DTFT and LTI systems

- The concept of eigen-function
  - Relationship with eigen-vector
- The frequency response of the system

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

- DTFT of the impulse response h[n]
- Recovering h[n] from the frequency response
  - synthesis

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DTFT and LTI systems

- Remarks
- BIBO stable system: existence of DTFT
  - Absolutely summable impulse response
- Representations of the frequency response

$$H(e^{j\omega}) = H_R(e^{j\omega}) + j H_I(e^{j\omega})$$

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j \measuredangle H(e^{j\omega})}$$

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DTFT and LTI systems

- Example: response to sinusoidal

$$x[n] = A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$

  – Response to the $x_1[n] = (A/2)e^{j\phi}e^{j\omega_0 n}$:

$$y_1[n] = H(e^{j\omega_0})\frac{A}{2}e^{j\phi}e^{j\omega_0 n}$$

  – Similarly

$$y_2[n] = H(e^{-j\omega_0})\frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$

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– Finally

$$y[n] = \frac{A}{2}[H(e^{j\omega_0})e^{j\phi}e^{j\omega_0 n} + H(e^{-j\omega_0})e^{-j\phi}e^{-j\omega_0 n}]$$

– Using DTFT properties:

$$y[n] = A|H(e^{j\omega_0})|\cos(\omega_0 n + \phi + \theta)$$

– The same frequency as the input
  - Different amplitude and phase
  - LTI systems: no changes in frequency

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DTFT and LTI systems

- IMPORTANT PROPERTY

$$x[n] \rightarrow \boxed{\text{LTI } h[n]} \rightarrow y[n] = x[n] * h[n]$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

- Synthesis: y[n]
- Every frequency component of x[n], i.e. $X(e^{j\omega})$, is affected by $H(e^{j\omega})$

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DTFT and LTI systems

- Example: MA

$$h[n] = \begin{cases} \dfrac{1}{M_1 + M_2 + 1}, & -M_1 \le n \le M_2, \\ 0, & \text{otherwise.} \end{cases}$$

- Frequency response

$$H(e^{j\omega}) = \frac{1}{M_1 + M_2 + 1} \sum_{n=-M_1}^{M_2} e^{-j\omega n} \quad = \frac{1}{M_1 + M_2 + 1} \frac{e^{j\omega M_1} - e^{-j\omega(M_2+1)}}{1 - e^{-j\omega}}$$

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DTFT and LTI systems

- Simplification:

$$H(e^{j\omega}) = \frac{1}{M_1 + M_2 + 1} \frac{e^{j\omega M_1} - e^{-j\omega(M_2+1)}}{1 - e^{-j\omega}}$$

$$= \frac{1}{M_1 + M_2 + 1} \frac{e^{j\omega(M_1+M_2+1)/2} - e^{-j\omega(M_1+M_2+1)/2}}{1 - e^{-j\omega}} e^{-j\omega(M_2-M_1+1)/2}$$

$$= \frac{1}{M_1 + M_2 + 1} \frac{e^{j\omega(M_1+M_2+1)/2} - e^{-j\omega(M_1+M_2+1)/2}}{e^{j\omega/2} - e^{-j\omega/2}} e^{-j\omega(M_2-M_1)/2}$$

$$= \frac{1}{M_1 + M_2 + 1} \frac{\sin[\omega(M_1 + M_2 + 1)/2]}{\sin(\omega/2)} e^{-j\omega(M_2-M_1)/2}.$$

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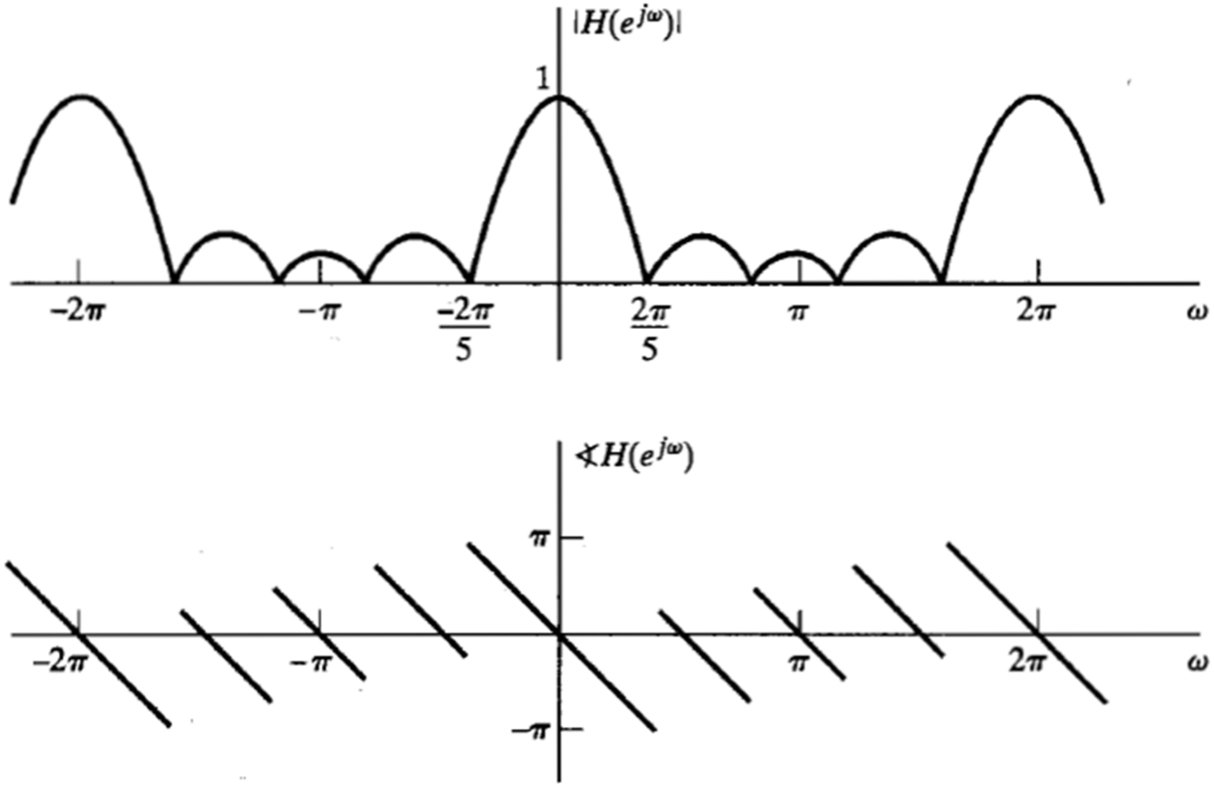
DTFT and LTI systems

- Magnitude/phase representation (Lowpass)



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DTFT and LTI systems

- Note:

$$x[n] \rightarrow \boxed{\text{LTI } h[n]} \rightarrow y[n] = x[n] * h[n]$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

  – Zero-input: zero output (LTI case)
  – Also,

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

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DTFT and LTI systems

- LCCDE for LTI systems (order N):

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

  - With non-zero a_0 and a_N
  - Taking DTFT of both sides:

$$\sum_{k=0}^{N} a_k e^{-j\omega k} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-j\omega k} X(e^{j\omega})$$

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DTFT and LTI systems

– Then,

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$= \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

- Remark: LCCDE & LTI
  - LCCDE solution y[n] for given x[n]:
    - Causal and non-causal solutions
    - Similar to CT case

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DTFT and LTI systems

- Initial rest assumption:
  - Zero output as long as input is zero
  - Unique causal LTI system
- Example: LTI system with initial rest

$$y[n] - ay[n-1] = x[n], \quad |a| < 1$$

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

- Causal system:

$$h[n] = a^n u[n]$$

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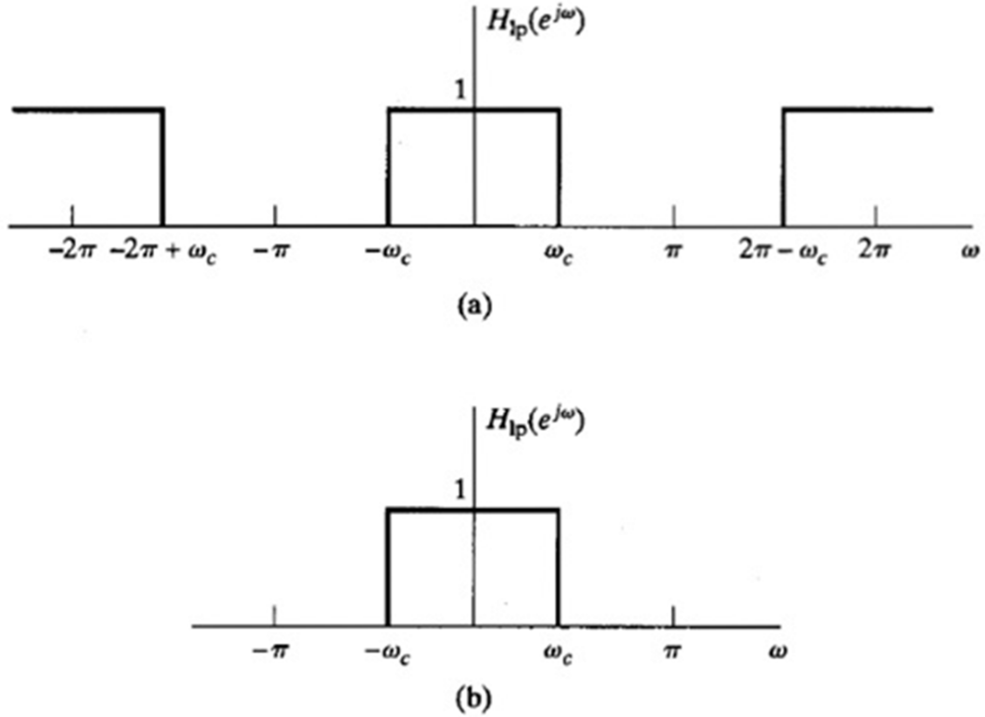
DTFT and LTI systems

- System examples: Lowpass filter (LPF)



$$H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi, \end{cases}$$

$$h_{lp}[n] = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty.$$

Ideal lowpass filter showing (a) periodicity of the frequency response and (b) one period of the periodic frequency response.

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DTFT and LTI systems

- Remarks on ideal LPF (not realizable)
  - Non-causal
  - Unstable
- The aim of filter design: designing approximation of the ideal filters

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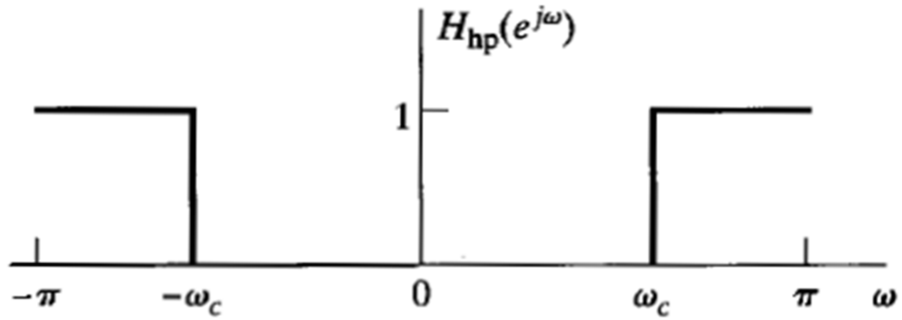
DTFT and LTI systems

- Highpass filter (HPF)



- Bandpass filter (BPF)