Digital Signal Processing (DSP)

Fall 2014
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DIGITAL SIGNAL PROCESSING (DSP)

Lecture 2
Discrete-time signals and systems

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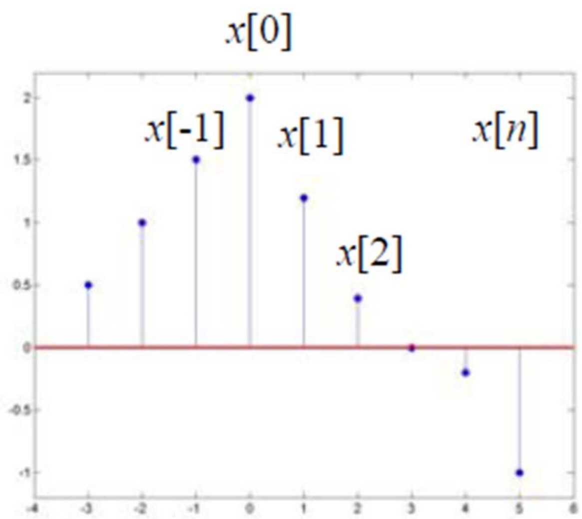
Discrete-time signals

- Discrete-time (DT) signals
  - A sequence of numbers: {x[n]}
    - Index n: integer
    - Sample x[n]: generally complex
  - CT signals & Sampling
  - x[n]=x(nTs)



http://kom.aau.dk/~zt/courses/DSP/MM1/Digital%20Signal%20Processing,%20MM1,%20version%200.6.pdf

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Discrete-time signals

- Operations
  - Sum of signals (element-wise)
  - Product of signals (element-wise)
  - Time-shift: delay
    - $y[n]=x[n-m]$
    - Integer m
  - Multiplication by the constant *a*
    - $ax[n]$

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Discrete-time signals

- Basic DT signals
  - Discrete-time impulse (unit sample sequence)

$$\delta[n] = \begin{cases} 0, & n \neq 0, \\ 1, & n = 0, \end{cases}$$

  - A way for describing arbitrary signal

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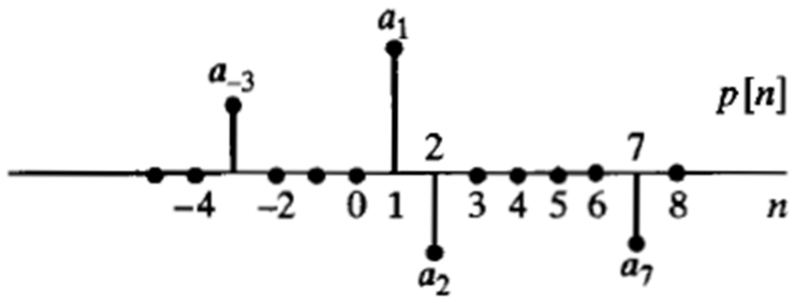
Discrete-time signals

– Example:



$$p[n] = a_{-3}\delta[n+3] + a_1\delta[n-1] + a_2\delta[n-2] + a_7\delta[n-7]$$

– General case

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

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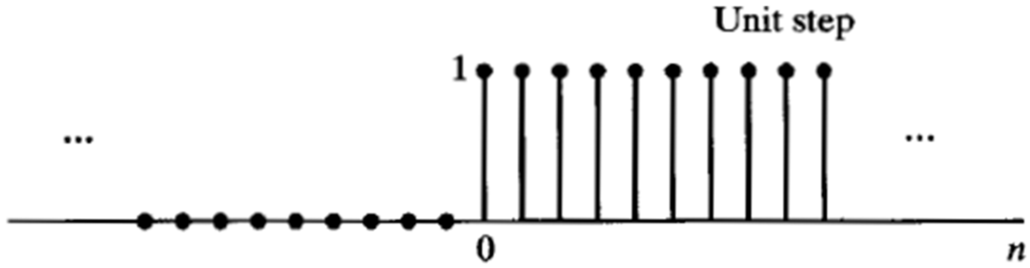
Discrete-time signals

– Unit step sequence

$$u[n]=\begin{cases}1, & n\ge 0,\\ 0, & n<0,\end{cases}$$



– Useful relationships

$$u[n]=\sum_{k=-\infty}^{n}\delta[k] \qquad\qquad \delta[n]=u[n]-u[n-1]$$

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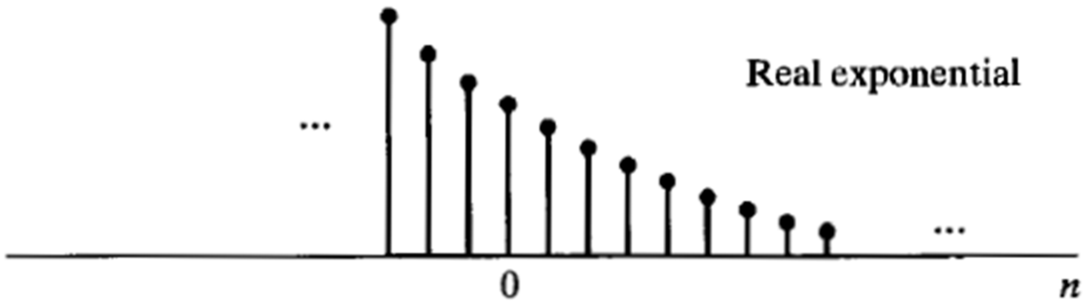
Discrete-time signals

– Exponential sequence: $x[n] = A\alpha^n$



$$x[n] = \begin{cases} A\alpha^n, & n \geq 0, \\ 0, & n < 0 \end{cases}$$

- Discuss various cases: real, complex, value of α

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Discrete-time signals

– A more detailed view

$$x[n] = A\alpha^n = |A|e^{j\phi}|\alpha|^n e^{j\omega_0 n}$$

$$= |A|\,|\alpha|^n e^{j(\omega_0 n+\phi)}$$

$$= |A|\,|\alpha|^n \cos(\omega_0 n + \phi) + j|A|\,|\alpha|^n \sin(\omega_0 n + \phi)$$

– Complex exponential sequence (CES): $|\alpha| = 1$

$$x[n] = |A|e^{j(\omega_0 n+\phi)} = |A|\cos(\omega_0 n + \phi) + j|A|\sin(\omega_0 n + \phi)$$

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Discrete-time signals

- Important properties of CES
  - Periodicity with respect to ω_0

$$x[n] = Ae^{j(\omega_0+2\pi)n}$$
$$= Ae^{j\omega_0 n}e^{j2\pi n} = Ae^{j\omega_0 n}$$

- The considered interval

$$-\pi < \omega_0 \leq \pi \text{ or } 0 \leq \omega_0 < 2\pi$$

- Difference with the CT case

Discrete-time signals

– Periodicity with respect to n

$$x[n] = x[n+N], \qquad \text{for all } n$$

– The period is N iff

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n}$$

– or

- Integer *k* $\omega_0 N = 2\pi k$
- Not periodic necessarily!!

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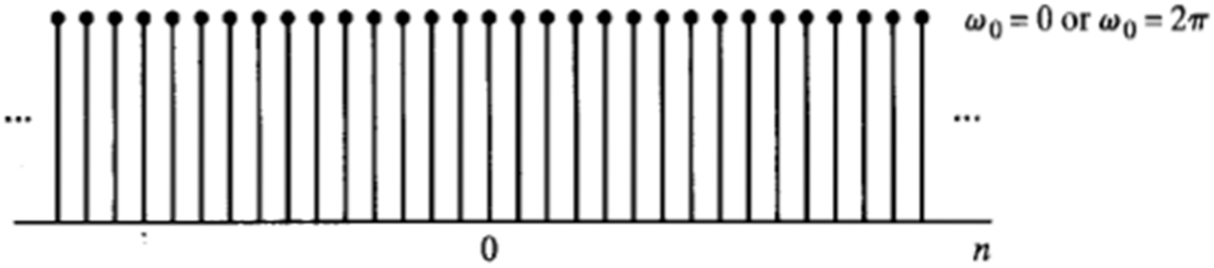
Discrete-time signals



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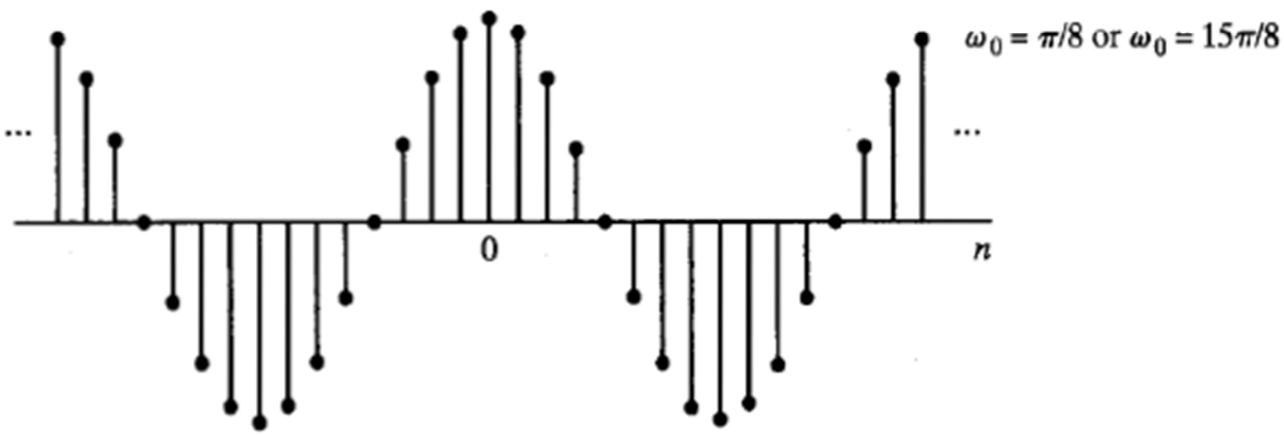

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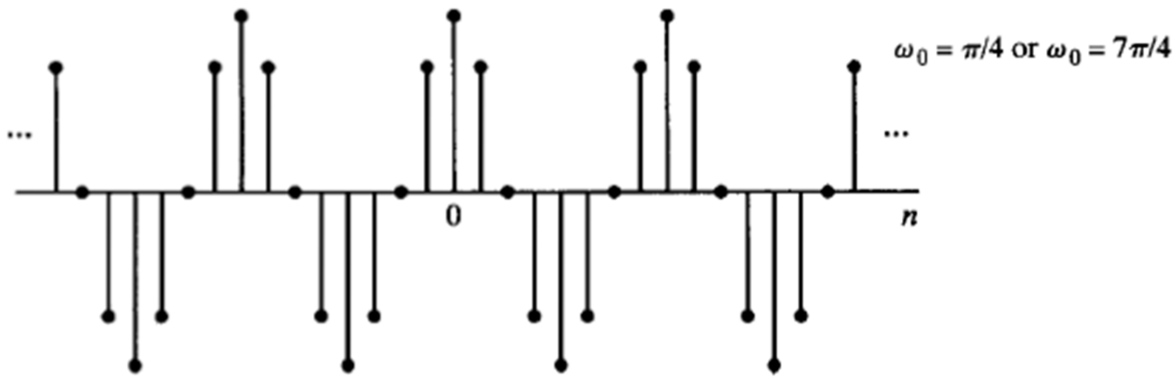
Discrete-time signals



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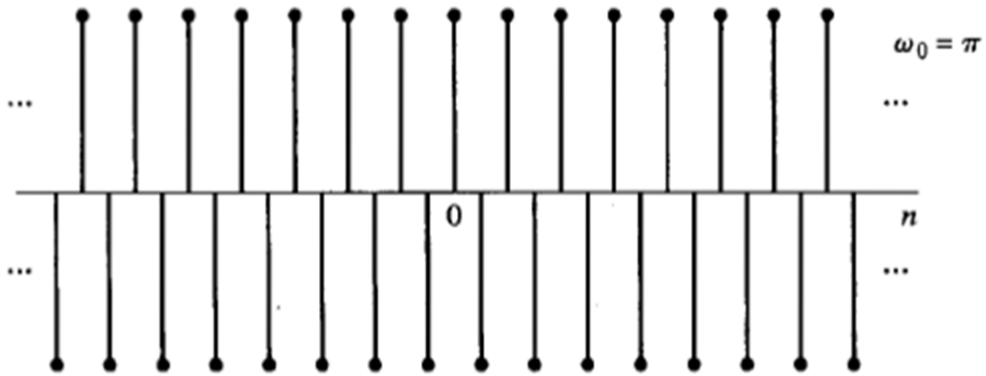

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Discrete-time signals

- Some definitions
  - Energy signals: limited energy

$$E_x \triangleq \sum_{n=-\infty}^{\infty} |x[n]|^2$$

  - Power signals: limited power

$$P_x \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

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Discrete-time signals

– Bounded signals

$$|x[n]| \leq B_x < \infty$$

– Absolutely summable signals

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

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Discrete-time signals

– Convolution

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- LTI systems

– Relation to correlation

- Important applications

$$r_{xx}[l] \triangleq \sum_{n=-\infty}^{\infty} x[n]\, x^*[n-l] = \sum_{n=-\infty}^{\infty} x[n+l]\, x^*[n],\ l = 0, \pm 1, \pm 2, \ldots .$$

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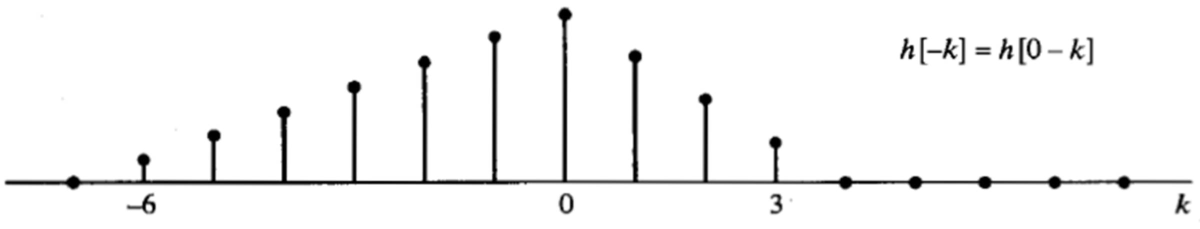
Discrete-time signals

(a)



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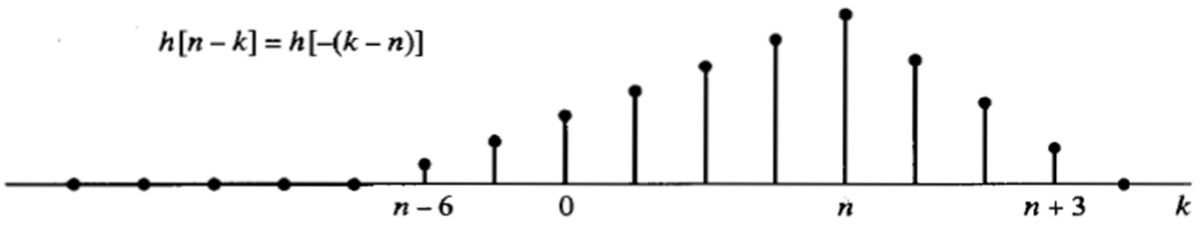

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Discrete-time systems

- Discrete-time systems
  - Maps the input to the output $y[n] = T\{x[n]\}$

$$\begin{array}{c}\text{Input Signal}\\ x[n]\end{array} \rightarrow \boxed{\text{Discrete-time system}} \rightarrow \begin{array}{c}\text{Output Signal}\\ y[n]\end{array}$$

  - Example: delay system

$$y[n] = x[n - n_d], \qquad -\infty < n < \infty,$$

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Discrete-time systems

– Time reversal

- y[n]=x[-n]

– Moving average (MA)

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

$$= \frac{1}{M_1 + M_2 + 1} \{x[n+M_1] + x[n+M_1-1] + \cdots + x[n] + x[n-1] + \cdots + x[n-M_2]\}.$$

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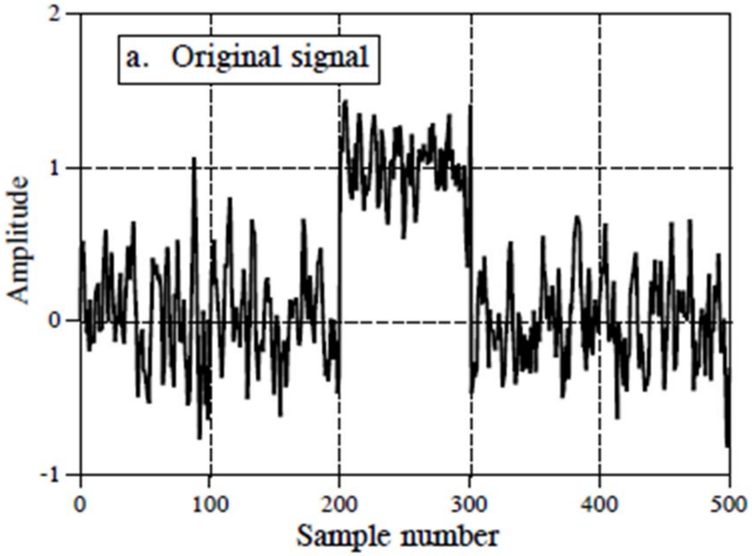
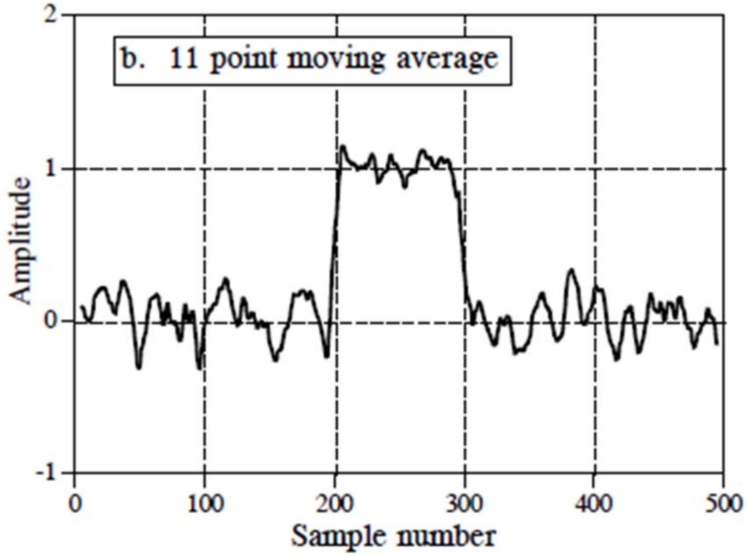
Discrete-time systems

- MA illustration:





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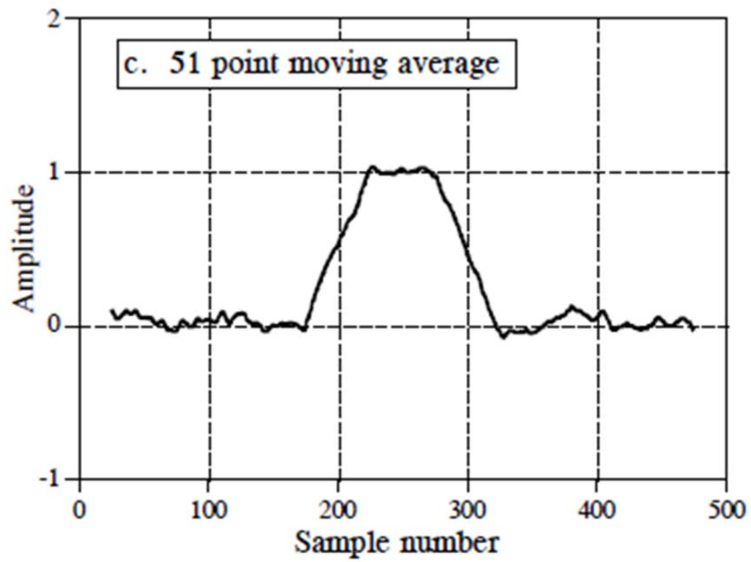
Discrete-time systems

- MA illustration:



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Discrete-time systems

- Basic properties
  - Linear

$$x[n] = Ax_1[n] + Bx_2[n] \rightarrow \boxed{\text{Discrete-time system}} \rightarrow y[n] = Ay_1[n] + By_2[n]$$

  - Memoryless
    - Current output depends only on the current input

$$y[n] = e^{x[n]}/\sqrt{n-2}$$

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Discrete-time systems

– Causal

$$y[n] = F\{x[n], x[n-1], x[n-2], \ldots\}$$

- Online/off-line processing

– Time-invariant:

$$x[n] \xrightarrow{\mathcal{T}} y[n] \quad \text{implies that} \quad x[n-k] \xrightarrow{\mathcal{T}} y[n-k]$$

– Behavior does not change with time shift

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Discrete-time systems

– Examples:
  - Moving average
  - Scaling: y[n]=x[3n]

- BIBO Stable (bounded input bounded output)

If $\exists M_x$ s.t. $|x[n]| \leq M_x < \infty \; \forall n$, then there must exist an M_y s.t. $|y[n]| \leq M_y < \infty \; \forall n$

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Discrete-time systems

- LTI: linear time-invariant systems
  - Physical systems
  - Modeling
  - Analysis tools

- Impulse response h[n]

$$\delta[n-k] \xrightarrow{\mathcal{T}} h_k[n]$$

  - System response to the discrete-impulse

A LTI system is characterized completely by its impulse response $h[n]$

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Discrete-time systems

- Input-output property

$$x[n] \rightarrow \boxed{\text{LTI } h[n]} \rightarrow y[n] = x[n] * h[n]$$

- Notes (LTI systems)
  - Causality

    An LTI system is **causal** iff its impulse response $h[n] = 0$ for all $n < 0$.

  - Stability: summability of impulse response

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Discrete-time systems

- Examples of LTI systems
  - Accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

  - Impulse response
    - Causal
    - Unstable

$$h[n] = \sum_{k=-\infty}^{n} \delta[k] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0, \end{cases} = u[n].$$

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Discrete-time systems

- Example

$$h[n] = a^n u[n] \text{ with } |a| < 1$$

  - Causal
  - Stable

$$S = \sum_{n=0}^{\infty} |a|^n = \frac{1}{1 - |a|} < \infty.$$

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Discrete-time systems

- FIR and IIR concepts
  - Finite impulse response

finite impulse response or **FIR**: only a finite number of $h[n]$ are nonzero

$$h[n] = \{h_0, h_1, h_2, \cdots, h_M, 0, 0, \cdots\}$$

$$\begin{aligned} h[n] &= 2\delta[n] + \delta[n-1] - \delta[n-3] \\ &= \{2, 1, 0, -1\} \end{aligned}$$

  - Infinite impulse response

infinite impulse response or **IIR**: an infinite number of $h[n]$ are nonzero

    - Impulse response of accumulator

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Discrete-time systems

- Connections of LTI systems

$$x[n] \rightarrow \boxed{h_1[n]} \rightarrow \boxed{h_2[n]} \rightarrow y[n]$$

$$x[n] \rightarrow \boxed{h_1[n] * h_2[n]} \rightarrow y[n]$$

$$x[n] \rightarrow \boxed{h_2[n] * h_1[n]} \rightarrow y[n]$$

$$x[n] \rightarrow \boxed{h_2[n]} \rightarrow \boxed{h_1[n]} \rightarrow y[n]$$

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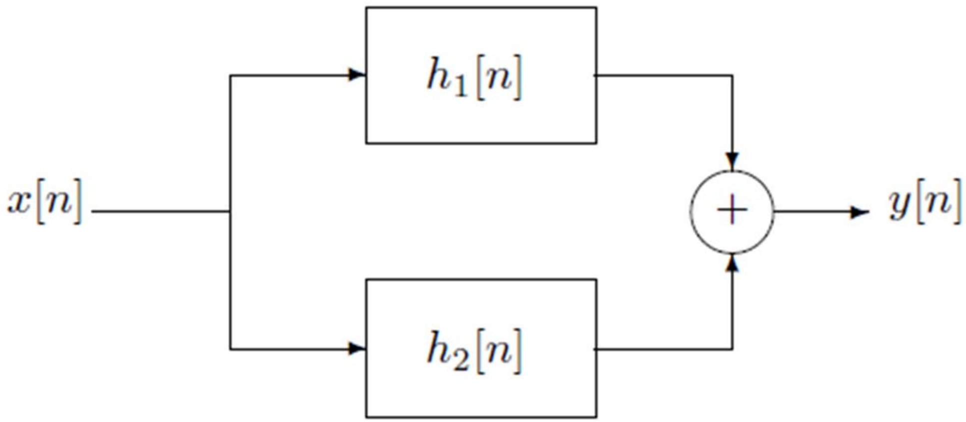
Discrete-time systems

- Connections of LTI systems



Distributive: $x[n] \rightarrow \boxed{h[n] = h_1[n] + h_2[n]} \rightarrow y[n]$

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Discrete-time systems

- Linear constant coefficient difference equations (LCCDE): a way for describing LTI systems

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

- With non-zero a_0 and a_N

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Discrete-time systems

- Example: accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

  - Clearly

$$y[n-1] = \sum_{k=-\infty}^{n-1} x[k]$$

  - and therefore,

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

  - Which leads to

$$y[n] - y[n-1] = x[n]$$

$$N = 1, a_0 = 1, a_1 = -1, M = 0, \text{ and } b_0 = 1$$