DEFLECTIONS: ENERGY METHODS

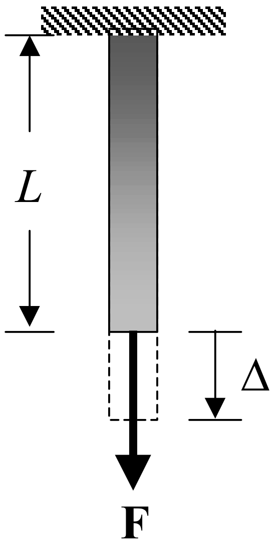
- External Work and Strain Energy
- Principle of Work and Energy
- Principle of Virtual Work
- Method of Virtual Work:
  - Trusses
  - Beams and Frames
- Castigliano's Theorem
  - Trusses
  - Beams and Frames

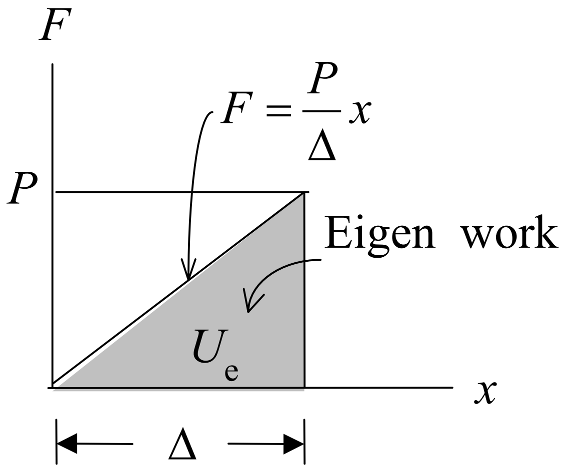
External Work and Strain Energy

Most energy methods are based on the *conservation of energy principle,* which states that the work done by all the external forces acting on a structure, U_e, is transformed into internal work or strain energy, U_i.

$$U_e = U_i$$

- **External Work-Force.**





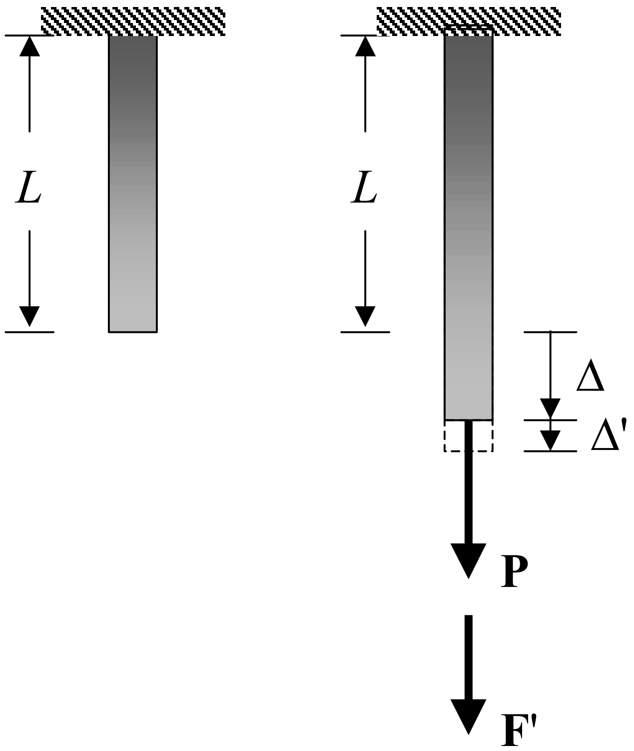
$$dU_e = F dx$$

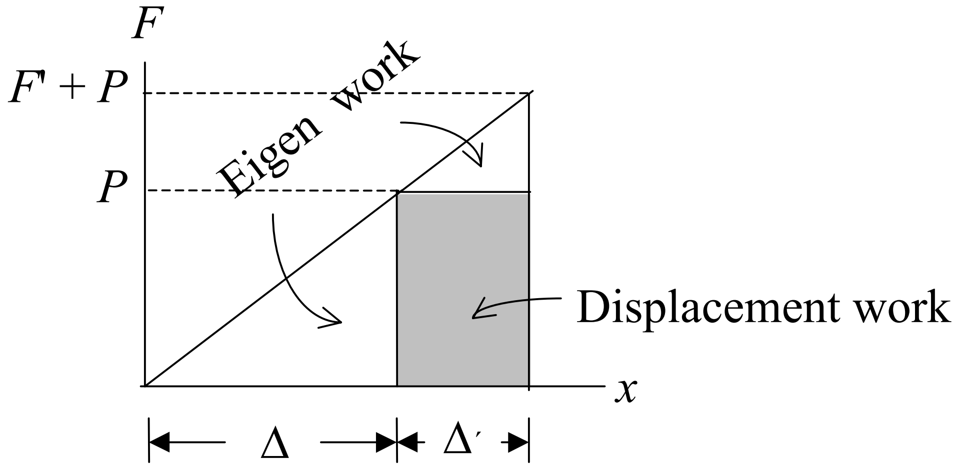
$$U_e = \int_0^x F dx$$

As the magnitude of **F** is *gradually* increased from zero to some limiting value $F = P$, the final elongation of the bar becomes Δ.

$$U_e = \int_0^\Delta \left(\frac{P}{\Delta} x\right) dx$$

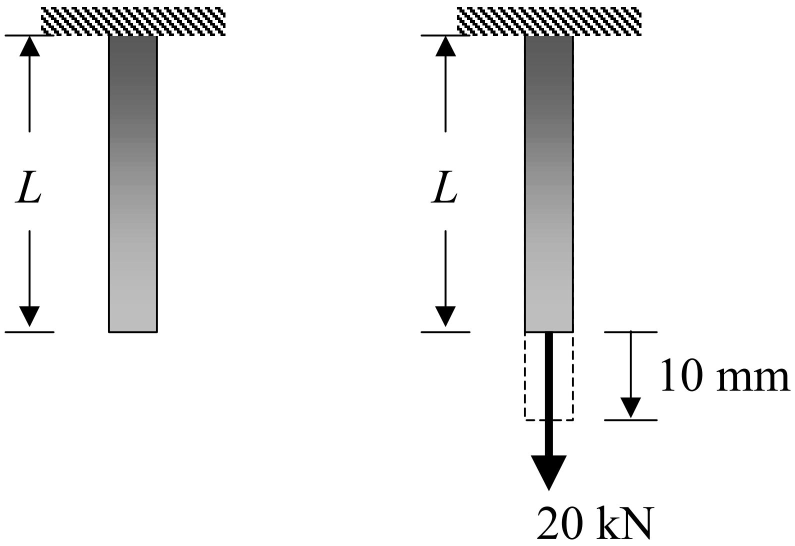
$$U_e = \left(\frac{P}{2\Delta} x^2\right)\Bigg|_0^\Delta = \underbrace{\frac{1}{2} P\Delta}_{\text{Eigen work}}$$

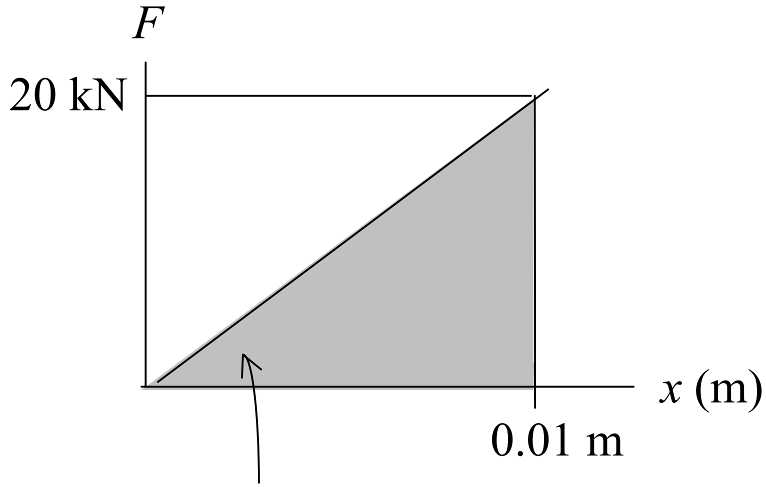




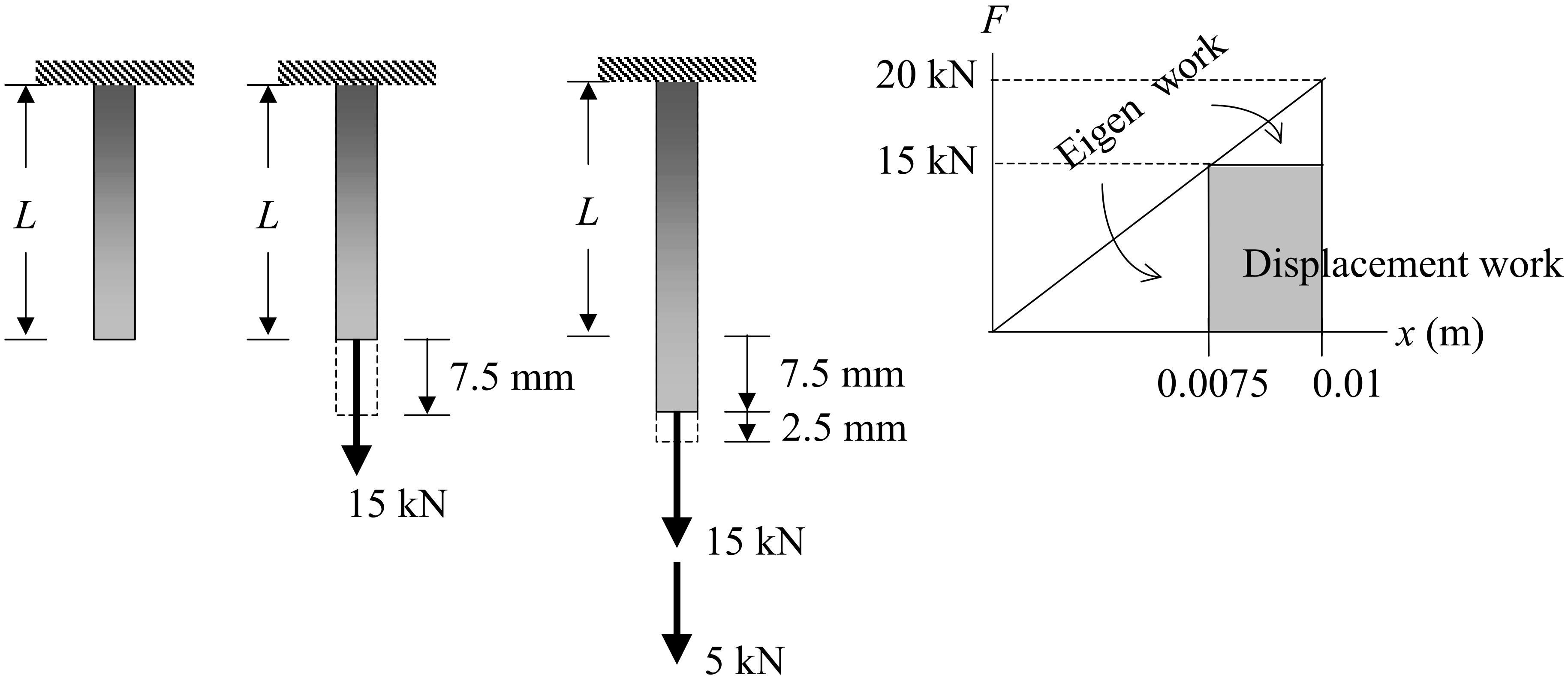
$$(U_e)_{Total} = (\text{Eigen Work})_P + (\text{Eigen Work})_{F'} + (\text{Displacement work})_P$$

$$(U_e)_{Total} = \frac{1}{2}(\Delta)(P) + \frac{1}{2}(\Delta')(F') + P(\Delta')$$





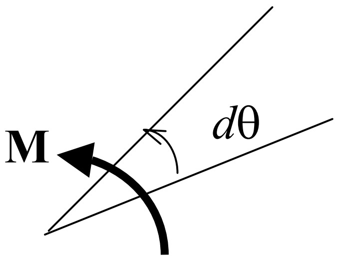
$$U_e = \frac{1}{2}(0.01)(20\times 10^3) = 100\ N \bullet m$$

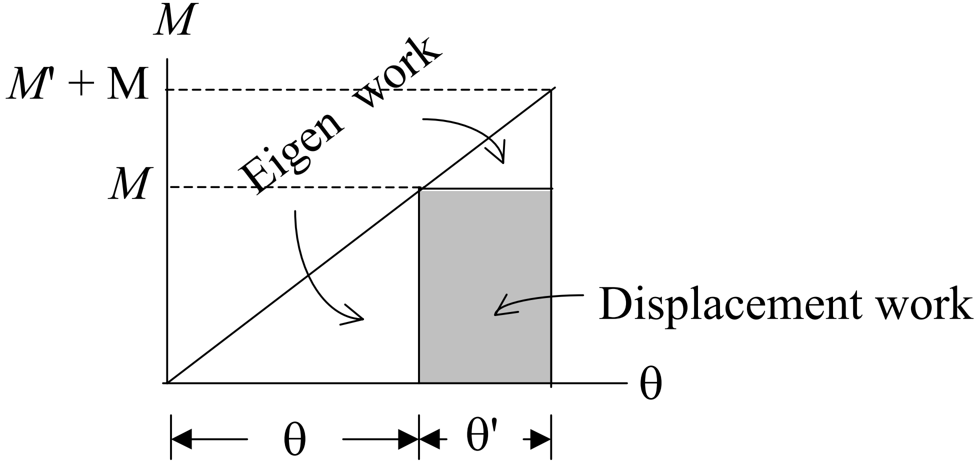


$$W = \frac{1}{2}(0.0075)(15\times10^3) + \frac{1}{2}(0.0025)(5\times10^3) + (0.0025)(15\times10^3)$$

$$= 56.25 + 6.25 + 37.50 = 100\ N \bullet m$$

- **External Work - Moment.**





$$dU_e = Md\theta$$

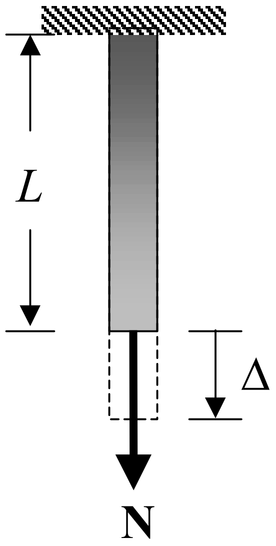
$$U_e = \int_0^\theta Md\theta \qquad \text{-----(8-12)}$$

$$U_e = \overbrace{\frac{1}{2}M\theta}^{\text{Eigen work}} \qquad \text{-----(8-13)}$$

$$(U_e)_{Total} = \frac{1}{2}\theta M + \frac{1}{2}\theta' M' + M\theta'$$

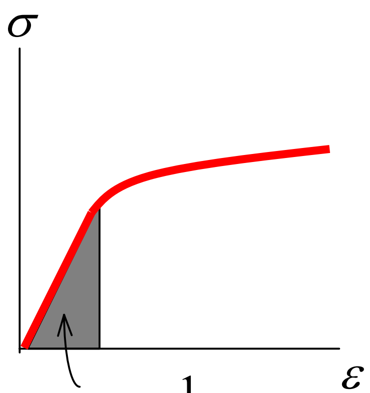
$$(U_e)_{Total} = \frac{1}{2}(M + M')(\theta + \theta') \qquad ---(8-14)$$

- **Strain Energy-Axial Force.**



$$E = \frac{\sigma}{\varepsilon}$$

$$\sigma = \frac{N}{A}$$



$$U_o = \frac{1}{2}\sigma\varepsilon$$

$$U_i = \int_V U_0 dV$$

$$= \int_V (\frac{1}{2}\sigma\varepsilon)(dV)$$

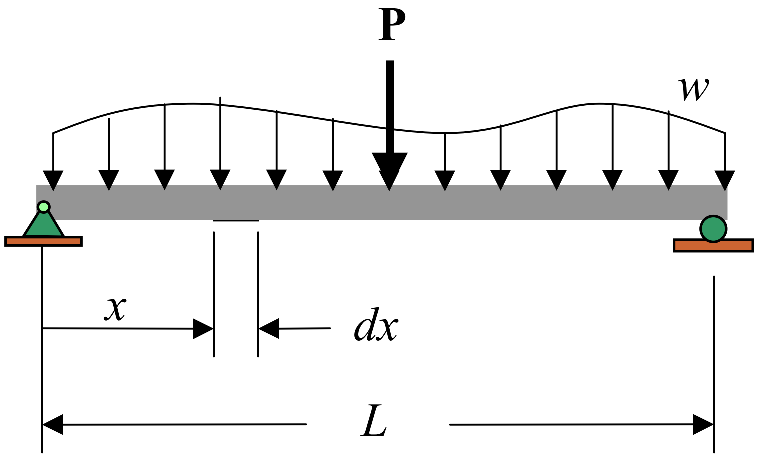
$$= \int_V \frac{1}{2}(\frac{\sigma^2}{E})dV$$

$$= \int_V \frac{1}{2E}(\frac{N}{A})^2 dV$$

$$= \int_L \frac{1}{2E}(\frac{N}{A})^2 Adx$$

$$= \int_L (\frac{N^2}{2EA})dx$$

- **Strain Energy-Bending**



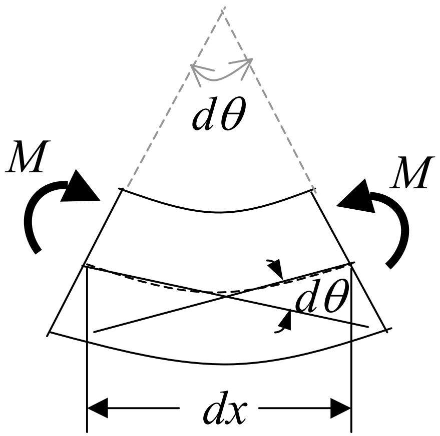
$$U_i = \int_V U_0 dV$$

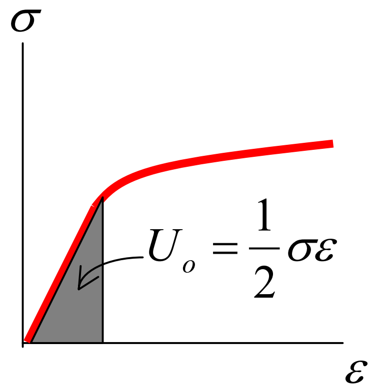
$$= \int_V \left(\frac{1}{2}\sigma\varepsilon\right)(dV)$$

$$= \int_V \frac{1}{2}\left(\frac{\sigma^2}{E}\right)dV$$

$$= \int_V \frac{1}{2E}\left(\frac{My}{I}\right)^2 dV$$

$$= \int_V \frac{1}{2E}\left(\frac{M^2 y^2}{I^2}\right)dV$$





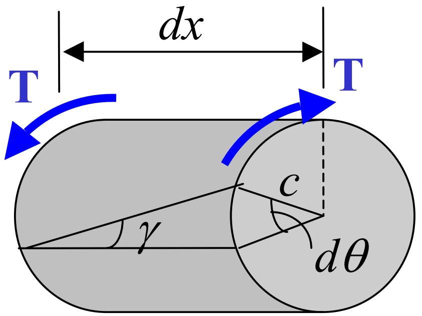
$$= \int_L \frac{1}{2E}\left(\frac{M^2}{I^2}\right)\left(\int_A y^2 dA\right)dx$$

$\left(\int_A y^2 dA\right) \rightarrow I$

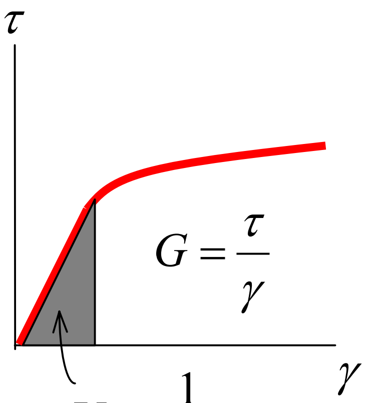
$$\boxed{= \int_L \left(\frac{M^2}{2EI}\right)dx}$$

$$\sigma = \frac{My}{I}$$

• Strain Energy-Torsion



$$\tau = \frac{T\rho}{J}$$



$$G = \frac{\tau}{\gamma}$$

$$U_o = \frac{1}{2}\tau\gamma$$

$$U_i = \int_V U_0 \, dV$$

$$= \int_V \left(\frac{1}{2}\tau\gamma\right) dV$$

$$= \int_V \frac{1}{2}\left(\frac{\tau^2}{G}\right) dV$$

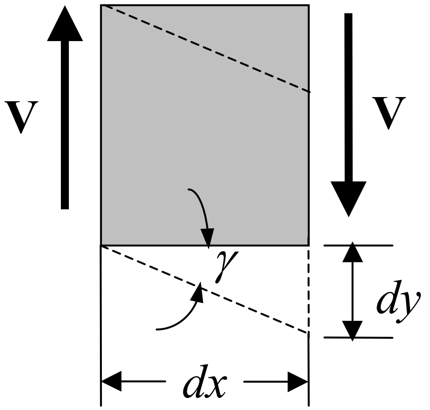
$$= \int_V \frac{1}{2G}\left(\frac{T\rho}{J}\right)^2 dV$$

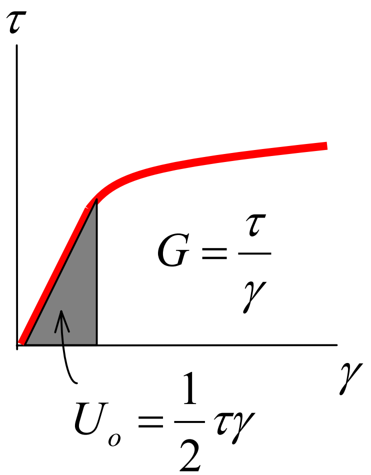
$$= \int_L \frac{1}{2G}\left(\frac{T^2}{J^2}\right)\left(\int_A \rho^2 \, dA\right) dx$$

($\int_A \rho^2 \, dA \rightarrow J$)

$$U_i = \int_L \frac{T^2}{2GJ} dx$$

- **Strain Energy-Shear**





$$U_i = \int_V U_0 dV$$

$$= \int_V (\frac{1}{2}\tau\gamma)(dV)$$

$$= \int_V \frac{1}{2}(\frac{\tau^2}{G}) dV$$

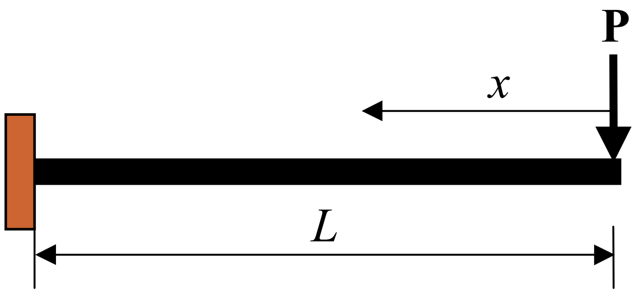
$$= \int \frac{1}{2G}(\frac{VQ}{It})^2 \, dV$$

$$= \int_L \frac{V^2}{2G}(\int_A \frac{Q^2}{It} dA)\, dx$$

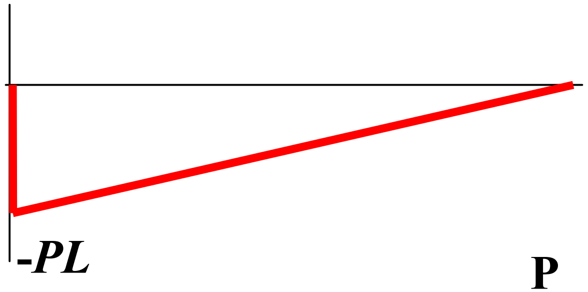
$\frac{K}{A}$

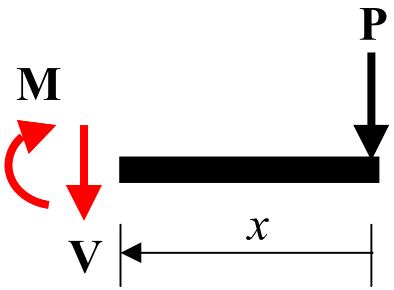
$$U_i = \int_L K \frac{V^2}{2GA} dx$$

Principle of Work and Energy



M diagram





$$+\circlearrowright \; \Sigma M_x = 0: \quad -M - Px = 0$$

$$M = -Px$$

$$U_e = U_i$$

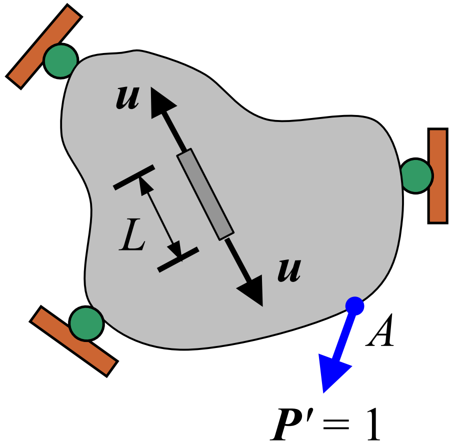
$$\frac{1}{2}P\Delta = \int_0^L \frac{M^2 dx}{2EI}$$

$$\frac{1}{2}P\Delta = \int_0^L \frac{(-Px)^2 dx}{2EI}$$

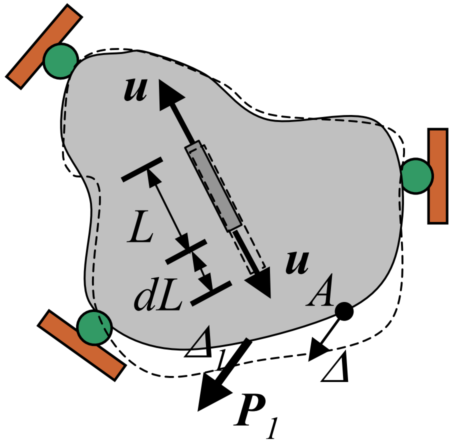
$$\frac{1}{2}P\Delta = \left.\frac{P^2 x^3}{6EI}\right|_0^L$$

$$\Delta = \frac{PL^3}{3EI}$$

Principle of Virtual Work



Apply virtual load ***P′*** first.



Then apply real load $\boldsymbol{P_1}$.

$$\delta U_e = \delta U_i$$

$$(\cancel{\frac{1}{2}P_1\Delta_1}) + 1 \bullet \Delta = \cancel{\int U_0 dV} + \Sigma u \bullet dL$$

↑ ↑ Real Work

Virtual loadings ↓ ↓

$$1 \bullet \Delta = \Sigma u \bullet dL$$

↑ ↑ Real displacements

In a similar manner,

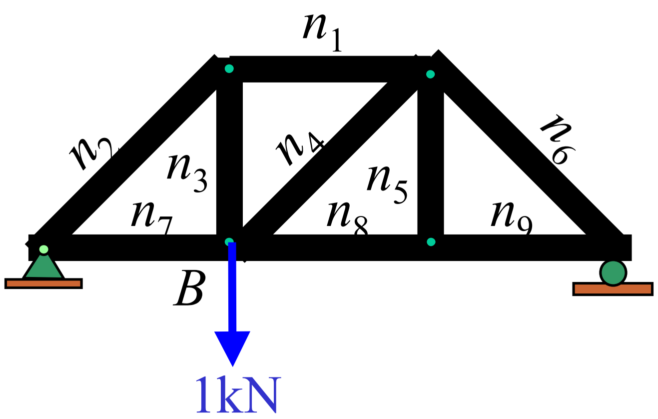
Virtual loadings ↓ ↓

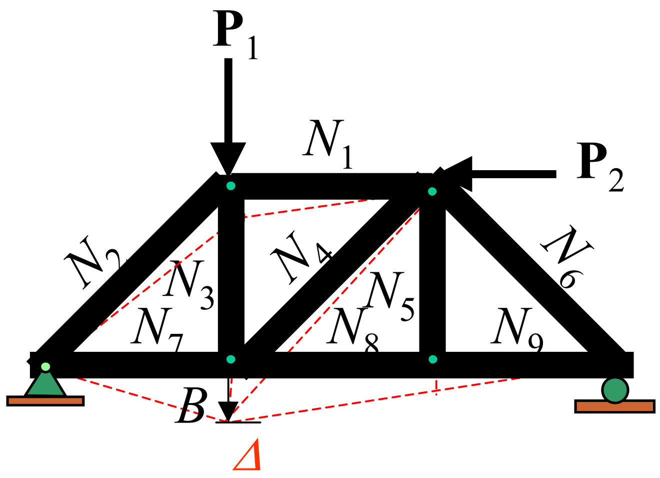
$$1 \bullet \theta = \Sigma u_\theta \bullet dL$$

↑ ↑ Real displacements

Method of Virtual Work : Truss

- **External Loading.**





$$1 \bullet \Delta = \Sigma \frac{nNL}{AE}$$

Where:

1 = external virtual unit load acting on the truss joint in the stated direction of Δ

n = internal virtual normal force in a truss member caused by the external virtual unit load

Δ = external joint displacement caused by the real load on the truss

N = internal normal force in a truss member caused by the real loads

L = length of a member

A = cross-sectional area of a member

E = modulus of elasticity of a member

- **Temperature**

$1 \cdot \Delta \quad = \quad \Sigma u \cdot dL$

dL

$$1 \bullet \Delta = \Sigma n \alpha (\Delta T) L$$

Where:

Δ = external joint displacement caused by the temperature change
α = coefficient of thermal expansion of member
ΔT = change in temperature of member

- **Fabrication Errors and Camber**

$1 \cdot \Delta \quad = \quad \Sigma u \cdot dL$

dL

$$1 \bullet \Delta = \Sigma n \Delta L$$

Where:

Δ = external joint displacement caused by the fabrication errors
ΔL = difference in length of the member from its intended size as caused by a fabrication error

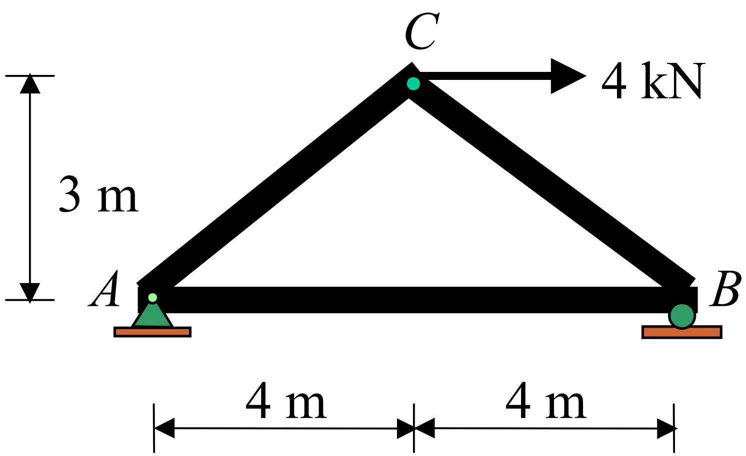
Example 8-15

The cross-sectional area of each member of the truss shown in the figure is $A = 400\ \text{mm}^2$ and $E = 200$ GPa.

(a) Determine the vertical displacement of joint *C* if a 4-kN force is applied to the truss at C.

(b) If no loads act on the truss, what would be the vertical displacement of joint *C* if member *AB* were 5 mm too short?

(c) If 4 kN force and fabrication error are both accounted, what would be the vertical displacement of joint *C*.

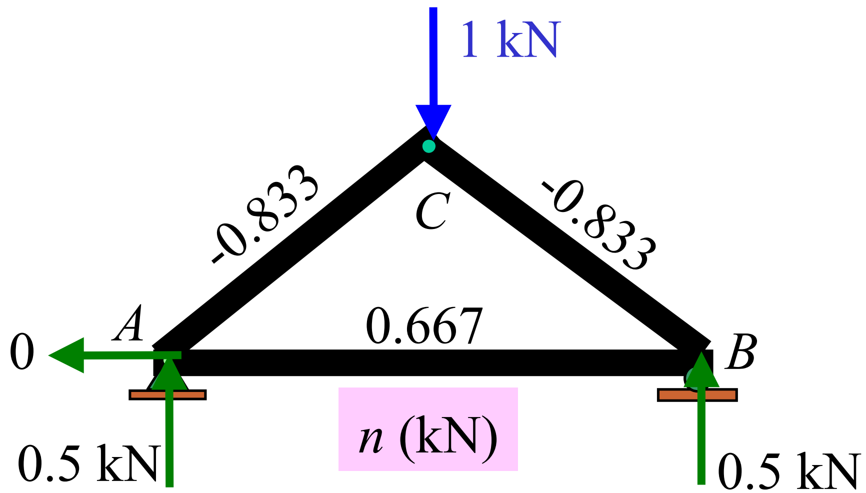


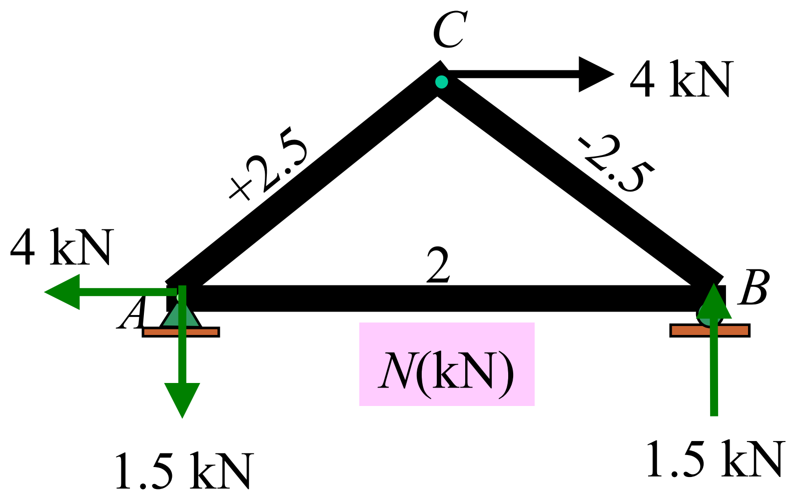
SOLUTION

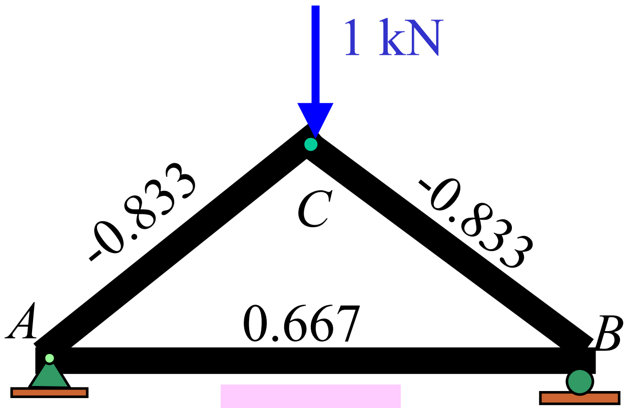
Part (a)

•***Virtual Force n.*** Since the *vertical displacement of joint C is to be determined, only a vertical 1 kN load is placed at joint C*. The *n* force in each member is calculated using the method of joint.

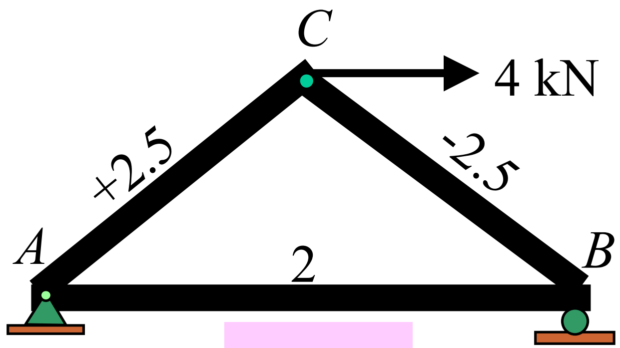
•***Real Force N.*** The *N* force in each member is calculated using the method of joint.



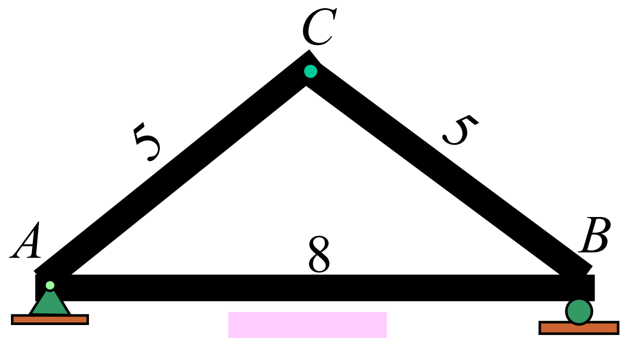




n (kN)

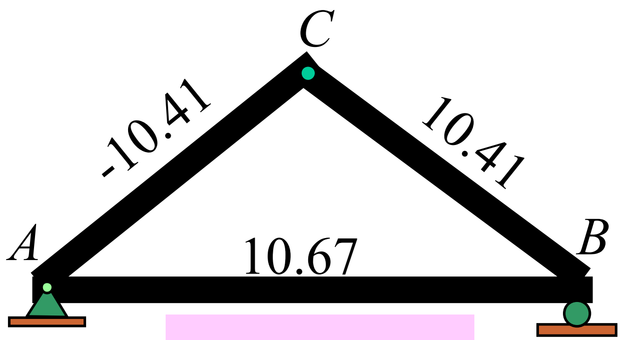


N (kN)



L (m)

||



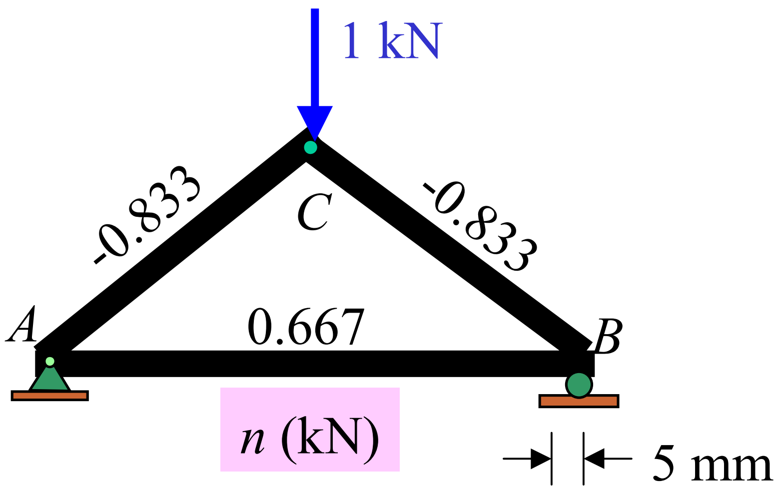
nNL (kN²•m)

$$(1\,kN)(\Delta_{CV}) = \sum \frac{nNL}{AE}$$

$$\Delta_{CV} = \frac{1}{AE}(-10.41+10.41+10.67) = \frac{10.67\,kN\bullet m}{(400\times10^{-6}\,m^2)(200\times10^{6}\,\frac{kN}{m^2})}$$

$$\Delta_{CV} = +0.133\text{ mm}, \downarrow$$

Part (b): The member AB were 5 mm too short



$$(1)(\Delta_{CV}) = \Sigma n(\Delta L)$$

$$\Delta_{CV} = (0.667)(-0.005)$$

$$\Delta_{CV} = \text{-3.33 mm}, \uparrow$$

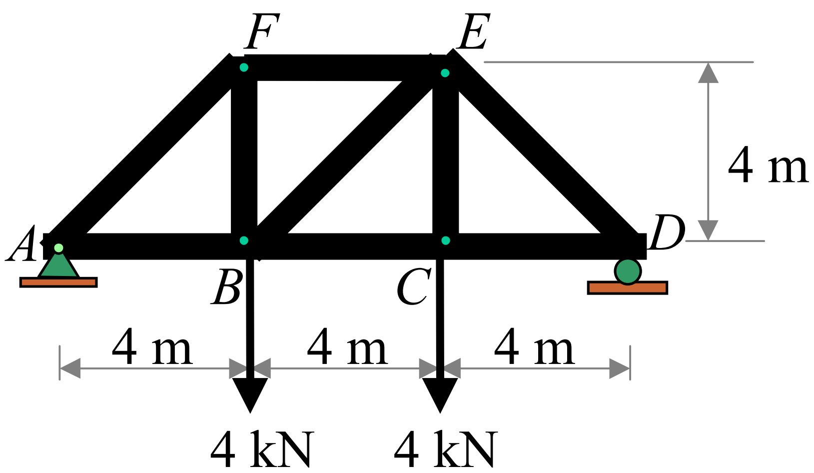
Part (c): The 4 kN force and fabrication error are both accounted.

$$\Delta_{CV} = 0.133 - 3.33 = \text{-3.20 mm}$$

$$\Delta_{CV} = \text{-3.20 mm}, \uparrow$$

Example 8-16

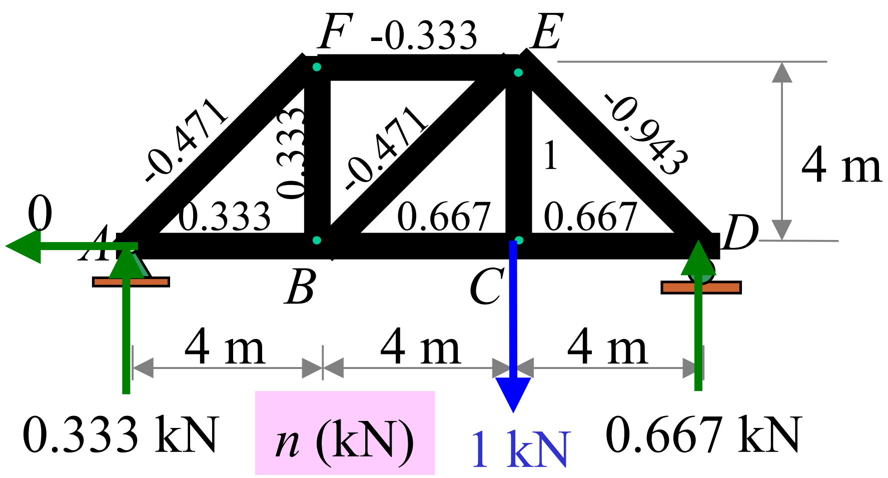
Determine the vertical displacement of joint C of the steel truss shown. The cross-section area of each member is $A = 400\ \text{mm}^2$ and $E = 200$ GPa.

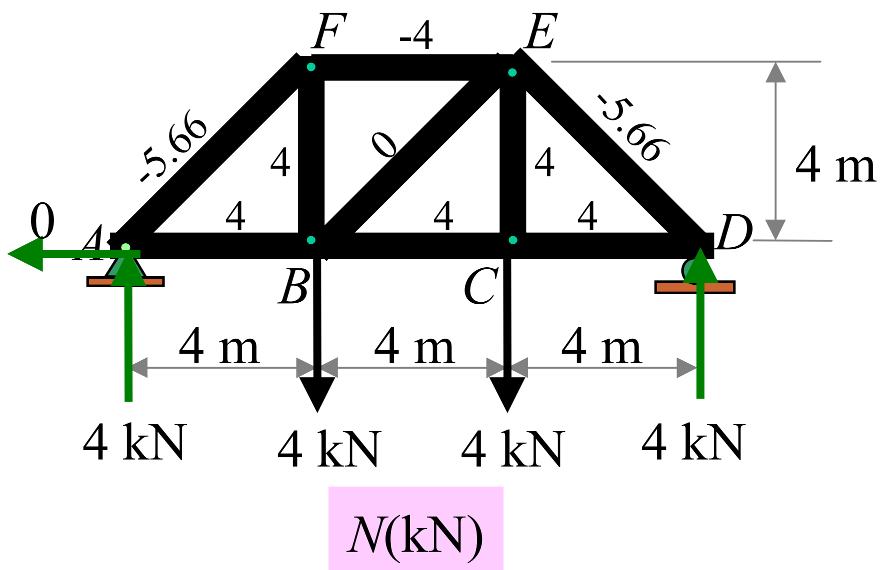


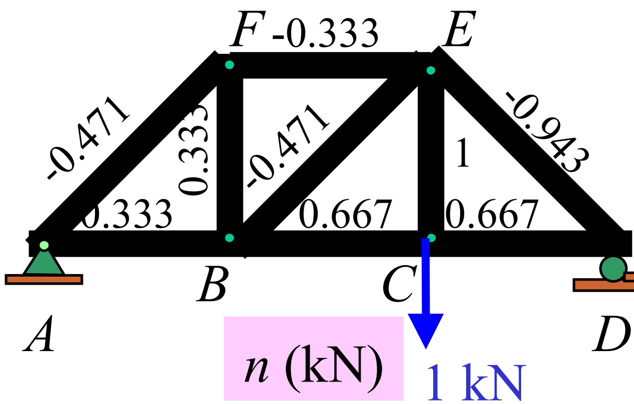
<u>SOLUTION</u>

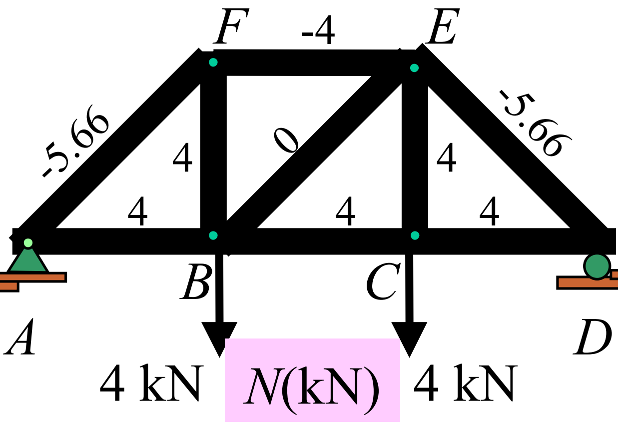
•***Virtual Force n.*** Since the *vertical displacement of joint C is to be determined, only a vertical 1 kN load is placed at joint C*. The *n* force in each member is calculated using the method of joint.

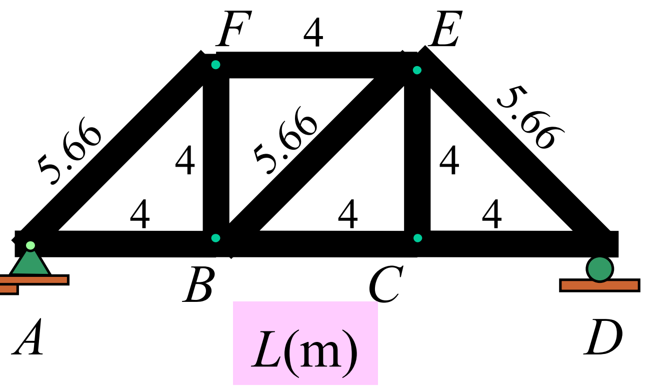
•***Real Force N.*** The *N* force in each member is calculated using the method of joint.











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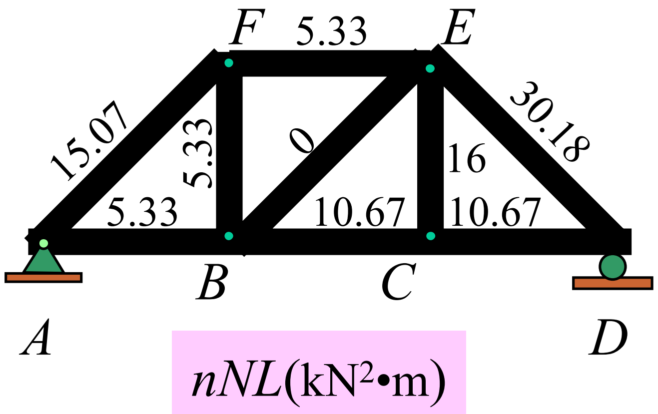


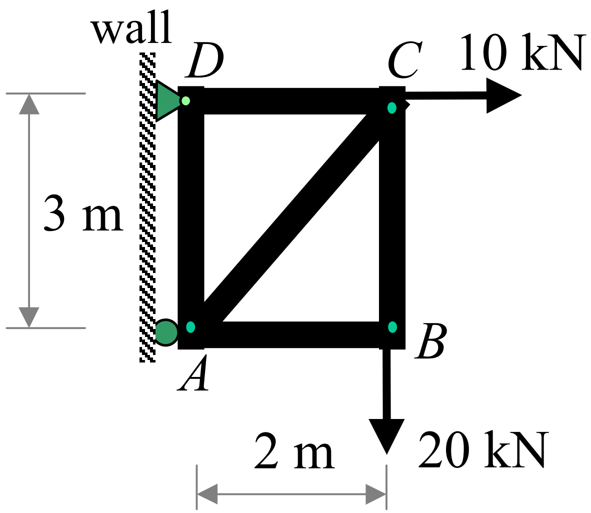
$$(1\,kN)(\Delta_{CV}) = \sum \frac{nNL}{AE}$$

$$\Delta_{CV} = \frac{1}{AE}[(15.07 + 3(5.33) + 2(10.67) + 16 + 30.18)] = \frac{72.4\ kN \bullet m}{(400\times 10^{-6} m^2)(200\times 10^6 \frac{kN}{m^2})}$$

$\Delta_{CV} = 1.23$ mm, $\downarrow$

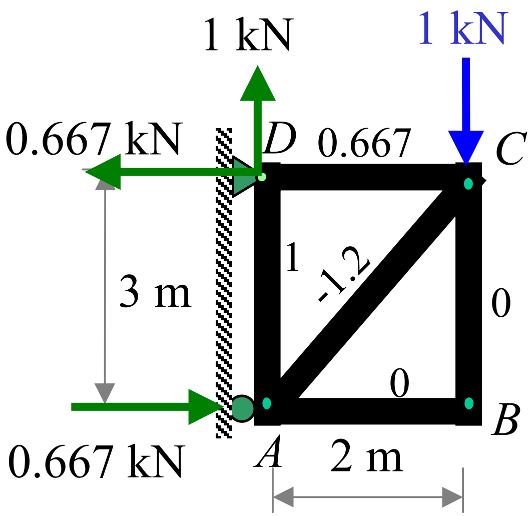
Example 8-17

Determine the vertical displacement of joint *C* of the steel truss shown. Due to radiant heating from the wall, members are subjected to a temperature change: member *AD* is increase +60°C, member *DC* is increase +40°C and member *AC* is decrease -20°C.Also member *DC* is fabricated 2 mm too short and member AC 3 mm too long. Take $\alpha = 12(10^{-6})$, the cross-section area of each member is A = 400 mm^2 and E = 200 GPa.

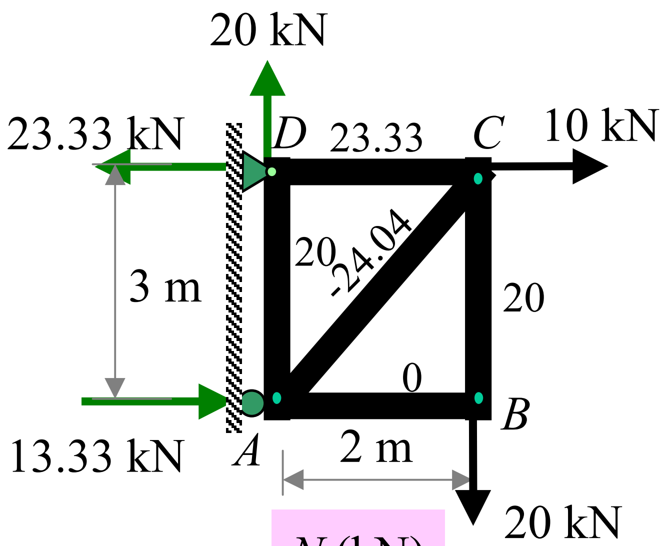


<u>**SOLUTION**</u>

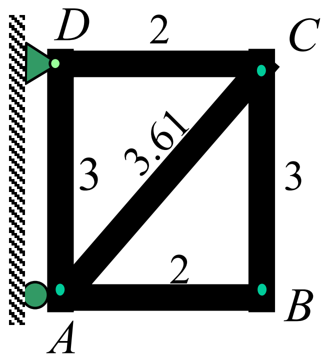
- **Due to loading forces.**



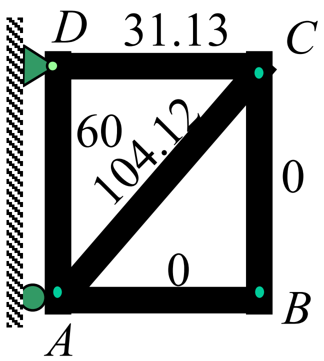
n (kN)



N (kN)



L (m)

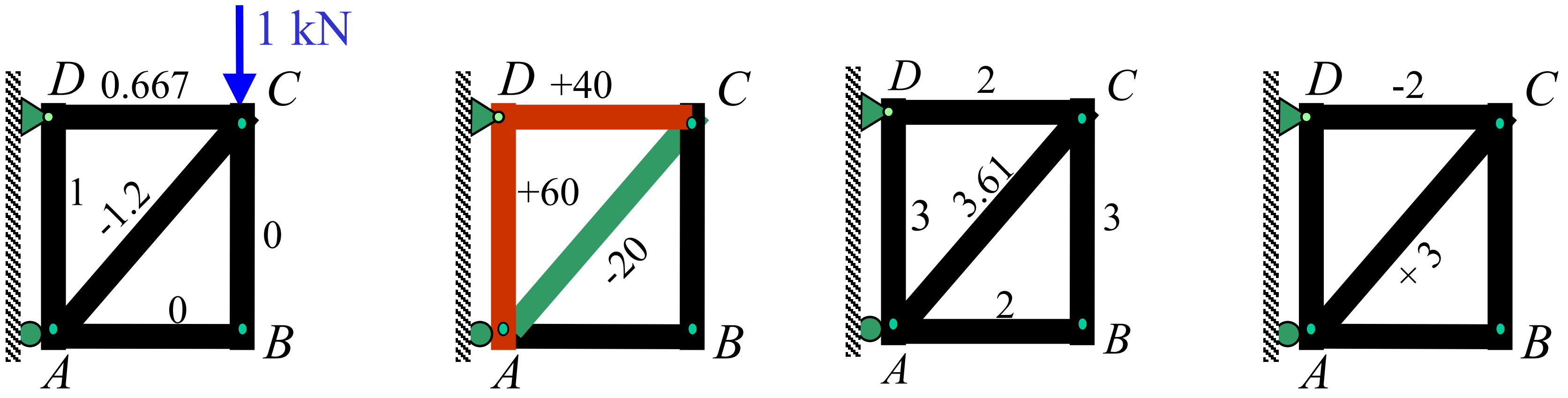


nNL(kN²•m)

$$(1\,kN)(\Delta_{CV}) = \sum \frac{nNL}{AE}$$

$$\Delta_{CV} = \frac{1}{(400)(200)}(60 + 31.13 + 104.12)$$

$$\Delta_{CV} = 2.44 \text{ mm}, \downarrow$$



n (kN) | ΔT (°C) | *L* (m) | Fabrication error (mm)

- **Due to temperature change.**

$$(1\,kN)(\Delta_{CV}) = \Sigma n\alpha(\Delta T)L$$

$$\Delta_{CV} = (12\times10^{-6})[(1)(60)(3)+(0.667)(40)(2)+(-1.2)(-20)(3.61)] = 3.84\;mm,\downarrow$$

- **Due to fabrication error.**

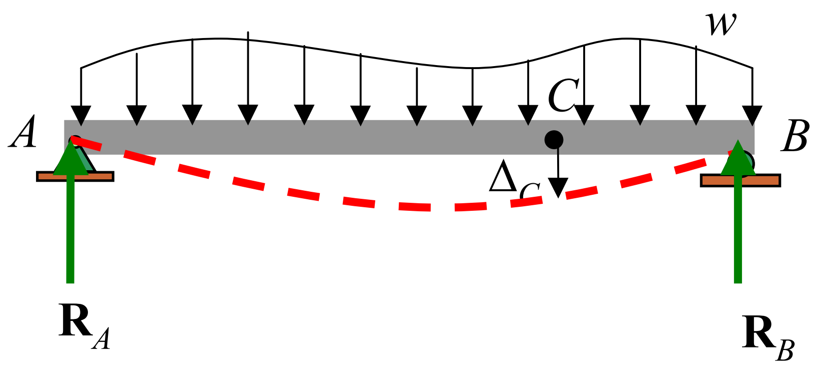
$$(1\,kN)(\Delta_{CV}) = \Sigma n(\Delta L)$$

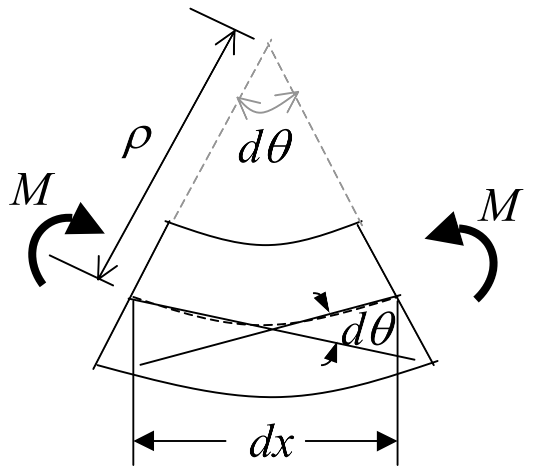
$$\Delta_{CV} = (0.667)(-0.002)+(-1.2)(0.003) = -4.93\;mm,\uparrow$$

- **Total displacement .**

$$(\Delta_{CV})_{Total} = 2.44+3.84-4.93 = 1.35\;mm,\downarrow$$

Method of Virtual Work : Bending





Virtual loadings

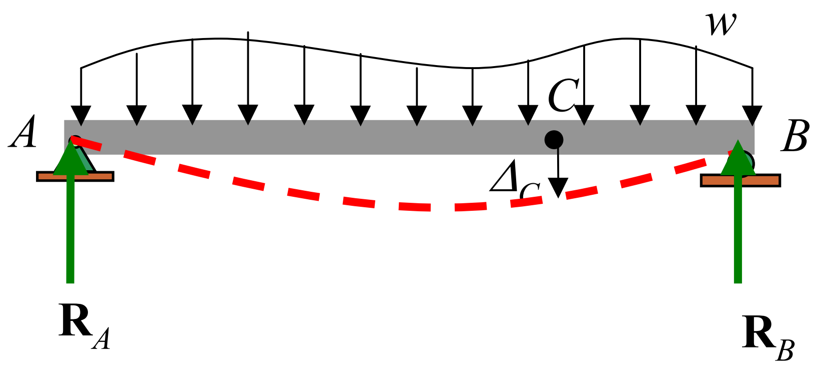
$$1 \bullet \Delta_C = \int (m\ )(d\theta) = \int_L (m_\Delta) \frac{M}{EI} dx$$

Real displacements

$$ds = \rho \, d\theta$$

$$d\theta = \frac{1}{\rho} ds \approx \frac{M}{EI} dx$$

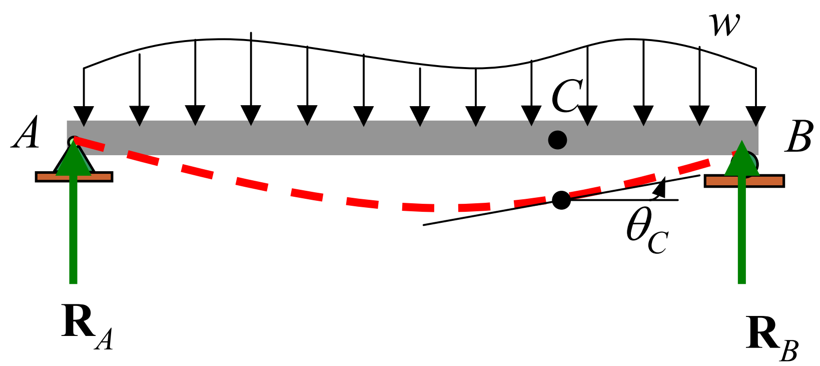
Method of Virtual Work : Beams and Frames



Virtual loadings

$$1 \bullet \Delta_C = \int (m)(d\theta) = \int_L (m_\Delta) \frac{M}{EI} dx$$

Real displacements



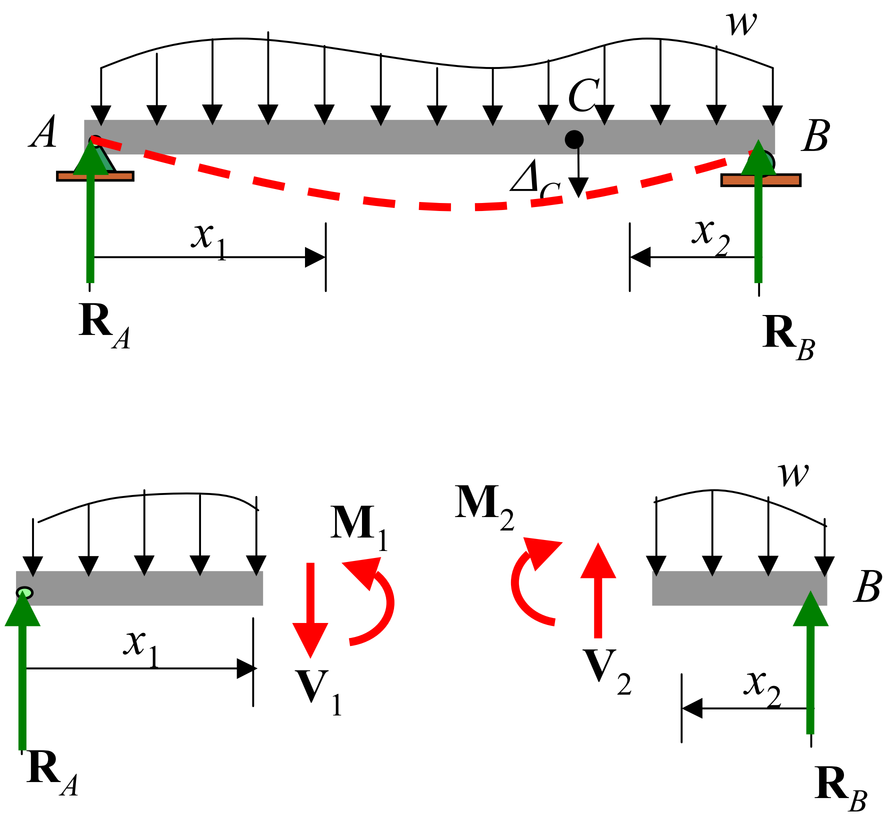
Virtual loadings

$$1 \bullet \theta_C = \int (m)(d\theta) = \int_L (m_\theta) \frac{M}{EI} dx$$

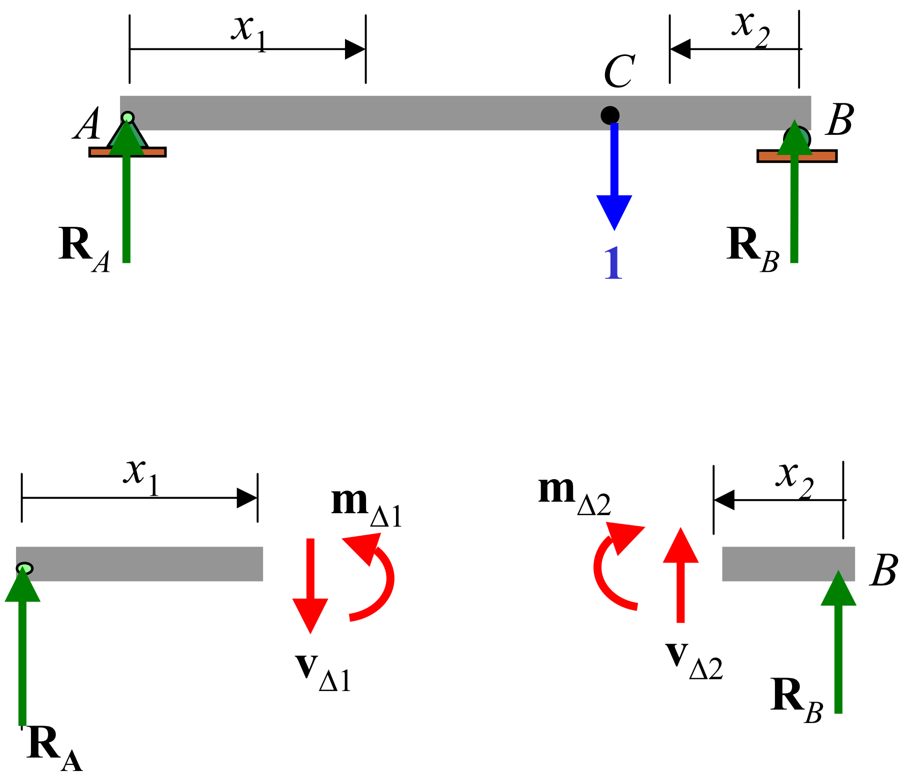
Real displacements

- ***Vertical Displacement***

Real load



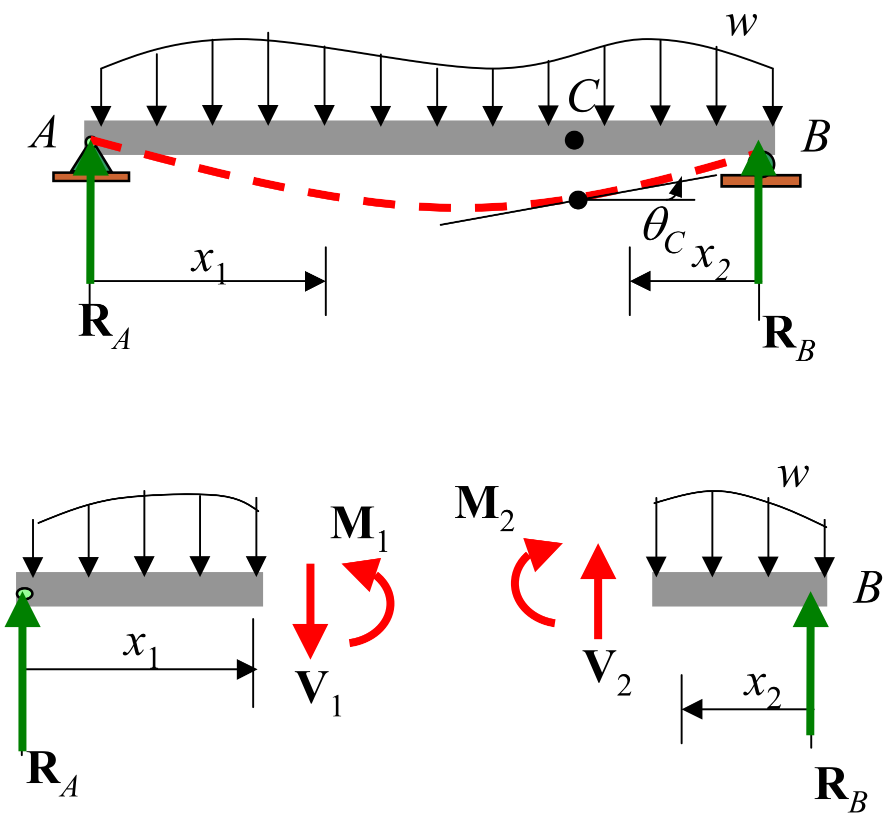
Virtual **unit load**



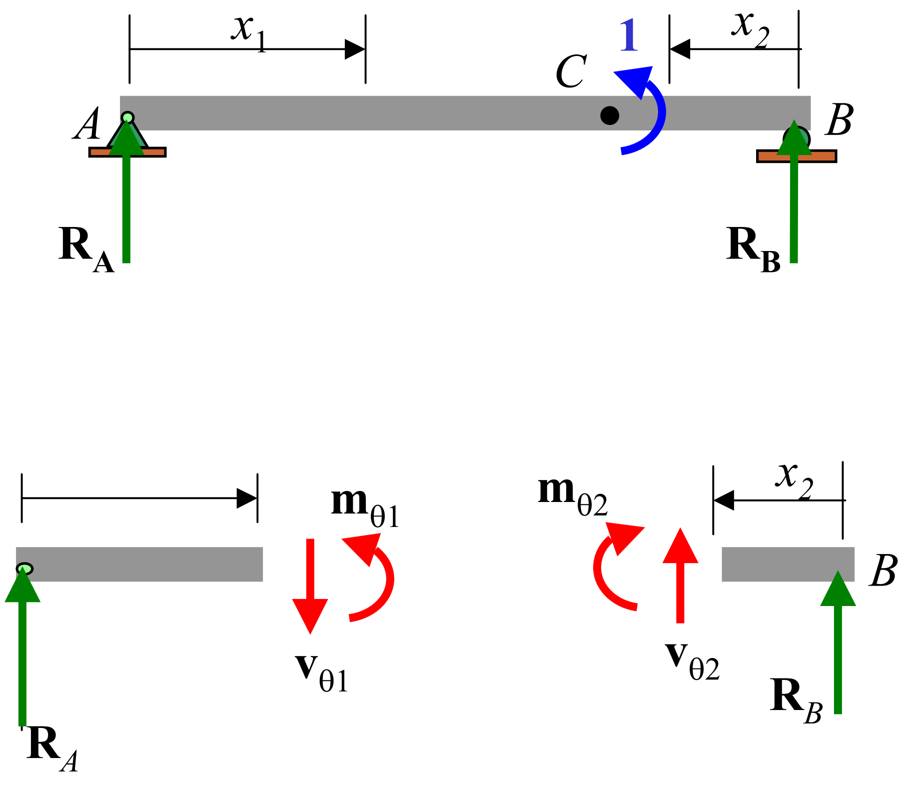
$$1 \bullet \Delta_C = \int_L (m_\Delta)\frac{M}{EI}\,dx$$

- ***Slope***

Real load



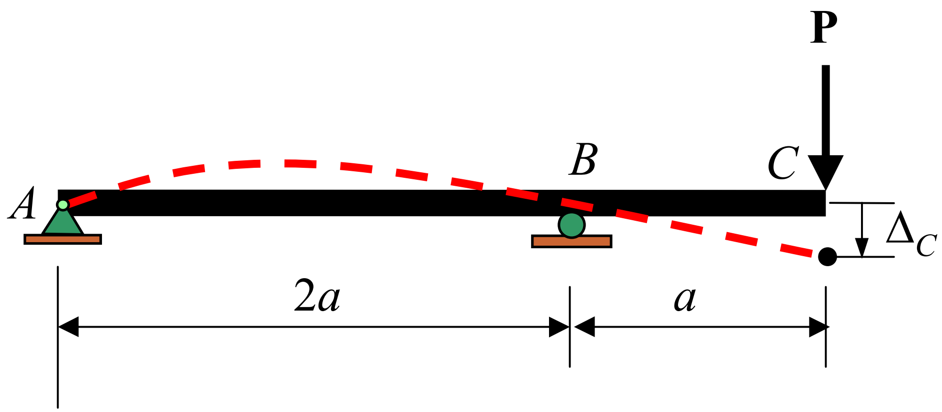
Virtual unit couple



$$1 \bullet \theta_C = \int_L (m_\theta)\frac{M}{EI}dx$$

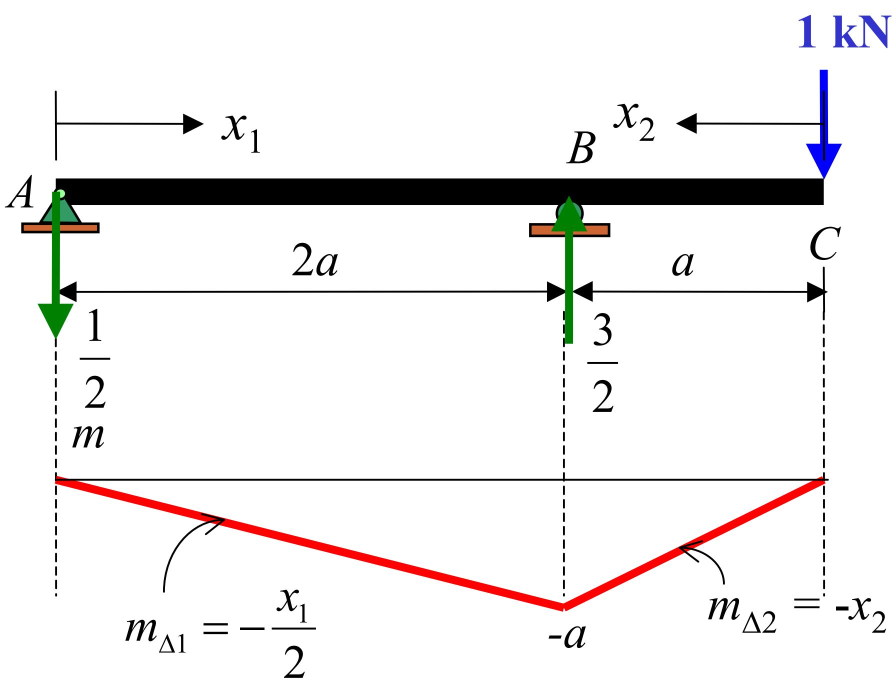
Example 8-18

The beam shown is subjected to a load **P** at its end. Determine the slope and displacement at *C*. *EI* is constant.

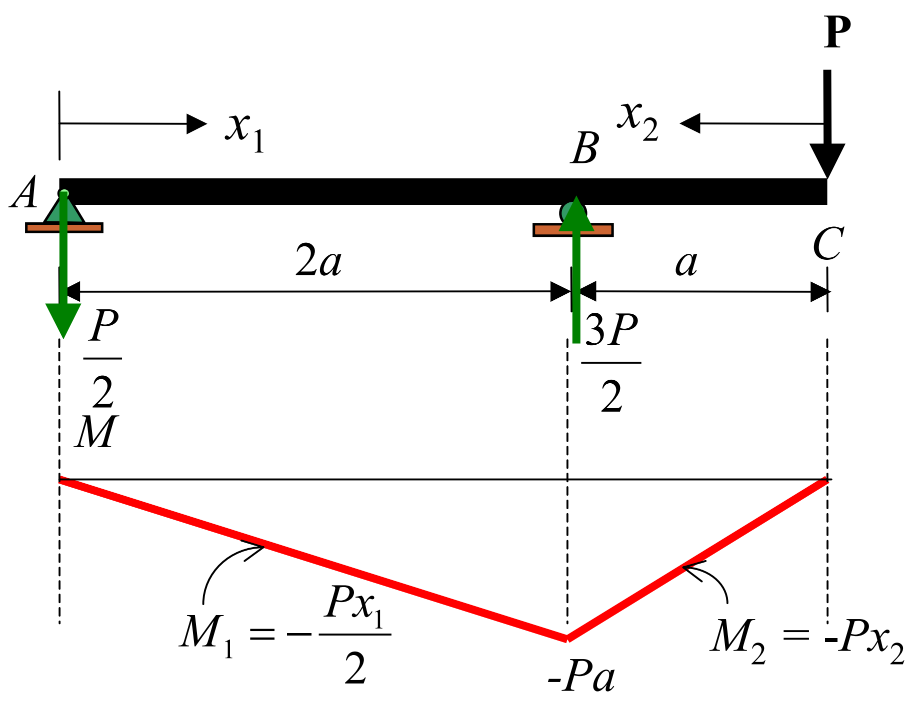


SOLUTION *Displacement at C*

•Virtual Moment m_Δ



•Real Moment M

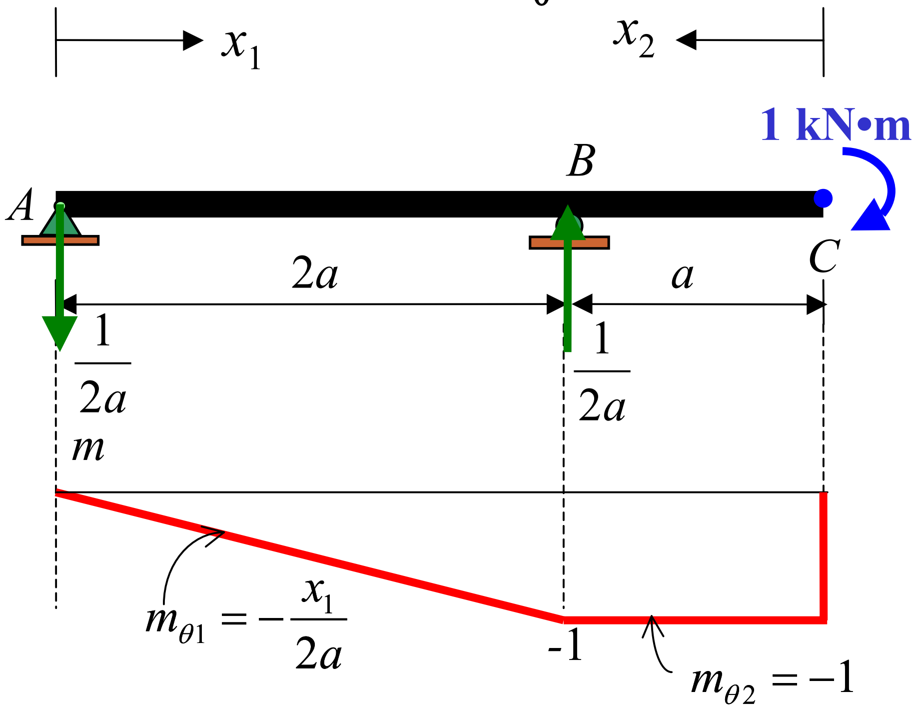


$$1 \bullet \Delta_C = \int_L \frac{m_\Delta M}{EI} dx = \frac{1}{EI} \int_0^{2a} (-\frac{x_1}{2})(-\frac{Px_1}{2}) dx_1 + \frac{1}{EI} \int_0^{a} (-x_2)(-Px_2) dx_2$$

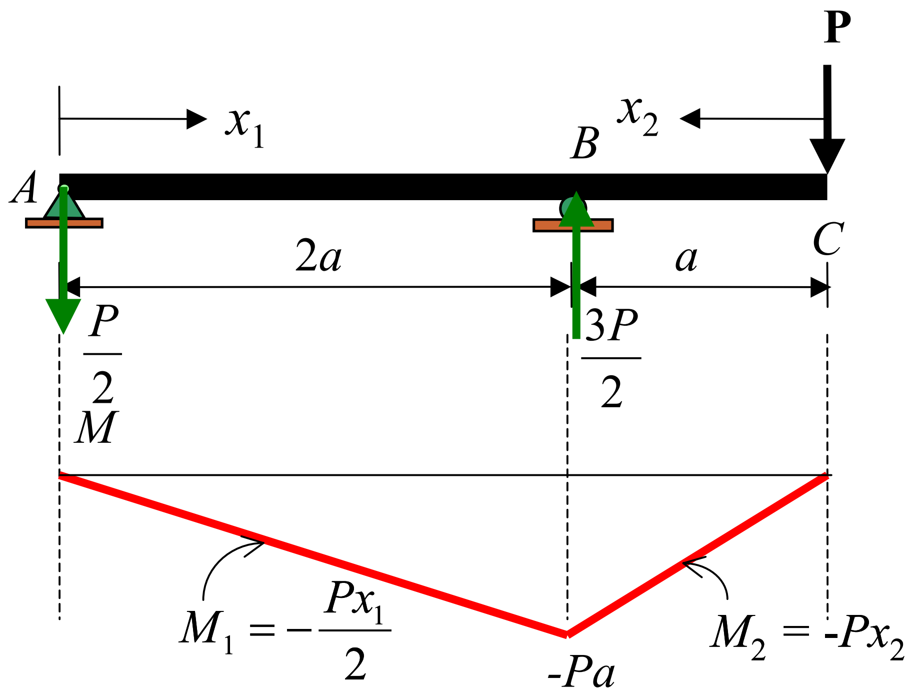
$$\Delta_C = (\frac{Px_1^3}{12EI})\Big|_a^{2a} + (\frac{Px_2^3}{3EI})\Big|_0^{a} = \frac{8Pa^3}{12EI} + \frac{Pa^3}{3EI} = \frac{Pa^3}{3EI} \downarrow$$

Slope at C

•*Virtual Moment* m_θ



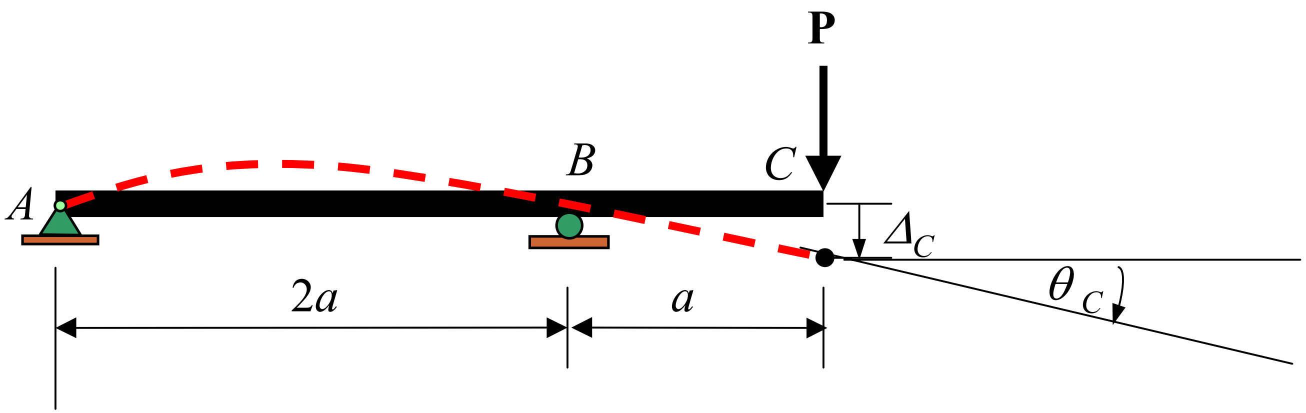
•*Real Moment M*



$$(1\,kN\bullet m)(\theta_C)=\int_0^L \frac{m_\theta M}{EI}dx=\frac{1}{EI}\int_0^{2a}(-\frac{x_1}{2a})(-\frac{Px_1}{2})dx_1+\frac{1}{EI}\int_0^{a}(-1)(-Px_2)dx_2$$

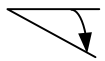
$$\theta_C=(\frac{1}{EI})(\frac{P}{4a})(\frac{x_1^3}{3})\Big|_0^{2a}+(\frac{1}{EI})(\frac{Px_2^2}{2})\Big|_0^{a}=(\frac{1}{EI})(\frac{P}{4a})(\frac{8a^3}{3})+(\frac{1}{EI})(\frac{Pa^2}{2})=\frac{7}{6}(\frac{Pa^2}{EI}),$$

•***Conclusion***



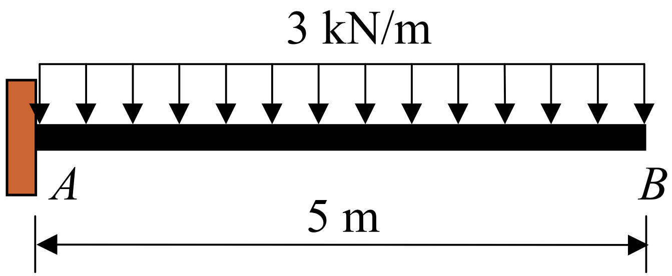
$$\Delta_C = \frac{Pa^3}{3EI} \downarrow$$

$$\theta_C = \frac{7}{6}\left(\frac{Pa^2}{EI}\right),$$



Example 8-19

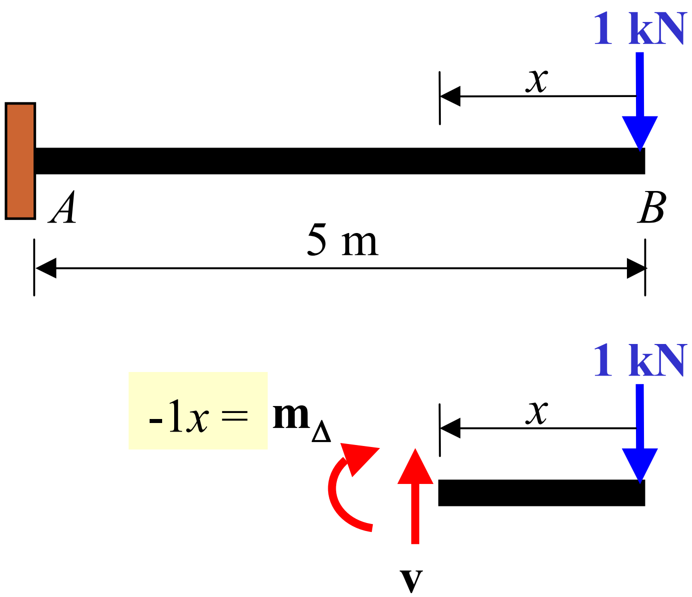
Determine the slope and displacement of point *B* of the steel beam shown in the figure below. Take $E = 200$ GPa, $I = 250(10^6)$ mm^4.



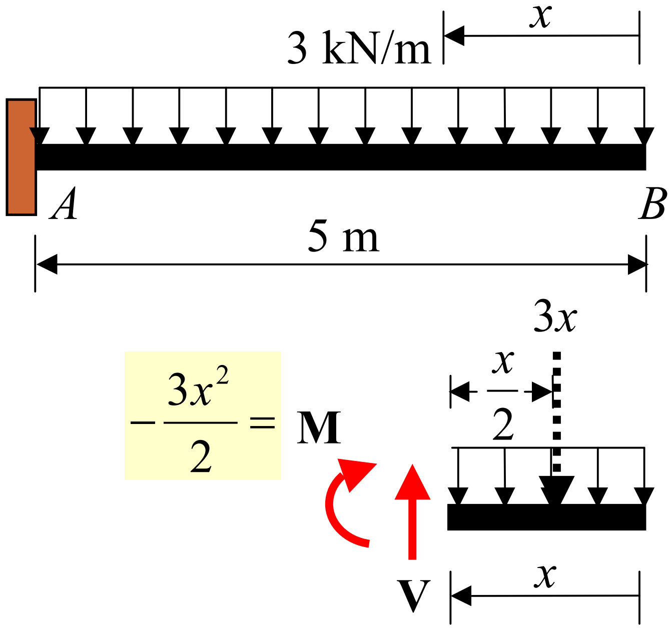
SOLUTION

Vertical Displacement at B

•***Virtual Moment*** m_Δ



•***Real Moment M***



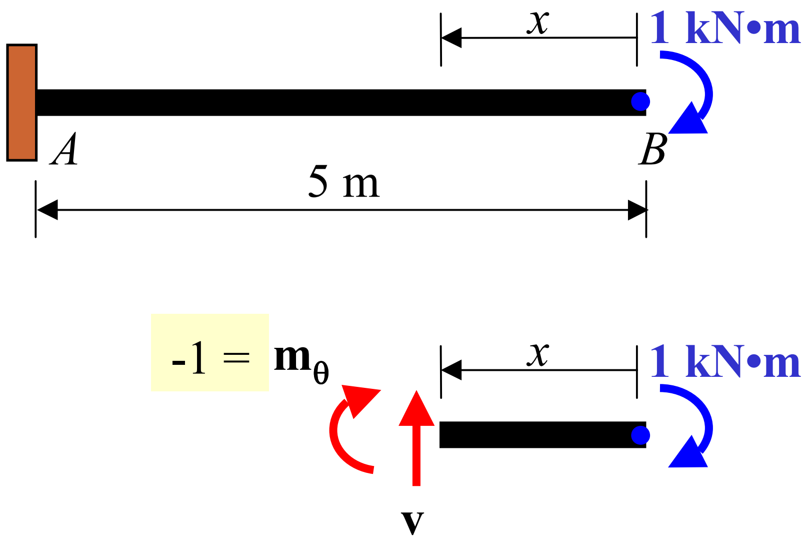
$$(1\,kN)(\Delta_B) = \int_0^L \frac{m_\Delta M}{EI}dx = \frac{1}{EI}\int_0^5 (-x)(-\frac{3x^2}{2})dx = \frac{1}{EI}\int_0^5 \frac{3x^3}{2} = \frac{1}{EI}(\frac{3x^4}{8}\Big|_0^5) = \frac{234.375\,kN^2 \bullet m^3}{EI}$$

$$\Delta_B = \frac{234.375\,kN \bullet m^3}{(200\times 10^6\,\frac{kN}{m})(250\times 10^{-6}\,m^4)} = 0.00469\,m = 4.69\,mm, \downarrow$$

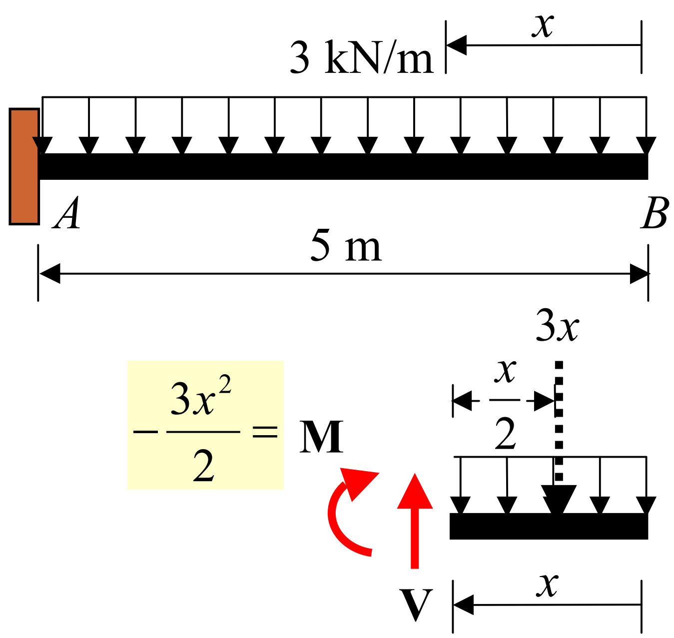
SOLUTION

Slope at B

•***Virtual Moment*** m_θ

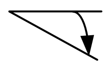


•***Real Moment M***



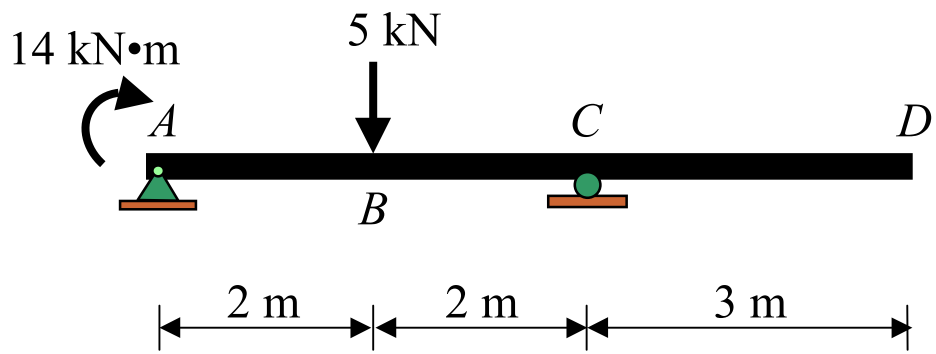
$$(1\,kN \bullet m)(\theta_B) = \int_0^L \frac{m_\theta M}{EI}dx = \frac{1}{EI}\int_0^5 (-1)(-\frac{3x^2}{2})dx = \frac{1}{EI}\int_0^5 \frac{3x^2}{2} = \frac{1}{EI}(\frac{3x^3}{6}\Big|_0^5) = \frac{62.5\,kN^2m^3}{EI}$$

$$\theta_B = \frac{62.5\,kN \bullet m^2}{(200\times10^6\,\frac{kN}{m})(250\times10^{-6}\,m^4)} = 0.00125\,rad,$$

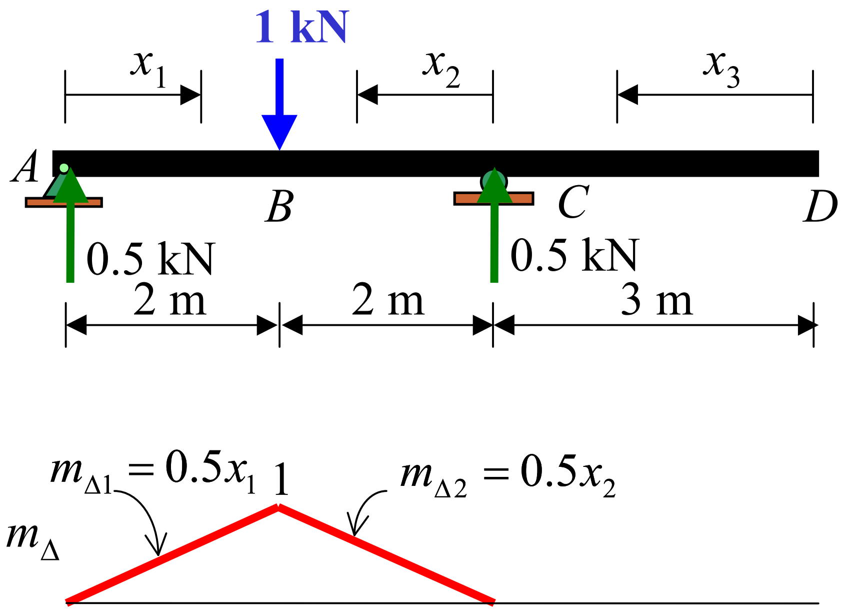


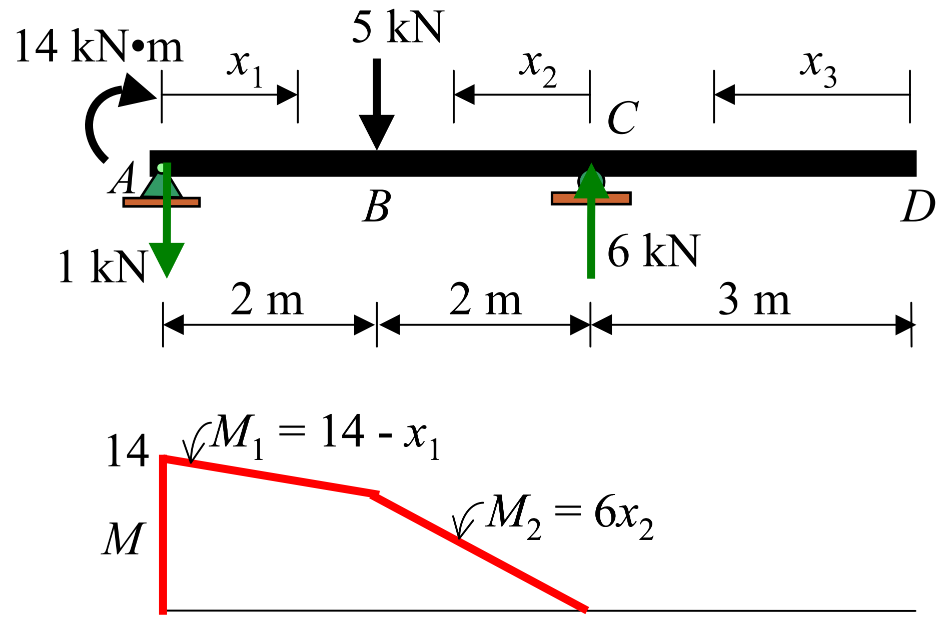
Example 8-20

Determine the slope and displacement of point B of the steel beam shown in the figure below. Take E = 200 GPa, I = 60(10^6) mm^4.



•Virtual Moment m_Δ **Displacement at B** **•Real Moment M**





$$(1\ kN)(\Delta_B) = \int_0^L \frac{m_\Delta M}{EI} dx$$

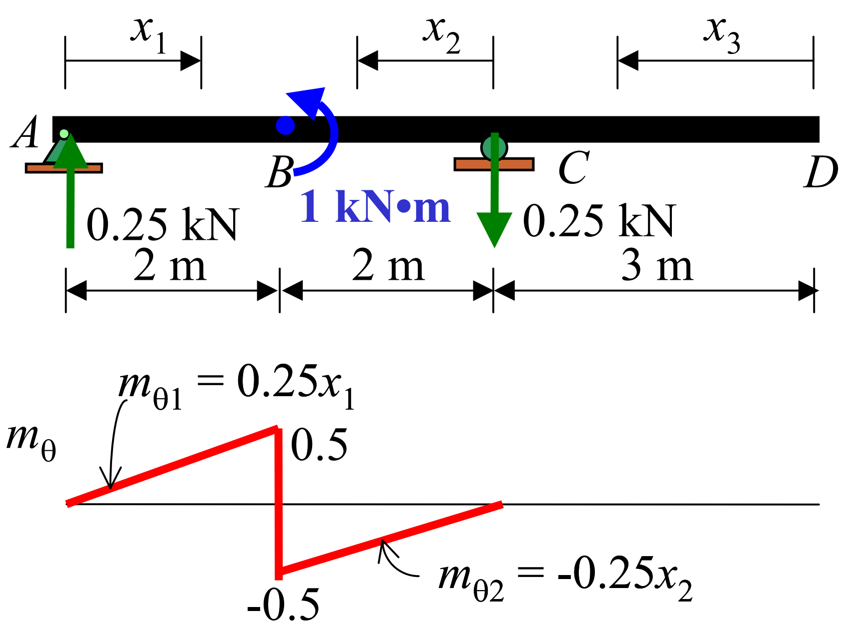
$$= \frac{1}{EI}\int_0^2 (0.5x_1)(14 - x_1)dx_1 + \frac{1}{EI}\int_0^2 (0.5x_2)(6x_2)dx_2 + \frac{1}{EI}\int_0^3 (0)(0)dx_3$$

$$= \frac{1}{EI}\int_0^2 (7x_1 - 0.5x_1^2)dx_1 + \frac{1}{EI}\int_0^2 (3x_2^2)dx_2 = (\frac{1}{EI})(\frac{7x_1^2}{2} - \frac{0.5x_1^3}{3}\Big|_0^2 + (\frac{1}{EI})(\frac{3x_2^3}{3})\Big|_0^2$$

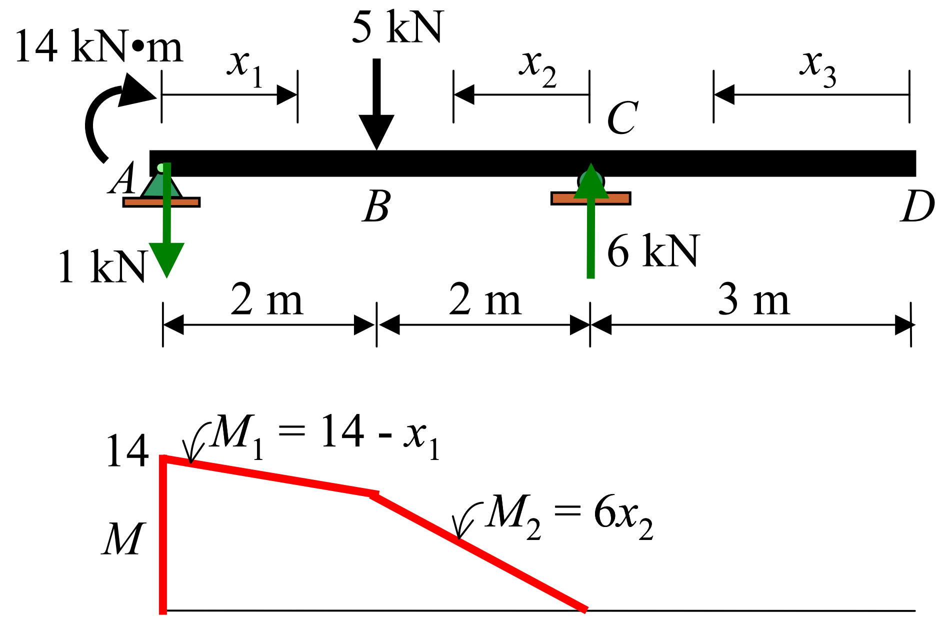
$$\Delta_B = \frac{20.667}{EI} = \frac{20.667}{(200)(60)} = 0.00172\ m = 1.72\ mm, \downarrow$$

Slide at B

•Virtual Moment m_θ



•Real Moment M

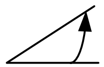


$$(1\,kN \bullet m)(\theta_B) = \int_0^L \frac{m_\theta M}{EI}dx \;= \frac{1}{EI}\int_0^2 (0.25x_1)(14 - x_1)dx_1 + \frac{1}{EI}\int_0^2 (-0.25x_2)(6x_2)dx_2 + \frac{1}{EI}\int_0^3 (0)(0)dx_3$$

$$= \frac{1}{EI}\int_0^2 (3.5x_1 - 0.25x_1^{\,2})dx_1 + \frac{1}{EI}\int_0^2 (-1.5x_2^{\,2})dx_2$$

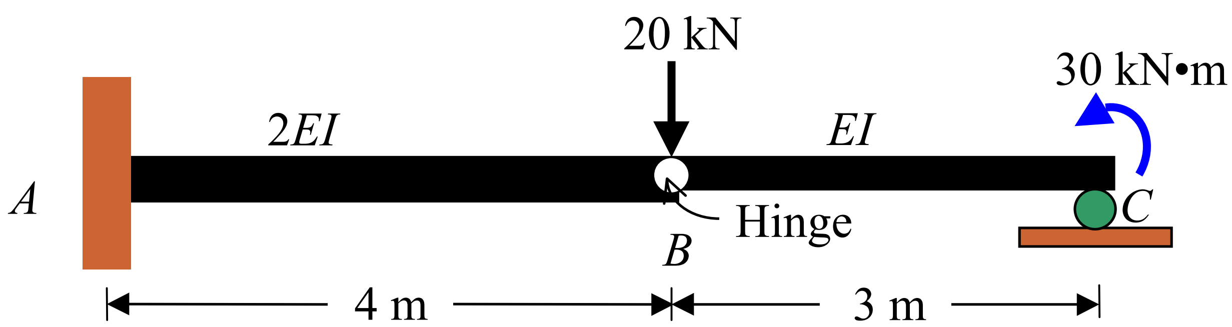
$$= \frac{1}{EI}\left(\frac{3.5x_1^{\,2}}{2} - \frac{0.25x_1^{\,3}}{3}\right)\Bigg|_0^2 + \frac{1}{EI}\left(-\frac{1.5x_2^{\,3}}{3}\right)\Bigg|_0^2$$

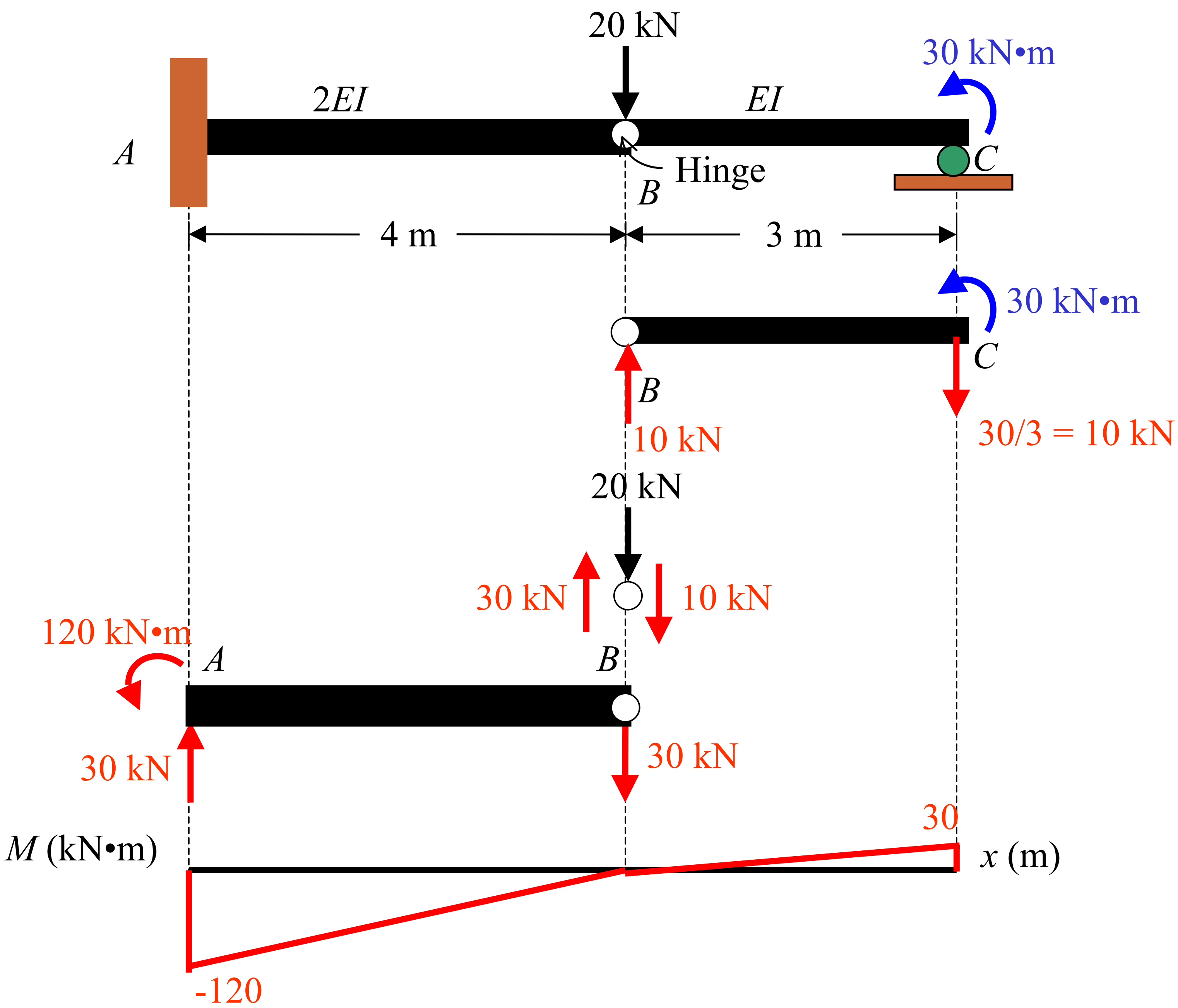
$$\theta_B = \frac{2.333}{EI} = \frac{2.333}{(200)(60)} = 0.000194\,rad,$$



Example 8-21

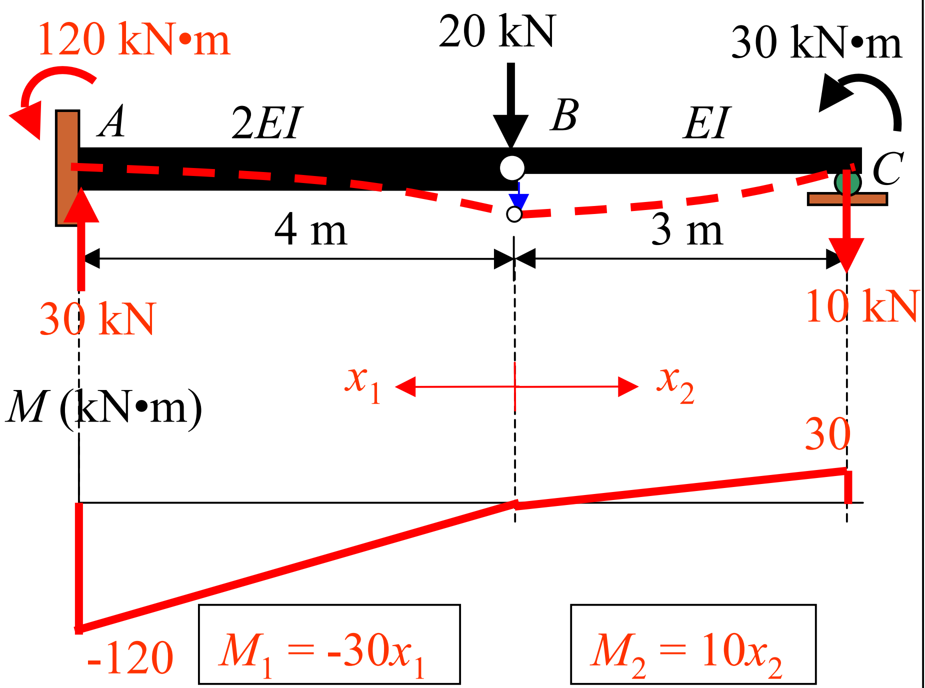
From the structure shown. Determine the slope and displacement at C. Take E = 200 GPa, $I = 200(10^6)$ mm^4.



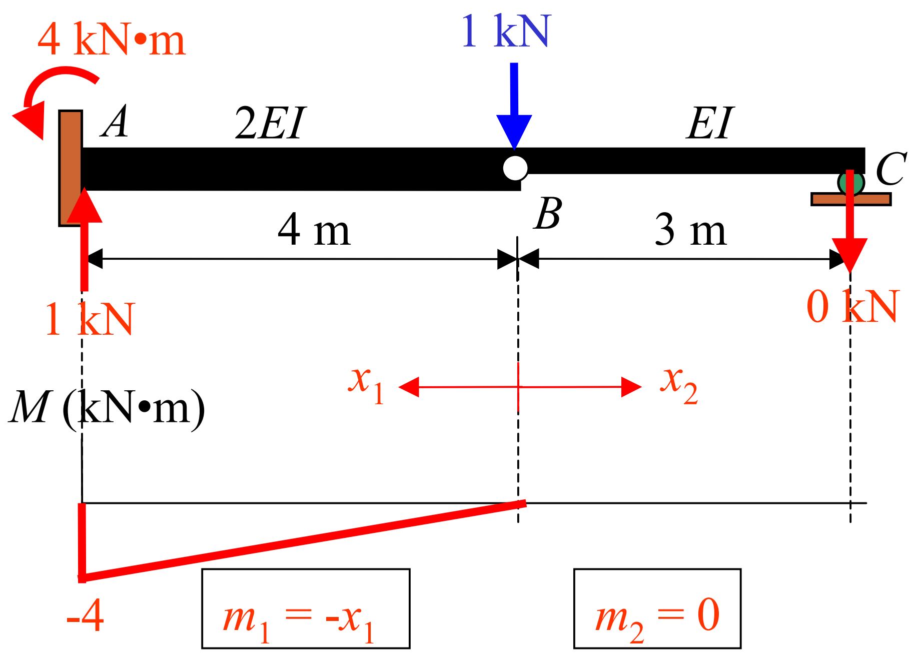

20 kN
30 kN•m
2EI
EI
A
Hinge
B
C
4 m
3 m
30 kN•m
C
B
10 kN
30/3 = 10 kN
20 kN
30 kN
10 kN
120 kN•m
A
B
30 kN
30 kN
30
M (kN•m)
x (m)
-120

Displacement at B

•*Real Moment M*



•*Virtual Moment m_Δ*

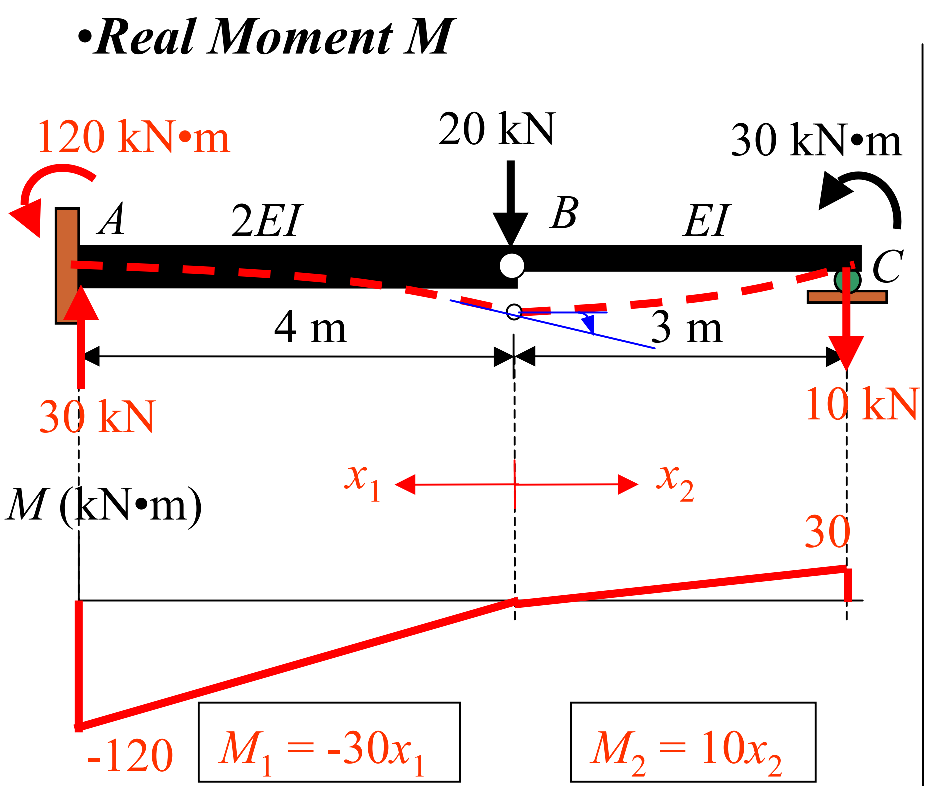


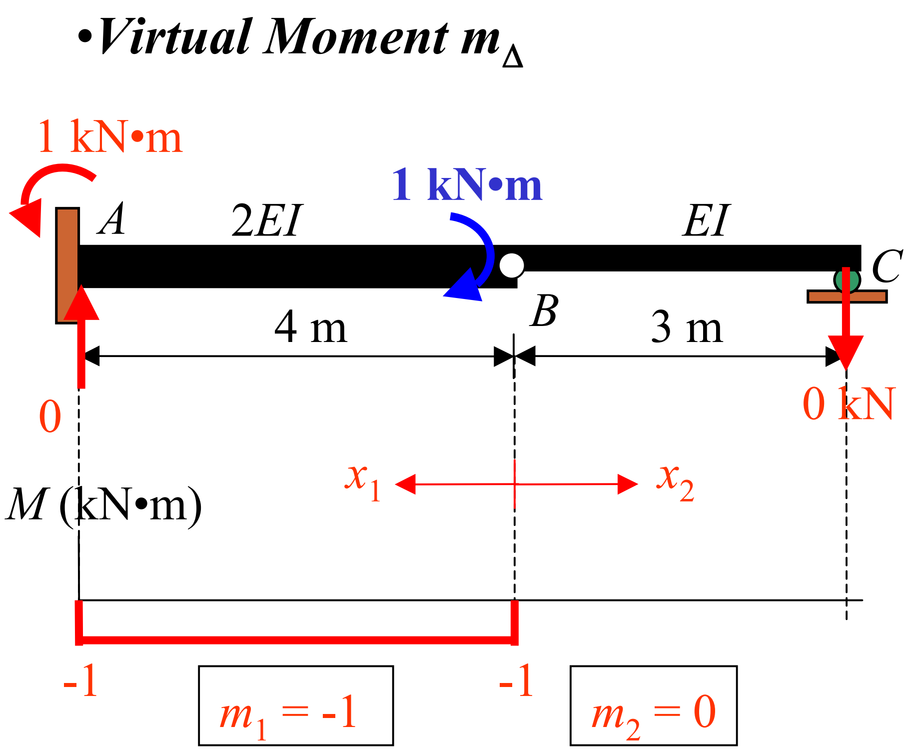
$$\Delta_B = \sum \int_L \frac{m_i M_i}{E_i I_i} dx = \int_0^4 (-x_1)(-30x_1)\frac{dx_1}{2EI} + 0$$

$$= \frac{1}{2EI}\left(\frac{30x^3}{3}\right)\Big|_0^4$$

$$= \frac{32}{EI} = \frac{32}{40\times10^3} = 0.008\ m \quad \downarrow$$

Slope at the left of B





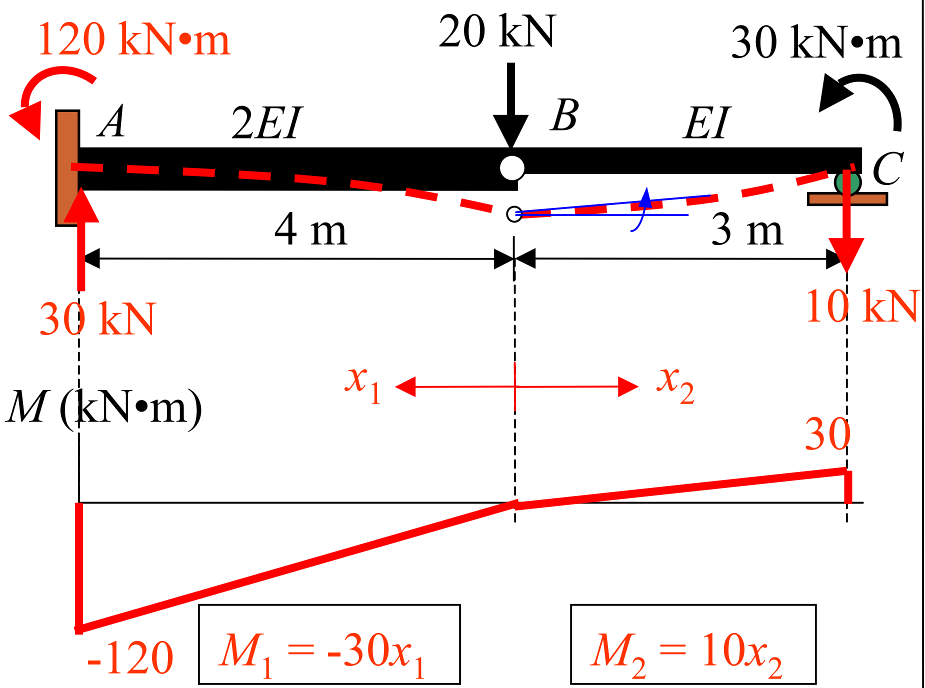
$$\theta_{BL} = \sum \int_L \frac{mM}{E_i I_i} dx_i = \int_0^4 (-1)(-30x_1)\frac{dx_1}{2EI} + 0$$

$$= \frac{1}{2EI}\left(\frac{30x^2}{2}\right)\Big|_0^4$$

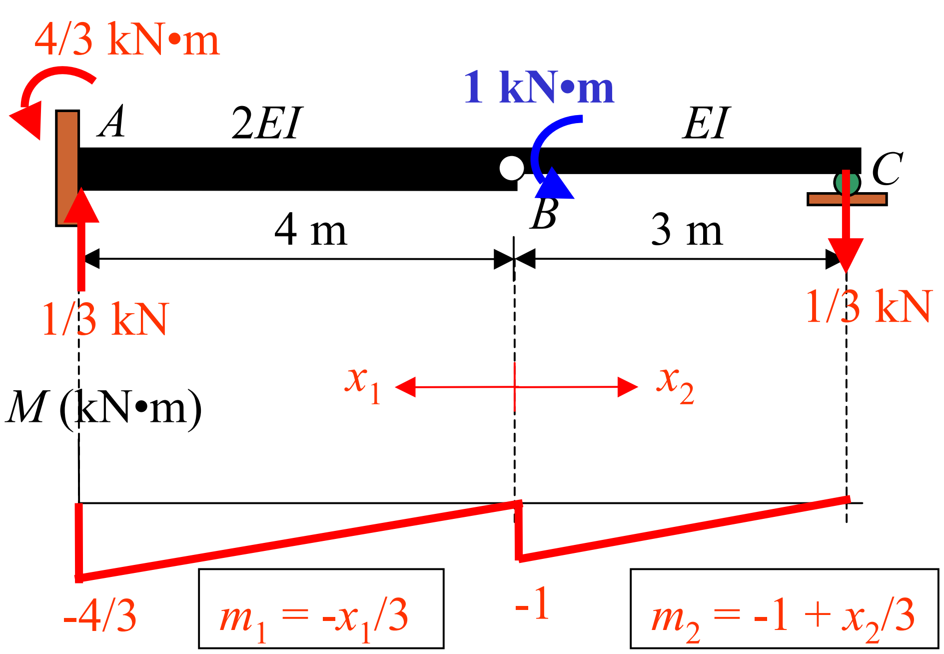
$$= \frac{120}{EI} = \frac{120}{40\times 10^3} = 0.003\ rad$$

Slope at the right of B

•*Real Moment M*



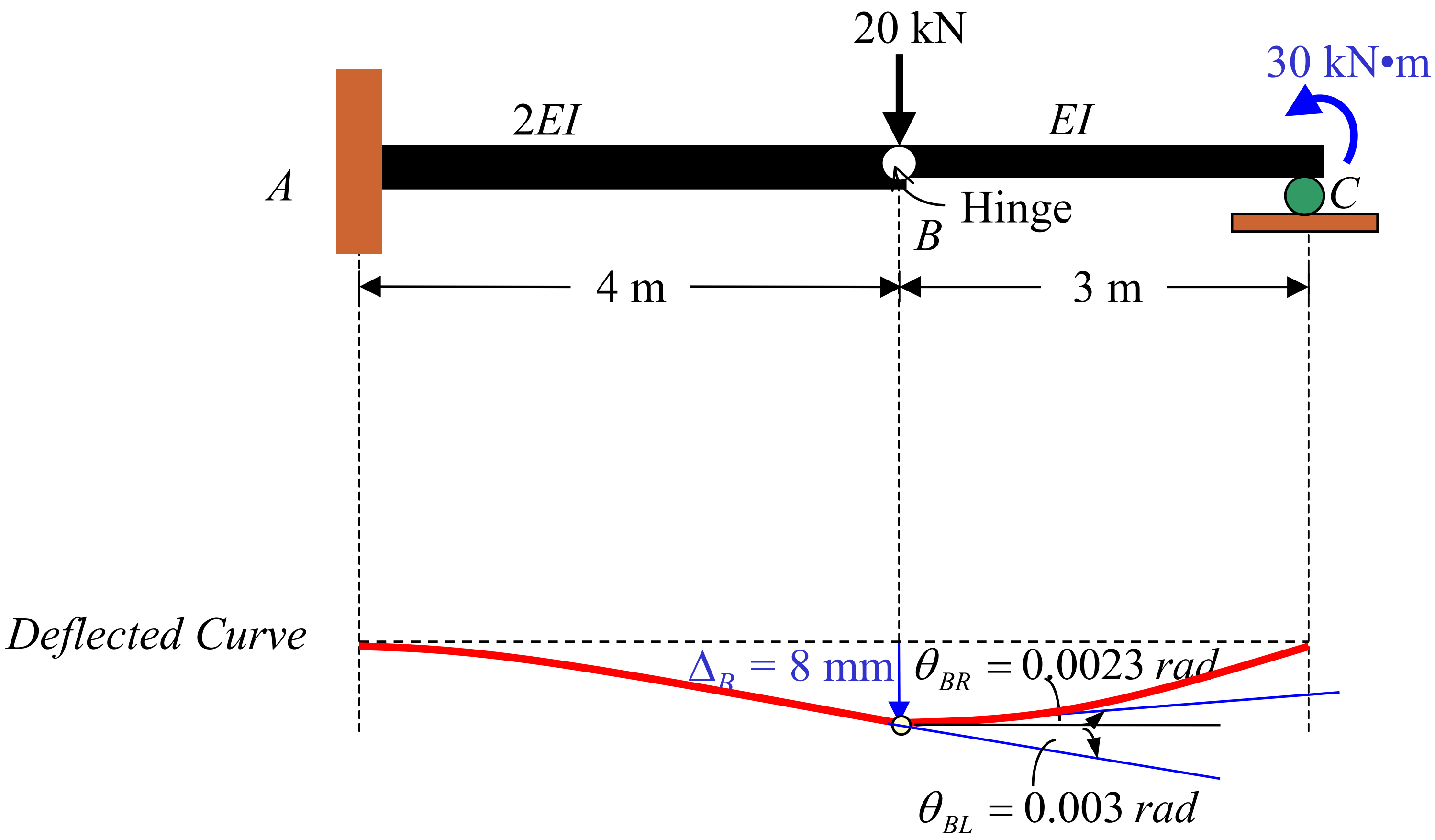
•*Virtual Moment m_Δ*



$$\theta_{BR} = \sum_i \int_L \frac{mM}{E_i I_i} dx = \int_0^4 (-\frac{x_1}{3})(-30x_1)\frac{dx_1}{2EI} + \int_0^3 (-1+\frac{x_2}{3})(10x_2)\frac{dx_2}{EI}$$

$$= \frac{1}{2EI}(\frac{10x_1^3}{3})\Big|_0^4 + \frac{1}{EI}(-\frac{10x_2^2}{2} + \frac{10x_2^3}{9})\Big|_0^3$$

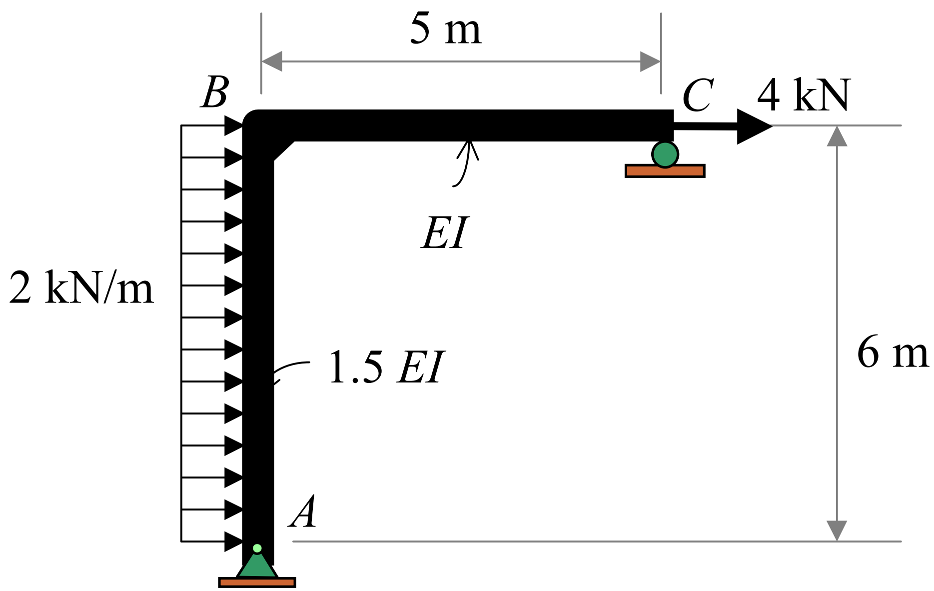
$$= \frac{106.67}{EI} + \frac{1}{EI}(-45+30) = \frac{91.67}{40\times10^3} = 0.0023\ rad$$


20 kN
30 kN•m
2EI
EI
A
Hinge
B
C
4 m
3 m
Deflected Curve
Δ_B = 8 mm
$\theta_{BR} = 0.0023\ rad$
$\theta_{BL} = 0.003\ rad$

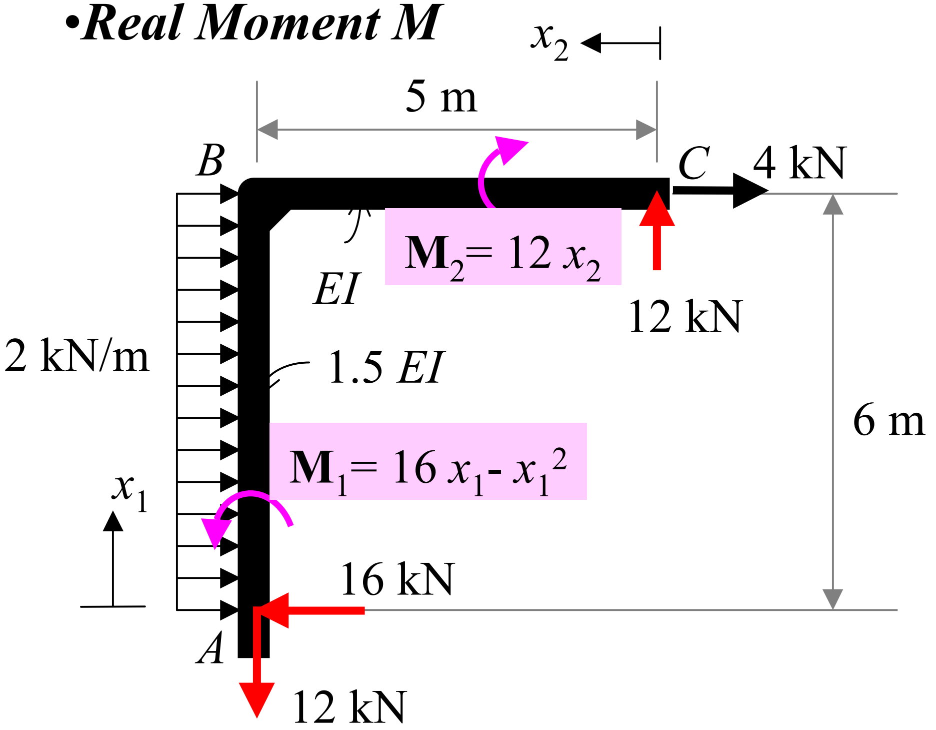
Example 8-22

(a) Determine the slope and the horizontal displacement of point ***C*** on the frame.
(b) Draw the bending moment diagram and deflected curve.

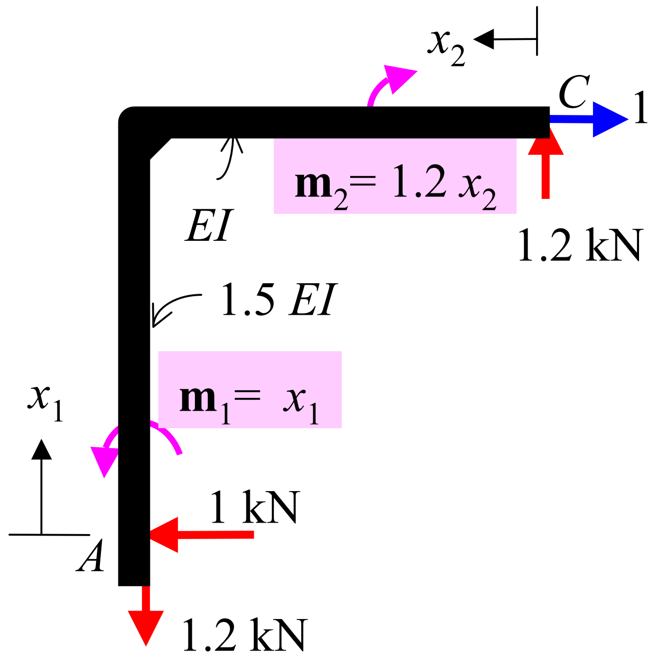
$E = 200$ GPa
$I = 200(10^6)$ mm^4



•Real Moment M



•Virtual Moment m_Δ

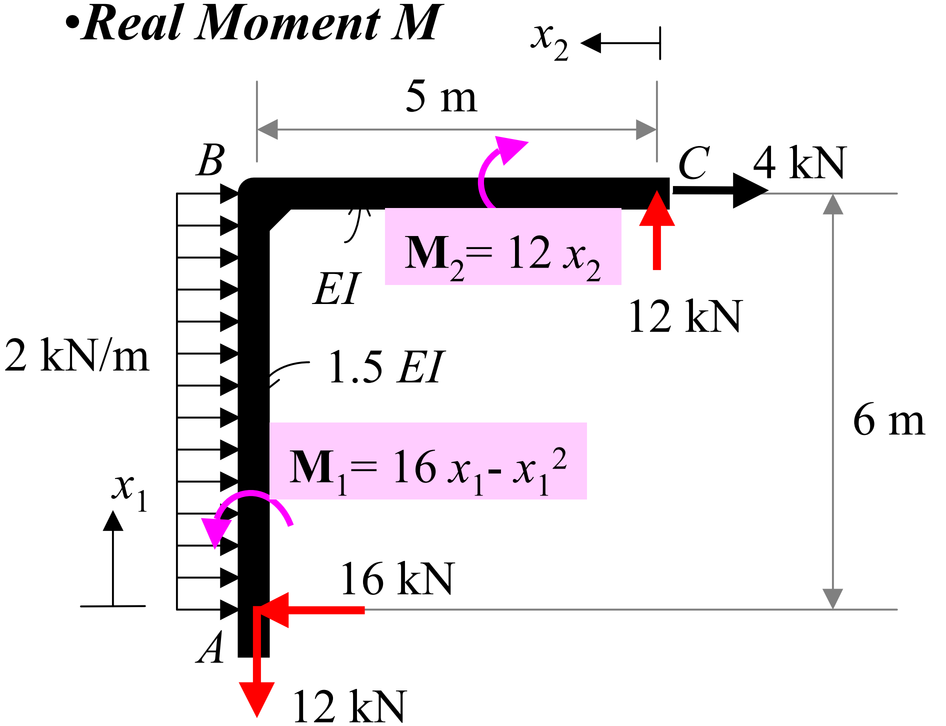


$$1 \bullet \Delta_{CH} = \int_L \frac{m_\Delta M}{EI} dx = \frac{1}{1.5EI}\int_0^6 (x_1)(16x_1 - x_1^2)dx_1 + \frac{1}{EI}\int_0^5 (1.2x_2)(12x_2)dx_2$$

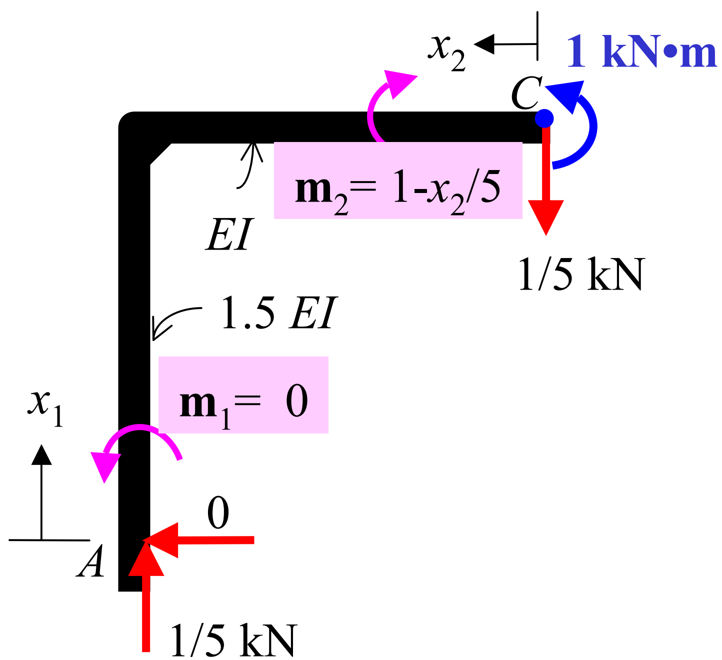
$$= \frac{1}{1.5EI}\int_0^6 (16x_1^{\ 2} - x_1^{\ 3})dx_1 + \frac{1}{EI}\int_0^5 (14.4x_2^{\ 2})dx_2$$

$$\Delta_{CH} = \frac{1}{1.5EI}\left(\frac{16x_1^{\ 3}}{3} - \frac{x_1^{\ 4}}{4}\right)\Bigg|_0^6 + \frac{1}{EI}\left(\frac{14.4x_2^{\ 3}}{3}\right)\Bigg|_0^5 = \frac{552}{EI} + \frac{600}{EI} = \frac{1152}{(200)(200)} = +28.8\ mm, \rightarrow$$

•***Real Moment M***



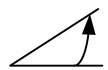
•***Virtual Moment*** m_θ

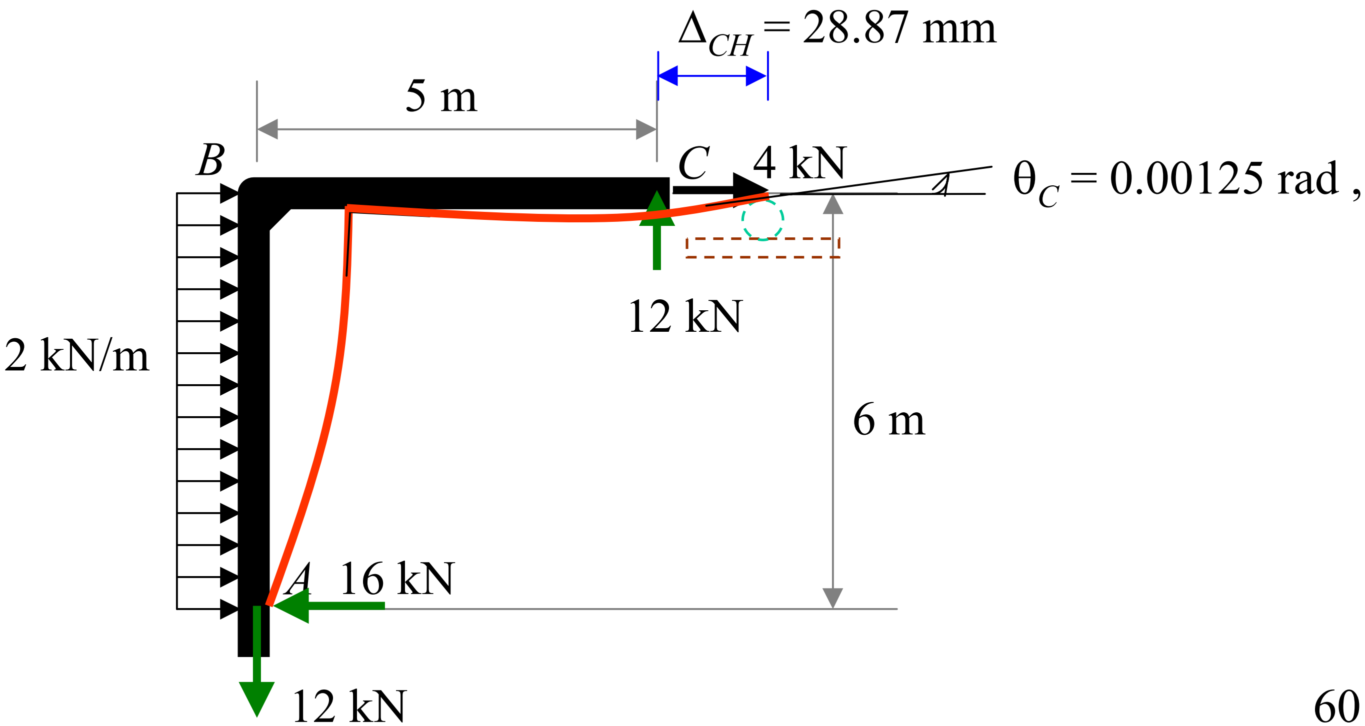


$$1 \bullet \theta_C = \int_L \frac{m_\theta M}{EI} dx = \frac{1}{1.5EI}\int_0^6 (0)(16x_1 - x_1^2)dx_1 + \frac{1}{EI}\int_0^5 (1 - \frac{x_2}{5})(12x_2)dx_2$$

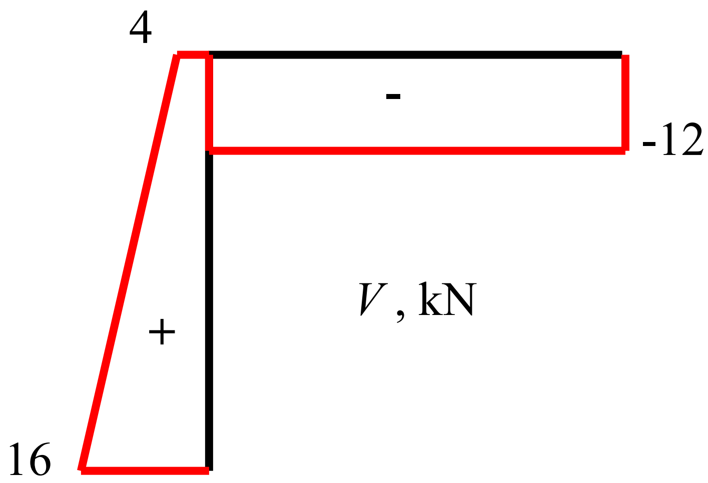
$$= 0 + \frac{1}{EI}\int_0^5 (12x_2 - \frac{12x_2^{\ 2}}{5})dx_2$$

$$\theta_C = \frac{1}{EI}\left(\frac{12x_2^{\ 2}}{2} - \frac{12x_2^{\ 3}}{5\times 3}\right)\Bigg|_0^5 = \frac{50}{EI} = \frac{50}{(200)(200)} = +0.00125\,rad,$$

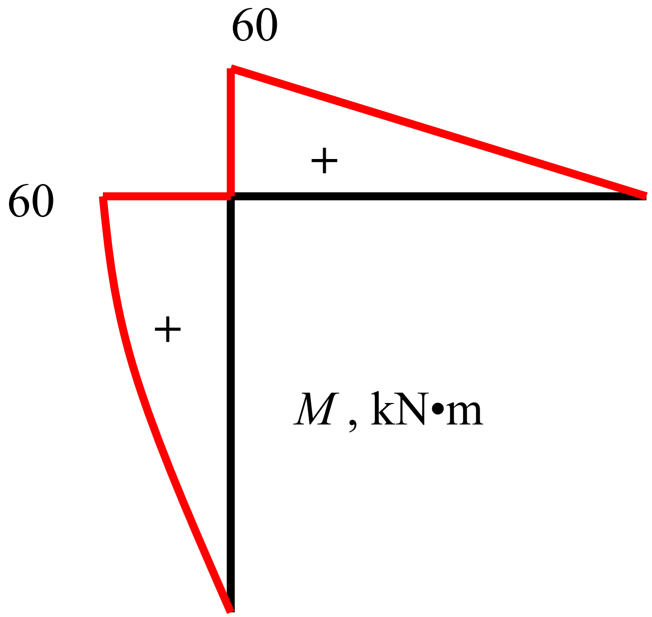




Δ_{CH} = 28.87 mm
5 m
B
C
4 kN
θ_C = 0.00125 rad ,
12 kN
2 kN/m
6 m
A
16 kN
12 kN



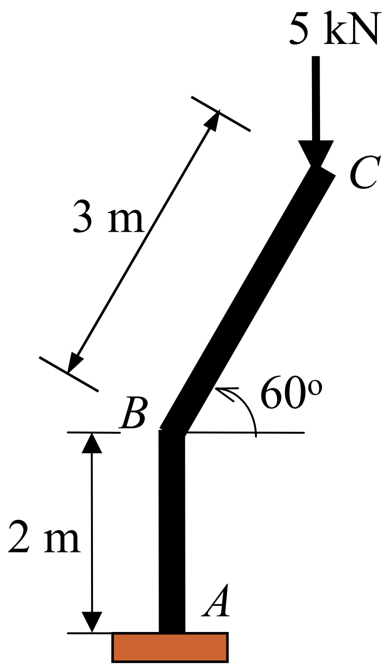
4
-
-12
+
V , kN
16



60
+
60
+
M , kN•m

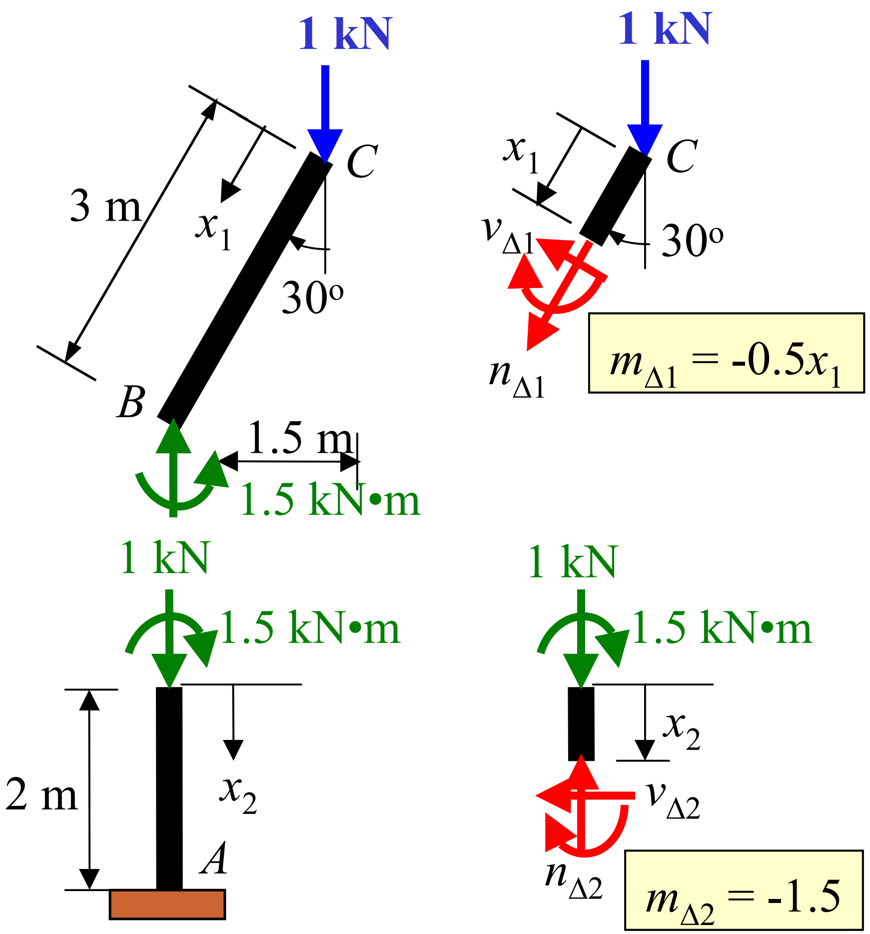
Example 8-23

Determine the slope and the vertical displacement of point ***C*** on the frame.
Take $E = 200$ GPa, $I = 15(10^6)$ mm^4.

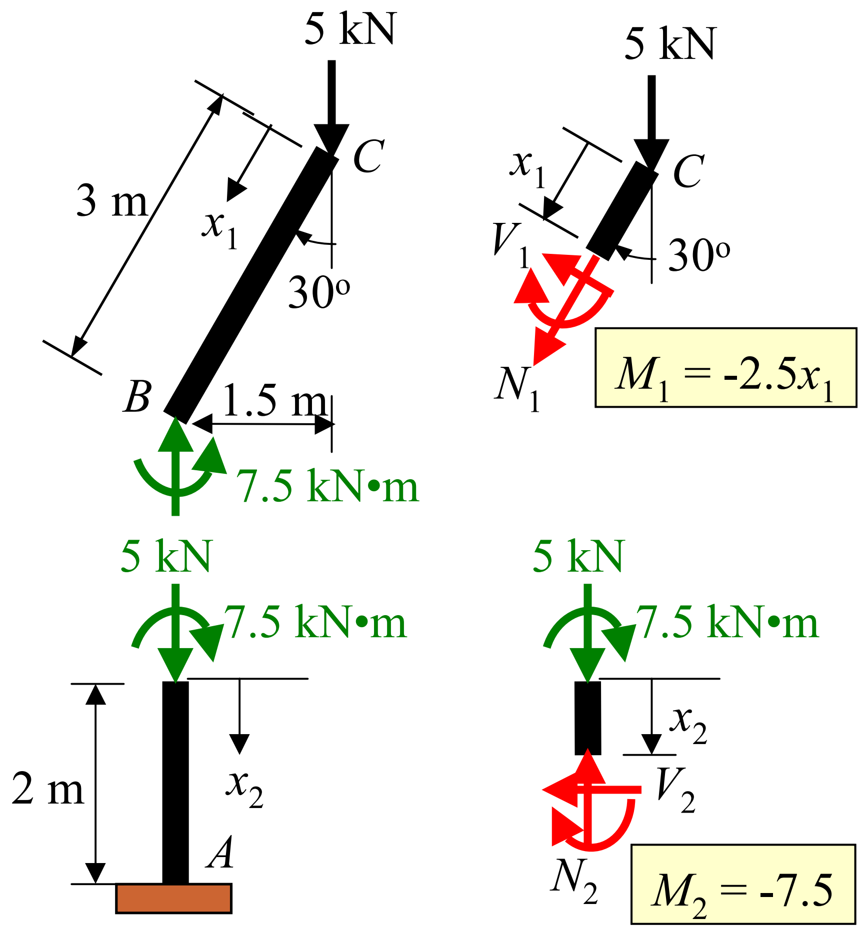


Displacement at C

•*Virtual Moment* m_Δ



•*Real Moment M*

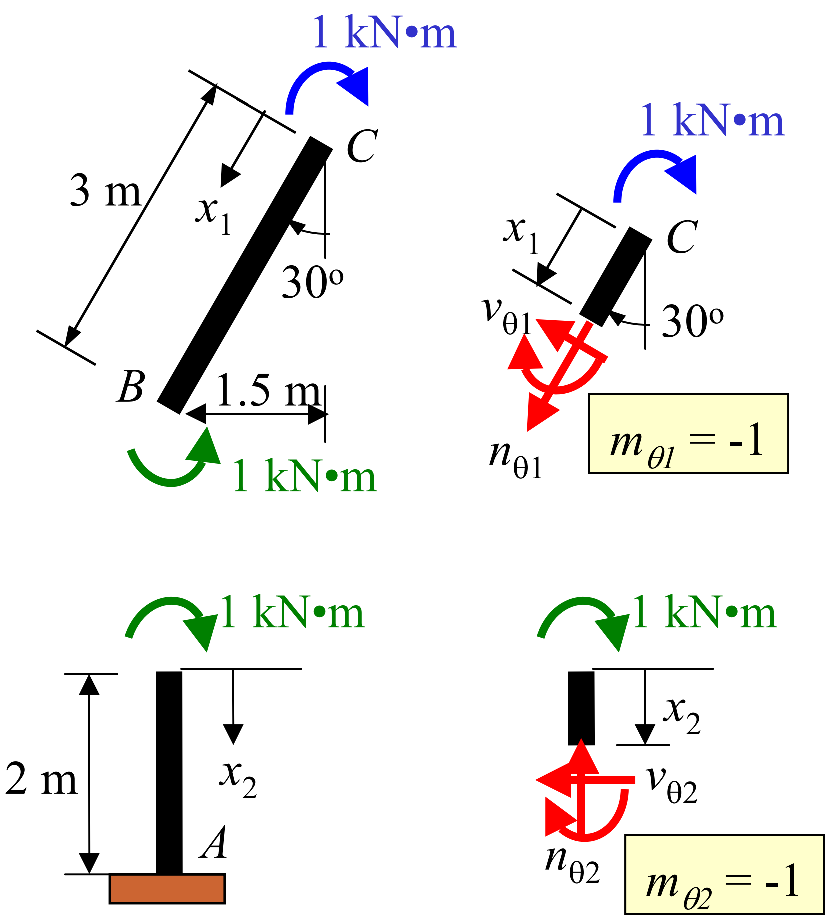


$$1\bullet\Delta_{CV}=\int_L \frac{m_\Delta M}{EI}dx = \frac{1}{EI}\int_0^3(-0.5x_1)(-2.5x_1)dx_1+\frac{1}{EI}\int_0^2(-1.5)(-7.5)dx_2$$

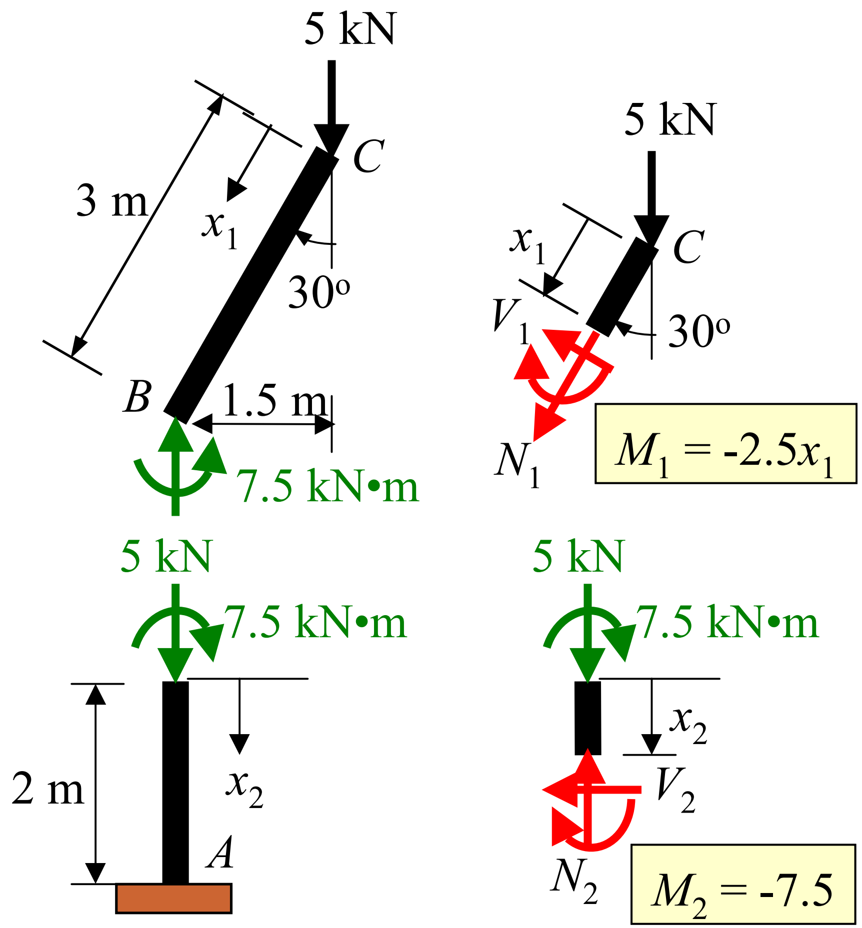
$$\Delta_{CV}=\frac{1}{EI}\left(\frac{1.25x_1^3}{3}\right)\Big|_0^3+\frac{1}{EI}(11.25x_2^2)\Big|_0^2=\frac{33.75}{EI}=\frac{33.75}{(200)(15)} = 11.25 \text{ mm}, \downarrow$$

Slope at C

•*Virtual Moment m_θ*



•*Real Moment M*



$$1\bullet\theta_C = \int_L \frac{m_\theta M}{EI}dx \qquad = \frac{1}{EI}\int_0^3 (-1)(-2.5x_1)dx_1 + \frac{1}{EI}\int_0^2 (-1)(-7.5)dx_2$$

$$\theta_C = \frac{1}{EI}\left(\frac{2.5x_1^2}{2}\right)\Big|_0^3 + \frac{1}{EI}(7.5x_2)\Big|_0^2 = \frac{26.25}{EI} = \frac{26.25}{(200)(15)} \quad = 0.00875 \text{ rad},$$

Virtual Strain Energy Caused by Axial Load, Shear, Torsion, and Temperature

- **Axial Load**

$$U_i = \int n\, d\Delta = \int_L n\left(\frac{N}{EA}\right)dx$$

(where $\frac{N}{EA}dx = d\Delta$)

Where

n = internal virtual axial load caused by the external virtual unit load
N = internal axial force in the member caused by the real loads
L = length of a member
A = cross-sectional area of a member
E = modulus of elasticity for the material

• **Bending**

$$U_i = \int m \, d\theta = \int_L m\left(\frac{M}{EI}\right)dx$$

(annotation: $\frac{M}{EI}dx \rightarrow d\theta$)

Where

n = internal virtual moment cased by the external virtual unit load
M = internal moment in the member caused by the real loads
L = length of a member
E = modulus of elasticity for the material
I = moment of inertia of cross-sectional area, computed about the the neutral axis

• **Torsion**

$$U_i = \int t\, d\theta = \int_L t\left(\frac{T}{GJ}\right)dx$$

(annotation: $\frac{T}{GJ}dx \rightarrow d\theta$)

Where

t = internal virtual torque caused by the external virtual unit load
T = internal torque in the member caused by the real loads
G = shear modulus of elasticity for the material
J = polar moment of inertia for the cross section, $J = \pi c^4/2$, where c is the radius of the cross-sectional area

• **Shear**

$$U_i = \int v\,d\upsilon = \int_L v\left(\frac{KV}{GA}\right)dx$$

(annotation: $\frac{KV}{GA}dx \rightarrow d\upsilon$)

Where

v = internal virtual shear in the member, expressed as a function of x and caused by the external virtual unit load

V = internal shear in the member expressed as a function of x and caused by the real loads

K = form factor for the cross-sectional area:

- K = 1.2 for rectangular cross sections
- K = 10/9 for circular cross sections
- $K \approx 1$ for wide-flange and I-beams, where A is the area of the web

G = shear modulus of elasticity for the material

A = cross-sectional area of a member

Temperature Displacement :

- **Axial** $$U_i = \int_L n(\alpha \Delta T)dx \quad \alpha \Delta T\,dx \rightarrow d\Delta$$

- **Bending** $$U_i = \int_L m(\alpha \frac{\Delta T}{2c})dx \quad \alpha \frac{\Delta T}{2c}dx \rightarrow d\theta$$

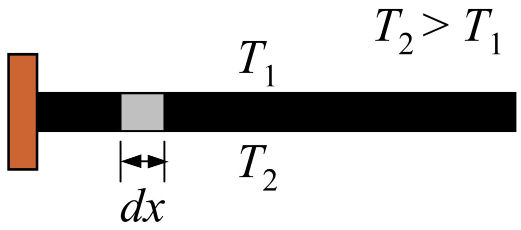
Where

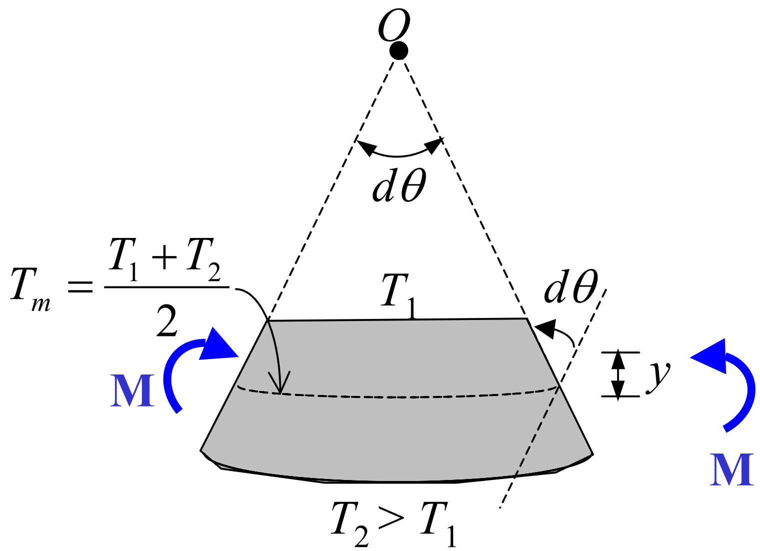
ΔT = Differential temperatures:

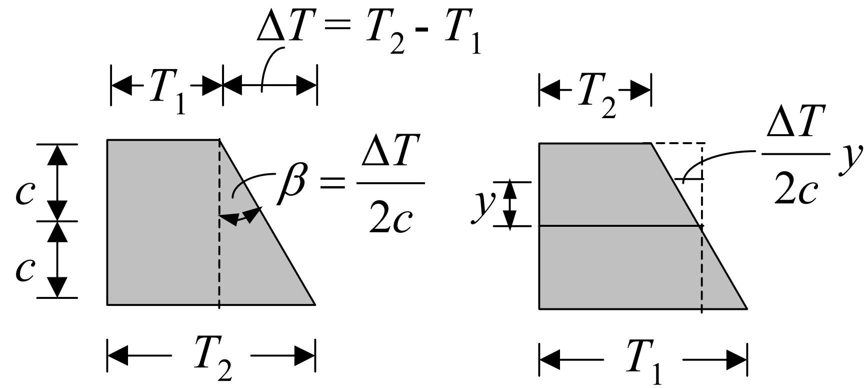
- between the neutral axis and room temperature, for axial
- between two extreme fibers, for bending

α = Coefficient of thermal expansion

• **Temperature**







$$(d\theta)\cancel{y} = \alpha(\frac{\Delta T}{2c}\cancel{y})dx$$

$$(d\theta) = \alpha(\frac{\Delta T}{2c})dx$$

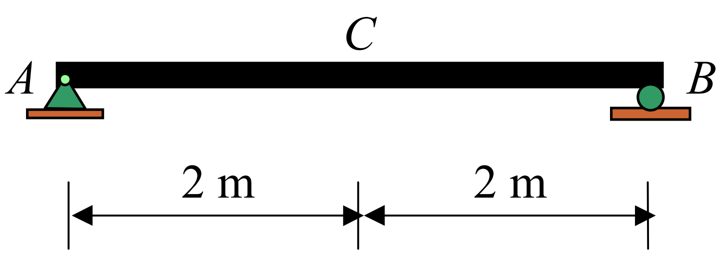
$$U_{temp} = \int m d\theta$$

$$U_{temp} = \int_0^L m(\alpha \frac{\Delta T}{2c})dx$$

Example 8-24

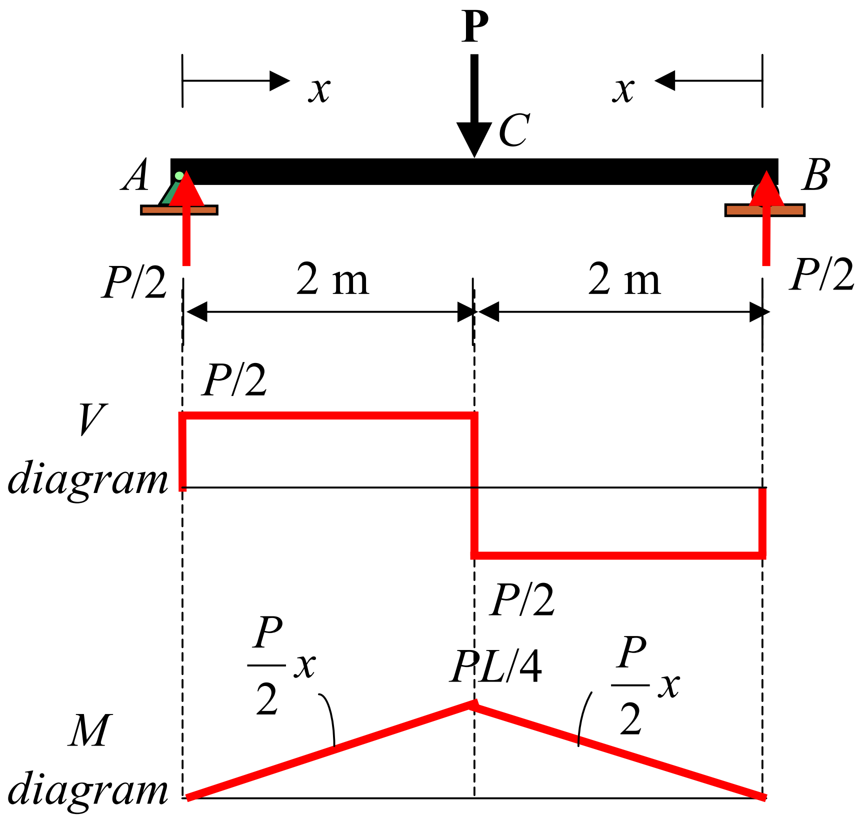
From the beam below Determine :
(a) If P = 60 kN is applied at the mid-span C, what would be the displacement at point C. Due to shear and bending moment.
(b) If the temperature at the top surface of the beam is 55 ^oC , the temperature at the bottom surface is 30 ^oC and the room temperature is 25 ^oC.
What would be the vertical displacement of the beam at its midpoint C and the the horizontal deflection of the beam at support B.
(c) if (a) and (b) are both accounted, what would be the vertical displacement of the beam at its midpoint C.

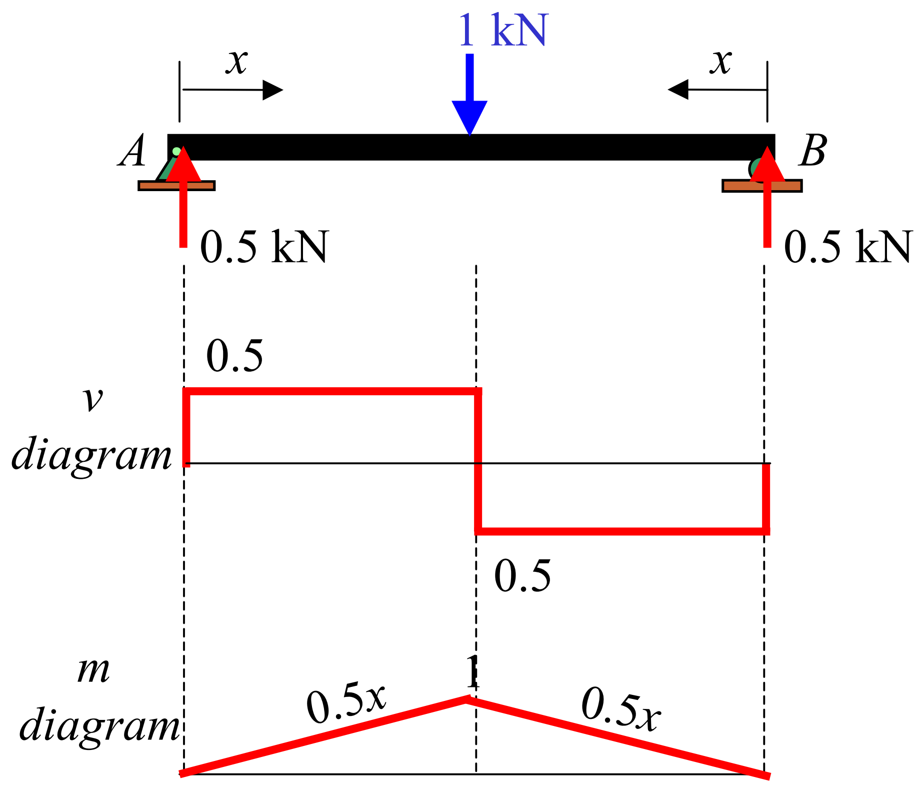
Take $\alpha = 12(10^{-6})/^oC$. E = 200 GPa, G = 80 GPa, I = $200(10^6)$ mm^4 and $A = 35$ (10^3) mm^2. The cross-section area is rectangular.



SOLUTION

• ***Part (a) :***





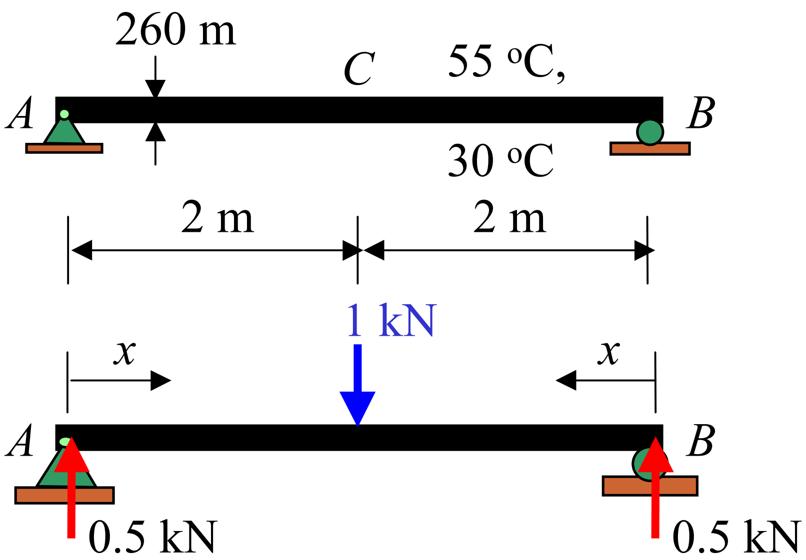
$$\Delta_{bending} = \int_L \frac{m_i M_i}{EI}dx = 2\int_0^{L/2} (\frac{x}{2})(\frac{Px}{2})\frac{dx}{EI} = \frac{2}{EI}(\frac{Px^3}{4\times 3})\bigg|_0^{L/2} = \frac{PL^3}{48EI} = \frac{60(4)^3}{48(200)(200)} = 2 \text{ mm}, \downarrow$$

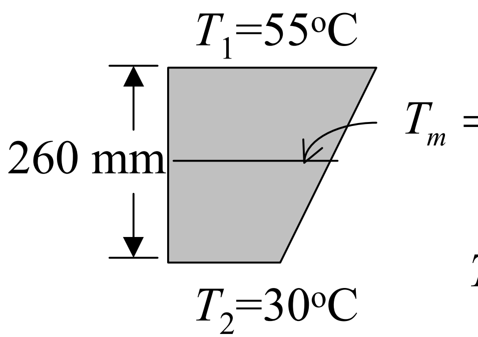
$$\Delta_{shear} = \int_L \frac{Kv_i V_i}{GA}dx = 2\int_0^{L/2} K(\frac{1}{2})(\frac{P}{2})\frac{dx}{GA} = \frac{KPx}{2GA}\bigg|_0^{L/2} = \frac{KPL}{4GA} = \frac{1.2(60)(4)}{4(80)(35000)} = 0.026 \text{ mm}, \downarrow$$

$$\Delta_C = \Delta_{bending} + \Delta_{shear} = 2 + 0.026 = 2.03 \text{ mm}, \downarrow$$

SOLUTION

•Part (b) : Vertical displacement at C



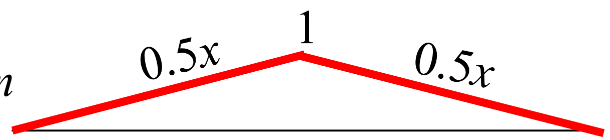


Temperature profile

$$T_m = \frac{55+30}{2} = 42.5$$

T_{room} = 25 °C ,

m diagram

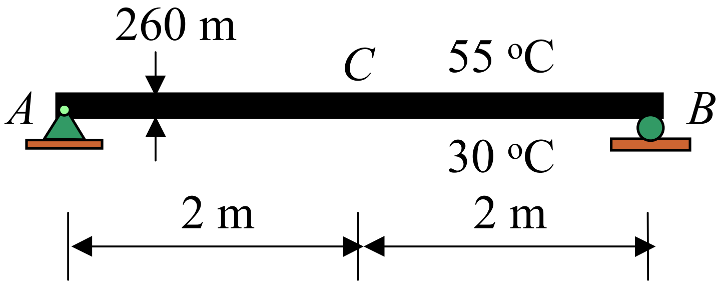


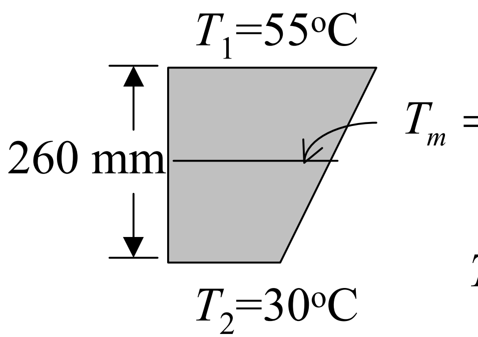
- *Bending*

$$(1\,kN)(\Delta_C) = \int_0^L \frac{m\alpha(\Delta T)}{2c}dx \quad = 2\frac{\alpha(\Delta T)}{2c}\int_0^2 (0.5x)dx \quad = 2\frac{(12\times10^{-6})(-25)}{(260\times10^{-3})}\left(\frac{0.5x^2}{2}\right)\Bigg|_0^2$$

Δ_C = -2.31 mm , ↑

• *Part (b) : Horizontal displacement at B*

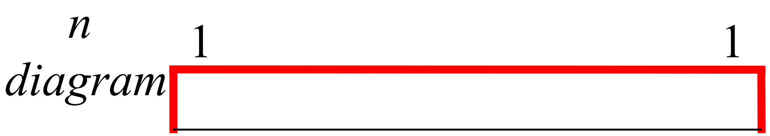




Temperature profile

$$T_m = \frac{55+30}{2} = 42.5$$

$$T_{room} = 25\ ^{o}\text{C}\ ,$$



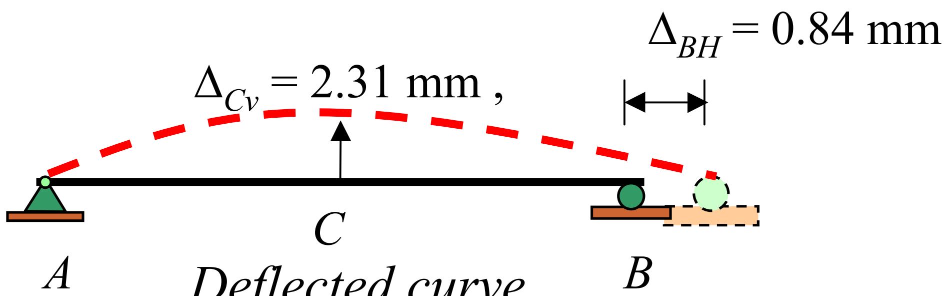
- *Axial*

$$(1\ kN)(\Delta_{BH}) = \int_L n\alpha(\Delta T)dx$$

$$= \alpha(\Delta T)\int_0^4 (1)dx$$

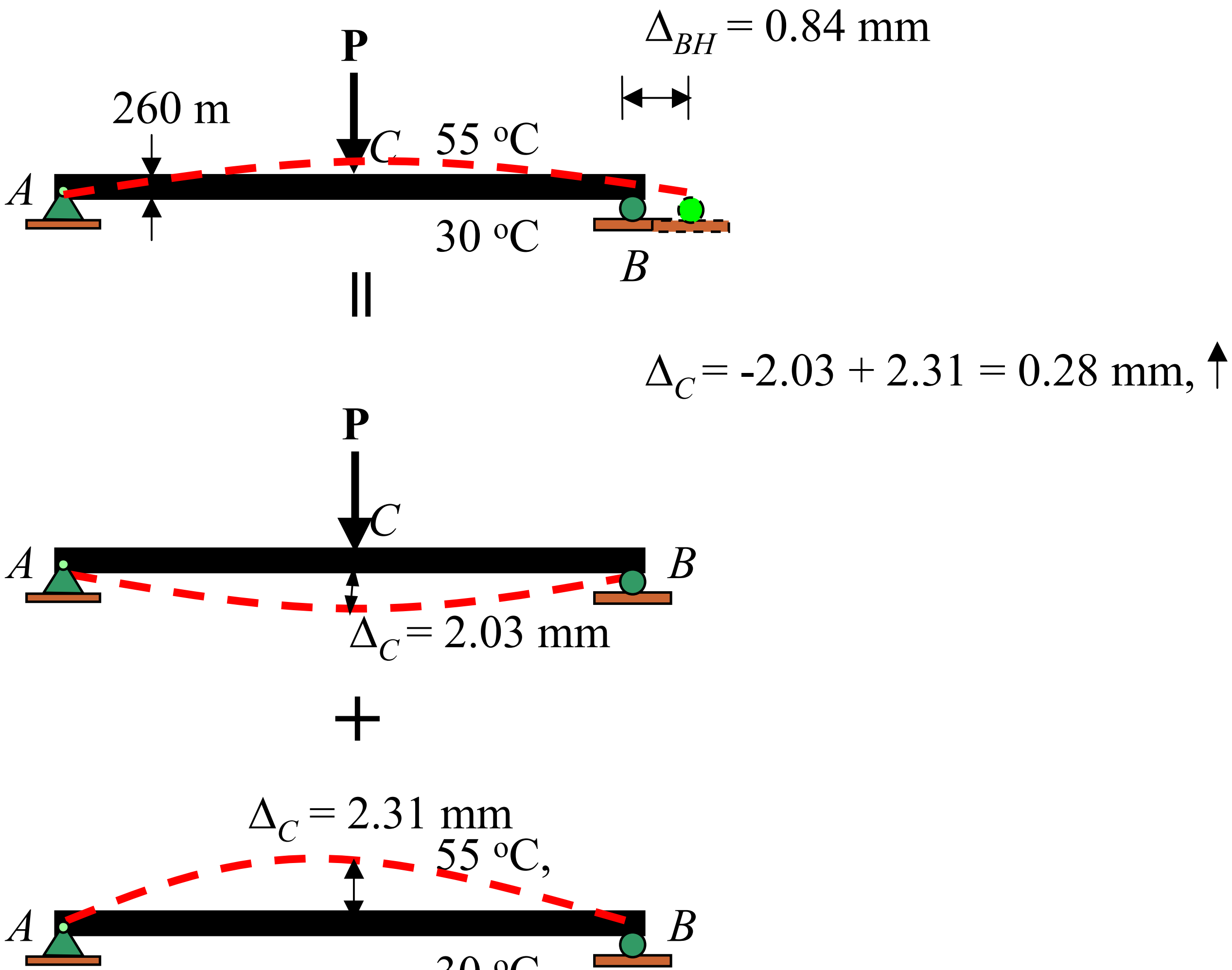
$$= (12\times 10^{-6})(42.5-25)(x)\Big|_0^4$$

$$\Delta_{BH} = 0.84\ \text{mm}\ , \longrightarrow$$



Deflected curve

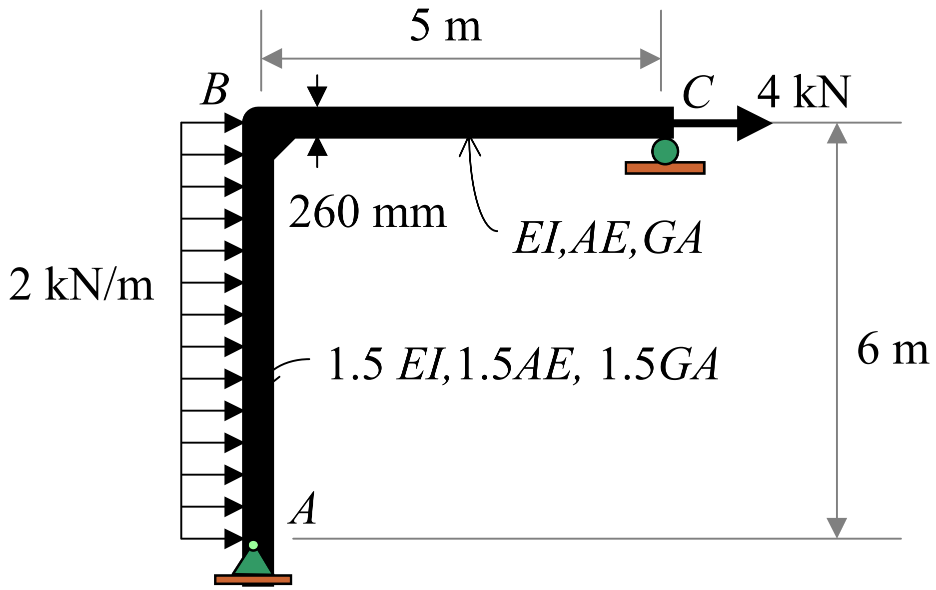
• ***Part (c) :***



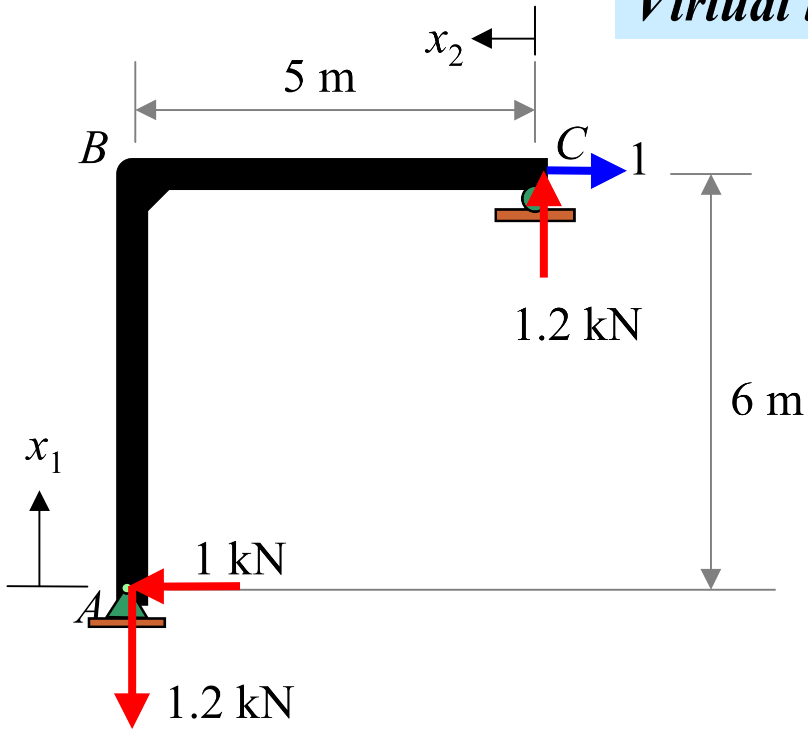
Δ_{BH} = 0.84 mm
P
260 m
C
55 °C
A
30 °C
B
=
Δ_C = -2.03 + 2.31 = 0.28 mm, ↑
P
C
A
B
Δ_C = 2.03 mm
+
Δ_C = 2.31 mm
55 °C,
A
B
30 °C

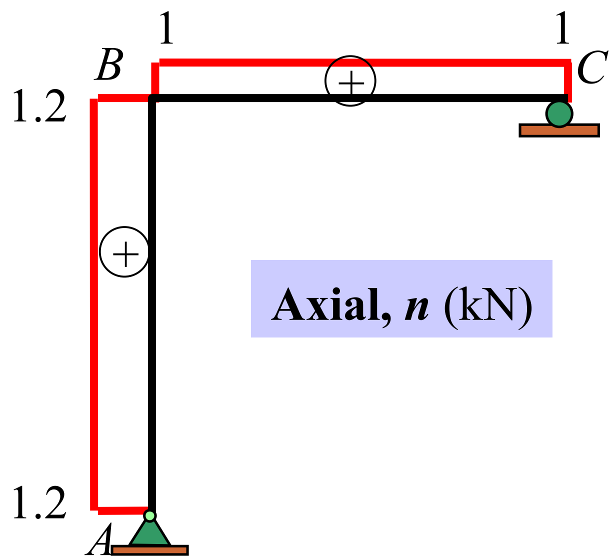
Example 8-25

Determine the horizontal displacement of point ***C*** on the frame.If the temperature at top surface of member *BC* is 30 °C , the temperature at the bottom surface is 55 °C and the room temperature is 25 °C.Take $\alpha = 12(10^{-6})/^{o}C$, $E = 200$ GPa, $G = 80$ GPa, $I = 200(10^6)$ mm^4 and $A = 35(10^3)$ mm^2 for both members. The cross-section area is rectangular. Include the internal strain energy due to axial load and shear.

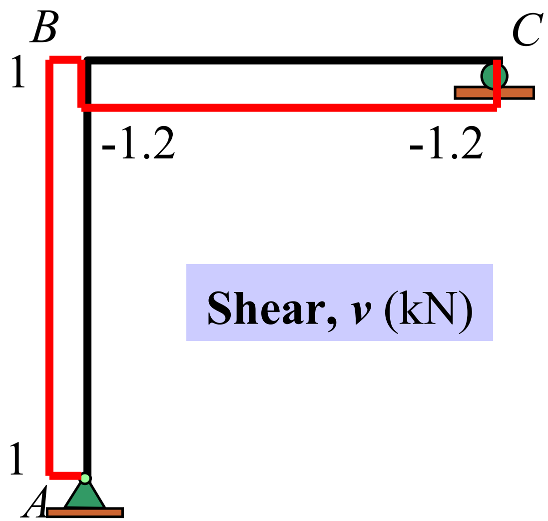


Virtual load

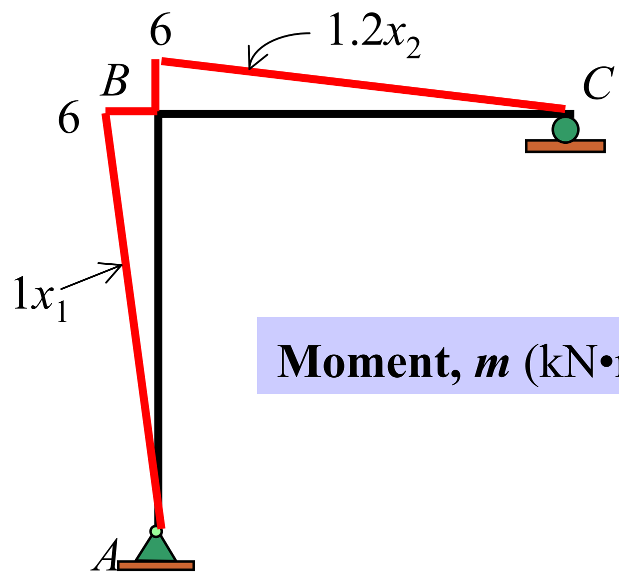




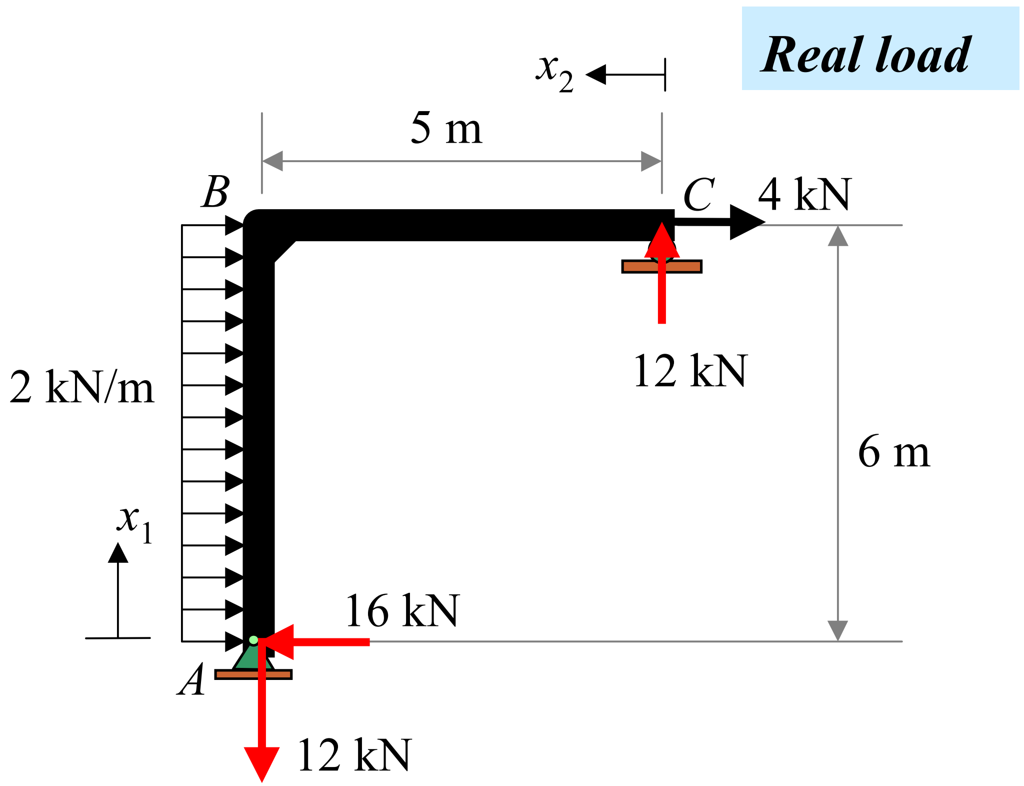
Axial, *n* (kN)

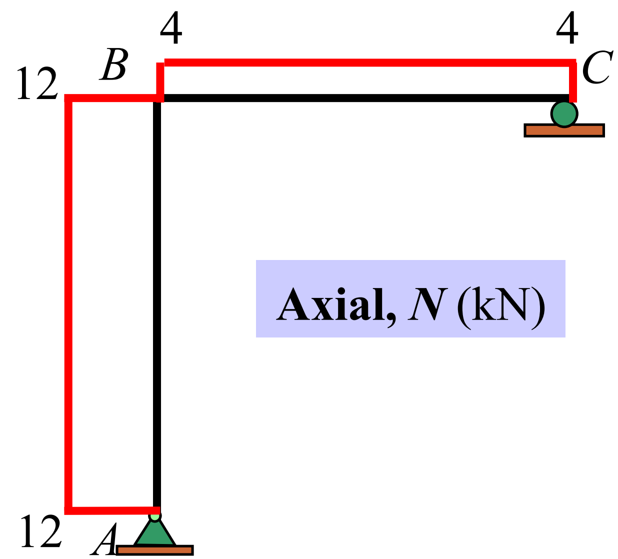


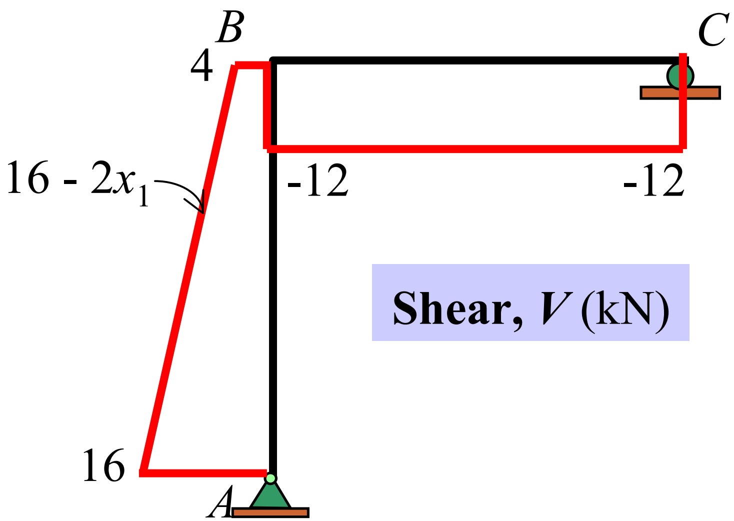
Shear, *v* (kN)

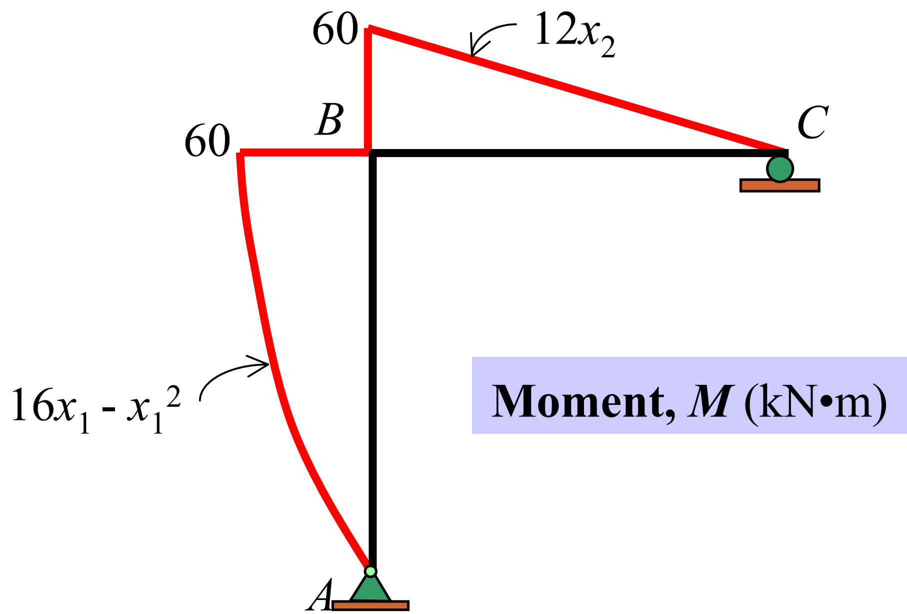


Moment, *m* (kN•m)

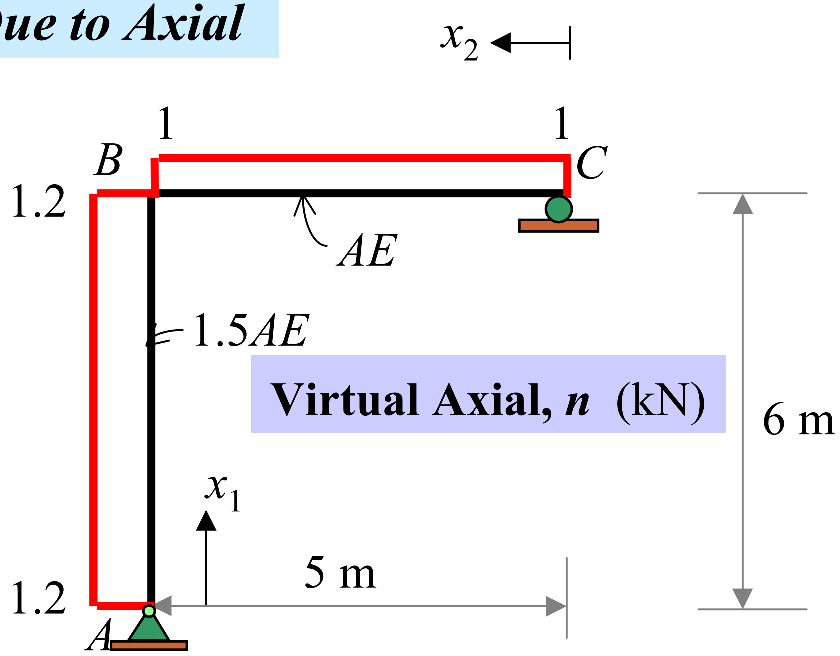

Real load
x_2
5 m
B
C
4 kN
2 kN/m
12 kN
6 m
x_1
16 kN
A
12 kN

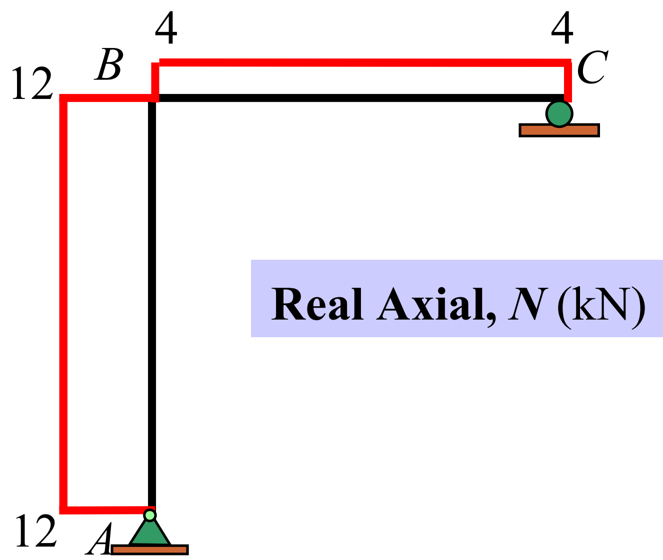

4
4
12
B
C
Axial, N (kN)
12
A


B
C
4
16 - $2x_1$
-12
-12
Shear, V (kN)
16
A


60
$12x_2$
60
B
C
$16x_1 - x_1^2$
Moment, M (kN•m)
A

•Due to Axial





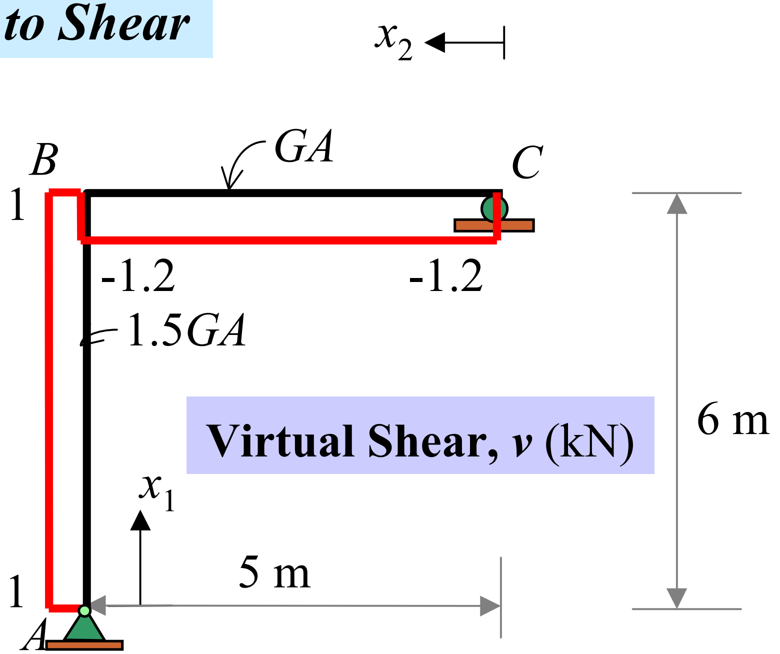
$$(1\,kN)(\Delta_{CH}) = \sum \frac{n_i N_i L_i}{A_i E_i}$$

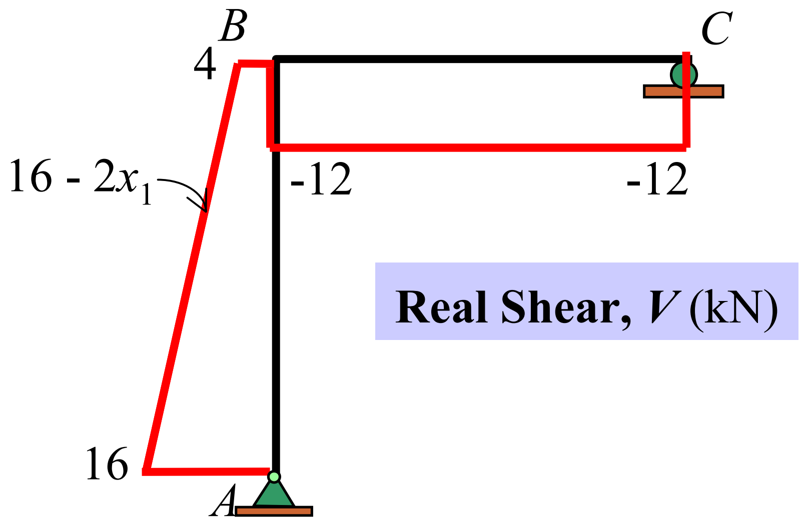
$$= \frac{(1.2)(12)(6)}{1.5AE} + \frac{(1)(4)(5)}{AE}$$

$$= \frac{77.6\,kN^2 \bullet m}{AE}$$

$$\Delta_{CH} = \frac{77.6\,kN \bullet m}{(35000\times 10^{-6}\,m^2)(200\times 10^6\,\frac{kN}{m^2})} = 1.109(10^{-5})\,m = 0.0111\,mm, \rightarrow$$

•*Due to Shear*





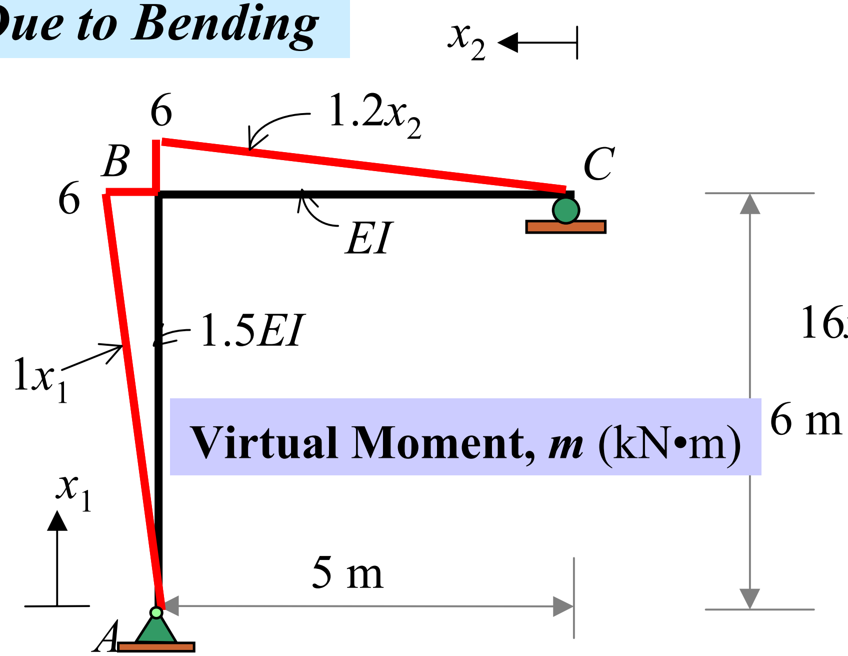
$$(1\,kN)(\Delta_{CH}) = \int_0^L K\left(\frac{\upsilon V}{GA}\right)dx$$

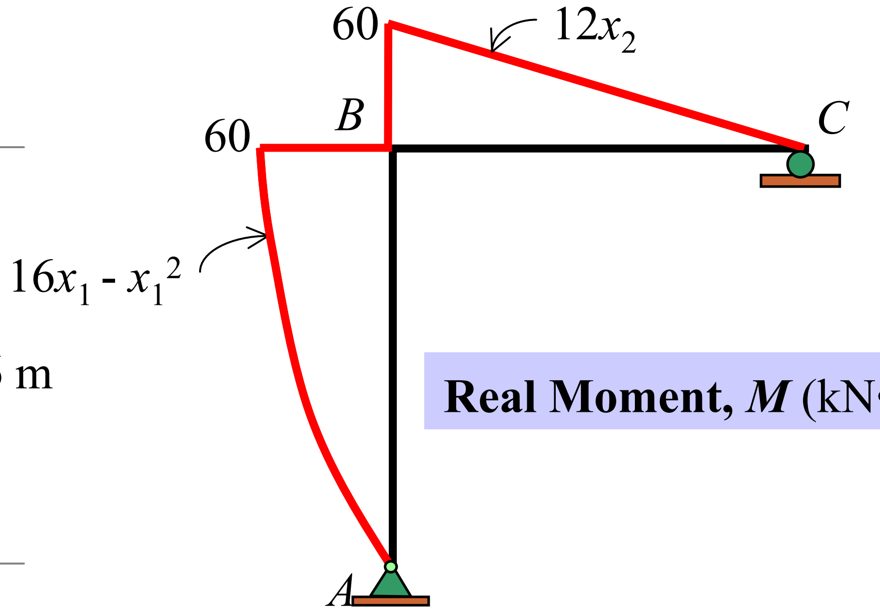
$$= \int_0^6 1.2\frac{(1)(16-2x_1)}{1.5GA}dx_1 + \int_0^5 1.2\frac{(-1.2)(-12)}{GA}dx_2$$

$$= \left(\frac{1.2}{1.5GA}\right)\left(16x_1 - \frac{2x_1^2}{2}\right)\Big|_0^6 + \left(\frac{1.2}{GA}\right)(14.4x_2)\Big|_0^5 = \frac{134.4\,kN^2 \bullet m}{GA}$$

$$\Delta_{CH} = \frac{134.4\,kN \bullet m}{(80\times10^6\,\frac{kN}{m^2})(35000\times10^{-6}\,m^2)} = 4.8(10^{-5})\,m = 0.048\,mm, \rightarrow$$

•*Due to Bending*





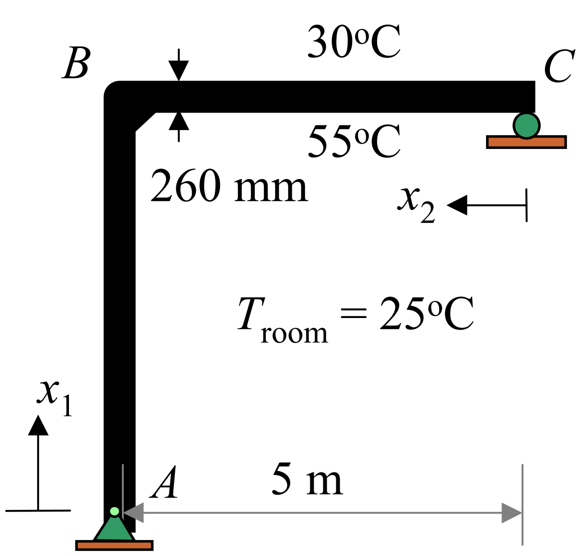
$$(1\,kN)(\Delta_{CH}) = \int_0^L \frac{mM}{EI}dx$$

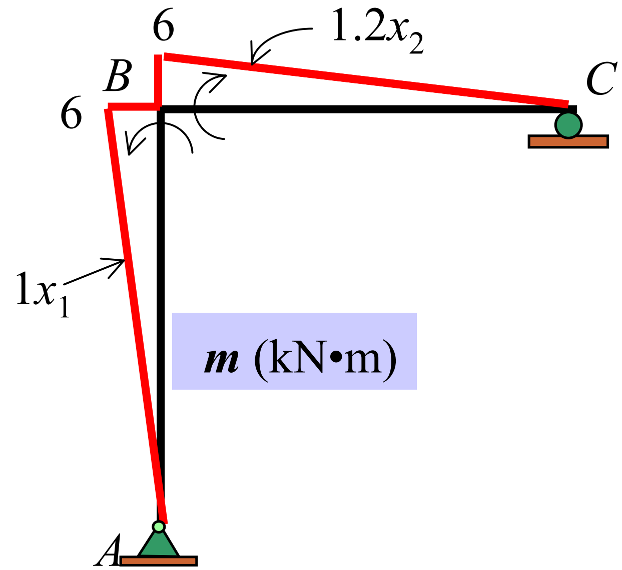
$$= \frac{1}{1.5EI}\int_0^6 (x_1)(16x_1 - x_1^{\,2})dx_1 + \frac{1}{EI}\int_0^5 (1.2x_2)(12x_2)dx_2$$

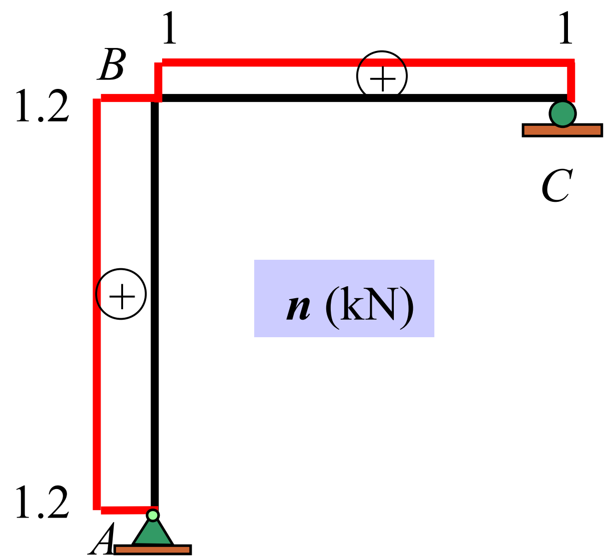
$$= \frac{1}{1.5EI}\left(\frac{16x_1^{\,3}}{3} - \frac{x_1^{\,4}}{4}\right)\Big|_0^6 + \frac{1}{EI}\left(\frac{14.4x_2^{\,3}}{3}\right)\Big|_0^5 = \frac{1152\,kN^2 \bullet m^3}{EI}$$

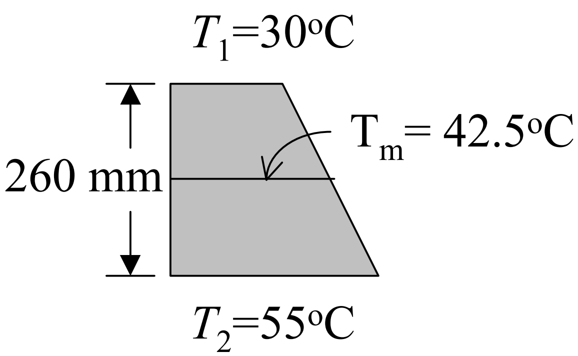
$$\Delta_{CH} = \frac{1152\,kN \bullet m^3}{(200\times10^6\,\frac{kN}{m^2})(200\times10^{-6}\,m^4)} = 0.0288\,m = +28.8\,mm, \rightarrow$$

•*Due to Temperature*









Temperature profile

- *Bending*

$$(1\,kN)(\Delta_{CH}) = \int_0^L \frac{m\alpha(\Delta T)}{2c}dx = \int_0^5 \frac{(1.2x_2)(12\times10^{-6})(55-30)}{(260\times10^{-3})}dx^2$$

$$\Delta_{CH} = 0.0173 \text{ m} = 17.3 \text{ mm}, \quad \longrightarrow$$

- *Axial*

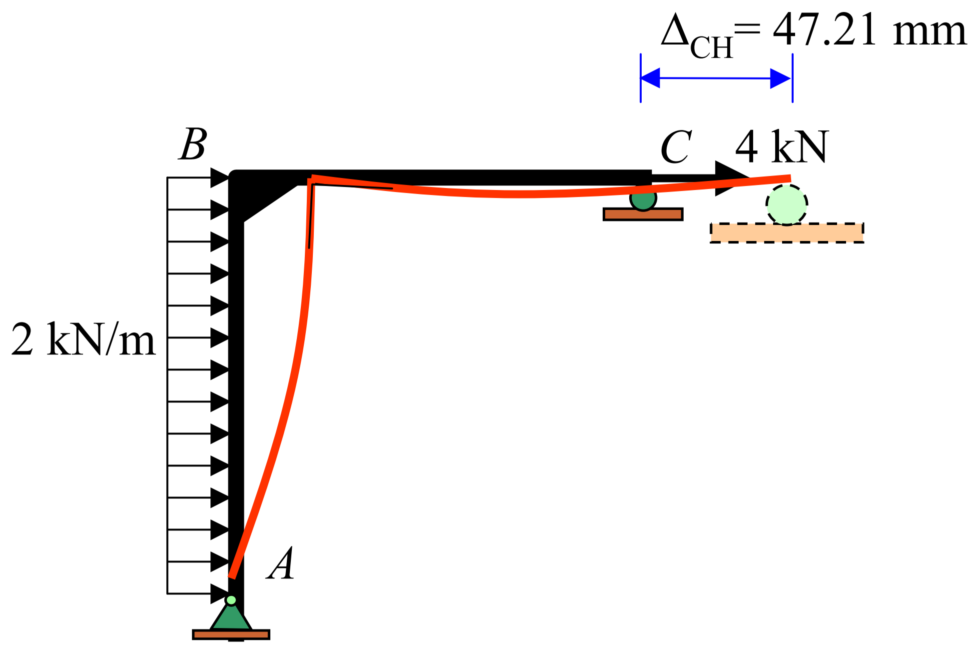
$$(1\,kN)(\Delta_{CH}) = \int_0^L n\alpha(\Delta T)dx = \int_0^5 (1)(12\times10^{-6})(42.5-25)dx_2$$

$$\Delta_{CH} = 0.00105 \text{ m} = 1.05 \text{ mm}, \quad \longrightarrow$$

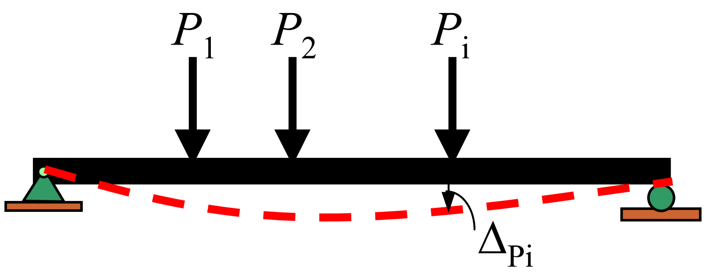
•*Total Displacement*

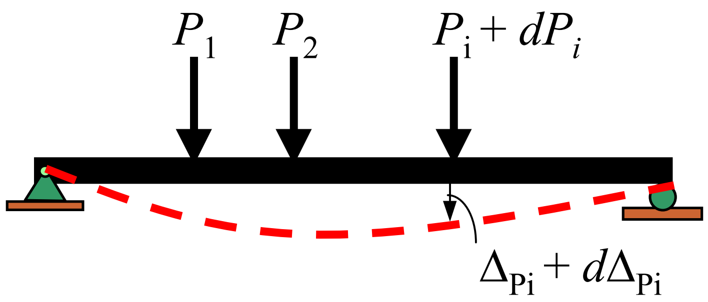
$$(\Delta_{CH})_{Total} = (\Delta_{CH})_{Axial} + (\Delta_{CH})_{Shear} + (\Delta_{CH})_{Bending} + (\Delta_{CH})_{Temp}$$

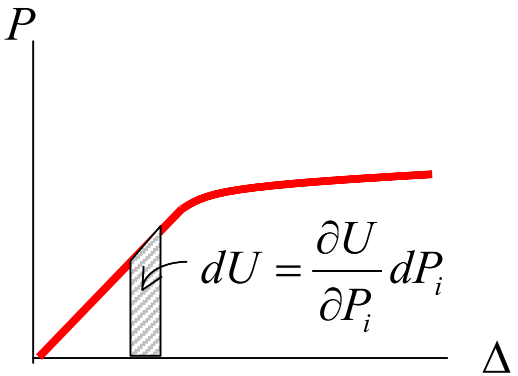
$$= 0.01109 + 0.048 + 28.8 + (17.3 + 1.05) = 47.21 \text{ mm}$$

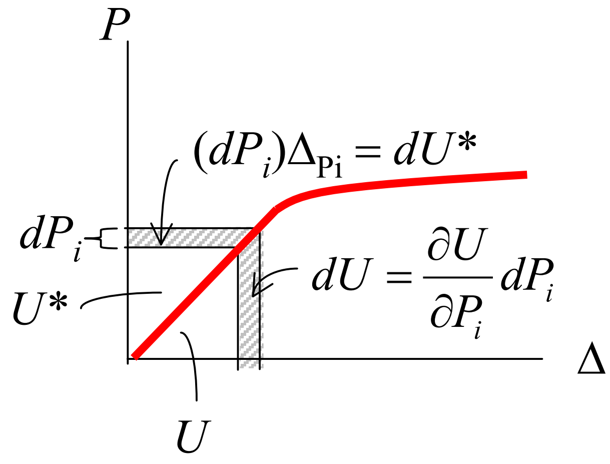


Castigliano's Theorem









$$U_i = f(P_1, P_2, \ldots, P_n)$$

$$U = U^*$$

$$dU = dU^*$$

$$\frac{\partial U}{\partial P_i} dP_i = (dP_i)\Delta_{Pi}$$

$$\Delta_{Pi} = \frac{\partial U}{\partial P_i}$$

Load Displacement :

- **Axial Load** $$\Delta_{Pi} = \frac{\partial}{\partial P_i}\left(\int_L \frac{N^2}{2AE}dx\right) = \int_L \underbrace{\left(\frac{\partial N}{\partial P_i}\right)}_{n_\Delta}\frac{N}{AE}dx$$

- **Bending** $$\Delta_{Pi} = \frac{\partial}{\partial P_i}\left(\int_L \frac{M^2}{2EI}dx\right) = \int \underbrace{\left(\frac{\partial M}{\partial P_i}\right)}_{m_\Delta}\frac{M}{EI}dx$$

- **Shear** $$\Delta_{Pi} = \frac{\partial}{\partial P_i}\left(\int_L \frac{KV^2}{2GA}dx\right) = \int K\underbrace{\left(\frac{\partial V}{\partial P_i}\right)}_{v_\Delta}\frac{V}{GA}dx$$

Where

Δ = external displacement of the truss, beam or frame
P = external force applied to the truss, beam or frame in the direction of Δ
N = internal axial force in the member caused by *both* the force P and the loads on the truss, beam or frame
M = internal moment in the beam or frame, expressed as a function of x and caused by both the force P and the real loads on the beam
V = internal moment in the beam or frame caused by both the force P and the real loads on the beam

Temperature Displacement :

- **Axial**

$$\Delta_{Pi} = \frac{\partial}{\partial P_i}\left(\int_L N(\alpha \Delta T)dx\right) = \int \underbrace{\left(\frac{\partial N}{\partial P_i}\right)}_{n_\Delta}(\alpha \Delta T)dx$$

- **Bending**

$$\Delta_{Pi} = \frac{\partial}{\partial P_i}\left(\int_L M\left(\alpha \frac{\Delta T}{2c}\right)dx\right) = \int \underbrace{\left(\frac{\partial M}{\partial P_i}\right)}_{m_\Delta}\left(\alpha \frac{\Delta T}{2c}\right)dx$$

Where

ΔT = Differential temperatures:

- between the neutral axis and room temperature, for axial
- between two extreme fibers, for bending

α = Coefficient of thermal expansion

Slope :

$$\theta_{Mi} = \frac{\partial U}{\partial M_i}$$

- **Bending**

$$\theta_{Mi} = \frac{\partial}{\partial M_i}\left(\int_L \frac{M^2}{2EI}dx\right) = \int_L \underbrace{\left(\frac{\partial M}{\partial M_i}\right)}_{m_\theta}\frac{M}{EI}dx$$

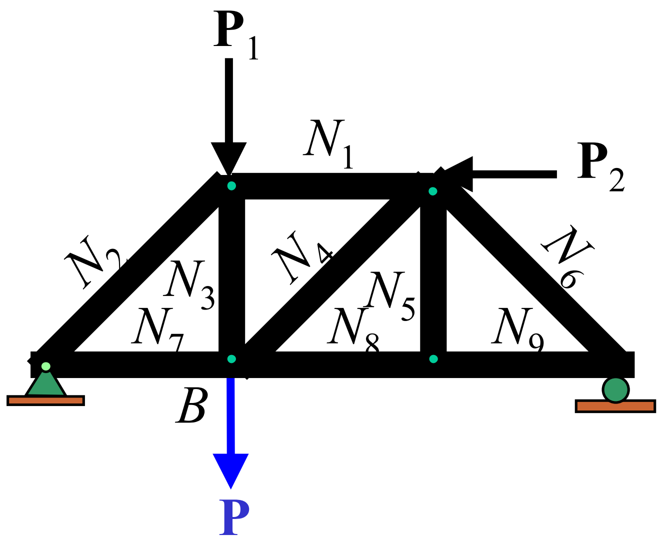
Where

θ = external slope of the beam or frame

M_i = external moment applied to the beam or frame in the direction of θ

M = internal moment in the beam or frame, expressed as a function of x and caused by both the force P and the real loads on the beam

Castigliano's Theorem : Truss



$$\Delta = \sum \left(\frac{\partial N_i}{\partial P}\right) \frac{N_i}{AE} L_i$$

Where:

Δ = external joint displacement of the truss

P = external force applied to the truss joint in the direction of Δ

N = internal force in a member cause by *both* the force P and the loads on the truss

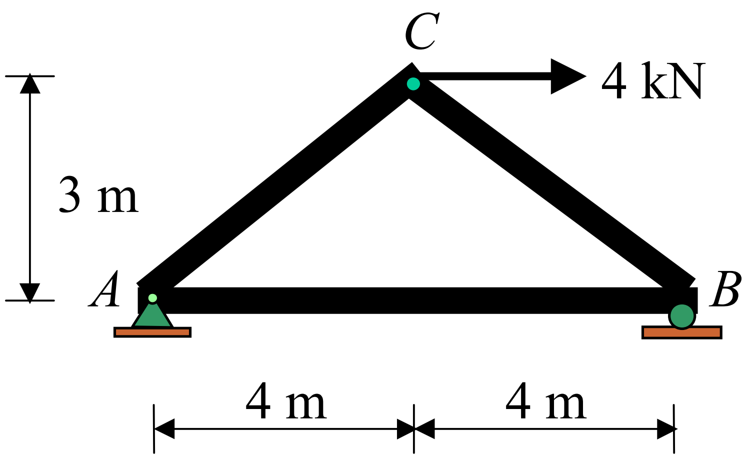
L = length of a member

A = cross-sectional area of a member

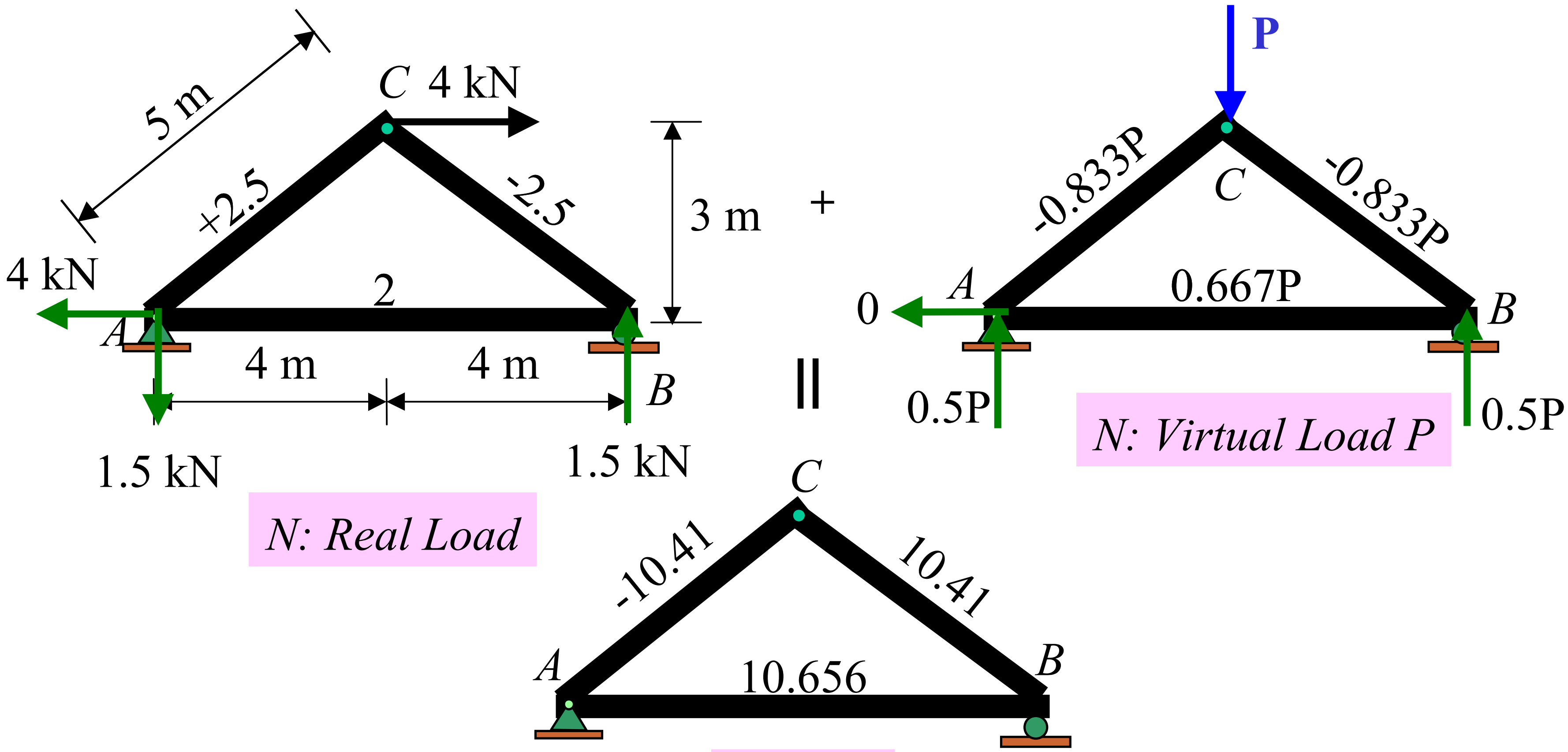
E = modulus of elasticity of a member

Example 8-26

Determine the vertical displacement of joint C of the truss shown in the figure below. The cross-sectional area of each member of the truss shown in the figure is $A = 400\ \text{mm}^2$ and $E = 200$ GPa.



SOLUTION



$$N\left(\frac{\partial N}{\partial P}\right)L$$

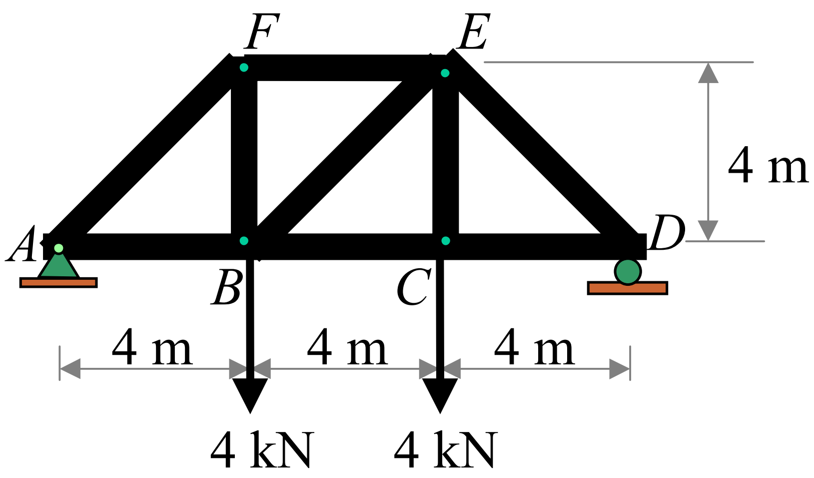
$$\Delta_{CV} = \sum N\left(\frac{\partial N}{\partial P}\right)\frac{L}{AE}$$

$$\Delta_{CV} = \frac{1}{AE}(-10.41+10.41+10.67) = \frac{10.67\ kN \bullet m}{(400\times10^{-6}\,m^2)(200\times10^6\,\frac{kN}{m^2})}$$

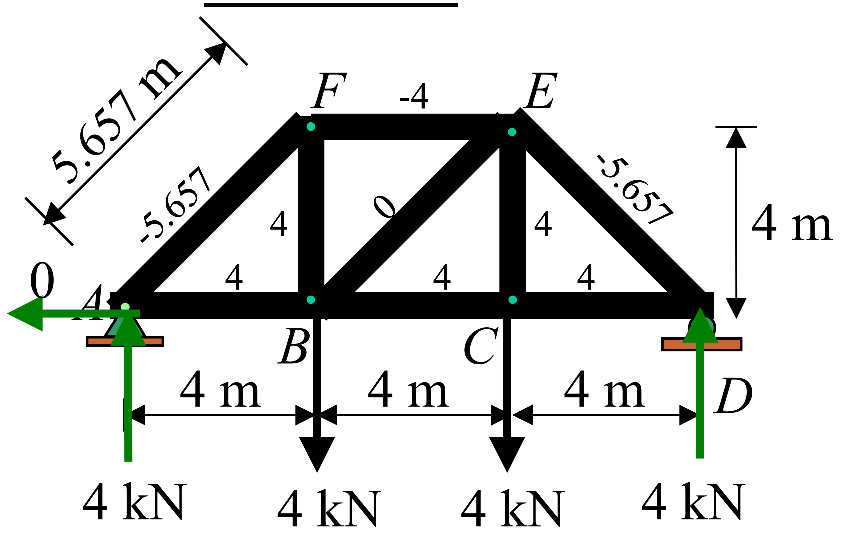
$$\Delta_{CV} = 0.133 \text{ mm}, \downarrow$$

Example 8-27

Determine the vertical displacement of joint C of the steel truss shown. The cross-section area of each member is A = 400 mm^2 and E = 200 GPa.

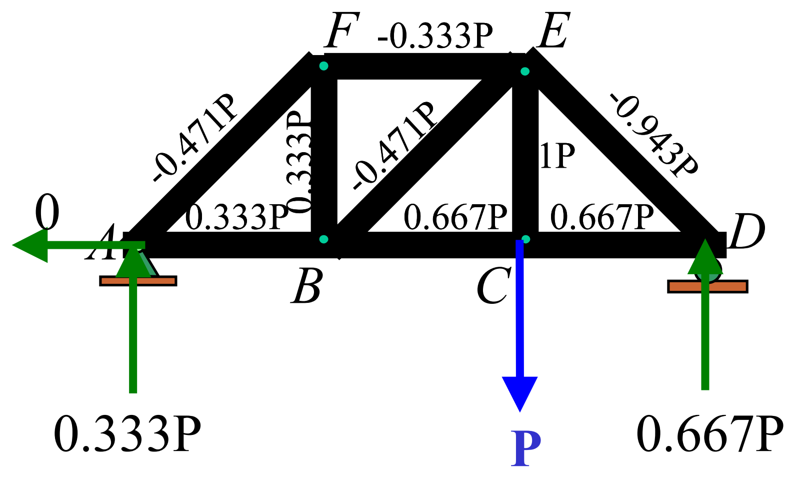


SOLUTION



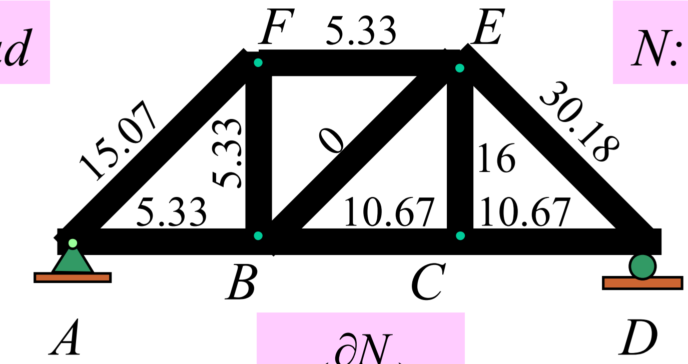
N: Real Load

+



N: Virtual Load P

||



$N(\frac{\partial N}{\partial P})L$

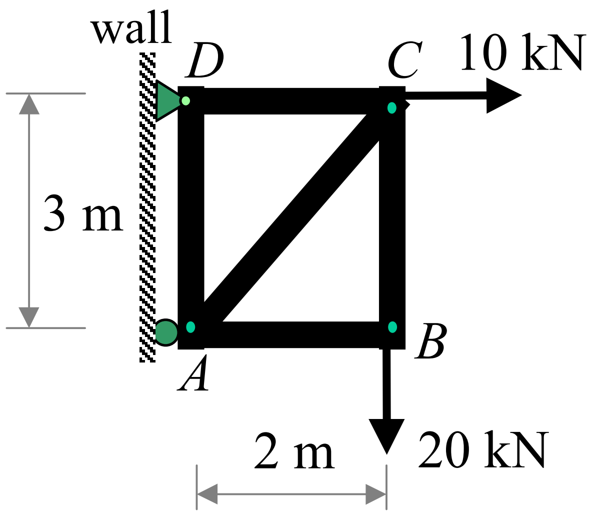
$$\Delta_{CV} = \sum N(\frac{\partial N}{\partial P})\frac{L}{AE}$$

$$\Delta_{CV} = \frac{1}{AE}[15.07 + 3(5.33) + 2(10.67) + 16 + 30.18] = \frac{72.4\ kN \bullet m}{(400\times10^{-6}\,m^2)(200\times10^{6}\,\frac{kN}{m^2})}$$

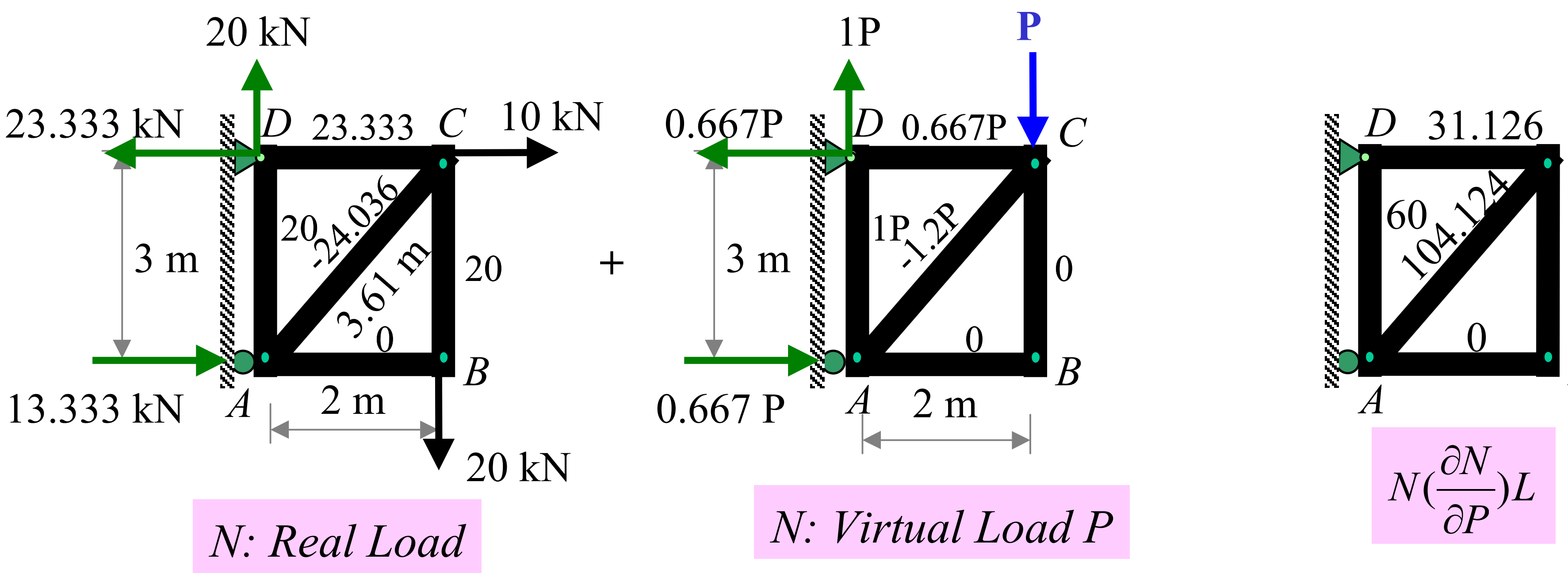
$$\Delta_{CV} = 1.23\ \text{mm},\ \downarrow$$

Example 8-28

Determine the vertical displacement of joint C of the steel truss shown. The cross-section area of each member is $A = 400\ \text{mm}^2$ and $E = 200$ GPa.



SOLUTION

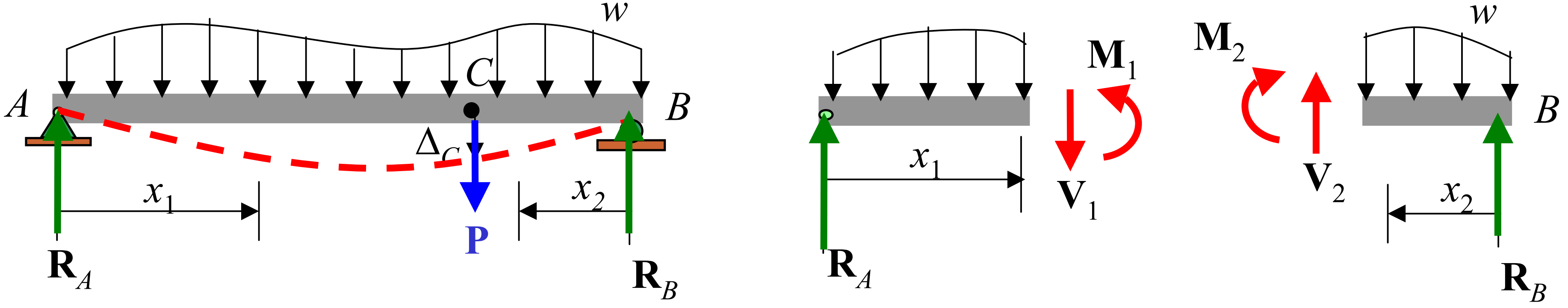


$$\Delta_{CV} = \sum N(\frac{\partial N}{\partial P})\frac{L}{AE}$$

$$\Delta_{CV} = \frac{1}{AE}(60 + 31.13 + 104.12) \;\; = \frac{195.25\; kN \bullet m}{(400\times 10^{-6} m^2)(200\times 10^6 \frac{kN}{m^2})}$$

Δ_{CV}= 2.44 mm, $\downarrow$

- ***Displacement***



$$\Delta = \int_L \left(\frac{\partial M}{\partial P}\right) \frac{M}{EI} dx$$

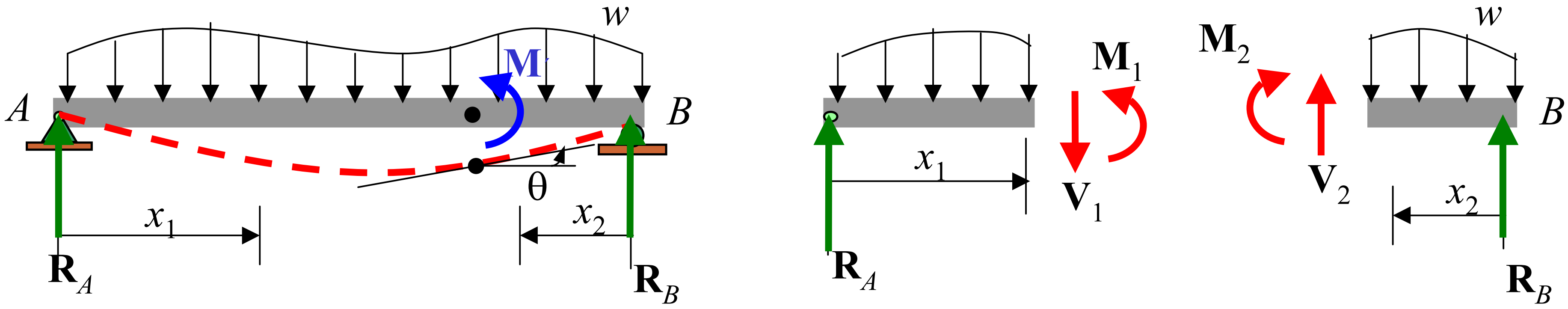
Where:

Δ = external displacement of the point caused by the real loads acting on the beam or frame

P = external force applied to the beam or frame in the direction of Δ

M = internal moment in beam or frame , expressed as a function of x and cause by *both* the force P and the loads on the beam or frame

• ***Slope***



$$\theta = \int_L \left(\frac{\partial M}{\partial M'}\right)\frac{M}{EI}dx$$

Where:

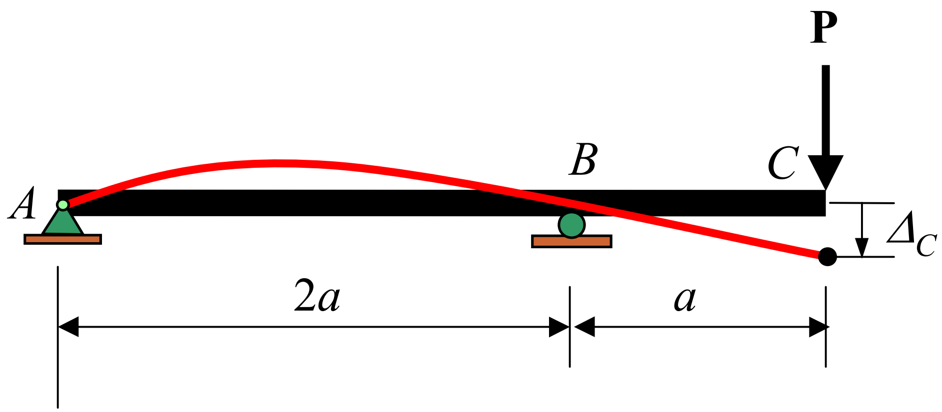
Δ = external displacement of the point caused by the real loads acting on the beam or frame

M' = external moment applied to the beam or frame in the direction of θ

M = internal moment in beam or frame , expressed as a function of x and cause by *both* the force P and the loads on the beam or frame

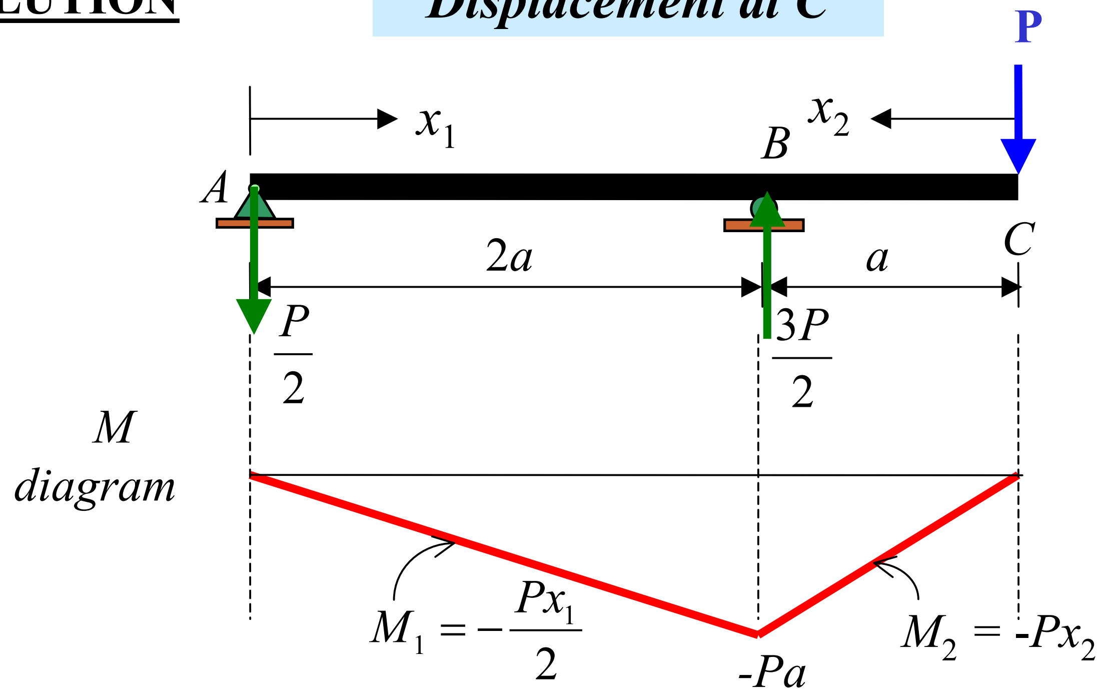
Example 8-29

The beam shown is subjected to a load **P** at its end. Determine the slope and displacement at *C*. *EI* is constant.



SOLUTION

Displacement at C

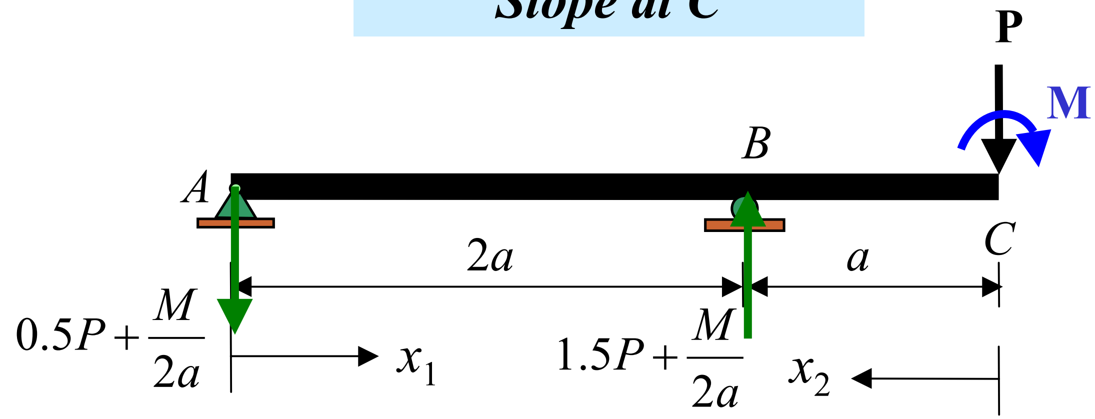


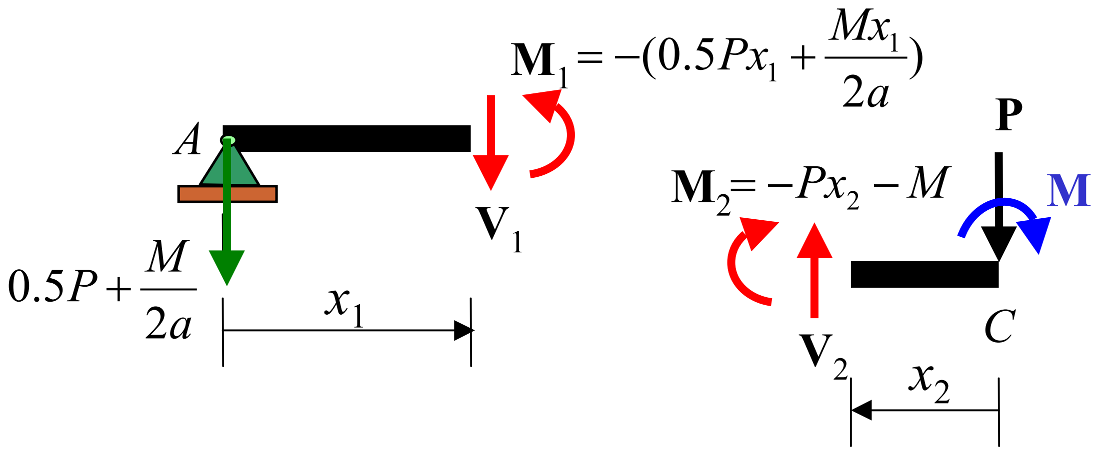
$$\Delta_C = \int_L \left(\frac{\partial M}{\partial P}\right)\frac{M}{EI}dx \qquad = \frac{1}{EI}\int_0^{2a}\left(\frac{\partial M_1}{\partial P}\right)(M_1)dx_1 + \frac{1}{EI}\int_0^{a}\left(\frac{\partial M_2}{\partial P}\right)(M_2)dx_2$$

$$= \frac{1}{EI}\int_0^{2a}\left(-\frac{x_1}{2}\right)\left(-\frac{Px_1}{2}\right)dx_1 + \frac{1}{EI}\int_0^{a}(-x_2)(-Px_2)dx_2$$

$$\Delta_C = \frac{1}{EI}\left(\frac{P}{4}\right)\left(\frac{x_1^3}{3}\right)\Bigg|_0^{2a} + \frac{1}{EI}(P)\left(\frac{x_2^3}{3}\right)\Bigg|_0^{a} = \frac{Pa^3}{EI}, \quad \downarrow$$

Slope at C





$$\theta_C=\frac{1}{EI}\int_0^{2a}(\frac{\partial M_1}{\partial M})(M_1)dx_1+\frac{1}{EI}\int_0^{a}(\frac{\partial M_2}{\partial M})(M_2)dx_2$$

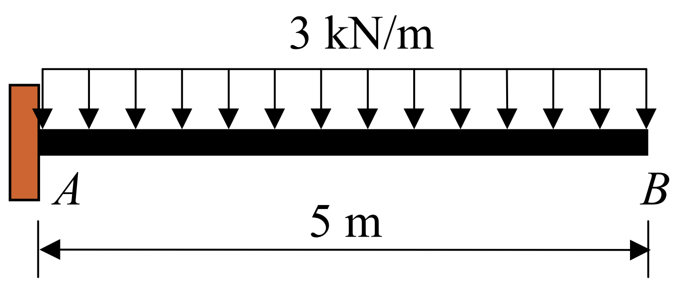
$$=\frac{1}{EI}\int_0^{2a}(-\frac{x_1}{2a})(-0.5Px_1-\frac{Mx_1}{2a})dx_1+\frac{1}{EI}\int_0^{a}(-1)(-Px_2-M)dx_2$$

(the terms $\frac{Mx_1}{2a}$ and M are struck through and marked 0)

$$\theta_C=\frac{1}{EI}(\frac{P}{4})(\frac{x_1^{\,3}}{3})\Big|_0^{2a}+\frac{1}{EI}(P)(\frac{x_2^{\,2}}{2})\Big|_0^{a}=\frac{2Pa^3}{3EI}+\frac{Pa^2}{2EI}=\frac{7Pa^3}{6EI},$$

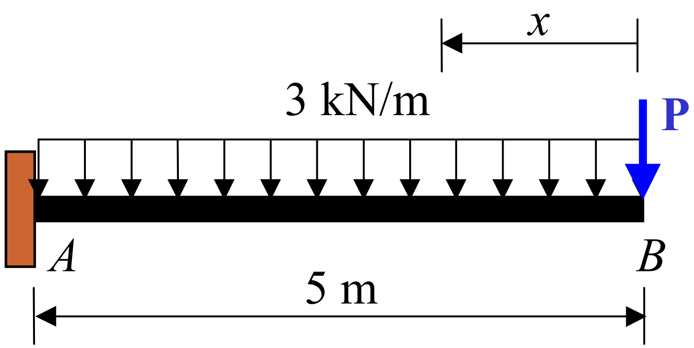
Example 8-30

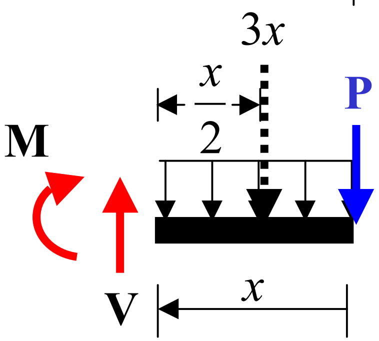
Determine the slope and displacement of point *B* of the steel beam shown in the figure below. Take $E = 200$ GPa, $I = 250(10^6)$ mm^4.



SOLUTION

Displacement at B





$$-Px - \frac{3x^2}{2} = \mathbf{M}$$

$$(\Delta_B) = \int_L \left(\frac{\partial M}{\partial P}\right)\frac{M}{EI}dx$$

$$= \frac{1}{EI}\int_0^5 (-x)\left(-Px - \frac{3x^2}{2}\right)dx \quad (P = 0)$$

$$= \frac{1}{EI}\int_0^5 \frac{3x^3}{2}$$

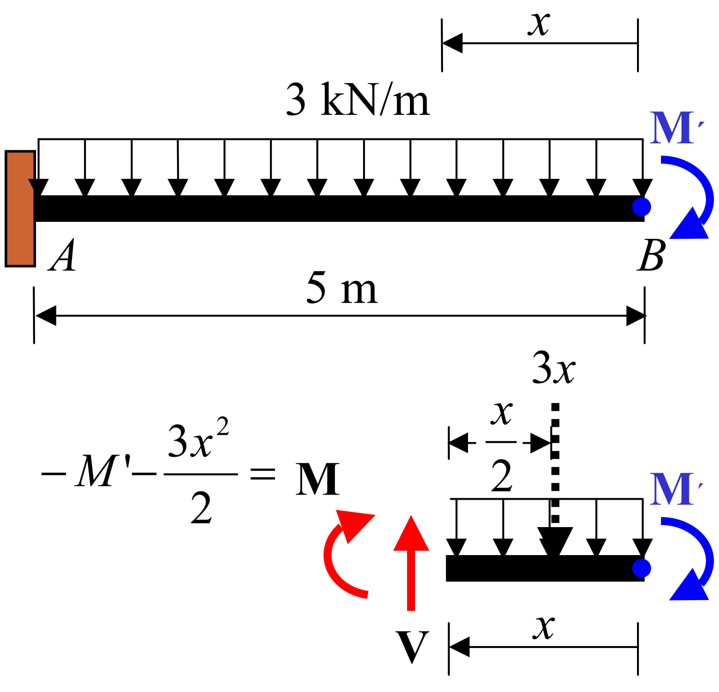
$$= \frac{1}{EI}\left(\frac{3x^4}{8}\Big|_0^5\right)$$

$$= \frac{234.375\ kN^2 \bullet m^3}{EI}$$

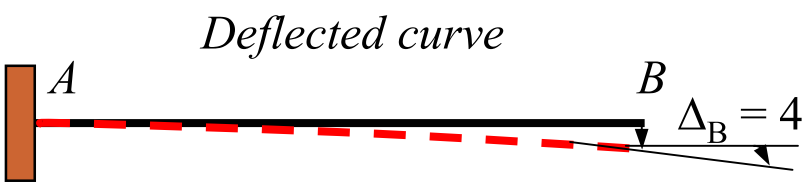
$$= \frac{234.375\ kN \bullet m^3}{(200\times10^6\ \frac{kN}{m})(250\times10^{-6}m^4)}$$

$$\Delta_B = 0.00469 \text{ m} = 4.69\text{mm}, \downarrow$$

slope at B



$$-M' - \frac{3x^2}{2} = \mathbf{M}$$



$\Delta_B = 4.69\text{mm},$

$\theta_B = 0.00125$ rad

$$\theta_B = \int_L \left(\frac{\partial M}{\partial M'}\right)\frac{M}{EI}dx$$

$$= \frac{1}{EI}\int_0^5 (-1)(-M' - \frac{3x^2}{2})dx \quad (M' = 0)$$

$$= \frac{1}{EI}\int_0^5 \frac{3x^2}{2}$$

$$= \frac{1}{EI}\left(\frac{3x^3}{6}\Big|_0^5\right)$$

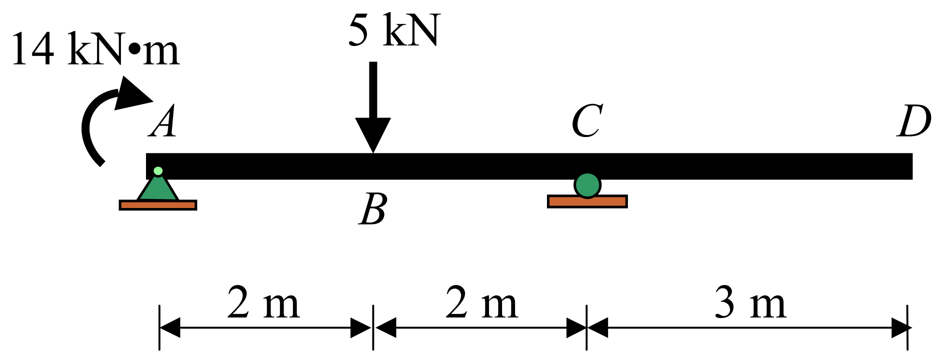
$$= \frac{62.5\ kN^2 \bullet m^3}{EI}$$

$$= \frac{62.5\ kN \bullet m^3}{(200\times10^6\ \frac{kN}{m})(250\times10^{-6}\ m^4)}$$

$\theta_B = 0.00125$ rad,

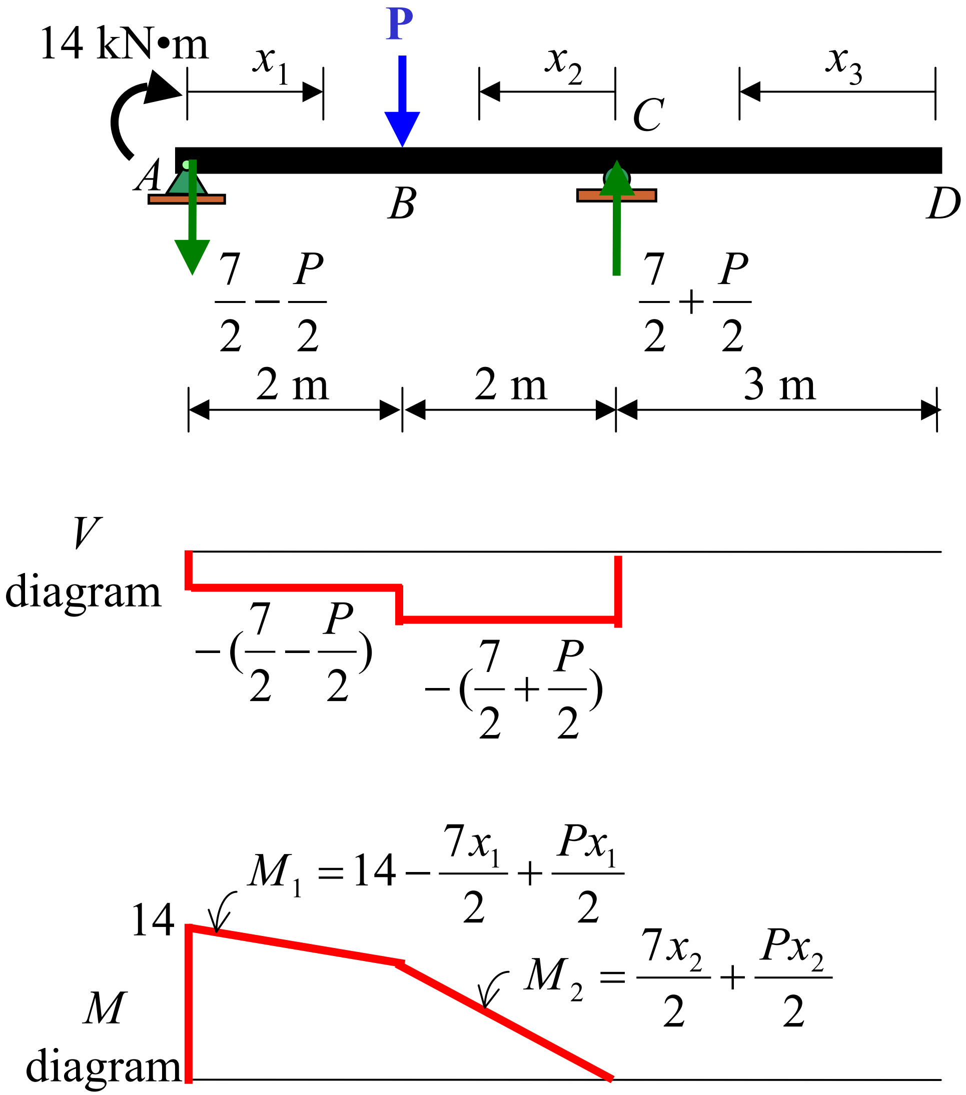
Example 8-31

Determine the slope and displacement of point B of the steel beam shown in the figure below. Take E = 200 GPa, I = 60(10^6) mm^4.



SOLUTION

Displacement at B



$$(\Delta_B) = \int_L \left(\frac{\partial M}{\partial P}\right)\frac{M}{EI}\,dx$$

$$= \frac{1}{EI}\int_0^2 \left(\frac{x_1}{2}\right)\left(14 - \frac{7x_1}{2} + \frac{x_1 \cancel{P}^{\,5}}{2}\right)dx_1$$

$$+ \frac{1}{EI}\int_0^2 \left(\frac{x_2}{2}\right)\left(\frac{7x_2}{2} + \frac{\cancel{P}^{\,5} x_2}{2}\right)dx_2$$

$$+ \int_0^3 (0)(0)\,dx_3$$

$$= \frac{1}{EI}\int_0^2 (7x_1 - 0.5x_1^{\,2})\,dx_1 + \frac{1}{EI}\int_0^2 (3x_2^{\,2})\,dx_2$$

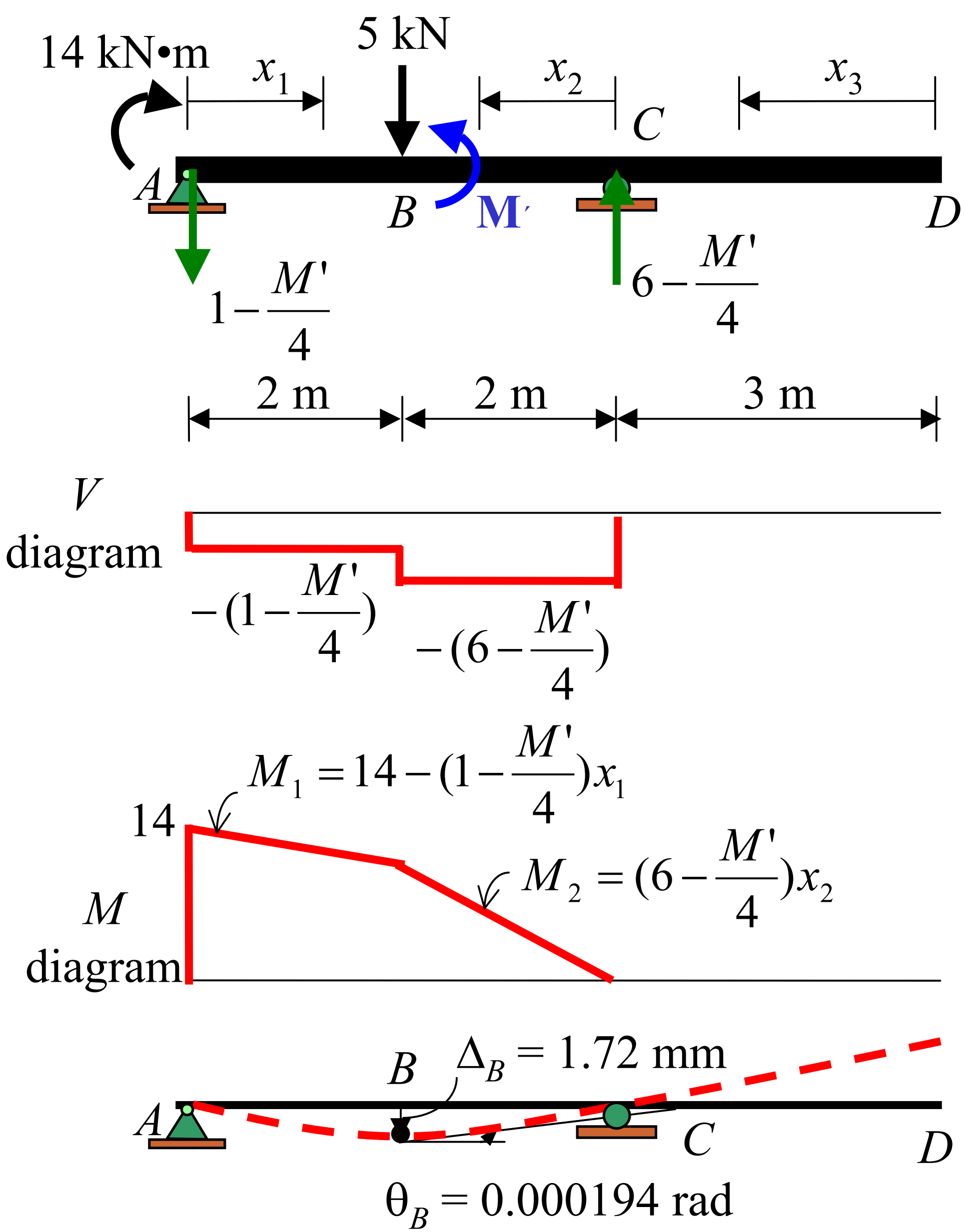
$$= \left(\frac{1}{EI}\right)\left(\frac{7x_1^{\,2}}{2} - \frac{0.5x_1^{\,3}}{3}\right)\Bigg|_0^2 + \left(\frac{1}{EI}\right)\left(\frac{3x_2^{\,3}}{3}\right)\Bigg|_0^2$$

$$= \frac{20.667}{EI} = \frac{20.667}{(200)(60)}$$

$$\Delta_B = 0.00172 \text{ m} = 1.72 \text{ mm}, \downarrow$$

SOLUTION

Slope at B



$$\theta_B = \int_0^L \left(\frac{\partial M}{\partial M'}\right)\frac{M}{EI}dx$$

$$= \frac{1}{EI}\int_0^2 \left(\frac{x_1}{4}\right)\left(14 - x_1 + \frac{M'}{4}\right)dx_1$$

$$+ \frac{1}{EI}\int_0^2 \left(-\frac{x_2}{4}\right)\left(6x_2 - \frac{M' x_2}{4}\right)dx_2$$

(with $M' = 0$)

$$+ \int_0^3 (0)(0)dx_3$$

$$= \frac{1}{EI}\int_0^2 (3.5x_1 - 0.25x_1^{\,2})dx_1$$

$$+ \frac{1}{EI}\int_0^2 (-1.5x_2^{\,2})dx_2$$

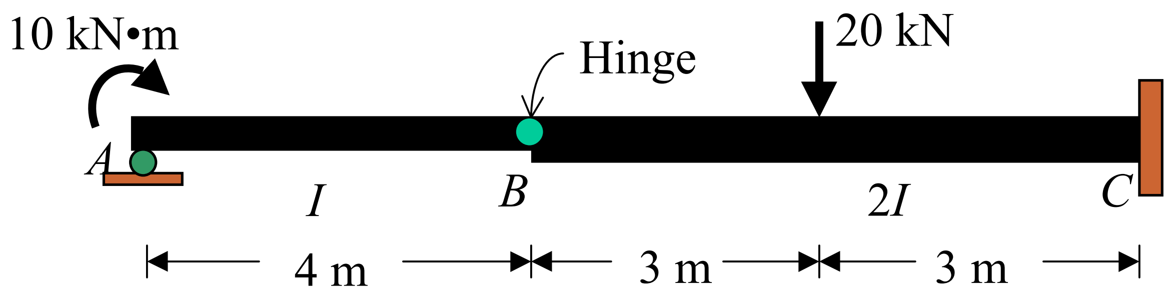
$$= \frac{1}{EI}\left(\frac{3.5x_1^{\,2}}{2} - \frac{0.25x_1^{\,3}}{3}\right)\Bigg|_0^2 + \frac{1}{EI}\left(-\frac{1.5x_2^{\,3}}{3}\right)\Bigg|_0^2$$

$$= \frac{2.333}{EI} = \frac{2.333}{(200)(60)}$$

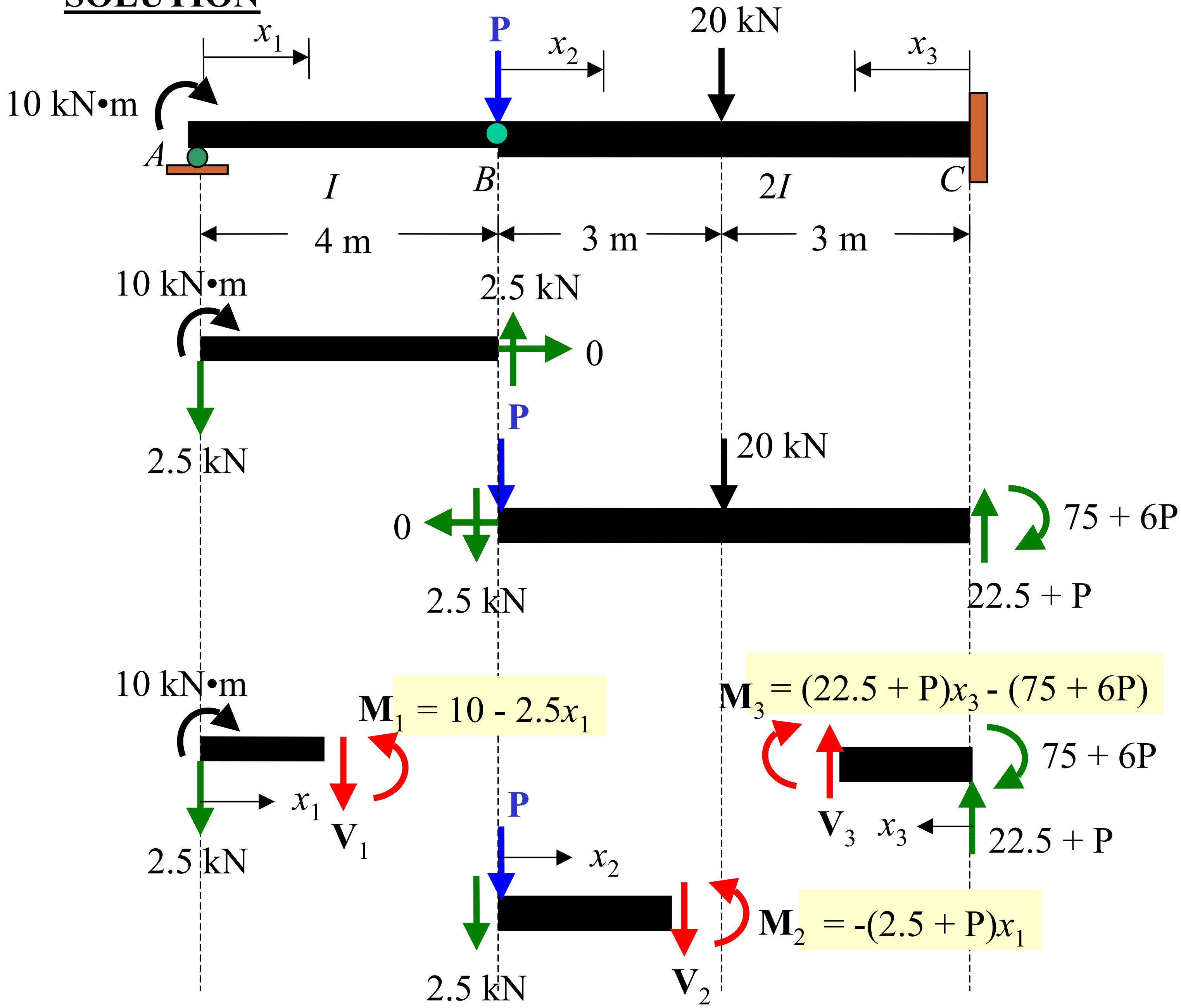
θ_B = 0.000194 rad,

Example 8-32

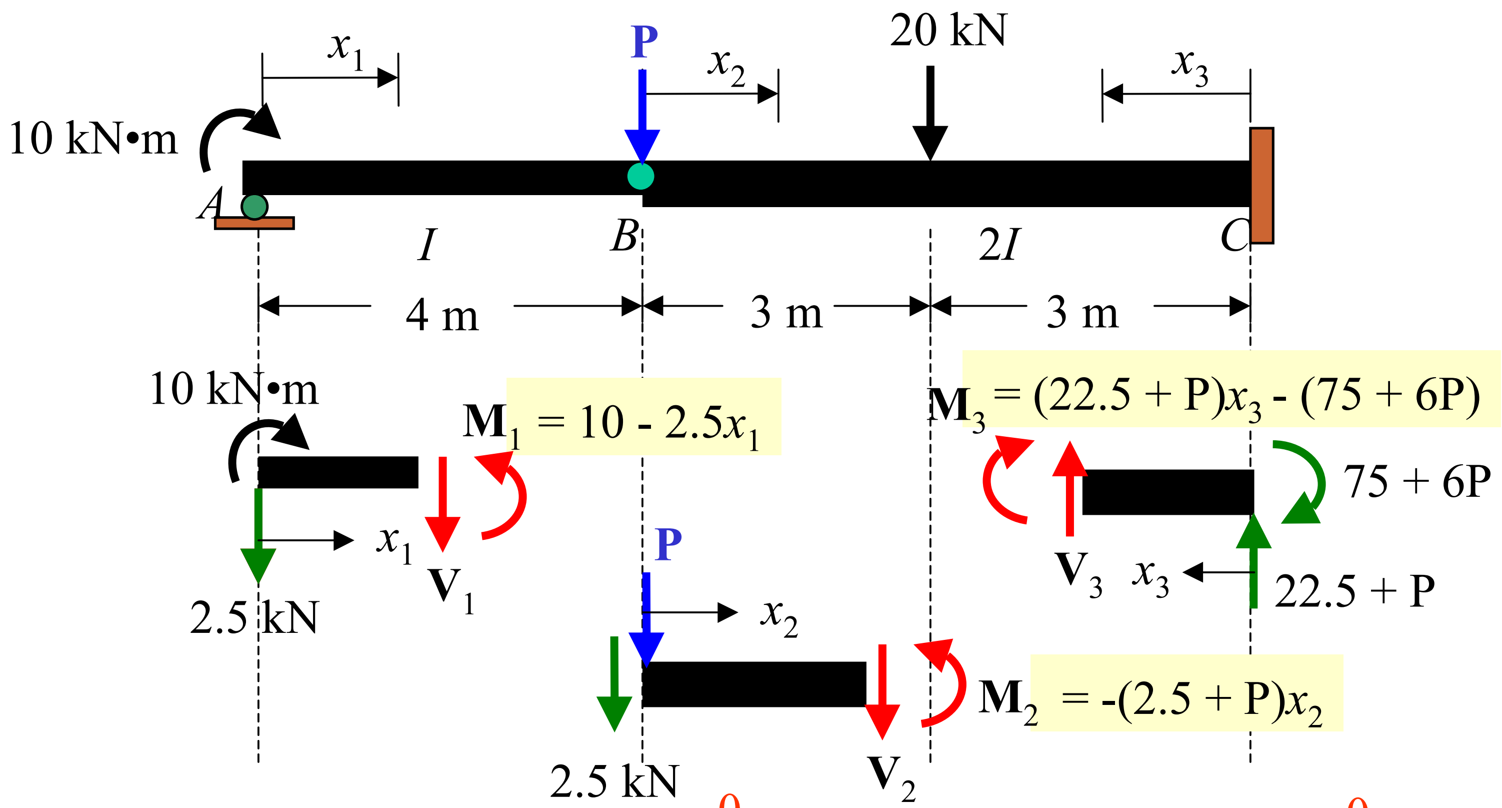
Determine the displacement of point *B* of the steel beam shown in the figure below. Take E = 200 GPa, $I = 200(10^6)$ mm^4.



SOLUTION



x1
P
x2
20 kN
x3
10 kN•m
A
B
I
2I
C
4 m
3 m
3 m
10 kN•m
2.5 kN
0
2.5 kN
P
20 kN
0
75 + 6P
2.5 kN
22.5 + P
10 kN•m
M1 = 10 - 2.5x1
M3 = (22.5 + P)x3 - (75 + 6P)
75 + 6P
x1
V1
V3
x3
22.5 + P
2.5 kN
P
x2
M2 = -(2.5 + P)x1
2.5 kN
V2



$$\Delta_B = \int_L \left(\frac{\partial M}{\partial P}\right)\frac{M}{EI}dx = \frac{1}{EI}\int_0^4 (0)(10 - 2.5x_1)dx_1 + \frac{1}{2EI}\int_0^3 (-x_2)(-2.5x_2 - x_2 P)dx_2$$

(P → 0)

$$+ \frac{1}{2EI}\int_0^3 (x_3 - 6)(22.5x_3 + x_3 P - 75 - 6P)dx_3$$

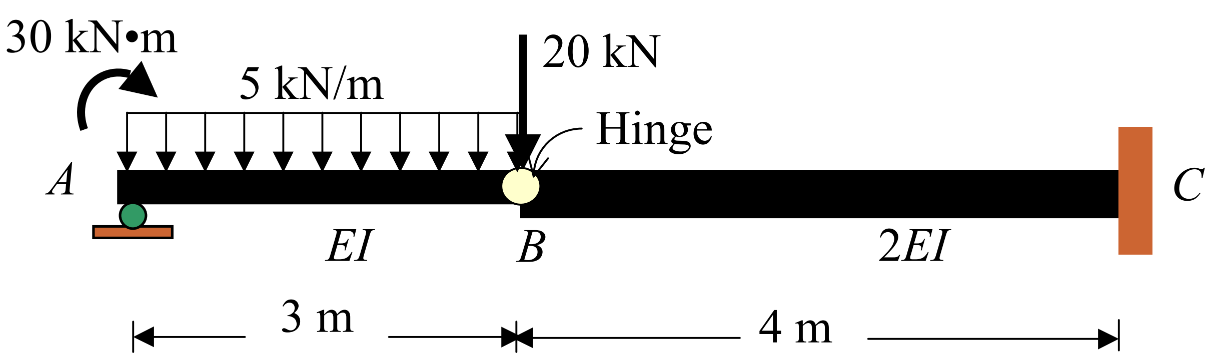
(P → 0)

$$= 0 + \frac{1}{2EI}\int_0^3 (2.5x_2{}^2)dx_2 + \frac{1}{2EI}\int_0^3 (22.5x_3{}^2 - 210x_3 + 450)dx_3$$

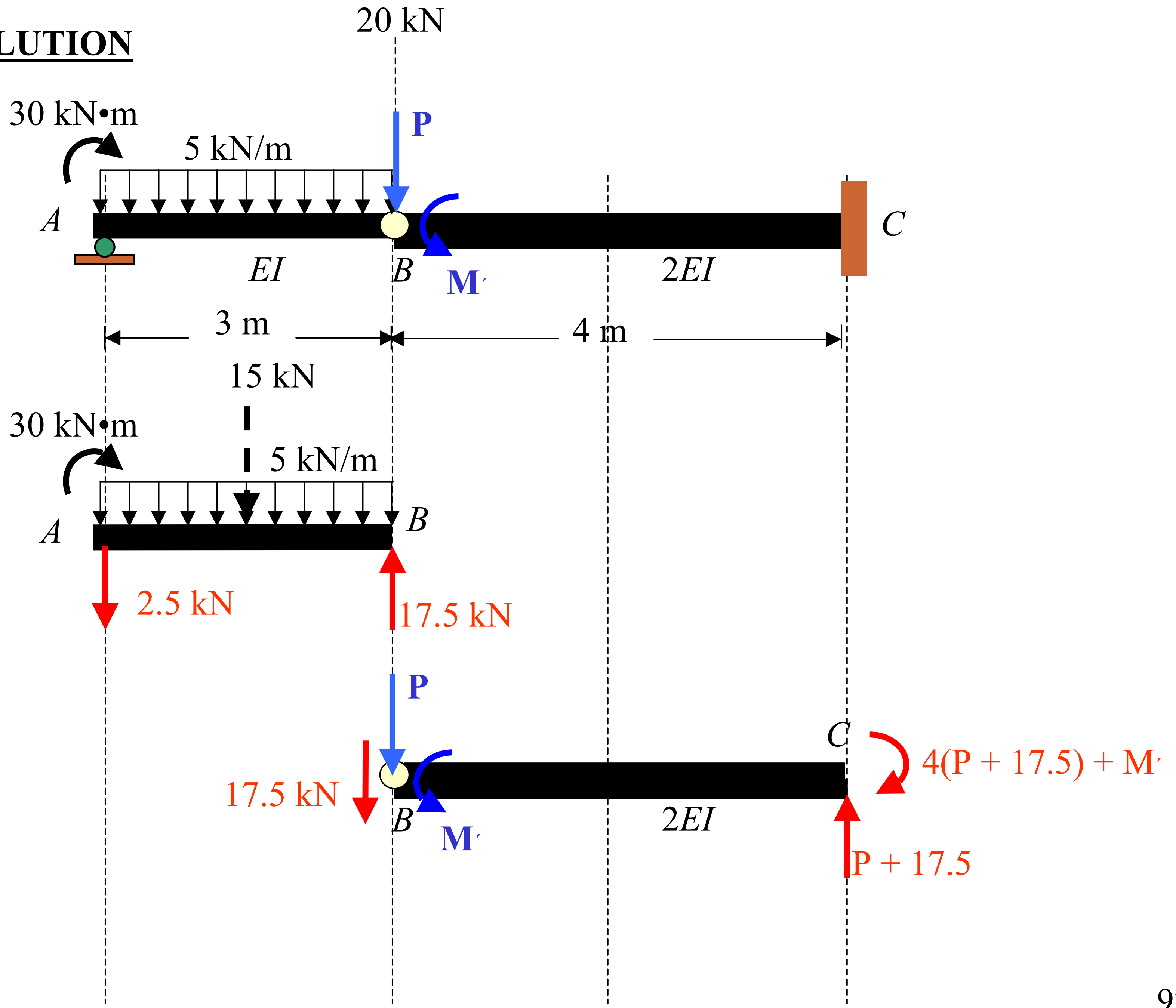
$$\Delta_B = \frac{11.25}{EI} + \frac{303.75}{EI} = \frac{315}{EI} = \frac{315}{(200)(200)} \quad = 7.875 \text{ mm}, \downarrow$$

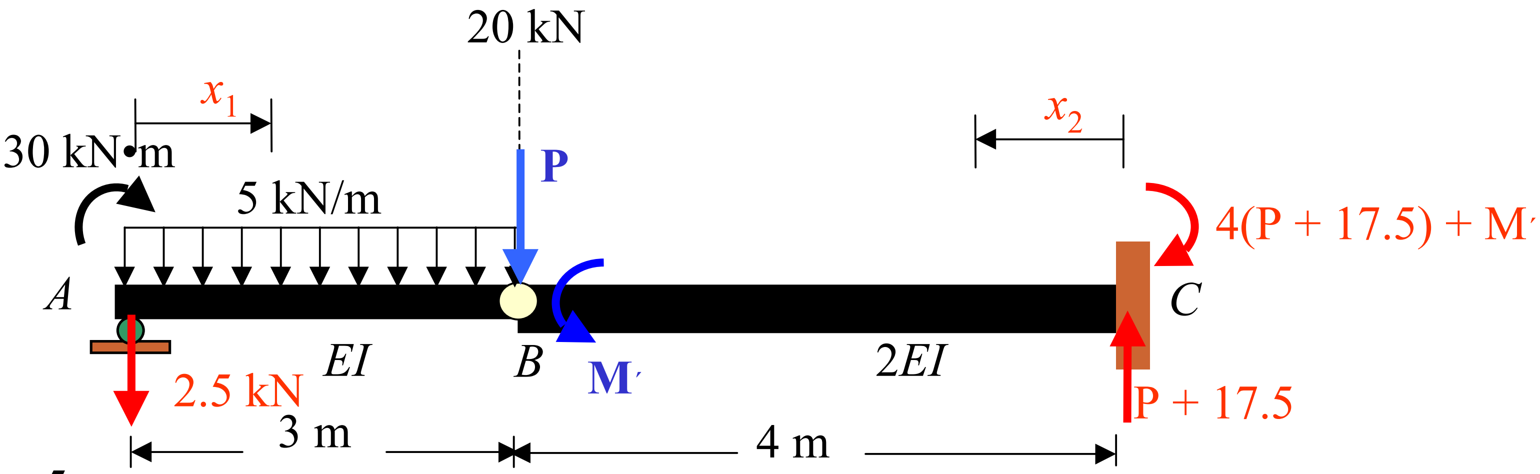
Example 8-33

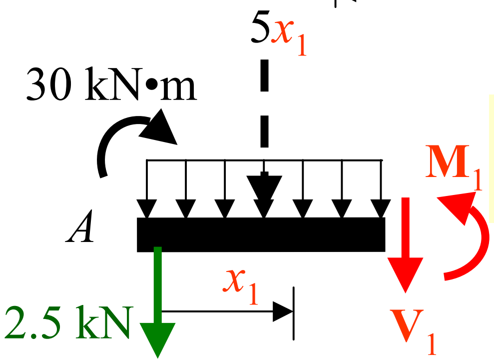
Determine the **displacement** of hinge *B* and the **slope** to the right of hinge *B* of the steel beam shown in the figure below.
Take $E = 200$ GPa, $I = 200(10^6)$ mm^4.



SOLUTION

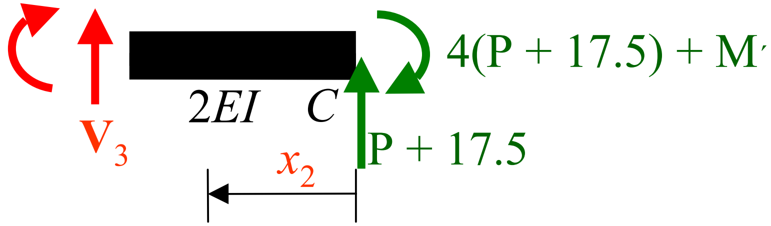

20 kN
30 kN•m
5 kN/m
P
A
C
EI
B
M´
2EI
3 m
4 m
15 kN
30 kN•m
5 kN/m
A
B
2.5 kN
17.5 kN
P
C
4(P + 17.5) + M´
17.5 kN
B
M´
2EI
P + 17.5





$$M_1 = 30 - \frac{5x_1^{\,2}}{2} - 2.5x_1$$

$$M_3 = (P + 17.5)x_2 - 4(P+17.5) - M'$$



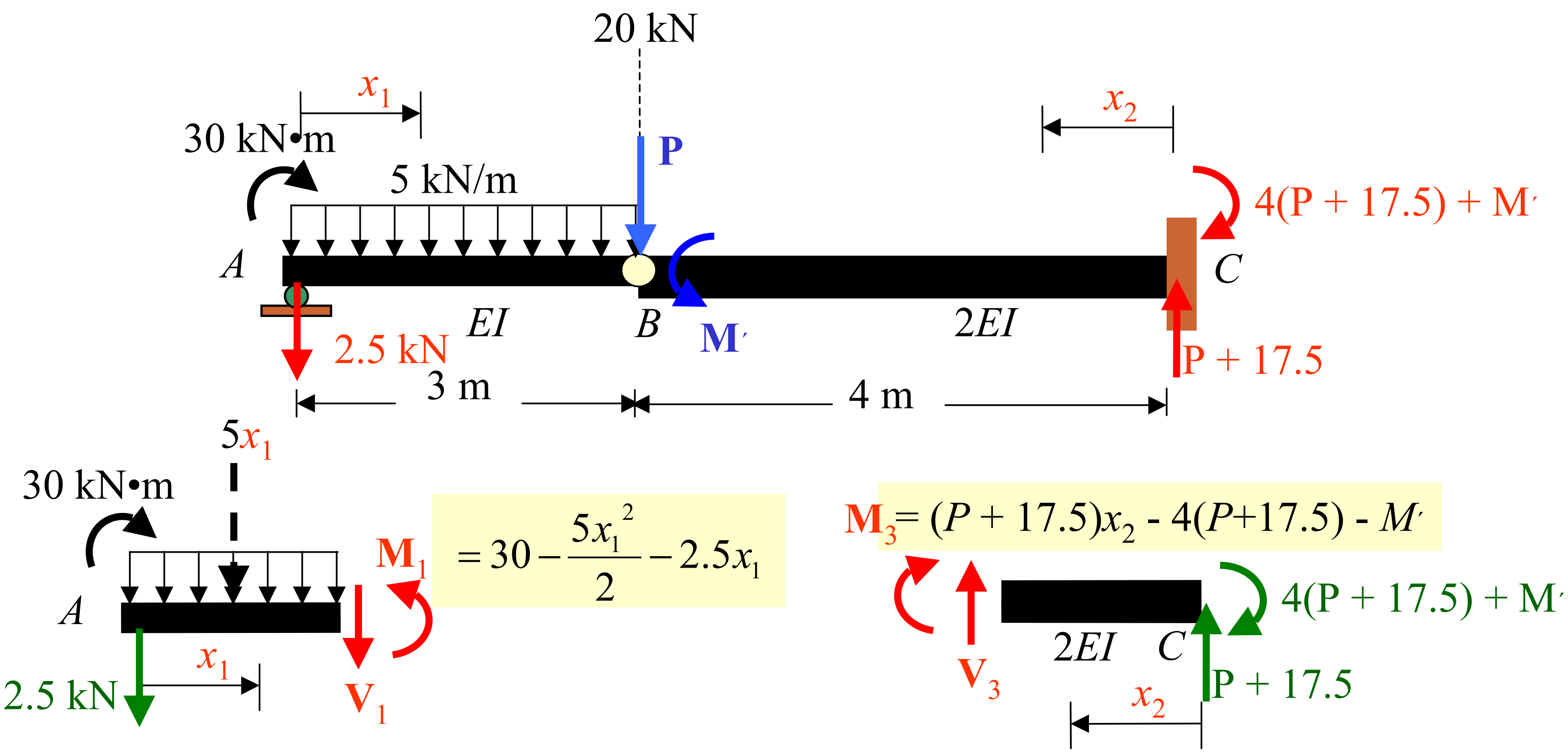
- **The displacement of hinge *B***

$$\Delta_B = \frac{1}{EI}\int_L M\left(\frac{\partial M}{\partial P}\right)dx$$

$$= 0 + \frac{1}{2EI}\int_0^4 (Px_2 + 17.5x_2 - 4P - 70 - M')(x_2 - 4)\,dx_2$$

(with $P \rightarrow 20$, $P \rightarrow 20$, $M' \rightarrow 0$)

$$= \frac{800}{2EI} = \frac{800}{2(200)(200)} = 0.01\,m = 10\,mm, \downarrow$$



- **The slope to the right of hinge *B***

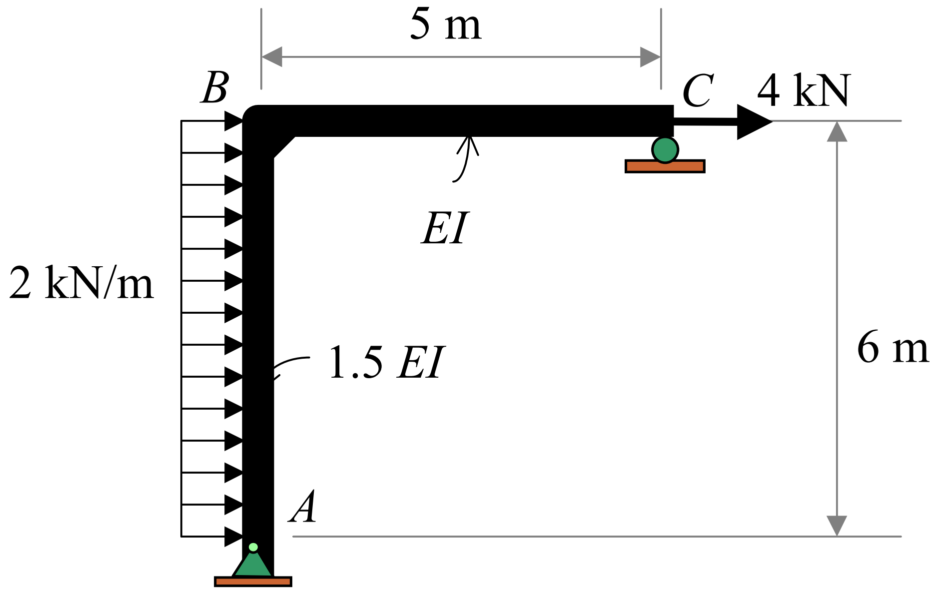
$$\Delta_B = \frac{1}{EI}\int_L M\left(\frac{\partial M}{\partial M'}\right)dx$$

$$= 0 + \frac{1}{2EI}\int_0^4 (Px_2 + 17.5x_2 - 4P - 70 - M')(-1)dx_2 \quad (P \to 20,\ P \to 20,\ M' \to 0)$$

$$= \frac{300}{2EI} = \frac{300}{2(200)(200)} = 3.75\times10^{-3}\ rad$$

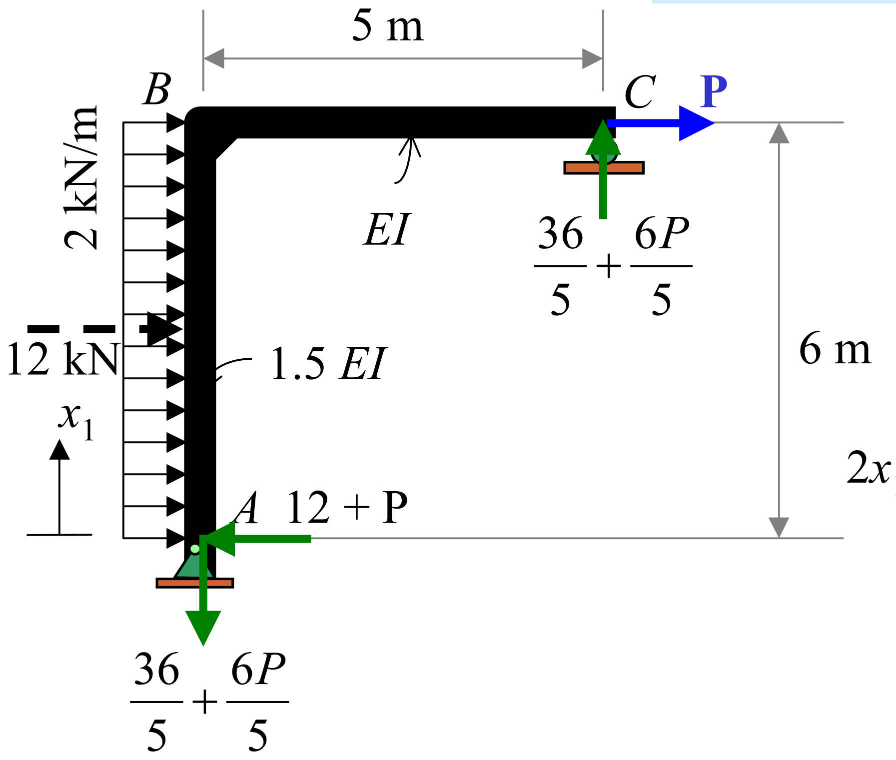
Example 8-34

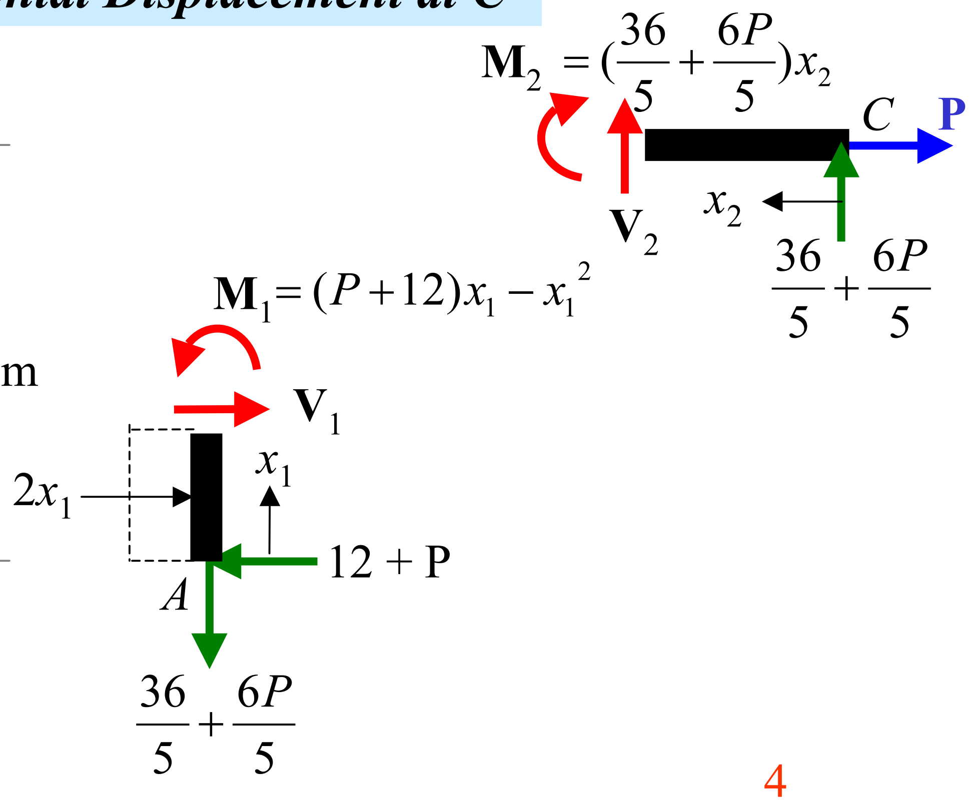
Determine the slope and the horizontal displacement of point ***C*** on the frame.
Take E = 200 GPa, I = 200(10^6) mm^4



SOLUTION

Horizontal Displacement at C



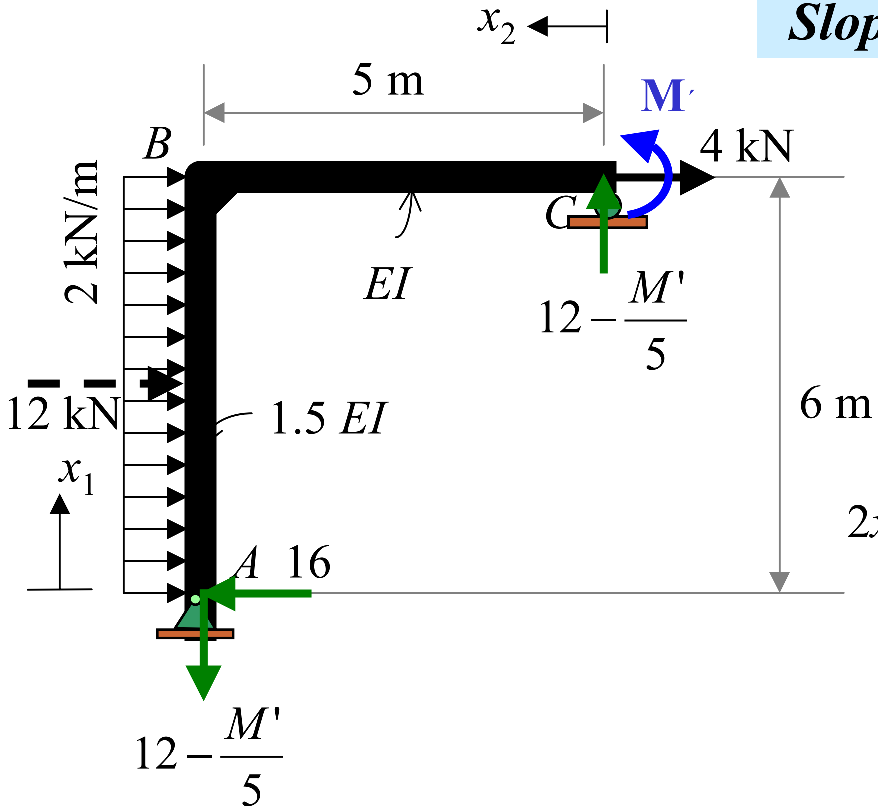


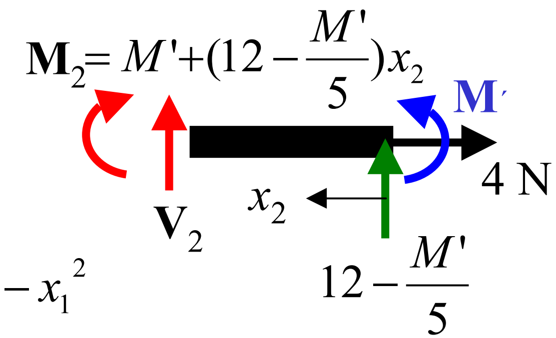
$$\Delta_{CH} = \int_L \left(\frac{\partial M_i}{\partial P}\right)\frac{M_i}{EI}dx = \frac{1}{1.5EI}\int_0^6 (x_1)(x_1 \overset{4}{\cancel{P}} + 12x_1 - x_1^2)dx_1 + \frac{1}{EI}\int_0^5 \left(\frac{6x_2}{5}\right)\left(\frac{36x_2}{5} + \frac{6\overset{4}{\cancel{P}}x_2}{5}\right)dx_2$$

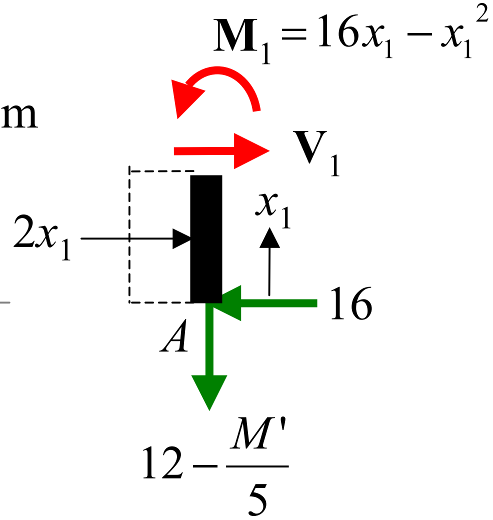
$$= \frac{1}{1.5EI}\int_0^6 (16x_1^2 - x_1^3)dx_1 + \frac{1}{EI}\int_0^5 (14.4x_2^2)dx_2$$

$$\Delta_{CH} = \frac{1}{1.5EI}\left(\frac{16x_1^3}{3} - \frac{x_1^4}{4}\right)\Bigg|_0^6 + \frac{1}{EI}\left(\frac{14.4x_2^3}{3}\right)\Bigg|_0^5 = \frac{552}{EI} + \frac{600}{EI} = \frac{1152}{(200)(200)} = +28.8 \text{ mm}, \rightarrow$$

Slope C







$$\theta_C = \int_0^L \left(\frac{\partial M_i}{\partial M'}\right)\frac{M_i}{EI}dx = \frac{1}{1.5EI}\int_0^6 (0)(16x_1 - x_1^2)dx_1 + \frac{1}{EI}\int_0^5 \left(1 - \frac{x_2}{5}\right)\left(M' + 12x_2 - \frac{M'x_2}{5}\right)dx_2$$

$$= 0 + \frac{1}{EI}\int_0^5 \left(12x_2 - \frac{12x_2^2}{5}\right)dx_2$$

$$\theta_C = \frac{1}{EI}\left(\frac{12x_2^2}{2} - \frac{12x_2^3}{5\times 3}\right)\Bigg|_0^5 = \frac{50}{EI} = \frac{50}{(200)(200)} = +0.00125 \text{ rad},$$