

SOME z-TRANSFORM PROPERTIES

Sequence	Transform	ROC
$x[n]$	$X(z)$	R_x
$x_1[n]$	$X_1(z)$	R_{x_1}
$x_2[n]$	$X_2(z)$	R_{x_2}
$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞

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- Properties (cont.)

$x^*[n]$	$X^*(z^*)$	R_x
$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
$\mathcal{J}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Initial-value theorem:

$$x[n] = 0, \quad n < 0 \qquad \lim_{z\to\infty} X(z) = x[0]$$

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- Inverse ZT
  - We need it for coming back from Z-domain
  - Convolution example
  - LTI systems

- Methods
- 1) using well-known pair/properties
  - We have seen already

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- Example:

$$X(z) = \left(\frac{1}{1 - \frac{1}{2} z^{-1}} \right), \qquad |z| > \frac{1}{2},$$

  - Note that

$$a^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1 - a z^{-1}}, \qquad |z| > |a|$$

  - thus:

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n]$$

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- 2) Partial fractional expansion (PFE)
  - rational X(z)

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.$$

  - M zeros
  - N poles

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- We can write

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})},$$

where the c_k's are the nonzero zeros of $X(z)$ and the d_k's are the nonzero poles of $X(z)$.

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- Which leads to (case i $M < N$)

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

  - with

$$A_k = (1 - d_k z^{-1}) X(z) \big|_{z=d_k}$$

  - First-order poles only

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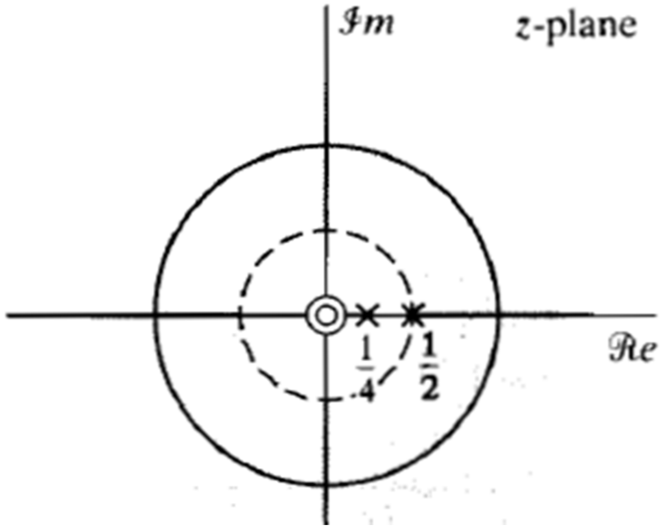
- Example:

$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}, \qquad |z| > \frac{1}{2}.$$



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– First-order poles:

$$X(z) = \frac{A_1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{A_2}{\left(1 - \frac{1}{2}z^{-1}\right)}$$

– with

$$A_1 = \left(1 - \tfrac{1}{4}z^{-1}\right) X(z)\Big|_{z=1/4} = -1,$$

$$A_2 = \left(1 - \tfrac{1}{2}z^{-1}\right) X(z)\Big|_{z=1/2} = 2\,.$$

– hence:

$$X(z) = \frac{-1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2}z^{-1}\right)} \Rightarrow x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$$

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- PFE case 2: $M \geq N$.

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}.$$

  – Obtain B_r's via divisions

- The situation with multiple-order poles

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1, k\neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m}$$

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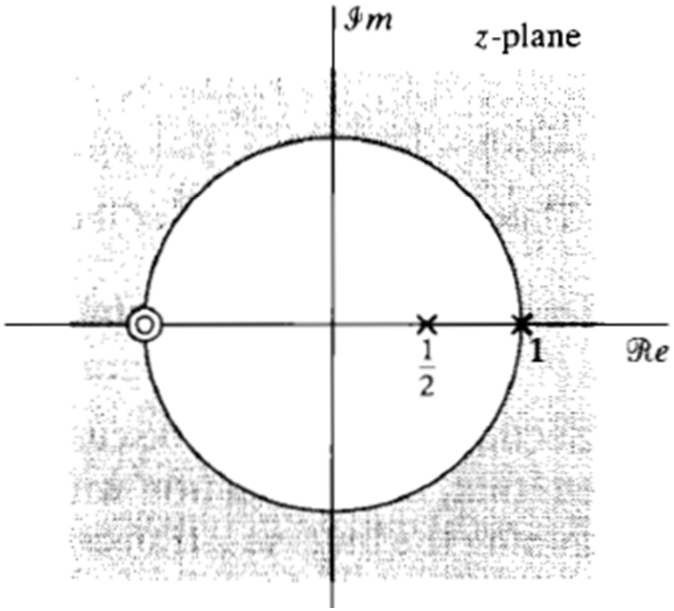
- Example:

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{(1 + z^{-1})^2}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}, \qquad |z| > 1$$



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– PFE

$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}.$$

– First-order poles

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}.$$

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– and,

$$A_1 = \left[\left(2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})} \right) \left(1 - \frac{1}{2}z^{-1} \right) \right]_{z=1/2} = -9,$$

$$A_2 = \left[\left(2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})} \right) (1 - z^{-1}) \right]_{z=1} = 8.$$

– resulting in

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}.$$

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– Finally

$$2 \overset{z}{\longleftrightarrow} 2\delta[n],$$

$$\frac{1}{1-\frac{1}{2}z^{-1}} \overset{z}{\longleftrightarrow} \left(\frac{1}{2}\right)^n u[n],$$

ROC is $|z| > 1$

$$\frac{1}{1-z^{-1}} \overset{z}{\longleftrightarrow} u[n].$$

– therefore,

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n].$$

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- 3) Power series expansion

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
$$= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots,$$

  – Useful for finite-length sequence

$$X(z) = z^2\left(1 - \tfrac{1}{2}z^{-1}\right)(1 + z^{-1})(1 - z^{-1}) = z^2 - \tfrac{1}{2}z - 1 + \tfrac{1}{2}z^{-1}.$$

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- so,

$$x[n] = \begin{cases} 1, & n = -2, \\ -\frac{1}{2}, & n = -1, \\ -1, & n = 0, \\ \frac{1}{2}, & n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$x[n] = \delta[n+2] - \tfrac{1}{2}\delta[n+1] - \delta[n] + \tfrac{1}{2}\delta[n-1].$$

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