CT Fourier transform

– 4) differentiation

$$\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(j\omega)$$

– 5) time-scaling

$$x(at) \xleftrightarrow{F} \frac{1}{|a|} X(\frac{j\omega}{a})$$

– useful

$$x(-t) \xleftrightarrow{F} X(-j\omega)$$

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CT Fourier transform

– 6) modulation

$$e^{j\omega_0 t}x(t) \stackrel{F}{\longleftrightarrow} X(j(\omega-\omega_0))$$

– 7) Parseval theorem

$$\int_{-\infty}^{\infty}|x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2 d\omega$$

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– 8) convolution

$$y(t) = h(t) * x(t) \overset{F}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega)$$

– 9) multiplication in time

$$r(t) = s(t)p(t) \longleftrightarrow R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))d\theta$$

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- Example: the differentiator system

$$y(t) = \frac{dx(t)}{dt}$$

  – Then,

$$Y(j\omega) = j\omega X(j\omega)$$

  – and

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = j\omega$$

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- Example: an LTI system with impulse response

$$h(t) = e^{-at}u(t), \qquad a > 0$$

  - And the input $x(t) = e^{-bt}u(t), \qquad b > 0$
  - We know

$$X(j\omega) = \frac{1}{b + j\omega} \qquad H(j\omega) = \frac{1}{a + j\omega}$$

  - Which leads to

$$Y(j\omega) = \frac{1}{(a + j\omega)(b + j\omega)}$$

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– Using partial fractional expansion ($a \neq b$)

$$Y(j\omega) = \frac{1}{b-a}\left[\frac{1}{a+j\omega} - \frac{1}{b+j\omega}\right]$$

– and,

$$y(t) = \frac{1}{b-a}\left[e^{-at}u(t) - e^{-bt}u(t)\right]$$

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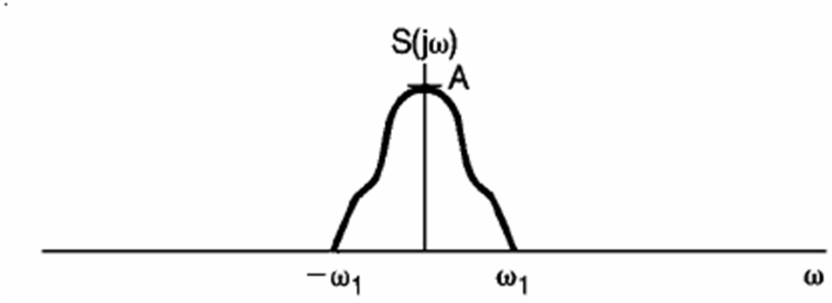
- An important example: s(t) with the following spectrum



  – Obtain the spectrum of $r(t) = s(t)\,p(t)$ where

$$p(t) = \cos\omega_0 t$$

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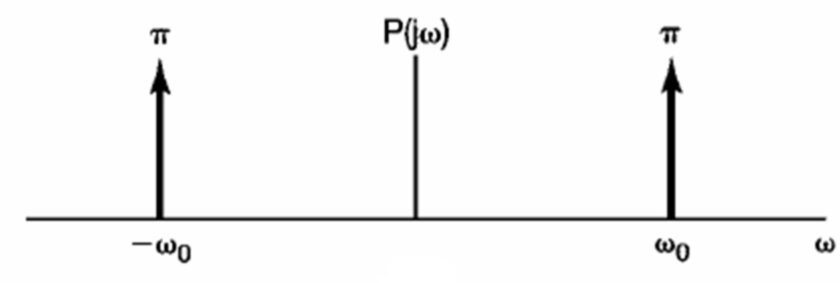
- We know that

$$P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



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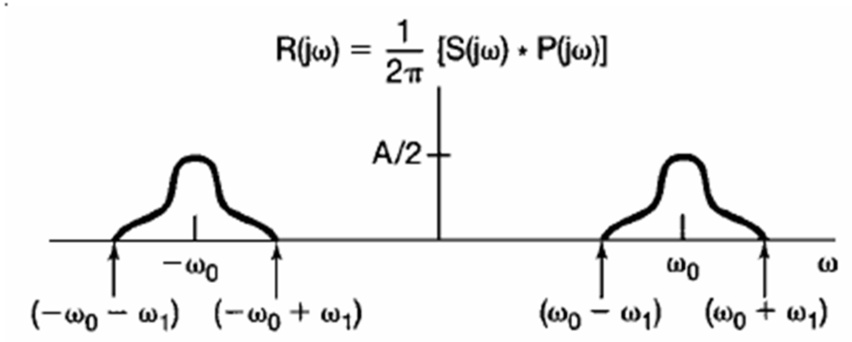
– Using multiplication property 8):

$$R(j\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}S(j\omega)P(j(\omega-\theta))d\theta$$

$$=\frac{1}{2}S(j\omega-\omega_0)+\frac{1}{2}S(j\omega+\omega_0)$$



- Derive this result using modulation property

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CT Fourier transform

- Table of properties

PROPERTIES OF THE FOURIER TRANSFORM

Property	Aperiodic signal	Fourier transform
	$x(t)$	$X(j\omega)$
	$y(t)$	$Y(j\omega)$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
Frequency Shifting	$e^{j\omega_0 t} x(t)$	$X(j(\omega - \omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|} X\left(\frac{j\omega}{a}\right)$

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- Table (cont.)

Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{Re}\{X(j\omega)\} = \mathcal{Re}\{X(-j\omega)\} \\ \mathcal{Im}\{X(j\omega)\} = -\mathcal{Im}\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd

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CT Fourier transform

- Some FT pairs

BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
$x(t) = 1$	$2\pi \delta(\omega)$

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- FT pairs (cont.)

$\dfrac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$
$\delta(t)$	1
$u(t)$	$\dfrac{1}{j\omega} + \pi\delta(\omega)$
$\delta(t - t_0)$	$e^{-j\omega t_0}$
$e^{-at}u(t), \mathcal{R}e\{a\} > 0$	$\dfrac{1}{a + j\omega}$

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CT Fourier transform

- Periodic signals
  - Theorem: the FT of the periodic signal x(t) is given by

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

  - Where

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt$$

  - And T is the period of the signal

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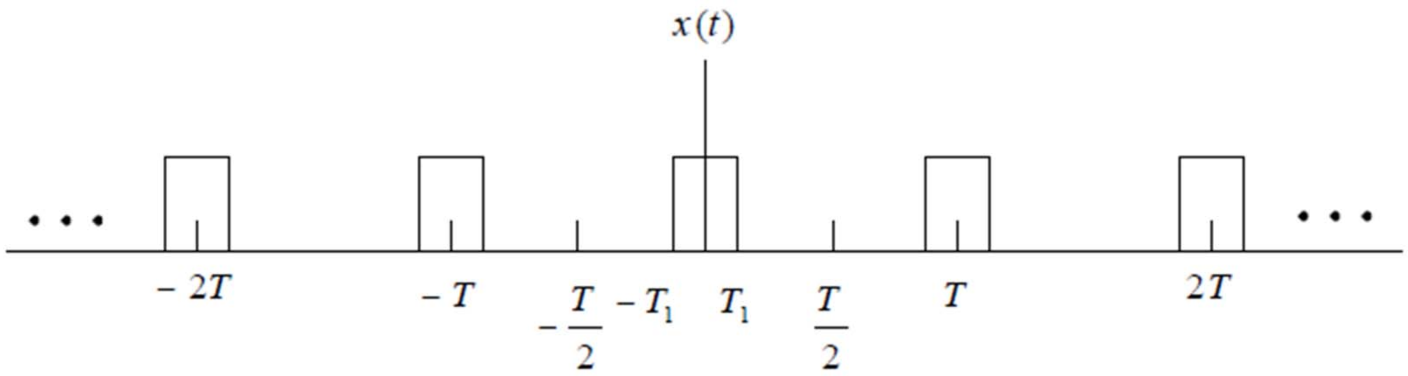
- Example: (power signal)



$$a_k = \frac{\sin k\omega_0 T_1}{\pi k}$$

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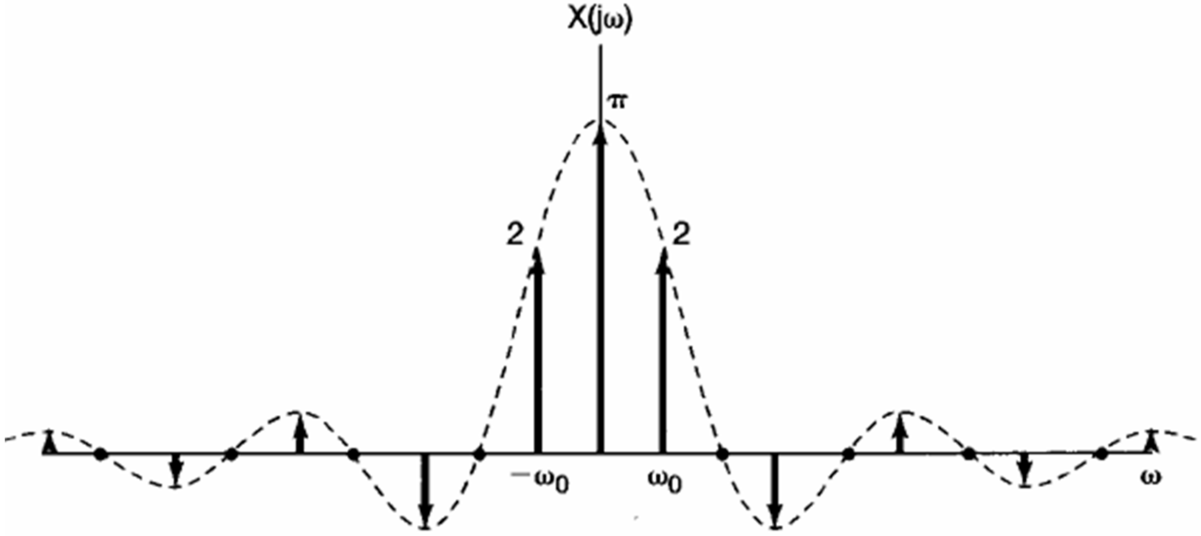
CT Fourier transform

– The FT

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$$



Fourier transform of a symmetric periodic square wave.

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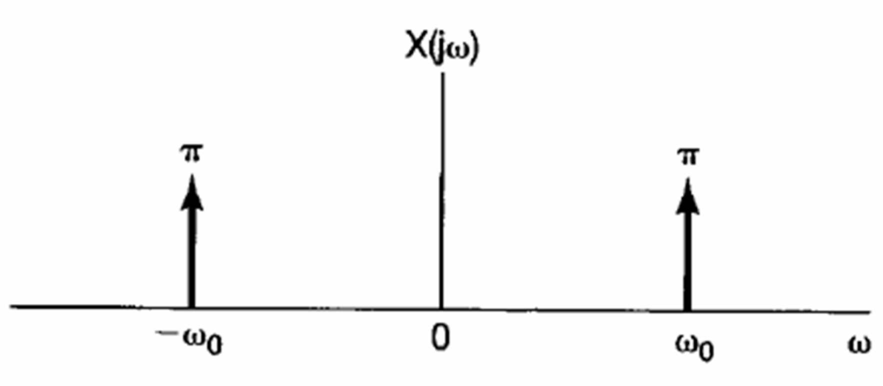
- Example: sin function is a special case

$$x(t) = \cos\omega_0 t$$



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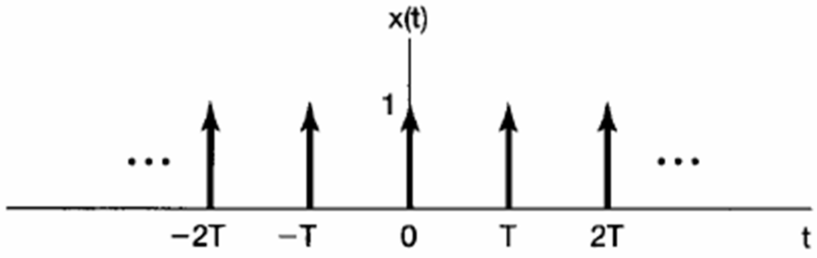
- A useful example: FT of the impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$



  - Periodic

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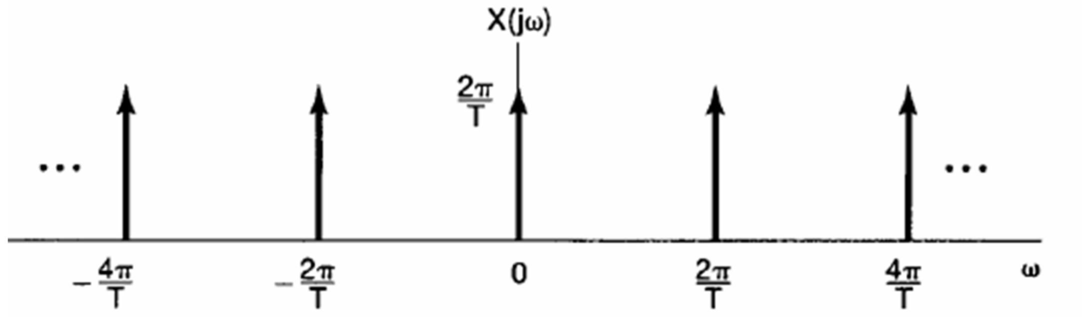
CT Fourier transform

- We begin by computing

$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2}\delta(t)e^{-j\omega_0 t} = \frac{1}{T}$$

- Leading to

$$X(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - \frac{2\pi k}{T}{}_0)$$



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