Digital Signal Processing (DSP)

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DIGITAL SIGNAL PROCESSING (DSP)

Lecture 4

Z-Transform (ZT)

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ZT

- Example: finite length sequence

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$$

- For ZT we have

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a},$$

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- ROC: the following term should be finite

$$\sum_{n=0}^{N-1} |az^{-1}|^n < \infty$$

  - Finite number of summing terms
  - The value of az^{-1} shall be finite
  - ROC: entire z-plane except origin *z=0*
- Pole-zero cancellation concept

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- Zeros: $z^N = a^N$
  - Which leads to

$$z_k = ae^{j(2\pi k/N)}, \qquad k = 0, 1, \ldots, N-1$$

  - Note that:

The zero at $k = 0$ cancels the pole at $z = a$.

  - and the remaining zeros:

$$z_k = ae^{j(2\pi k/N)}, \qquad k = 1, \ldots, N-1$$

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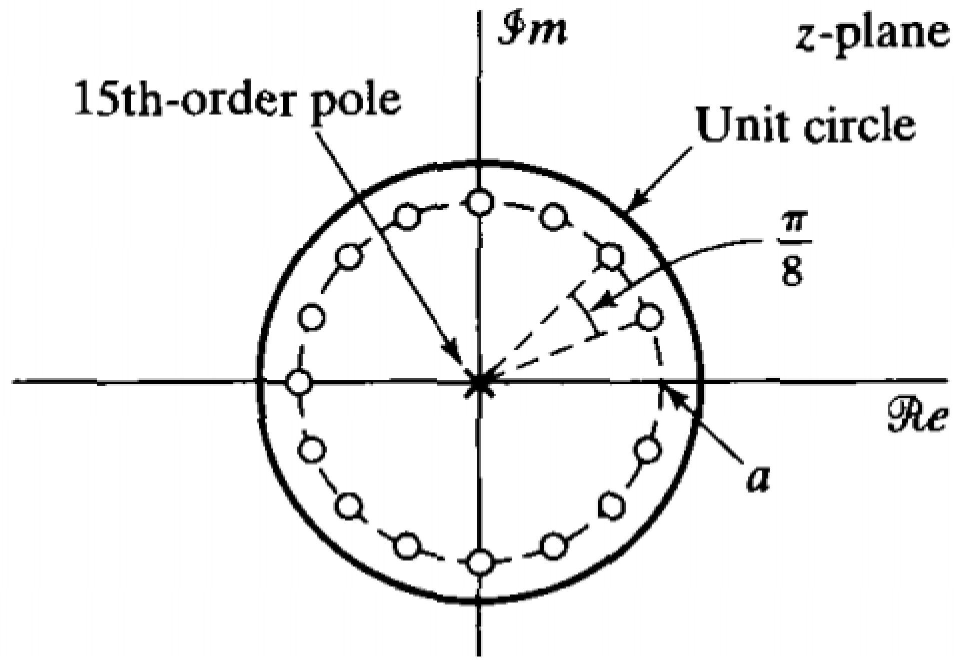
- Zero-pole plot for *N=16*

$$\frac{1}{z^{N-1}}\frac{z^N - a^N}{z - a}$$



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- Some commons ZT pairs: be careful for ROC

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$

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- Properties of ROC of the ZT

PROPERTY 1: The ROC is a ring or disk in the z-plane centered at the origin; i.e., $0 \le r_R < |z| < r_L \le \infty$.

PROPERTY 2: The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the unit circle.

PROPERTY 3: The ROC cannot contain any poles.

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- Properties of ROC of the ZT (cont.)

PROPERTY 4: If $x[n]$ is a *finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \leq n \leq N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.

PROPERTY 5: If $x[n]$ is a *right-sided sequence*, i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

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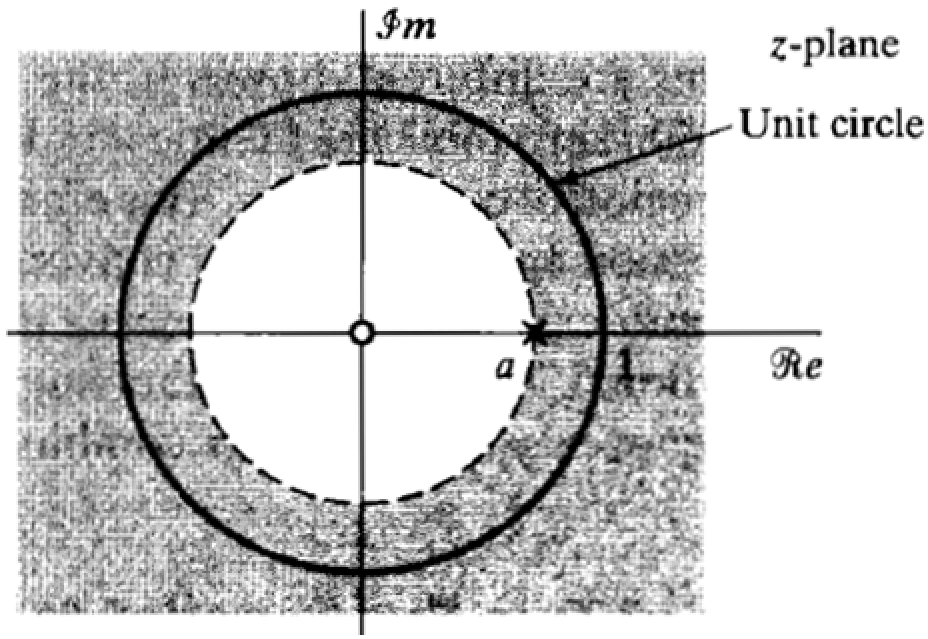
- Remember $x[n] = a^n u[n]$



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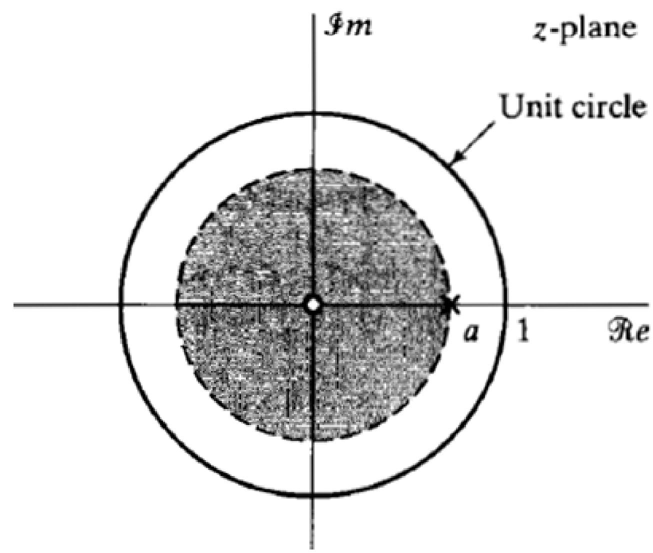
- Properties of ROC of the ZT (cont.)

PROPERTY 6: If $x[n]$ is a *left-sided sequence*, i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the *innermost* (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

- Remember:

$$x[n] = -a^n u[-n-1].$$



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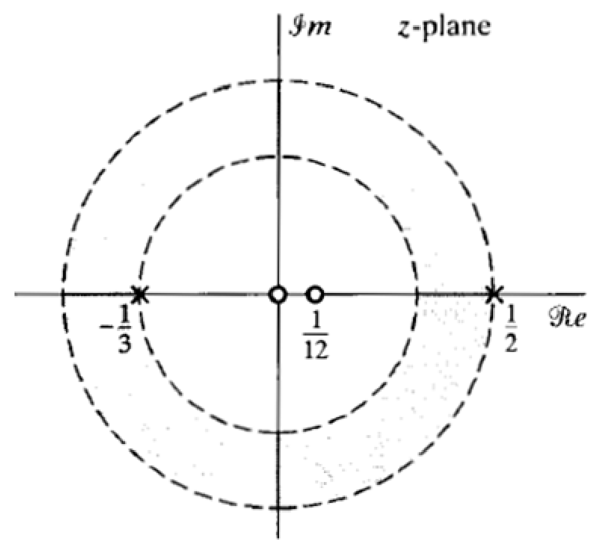
- Properties of ROC of the ZT (cont.)

PROPERTY 7: A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with property 3, not containing any poles.

- Remember:

$$x[n] = \left(-\tfrac{1}{3}\right)^n u[n] - \left(\tfrac{1}{2}\right)^n u[-n-1].$$



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- Intuitive observations
- Trick for the proofs
  - Remember

$$z = re^{j\omega}$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n},$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}$$

  - Convergence of *X(z)*: absolutely summable $x[n]r^{-n}$
- generally $r_R < |z| < r_L$

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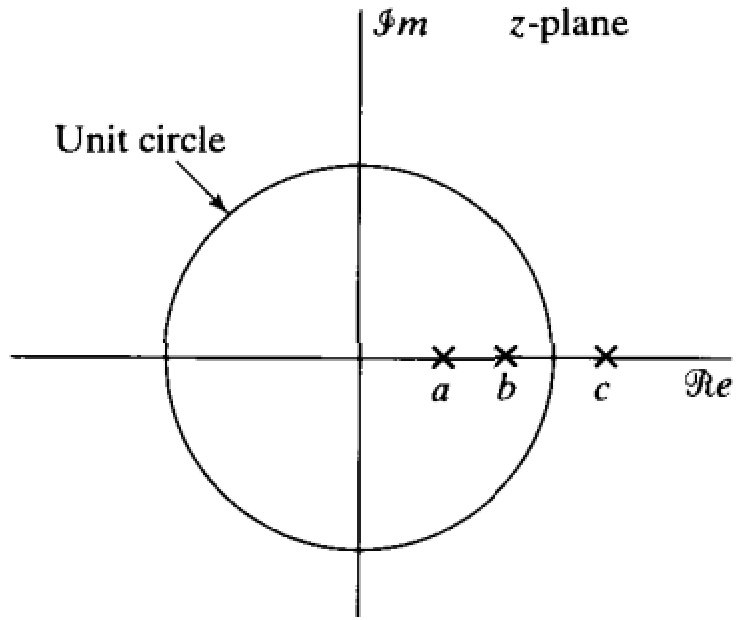
- Example: given zero/poles



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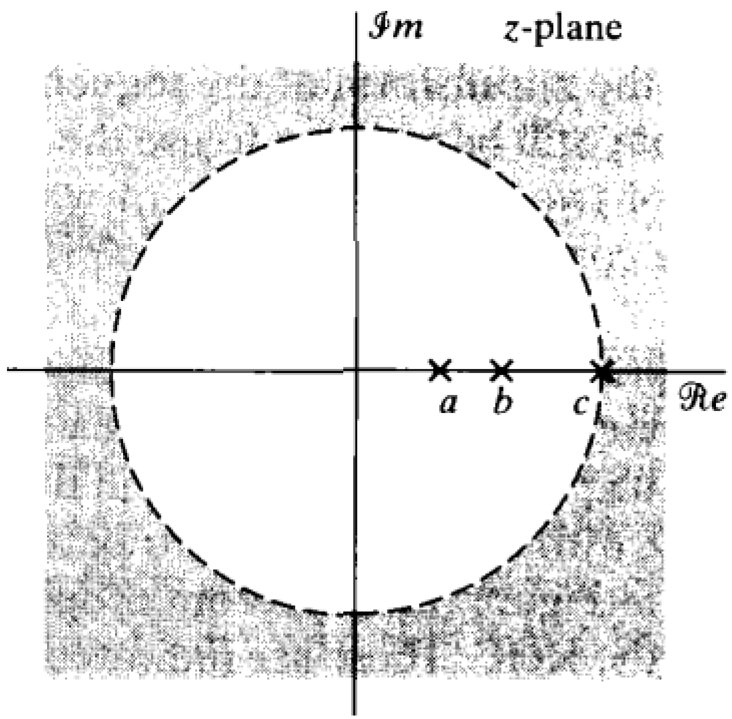
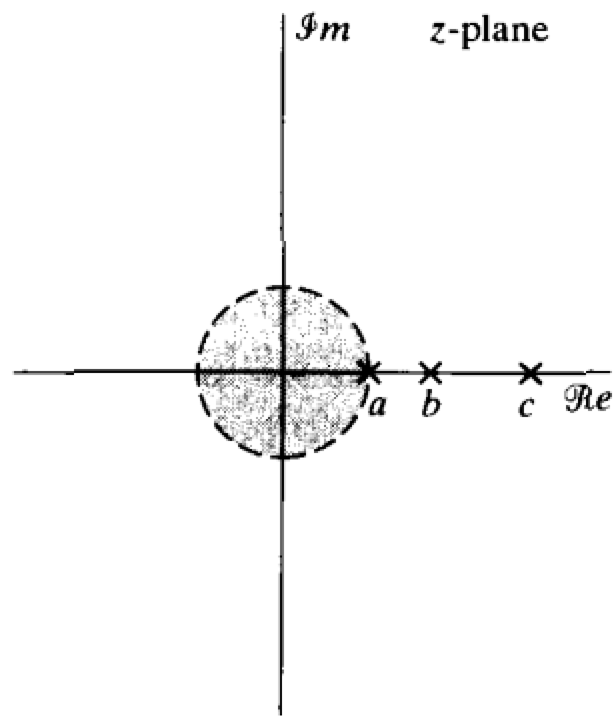
- Example (cont.):





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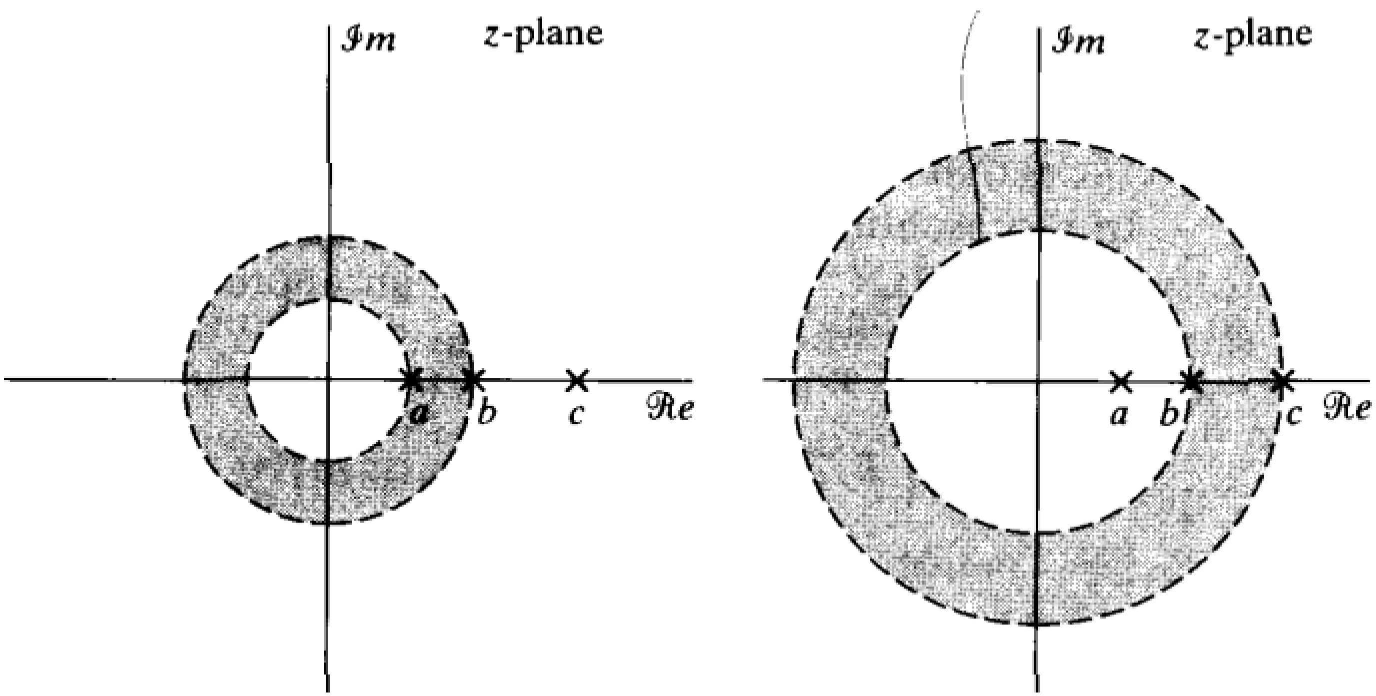
- Example (cont.):



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- Interesting example!

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] - \left(-\tfrac{1}{3}\right)^n u[-n-1]$$

  - Two sided signal
    - No intersection between ROC of these two terms
    - ZT does not exist!
  - We may consider each term separately
- **ZT is linked to the corresponding ROC**

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- Properties of ZT
  - Let

$$x_1[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z), \qquad \text{ROC} = R_{x_1}$$

$$x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_2(z), \qquad \text{ROC} = R_{x_2}$$

  - Linearity

$$ax_1[n] + bx_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} aX_1(z) + bX_2(z), \qquad \text{ROC contains } R_{x_1} \cap R_{x_2}$$

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- Example:

$$x[n] = a^n u[n] - a^n u[n-N]$$

  - Both terms have poles at *z=a*
  - We have seen that

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$$

  - and

$$X(z) = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}$$

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  - Remember: pole-zero cancellation
  - The results has no pole at $z=a$
- **Observation:**
- *Pole-zero cancellation may occur when using linearity property of ZT. In such cases, the ROC may be larger than the intersection of the ROCs for the associated terms.*

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- Time-shift

$$x[n-n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0}X(z), \qquad \text{ROC} = R_x \text{(except for the possible addition or deletion of } z=0 \text{ or } z=\infty)$$

- A simple proof: $y[n] = x[n-n_0]$

$$Y(z) = \sum_{n=-\infty}^{\infty} x[n-n_0]z^{-n} = \sum_{m=-\infty}^{\infty} x[m]z^{-(m+n_0)} = z^{-n_0} \sum_{m=-\infty}^{\infty} x[m]z^{-m}$$

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- Example:

$$X(z) = \frac{z^{-1}}{1 - \frac{1}{4}z^{-1}}, \qquad |z| > \frac{1}{4}.$$

  - We can write

$$X(z) = z^{-1}\left(\frac{1}{1 - \frac{1}{4}z^{-1}}\right), \qquad |z| > \frac{1}{4}.$$

  - Therefore,

$$x[n] = \left(\frac{1}{4}\right)^{n-1} u[n-1]$$

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- Multiplication by exponential sequence

$$z_0^n x[n] \stackrel{z}{\longleftrightarrow} X(z/z_0), \qquad \text{ROC} = |z_0| R_x$$

The notation $\text{ROC} = |z_0| R_x$ denotes that the ROC is R_x scaled by $|z_0|$; i.e., if R_x is the set of values of z such that $r_R < |z| < r_L$, then $|z_0| R_x$ is the set of values of z such that $|z_0| r_R < |z| < |z_0| r_L$.

  - When DTFT exist:

$$e^{j\omega_0 n} x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})$$

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- Note: if $X(z)$ has a pole at $z = z_1$
  - Then: $X(z_0^{-1} z)$ will have a pole at $z = z_0 z_1$

- Example:

$$x[n] = r^n \cos(\omega_0 n) u[n]$$

  - Using Euler equation

- $$x[n] = \frac{1}{2}(re^{j\omega_0})^n u[n] + \frac{1}{2}(re^{-j\omega_0})^n u[n]$$

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– We know that

$$u[n] \overset{z}{\longleftrightarrow} \frac{1}{1 - z^{-1}}, \qquad |z| > 1$$

– Consequently,

$$\tfrac{1}{2}(re^{j\omega_0})^n u[n] \overset{z}{\longleftrightarrow} \frac{\frac{1}{2}}{1 - re^{j\omega_0} z^{-1}}, \qquad |z| > r,$$

$$\tfrac{1}{2}(re^{-j\omega_0})^n u[n] \overset{z}{\longleftrightarrow} \frac{\frac{1}{2}}{1 - re^{-j\omega_0} z^{-1}}, \qquad |z| > r.$$

– Then simply use linearity

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- Differentiation

$$nx[n] \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \qquad \text{ROC} = R_x$$

  – Simple proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n},$$

$$-z\frac{dX(z)}{dz} = -z\sum_{n=-\infty}^{\infty}(-n)x[n]z^{-n-1}$$

$$= \sum_{n=-\infty}^{\infty} nx[n]z^{-n} = \mathcal{Z}\{nx[n]\}.$$

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- Example:

$$x[n] = na^n u[n] = n(a^n u[n])$$

- Using the differentiation property:

$$X(z) = -z\frac{d}{dz}\left(\frac{1}{1 - az^{-1}}\right), \qquad |z| > |a|$$

$$= \frac{az^{-1}}{(1 - az^{-1})^2}, \qquad |z| > |a|.$$

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- Conjugate

$$x^*[n] \overset{z}{\longleftrightarrow} X^*(z^*), \qquad \text{ROC} = R_x$$

- Time-reversal

$$x^*[-n] \overset{z}{\longleftrightarrow} X^*(1/z^*), \qquad \text{ROC} = \frac{1}{R_x}$$

The notation $\text{ROC} = 1/R_x$ implies that R_x is inverted; i.e., if R_x is the set of values of z such that $r_R < |z| < r_L$, then the ROC is the set of values of z such that $1/r_L < |z| < 1/r_R$. Thus, if z_0 is in the ROC for $x[n]$, then $1/z_0^*$ is in the ROC for the z-transform of $x^*[-n]$.

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– Result

$$x[-n] \stackrel{z}{\longleftrightarrow} X(1/z), \qquad \text{ROC} = \frac{1}{R_x}$$

- Example:

$$x[n] = a^{-n}u[-n],$$

– The time-reversed version of the following sequence

$$x[n] = a^n u[n]$$

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– Applying time-reversal property:

$$X(z) = \frac{1}{1 - az^{-1}} \qquad |z| > |a|$$

– This yields

$$X(z) = \frac{1}{1 - az} = \frac{-a^{-1}z^{-1}}{1 - a^{-1}z^{-1}}, \qquad |z| < |a^{-1}|.$$

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