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- Symmetry property (Hermitian symmetry)
  - Let x[n] is real, then

$$x[n] = x^*[n]$$

    - which leads to the following symmetry

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$

    - Proof : straightforward
    - Useful/important results

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- Separation of odd/even components

$$x[n] = x_e[n] + x_o[n]$$

  – where

$$x_e[n] = \tfrac{1}{2}(x[n] + x^*[-n]) = x_e^*[-n]$$

$$x_o[n] = \tfrac{1}{2}(x[n] - x^*[-n]) = -x_o^*[-n]$$

  – Conjugate-symmetric/conjugate-antisymmetric

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– Similarly

$$X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega})$$

– with

$$X_e(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) + X^*(e^{-j\omega})]$$

$$X_o(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) - X^*(e^{-j\omega})].$$

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- Some properties

SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$

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- Cont. (for real signals)

The following properties apply only when $x[n]$ is real:

7.	Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8.	Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9.	Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10.	Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
11.	Any real $x[n]$	$\sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega})$ (phase is odd)
12.	$x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13.	$x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

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- Clues for the proofs

$$\mathcal{R}e\{x[n]\} = \frac{1}{2}(x[n] + x^*[n])$$

$$j\mathcal{J}m\{x[n]\} = \frac{1}{2}(x[n] - x^*[n])$$

  - The real case

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$

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– and,

$$X_R(e^{j\omega}) + jX_I(e^{j\omega}) = X_R(e^{-j\omega}) - jX_I(e^{-j\omega})$$

– then,

$$X_R(e^{j\omega}) = X_R(e^{-j\omega})$$

$$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$$

– similarly,

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\sphericalangle X(e^{j\omega})}$$

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- Which leads to

$$|X(e^{j\omega})| = \sqrt{X_R^2(e^{j\omega}) + X_I^2(e^{j\omega})} = \sqrt{X_R^2(e^{-j\omega}) + X_I^2(e^{-j\omega})} = |X(e^{-j\omega})|$$

- and

$$\begin{aligned}\measuredangle X(e^{j\omega}) &= tan^{-1}\frac{X_I(e^{j\omega})}{X_R(e^{j\omega})} \\ &= tan^{-1}\frac{-X_I(e^{-j\omega})}{X_R(e^{-j\omega})} \\ &= -tan^{-1}\frac{X_I(e^{-j\omega})}{X_R(e^{-j\omega})} \\ &= -\measuredangle X(e^{-j\omega})\end{aligned}$$

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- Example: $x[n] = a^n u[n]$ with $X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$ if $|a| < 1$

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} = X^*(e^{-j\omega}) \qquad \text{(property 7)},$$

$$X_R(e^{j\omega}) = \frac{1 - a\cos\omega}{1 + a^2 - 2a\cos\omega} = X_R(e^{-j\omega}) \qquad \text{(property 8)},$$

$$X_I(e^{j\omega}) = \frac{-a\sin\omega}{1 + a^2 - 2a\cos\omega} = -X_I(e^{-j\omega}) \qquad \text{(property 9)},$$

$$|X(e^{j\omega})| = \frac{1}{(1 + a^2 - 2a\cos\omega)^{1/2}} = |X(e^{-j\omega})| \qquad \text{(property 10)},$$

$$\sphericalangle X(e^{j\omega}) = \tan^{-1}\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right) = -\sphericalangle X(e^{-j\omega}) \qquad \text{(property 11)}.$$

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- Difference

$$\{x[n] - x[n-1]\} \leftrightarrow (1 - e^{-j\omega})X(e^{j\omega})$$

- Summation (skip the proof)

$$\left\{\sum_{m=-\infty}^{n} x[m]\right\} \leftrightarrow \frac{1}{1 - e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega + 2\pi k)$$

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- Differentiation in frequency domain
  - Observe that

$$\frac{d}{d\omega}X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} -jnx[n]e^{-j\omega n}$$

  - And therefore,

$$\{nx[n]\} \leftrightarrow j\frac{d}{d\omega}X(e^{j\omega})$$

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- Parseval theorem (important)

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(e^{j\omega})|^2 d\omega$$

- Energy is kept!!!
- Application: denoising in frequency domain

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- Convolution property
  - If

$$\{y[n]\} = \{h[n]\} \star \{x[n]\}$$

  - Then,

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

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- Summary of properties

FOURIER TRANSFORM THEOREMS

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$

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- Cont. (properties)

6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

9. $$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

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- DTFT pairs

FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
4. $a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{1 - ae^{-j\omega}}$

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- Example: $x[n] = a^n u[n-5]$
  - let $x_1[n] = a^n u[n]$
  - Then

$$X_1(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

  - Observe that $x_2[n] = x_1[n-5]$ with $x[n] = a^5 x_2[n]$
  - Hence,

$$X_2(e^{j\omega}) = \frac{e^{-j5\omega}}{1 - ae^{-j\omega}}$$

  - Finally,

$$X(e^{j\omega}) = \frac{a^5 e^{-j5\omega}}{1 - ae^{-j\omega}}$$

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- Example: compute the sequence

$$X(e^{j\omega}) = \frac{1}{(1-ae^{-j\omega})(1-be^{-j\omega})}$$

- Using partial fraction expansion

$$X(e^{j\omega}) = \frac{a/(a-b)}{1-ae^{-j\omega}} - \frac{b/(a-b)}{1-be^{-j\omega}}$$

- Therefore,

$$x[n] = \left(\frac{a}{a-b}\right) a^n u[n] - \left(\frac{b}{a-b}\right) b^n u[n].$$

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DTFT and LTI systems

- LTI systems

$$x[n] \rightarrow \boxed{\text{LTI } h[n]} \rightarrow y[n] = x[n] * h[n]$$

- Output: convolution

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- Impulse response h[n]
  - System response to the discrete-impulse

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DTFT and LTI systems

- Note: response to CES, viz. $x[n] = e^{j\omega n}$
- The output

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)}$$

$$= e^{j\omega n} \left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \right)$$

- Let $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$, then

$$y[n] = H(e^{j\omega}) e^{j\omega n}$$

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DTFT and LTI systems

- The concept of eigen-function
  - Relationship with eigen-vector
- The frequency response of the system

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

- DTFT of the impulse response h[n]
- Recovering h[n] from the frequency response
  - synthesis

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