Digital Signal Processing (DSP)

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DIGITAL SIGNAL PROCESSING (DSP)

Lecture 4

Z-Transfrom

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ZT

- Z-transform (ZT)
  - By Prof. Zadeh
  - A generalization of DTFT
    - Limitations of DTFT
    - Existence/ convergence
    - Can be applied to a large class of signals
  - DT version of Laplace transform

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ZT

- ZT: a mapping from DT signal to a function of the (complex) variable *z:*

$$x[n] \overset{z}{\longleftrightarrow} X(z) \qquad \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z)$$

- Remember:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

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- Domain of X(z): The set of complex z for which X(z) absolutely converges:

$$\sum_{n=-\infty}^{\infty} |x[n]|\,|z|^{-n} < \infty$$

- ROC: region of convergence (ROC)
- ROC: only depends on abs(z)

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- Note:

$$z = re^{j\omega}$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n},$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}$$

- Special case:
  - DTFT for $r = 1$
- Adjust *r* for convergence

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- Example: $x[n] = a^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} = a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$$

  - For convergence:

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty.$$

  - Leading to: $|az^{-1}| < 1$

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- In sum:

$$X(z) = \sum_{n=0}^{\infty}(az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|$$

  - DTFT exists if $|a| < 1$
  - Unit step function $a = 1$

$$X(z) = \frac{1}{1-z^{-1}}, \qquad |z| > 1$$

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- Specifying an ZT: both X(z) and ROC
- Unit circle: DTFT
  - ROC includes unit circle means that DTFT exists
- Rational X(z):

$$X(z) = \frac{P(z)}{Q(z)}$$

  - Zero: $X(z) = 0$
  - Pole: infinite value

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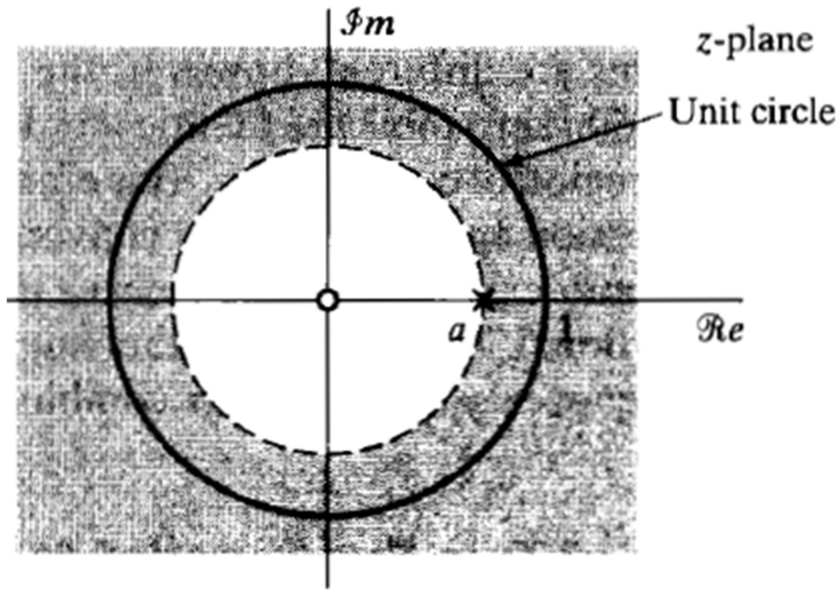
- Graphical representation (zero-pole plot):
  - Right-hand side signal
  - ROC: outside a circle



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- Example: $x[n] = -a^n u[-n-1]$

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n.$$

- And for $|a^{-1} z| < 1$

$$X(z) = 1 - \frac{1}{1 - a^{-1} z} = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}, \qquad |z| < |a|.$$

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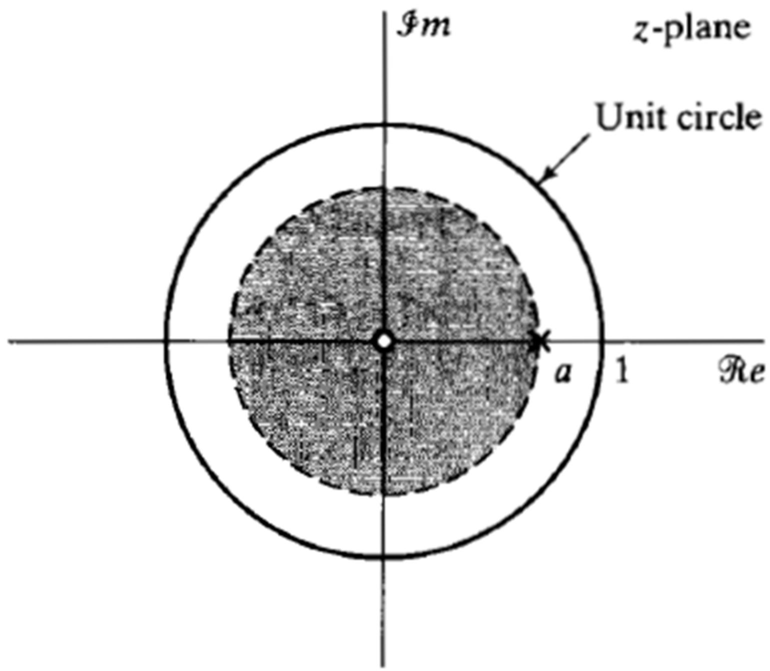
- Zero-pole plot:
  - left-hand side signal
  - ROC: inside a circle



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- Example: $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n].$

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \right\} z^{-n} \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{3}\right)^n u[n] z^{-n} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n \\
&= \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{3} z^{-1}} = \frac{2\left(1 - \frac{1}{12} z^{-1}\right)}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 + \frac{1}{3} z^{-1}\right)} \\
&= \frac{2z\left(z - \frac{1}{12}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)}.
\end{aligned}
$$

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- Convergence
  - The first term: $\left|\frac{1}{2}z^{-1}\right| < 1$
  - The second term: $\left|\left(-\frac{1}{3}\right)z^{-1}\right| < 1$
  - Or equivalently:

$$|z| > \frac{1}{2} \text{ and } |z| > \frac{1}{3}$$

  - Which leads to

$$|z| > \frac{1}{2}$$

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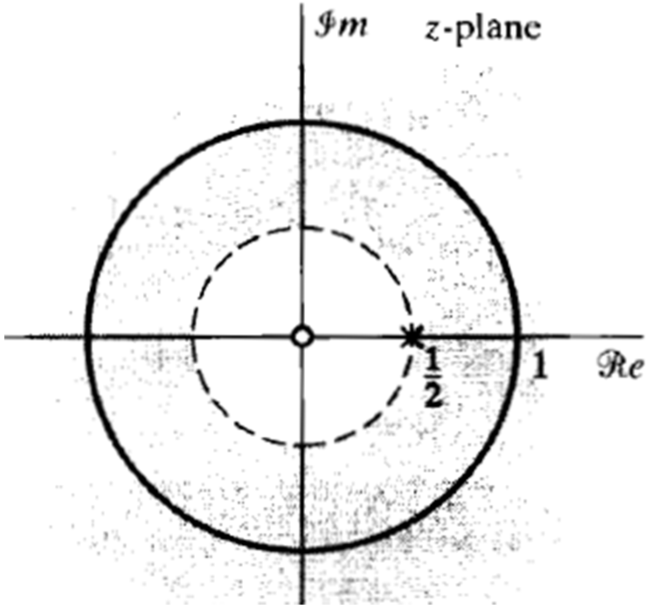
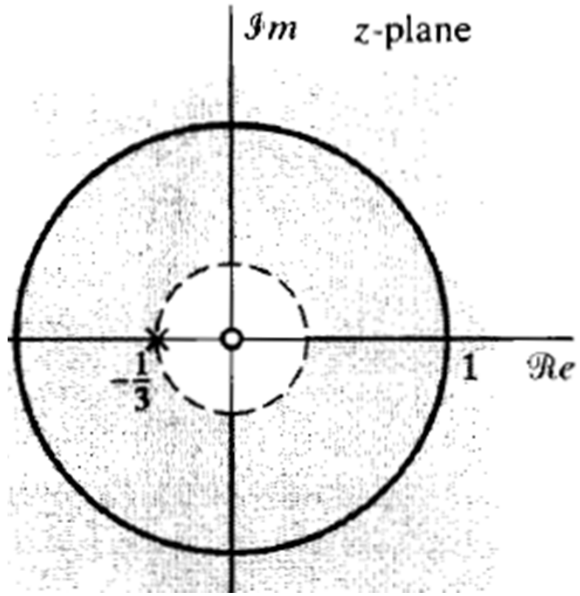
- Illustration: zero-poles of the terms





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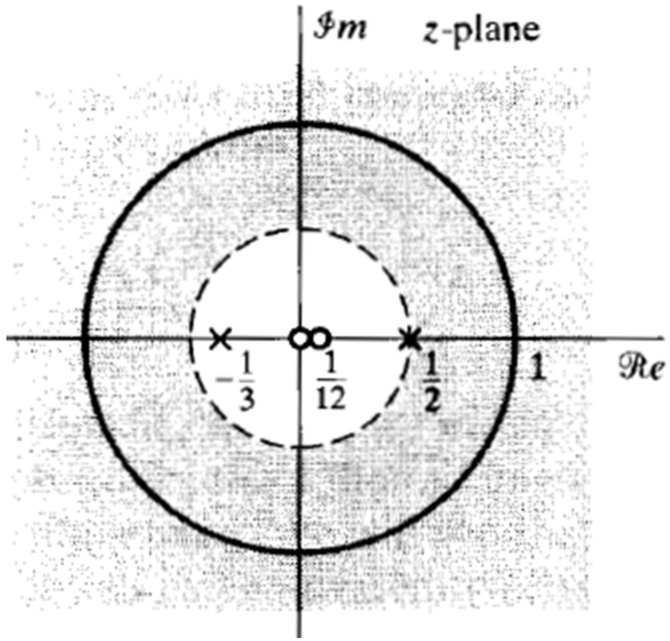
- Zero-poles of the sum
  - ROC includes no pole



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- Example: a two-sided signal

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1].$$

- Using previous results:

$$\left(-\frac{1}{3}\right)^n u[n] \stackrel{z}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |z| > \frac{1}{3},$$

$$-\left(\frac{1}{2}\right)^n u[-n-1] \stackrel{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| < \frac{1}{2}.$$

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- Therefore,

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \frac{1}{3} < |z|, \quad |z| < \frac{1}{2},$$

$$= \frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}.$$

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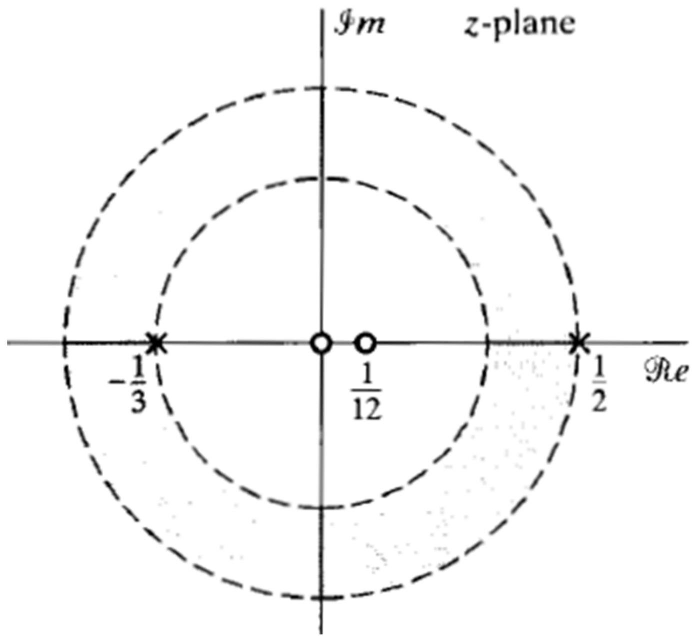
- Zero-pole plot:
  - Two-sided signal
  - ROC: a ring



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- Example:

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise}. \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a},$$

- The term $\sum_{n=0}^{N-1} |az^{-1}|^n < \infty.$ should be finite

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