دانشگاه صنعتی اصفهان

دانشکده مکانیک

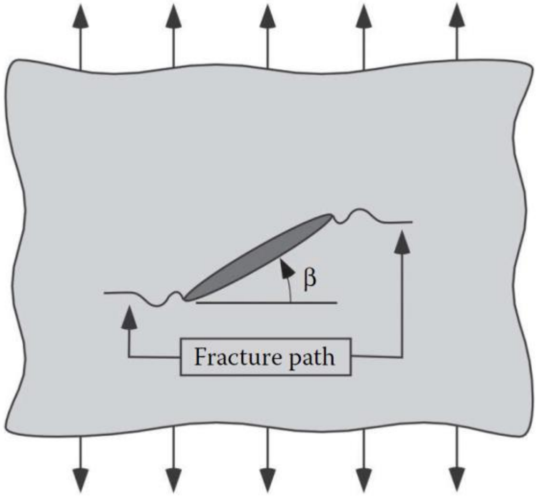
# Mixed-Mode Fracture

## Mixed-Mode Fracture

- When two or more modes of loading are present, $G = \frac{K_I^{\ 2}}{E'} + \frac{K_{II}^{\ 2}}{E'} + \frac{K_{III}^{\ 2}}{2G}$

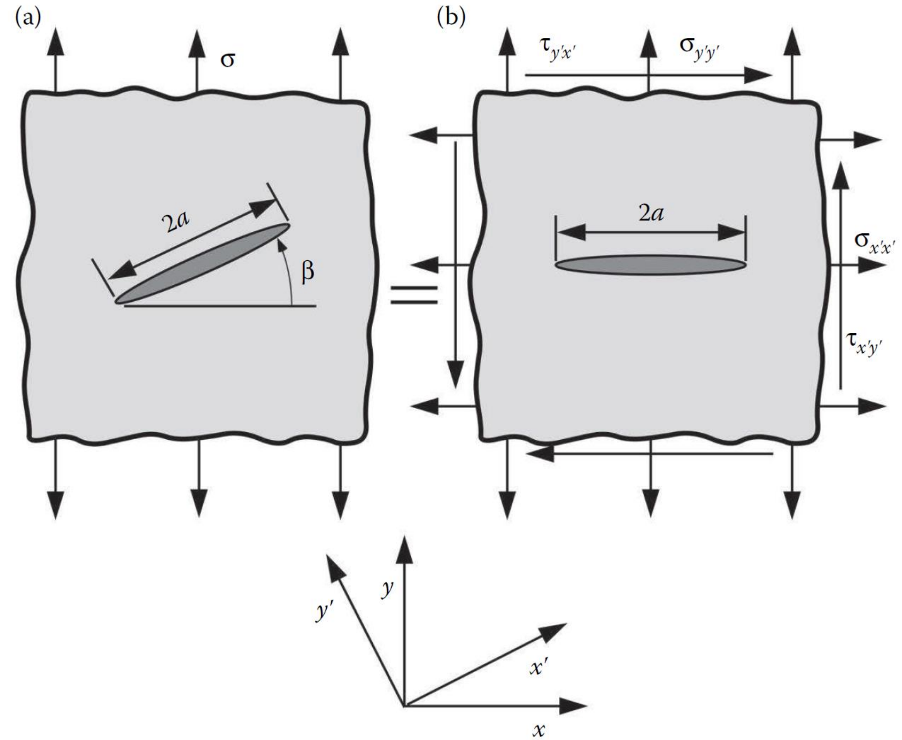
indicates that energy release rate contributions from each mode are additive. This equation, however, assumes self-similar crack growth.

- Typical propagation from an initial crack that is not orthogonal to the applied normal stress. The loading for the initial angled crack is a combination of Modes I and II, but the crack tends to propagate normal to the applied stress, resulting in pure Mode I loading.



## Mixed-Mode Fracture

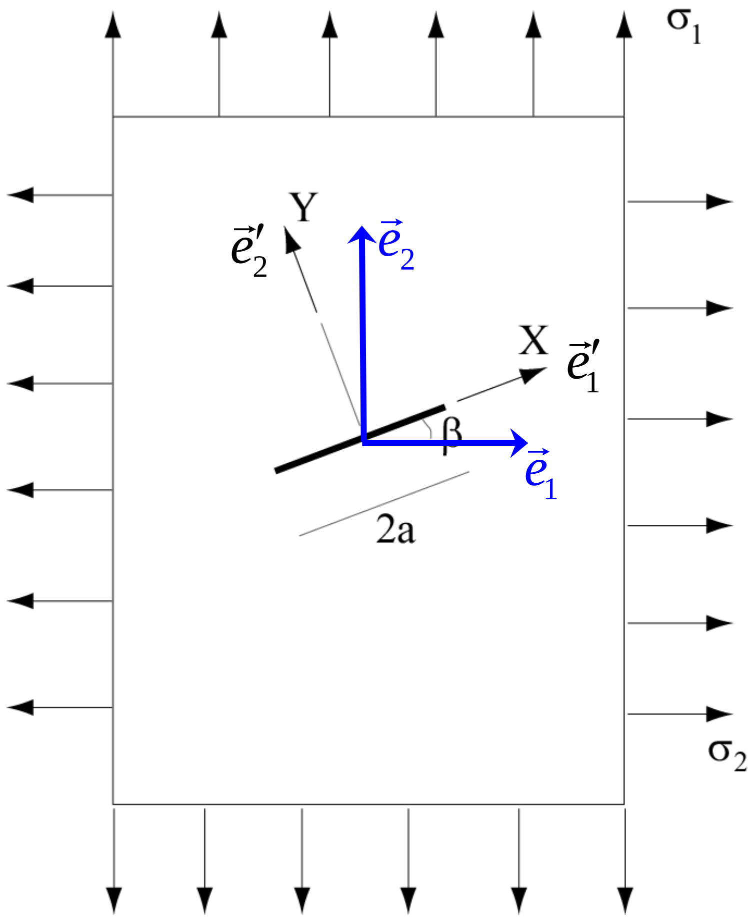
- Through crack in an infinite plate for the general case where the principal stress is not perpendicular to the crack plane.




- A propagating crack seeks the path of least resistance (or the path of maximum driving force) and need not be confined to its initial plane. If the material is isotropic and homogeneous, the crack will propagate in such a way as to maximize the energy release rate.

Mixed-Mode Fracture

biaxial loading




$$K_{I(0)} = \sigma_1\sqrt{\pi a}$$

$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$ in global frame ( $\vec{e}_1 \ \ \vec{e}_2$ )

→ expressed in local frame ( $\vec{e}'_1 \ \ \vec{e}'_2$ )

$$\vec{e}'_1 = \cos\beta\,\vec{e}_1 + \sin\beta\,\vec{e}_2$$

$$\vec{e}'_2 = -\sin\beta\,\vec{e}_1 + \cos\beta\,\vec{e}_2$$

$$(\mathbf{Q}) = \begin{pmatrix} \vec{e}_1\cdot\vec{e}'_1 & \vec{e}_1\cdot\vec{e}'_2 \\ \vec{e}_2\cdot\vec{e}'_1 & \vec{e}_2\cdot\vec{e}'_2 \end{pmatrix}$$

**Q** = Rotation tensor

Thus, $(\mathbf{Q}) = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$

Stress tensor components :

$$\begin{pmatrix} \sigma'_{11} & \sigma'_{12} \\ \sigma'_{21} & \sigma'_{22} \end{pmatrix} = (Q)^T \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} (Q)$$

Mixed-Mode Fracture

$$\begin{pmatrix} \sigma'_{11} & \sigma'_{12} \\ \sigma'_{21} & \sigma'_{22} \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$= \begin{bmatrix} c^2\sigma_{11} + 2cs\sigma_{12} + s^2\sigma_{22} & -cs\sigma_{11} + (c^2 - s^2)\sigma_{12} + cs\sigma_{22} \\ -cs\sigma_{11} + (c^2 - s^2)\sigma_{12} + cs\sigma_{22} & s^2\sigma_{11} - 2cs\sigma_{12} + c^2\sigma_{22} \end{bmatrix}$$

crack tip stresses

$$\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} f_{Iij}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} f_{IIij}(\theta)$$

with:

$$K_I = \varphi\sigma'_{22}\sqrt{\pi a}; \qquad K_{II} = \gamma\sigma'_{12}\sqrt{\pi a}$$

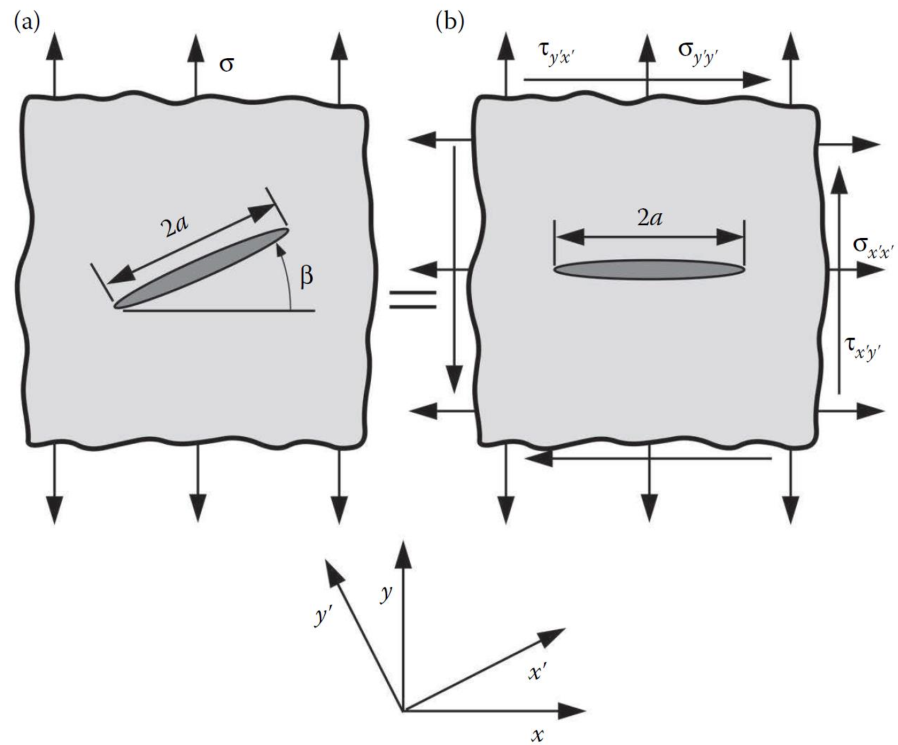
$\sigma'_{11}$ “does not do anything”

## Mixed-Mode Fracture

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_{22} \end{bmatrix} \Longrightarrow \boldsymbol{\sigma}' = \begin{bmatrix} s^2\sigma & cs\sigma \\ cs\sigma & c^2\sigma \end{bmatrix}$$

$$K_I = K_{I(0)} \cos^2 \beta$$

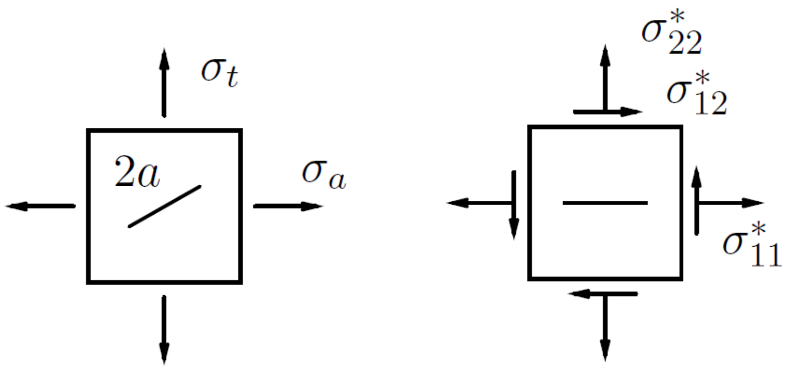
$$K_{II} = K_{I(0)} \cos \beta \sin \beta$$

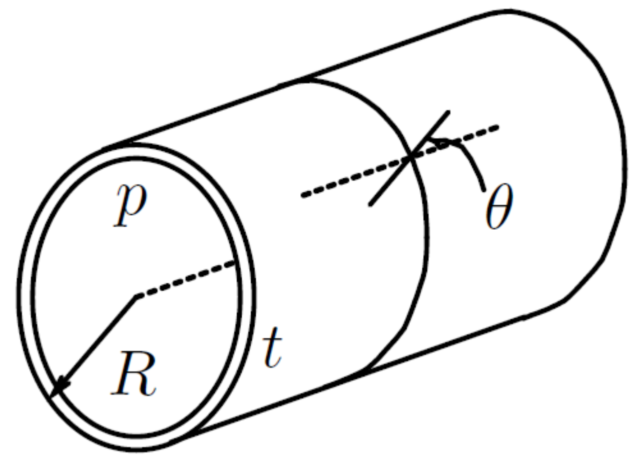



$K_{I(0)}$ is the Mode I stress intensity when $\beta = 0$. $K_{I(0)} = \sigma\sqrt{\pi a}$

## Example multi-mode load

Crack in tube wall at an angle $\theta$ with the tube axis.




$$\sigma_t = \frac{pR}{t} = \sigma \quad ; \quad \sigma_a = \frac{pR}{2t} = \tfrac{1}{2}\sigma$$

$$\sigma_{22}^* = s^2\,\tfrac{1}{2}\,\sigma + c^2\sigma \quad ; \quad \sigma_{12}^* = cs(1-\tfrac{1}{2})\sigma = \tfrac{1}{2}\,cs\,\sigma$$

$$K_I = \sigma_{22}^*\sqrt{\pi a} = (\tfrac{1}{2}s^2 + c^2)\sigma\sqrt{\pi a} = (\tfrac{1}{2}s^2 + c^2)\frac{pR}{t}\sqrt{\pi a}$$

$$K_{II} = \sigma_{12}^*\sqrt{\pi a} = \tfrac{1}{2}\,cs\,\sigma = \tfrac{1}{2}\,cs\,\frac{pR}{t}\sqrt{\pi a}$$

## Crack Growth Direction

- criteria for crack growth direction :
  - maximum tangential stress (MTS) criterion
  - strain energy density (SED) criterion

requirement : crack tip stresses in cylindrical coordinates

## Mixed-Mode Fracture

The crack tip stress fields (in polar coordinates) for the Mode I portion of the loading:

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{5}{4}\cos\left(\frac{\theta}{2}\right) - \frac{1}{4}\cos\left(\frac{3\theta}{2}\right)\right]$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{3}{4}\cos\left(\frac{\theta}{2}\right) + \frac{1}{4}\cos\left(\frac{3\theta}{2}\right)\right]$$

$$\tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{1}{4}\sin\left(\frac{\theta}{2}\right) + \frac{1}{4}\sin\left(\frac{3\theta}{2}\right)\right]$$

The singular stress fields for Mode II:

$$\sigma_{rr} = \frac{K_{II}}{\sqrt{2\pi r}}\left[-\frac{5}{4}\sin\left(\frac{\theta}{2}\right) + \frac{3}{4}\sin\left(\frac{3\theta}{2}\right)\right]$$

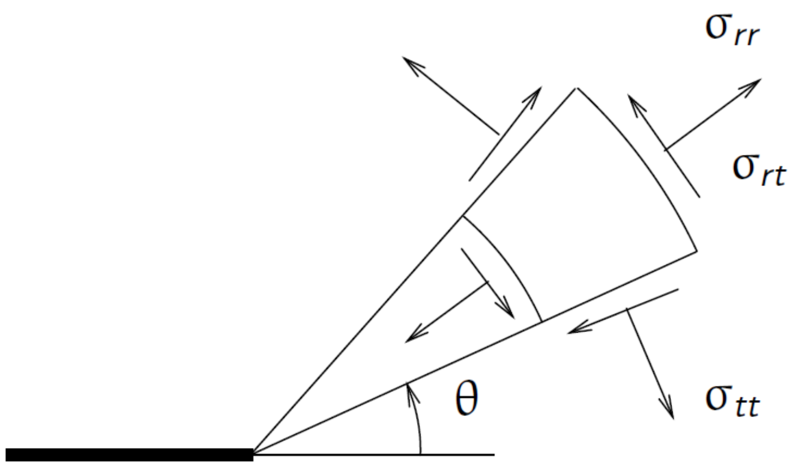
$$\sigma_{\theta\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left[-\frac{3}{4}\sin\left(\frac{\theta}{2}\right) - \frac{3}{4}\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\tau_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{1}{4}\cos\left(\frac{\theta}{2}\right) + \frac{3}{4}\cos\left(\frac{3\theta}{2}\right)\right]$$

## Crack Growth Direction

- Maximum tangential stress criterion

Erdogan & Sih (1963):




Hypothesis: *crack growth towards local maximum of* $\sigma_{tt}$

$$\frac{\partial \sigma_{tt}}{\partial \theta} = 0 \qquad and \qquad \frac{\partial^2 \sigma_{tt}}{\partial \theta^2} < 0 \qquad \rightarrow \qquad \theta_c$$

$$\sigma_{tt}(\theta = \theta_c) = \sigma_{tt}(\theta = 0) = \frac{K_{Ic}}{\sqrt{2\pi r}} \qquad \rightarrow \qquad crack \;\; growth$$

## معیار بیشینه تنش جهت گسترش ترک در مود ترکیبی

- Maximum tangential stress criterion

$$\frac{\partial \sigma_{tt}}{\partial \theta}=0 \quad \rightarrow$$

$$\frac{3}{2}\frac{K_I}{\sqrt{2\pi r}}\left[-\frac{1}{4}\sin(\frac{\theta}{2})-\frac{1}{4}\sin(\frac{3\theta}{2})\right]+\frac{3}{2}\frac{K_{II}}{\sqrt{2\pi r}}\left[-\frac{1}{4}\cos(\frac{\theta}{2})-\frac{3}{4}\cos(\frac{3\theta}{2})\right]=0$$

$$K_I \sin\theta + K_{II}[3\cos\theta - 1]=0$$

$$\frac{\partial^2 \sigma_{tt}}{\partial \theta^2}<0 \quad \rightarrow$$

$$\frac{3}{4}\frac{K_I}{\sqrt{2\pi r}}\left[-\frac{1}{4}\cos(\frac{\theta}{2})-\frac{3}{4}\cos(\frac{3\theta}{2})\right]+\frac{3}{4}\frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{1}{4}\sin(\frac{\theta}{2})+\frac{9}{4}\sin(\frac{3\theta}{2})\right]<0$$

$$\sigma_{tt}(\theta=\theta_c)=\frac{K_{Ic}}{\sqrt{2\pi r}} \quad \rightarrow$$

$$\frac{1}{4}\frac{K_I}{K_{Ic}}\left[3\cos(\frac{\theta_c}{2})+\cos(\frac{3\theta_c}{2})\right]+\frac{1}{4}\frac{K_{II}}{K_{Ic}}\left[-3\sin(\frac{\theta_c}{2})-3\sin(\frac{3\theta_c}{2})\right]=1$$

- Maximum tangential stress criterion

- Mode $I$: $K_{II} = 0$

$$\frac{\partial \sigma_{tt}}{\partial \theta} = K_I \sin(\theta) = 0 \quad \rightarrow \quad \theta_c = 0$$

$$\left.\frac{\partial^2 \sigma_{tt}}{\partial \theta^2}\right|_{\theta_c} < 0$$

$$\sigma_{tt}(\theta_c) = \frac{K_{Ic}}{\sqrt{2\pi r}} \quad \rightarrow \quad K_I = K_{Ic}$$

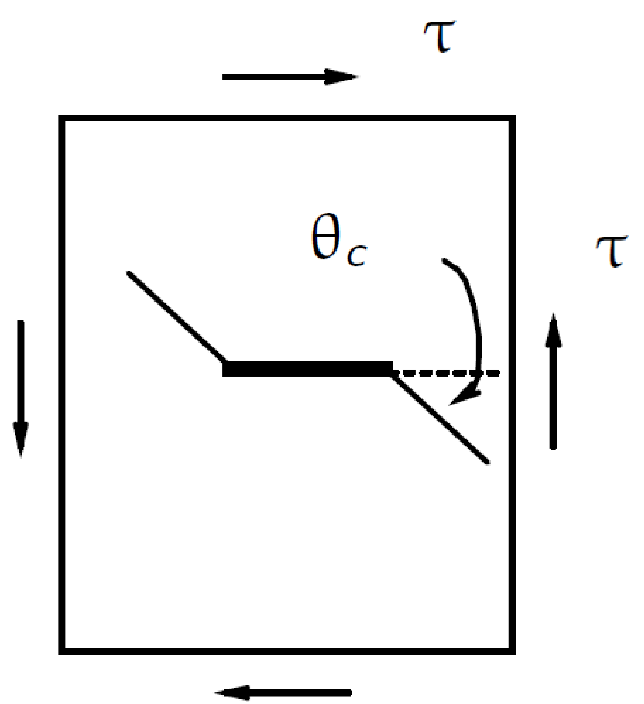
❖ Maximum tangential stress criterion

➢ Mode $II$: $K_I = 0$

$$\frac{\partial \sigma_{tt}}{\partial \theta} = K_{II}[3\cos\theta_c - 1] = 0 \quad \rightarrow \quad \theta_c = \pm \arccos(\frac{1}{3}) = \pm 70.6^{\text{O}}$$

$$\left. \frac{\partial^2 \sigma_{tt}}{\partial \theta^2} \right|_{\theta_c = -70.6} < 0$$

$$\sigma_{tt}(\theta_c) = \frac{K_{Ic}}{\sqrt{2\pi r}} \quad \rightarrow \quad K_{IIc} = \sqrt{\frac{3}{4}} K_{Ic}$$



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- Maximum tangential stress criterion

Multi-mode load

$$K_I\left[-\sin(\frac{\theta}{2})-\sin(\frac{3\theta}{2})\right]+K_{II}\left[-\cos(\frac{\theta}{2})-3\cos(\frac{3\theta}{2})\right]=0$$

$$K_I\left[-\cos(\frac{\theta}{2})-3\cos(\frac{3\theta}{2})\right]+K_{II}\left[\sin(\frac{\theta}{2})+9\sin(\frac{3\theta}{2})\right]<0$$

$$K_I\left[3\cos(\frac{\theta}{2})+\cos(\frac{3\theta}{2})\right]+K_{II}\left[-3\sin(\frac{\theta}{2})-3\sin(\frac{3\theta}{2})\right]=4K_{Ic}$$

$$\begin{cases}-K_I f_1-K_{II} f_2=0\\ -K_I f_2+K_{II} f_3<0\\ K_I f_4-3K_{II} f_1=4K_{Ic}\end{cases}\Rightarrow\begin{cases}-\left(\frac{K_I}{K_{Ic}}\right)f_1-\left(\frac{K_{II}}{K_{Ic}}\right)f_2=0\\ -\left(\frac{K_I}{K_{Ic}}\right)f_2+\left(\frac{K_{II}}{K_{Ic}}\right)f_3<0\\ \left(\frac{K_I}{K_{Ic}}\right)f_4-3\left(\frac{K_{II}}{K_{Ic}}\right)f_1=4\end{cases}$$

# معیار بیشینه تنش جهت گسترش ترک در مود ترکیبی

- Maximum tangential stress criterion

$$\frac{\partial \sigma_{tt}}{\partial \theta} = 0 \quad \rightarrow \quad K_I \sin\theta + K_{II}[3\cos\theta - 1] = 0$$

This Eq. can be solved by writing:

$$2K_I \sin\frac{\theta}{2}\cos\frac{\theta}{2} + 3K_{II}[\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}] - K_{II}[\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}] = 0$$

which yields

$$2K_{II}\tan^2\frac{\theta}{2} - K_I \tan\frac{\theta}{2} - K_{II} = 0$$

$$\left(\tan\frac{\theta}{2}\right)_{1,2} = \frac{1}{4}\frac{K_I}{K_{II}} \pm \sqrt{\frac{1}{4}\left(\frac{K_I}{K_{II}}\right)^2 + 8} \quad \rightarrow \quad \theta_c$$

$$\sigma_{tt}(\theta = \theta_c) = \frac{1}{\sqrt{2\pi r}}\cos^2\frac{\theta_c}{2}\left(K_I \cos\frac{\theta_c}{2} - 3K_{II}\sin\frac{\theta_c}{2}\right)$$

# معیار بیشینه تنش جهت گسترش ترک در مود ترکیبی

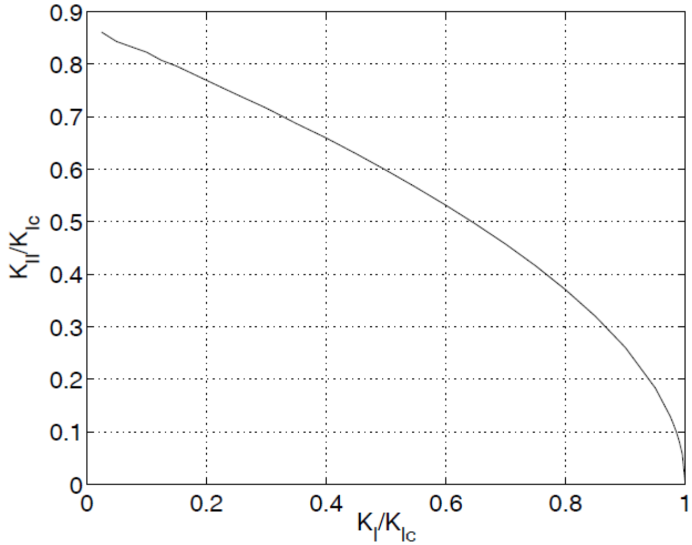
- Maximum tangential stress criterion

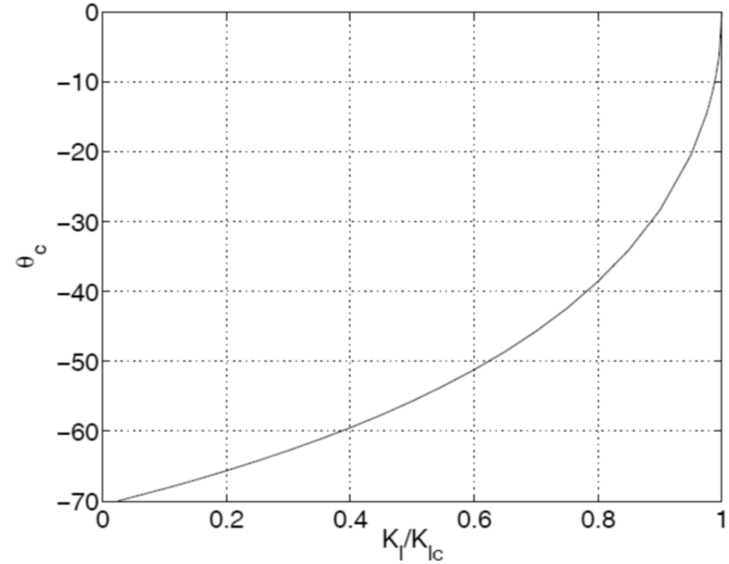
$$\sigma_{tt}(\theta_c) = \frac{K_{Ic}}{\sqrt{2\pi r}} \quad (\theta = 0) \quad \rightarrow \qquad K_{Ic} = K_I \cos^3\frac{\theta_c}{2} - 3K_{II}\cos^2\frac{\theta_c}{2}\sin\frac{\theta_c}{2}$$

- Maximum tangential stress criterion

Multi-mode load

Determining the crack growth direction for multi-mode loading is only possible using numerical calculations. The relations which have to be satisfied for crack growth in a certain direction are written is terms of the parameter $K_I/K_{Ic}$.

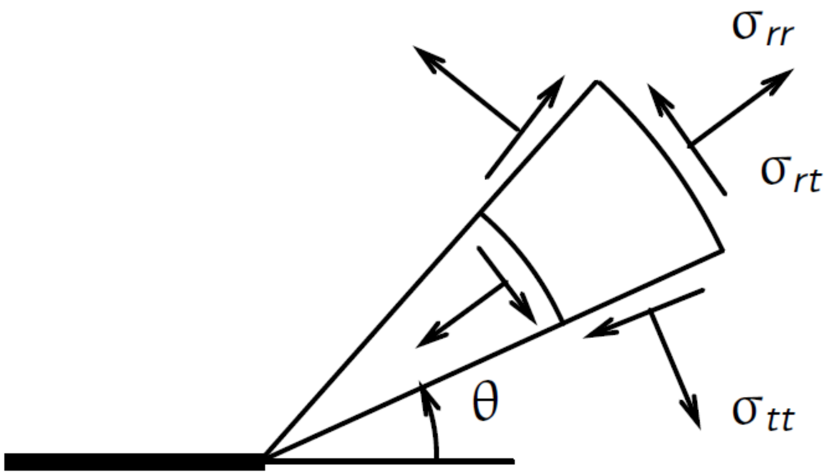



In the first one $K_{II}/K_{Ic}$ is plotted as a function of $K_I/K_{Ic}$ and it can be used to determine which combination leads to crack growth. From the second plot the crack growth angle $\theta_c$ can then be determined.

- Strain energy density (SED) criterion

Sih (1973)




$U_i$ = Strain Energy Density (Function)

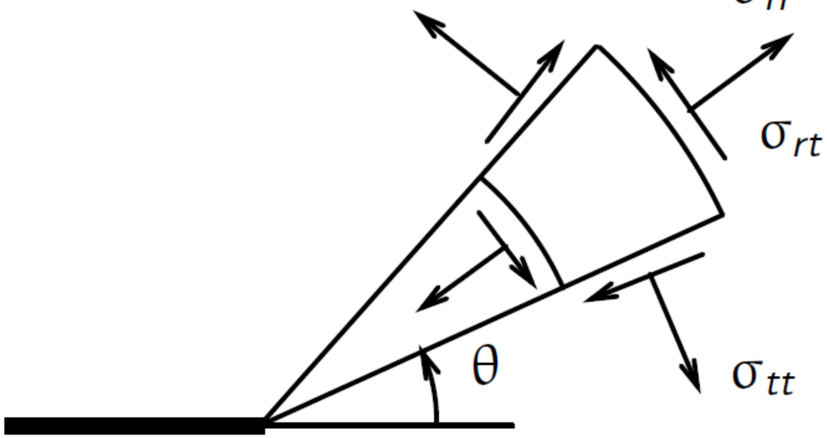
$S$ = Strain Energy Density Factor= $rU_i$ =S($K_I$, $K_{II}$, $\theta$ )

The strain energy density $U_i$ is the stored elastic energy per unit of volume. For linear elastic material behavior this specific energy is easily calculated in the crack tip region and appears to be inverse proportional to the distance to the crack tip. The strain energy density factor S is now defined as the product $rU_i$, as such being independent of $r$.

- Strain energy density (SED) criterion

Sih (1973)

$$S = rU_i = r\int_0^{\varepsilon_{ij}} \sigma d\,\varepsilon_{ij}$$




Hypothesis: *crack growth towards local minimum of SED*

$$\frac{\partial S}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 S}{\partial \theta^2} > 0 \quad \rightarrow \quad \theta_c$$

$$S(\theta = \theta_c) = S(\theta = 0,\ \text{pl.strain}) = S_c \quad \rightarrow \quad \textit{crack growth}$$

# معیار کمینه ضریب چگالی انرژی کرنشی جهت گسترش ترک در مود ترکیبی

❖ Strain energy density (SED) criterion

$$U_i = \frac{1}{2E}(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2) - \frac{\nu}{E}(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}) + \frac{1}{2G}(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2)$$

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}}\cos(\frac{\theta}{2})\left[1 - \sin(\frac{\theta}{2})\sin(\frac{3\theta}{2})\right] - \frac{K_{II}}{\sqrt{2\pi r}}\sin(\frac{\theta}{2})\left[2 + \cos(\frac{\theta}{2})\cos(\frac{3\theta}{2})\right]$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}\cos(\frac{\theta}{2})\left[1 + \sin(\frac{\theta}{2})\sin(\frac{3\theta}{2})\right] + \frac{K_{II}}{\sqrt{2\pi r}}\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})\cos(\frac{3\theta}{2})$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}}\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})\cos(\frac{3\theta}{2}) + \frac{K_{II}}{\sqrt{2\pi r}}\cos(\frac{\theta}{2})\left[1 - \sin(\frac{\theta}{2})\sin(\frac{3\theta}{2})\right]$$

➡ $S = rU_i = S(K_I, K_{II}, \theta)$

- Strain energy density (SED) criterion

$$S = rU_i = S(K_I, K_{II}, \theta) = a_{11}k_I^2 + 2a_{12}k_I k_{II} + a_{22}k_{II}^2$$

$$a_{11} = \frac{1}{16G}(1+\cos\theta)(\kappa - \cos\theta)$$

$$a_{12} = \frac{1}{16G}\sin\theta[2\cos\theta - (\kappa - 1)]$$

$$a_{22} = \frac{1}{16G}[(\kappa+1)(1-\cos\theta) + (1+\cos\theta)(3\cos\theta - 1)]$$

$$k_i = K_I / \sqrt{\pi}$$

- Strain energy density (SED) criterion

Mode I: $K_{II} = 0, \quad K_I = \sigma\sqrt{\pi a}$

$$S = a_{11} k_I^2 = \frac{\sigma^2 a}{16G}(1+\cos\theta)(\kappa - \cos\theta)$$

$$\frac{\partial S}{\partial \theta} = \sin\theta[2\cos\theta - (\kappa - 1)] = 0 \quad \rightarrow \quad \theta_c = 0 \quad \text{or} \quad \theta_c = \arccos(\frac{\kappa - 1}{2})$$

$$\frac{\partial^2 S}{\partial \theta^2} = 2\cos 2\theta - (\kappa - 1)\cos\theta > 0 \quad \rightarrow \quad \theta_c = 0$$

$$S(\theta_c) = \frac{\sigma^2 a}{16G}(2)(\kappa - 1) = \frac{\sigma^2 a}{8G}(\kappa - 1)$$

$$S_c = S(\theta_c, Pl.\,Strain) = \frac{(1+\nu)(1-2\nu)\sigma^2 a}{2\pi E} K_{Ic}^2$$

❖ Strain energy density (SED) criterion

Mode II: $K_I = 0, \qquad K_{II} = \tau\sqrt{\pi a}$

$$S = a_{22}k_{II}^2 = \frac{\tau^2 a}{16G}[(\kappa+1)(1-\cos\theta)+(1+\cos\theta)(3\cos\theta-1)]$$

$$\left.\begin{aligned}\frac{\partial S}{\partial\theta} &= \sin\theta[-6\cos\theta+(\kappa-1)]=0\\ \frac{\partial^2 S}{\partial\theta^2} &= 6-\cos^2\theta+(\kappa-1)\cos\theta>0\end{aligned}\right\} \rightarrow \qquad \theta_c = \pm\arccos\left(\frac{\kappa-1}{6}\right)$$

$$S(\theta_c) = \frac{\tau^2 a}{16G}\left[\frac{1}{12}(-\kappa^2+14\kappa-1)\right]$$

$$S(\theta_c) = S_c \qquad \tau_c = \frac{1}{\sqrt{a}}\sqrt{\frac{192GS_c}{-\kappa^2+14\kappa-1}}$$

- Strain energy density (SED) criterion

Multi-mode load

It is again not possible to determine the crack growth direction for a general multi-mode loading. Numerically it can be done, however.

For this situation the scaled stress intensity factors $k_I$ and $k_{II}$ can be calculated, from which follows the expression for $S$ as a function of $\beta$ and $\theta$. The requirements according to the SED criterion can be used to test a sequence of $\theta_c$ values for a sequence of β values and a plot can be made.

$$k_I = \sigma\sqrt{a}\sin^2\beta \quad ; \qquad k_{II} = \sigma\sqrt{a}\sin\beta\cos\beta$$

$$S = S(K_I, K_{II}, \theta, \beta) = \sigma^2 a \sin^2\beta[a_{11}\sin^2\beta + 2a_{12}\sin\beta\cos\beta + a_{22}\cos^2\beta]$$

$$\frac{\partial S}{\partial \theta} = (\kappa - 1)\sin(\theta_c - 2\beta) - 2\sin 2(\theta_c - \beta) - \sin 2\theta_c = 0$$

$$\frac{\partial^2 S}{\partial \theta^2} = (\kappa - 1)\cos(\theta_c - 2\beta) - 4\cos 2(\theta_c - \beta) - 2\cos 2\theta_c > 0$$

## Propagation criteria

## General criteria:

$$\Omega\left(K_I, K_{II}, K_{IC}, K_{IIC}, \beta_i, ......\right) = 0$$ explicit form obtained experimentally

**Examples in Modes I and II**

$$\left(\frac{K_I}{K_{IC}}\right)^m + C_0\left(\frac{K_{II}}{K_{IIC}}\right)^n = 1$$ m , n and $C_0$ parameters determined experimentally

Erdogan / Shih criterion (1963):

Crack growth occurs on directions normal to the maximum principal stress

$$K_I\left[\sin\frac{3\theta}{2} + \sin\frac{\theta}{2}\right] + K_{II}\left[\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\right] = 0$$

Condition to obtain the crack direction

## Interaction of Multiple Cracks

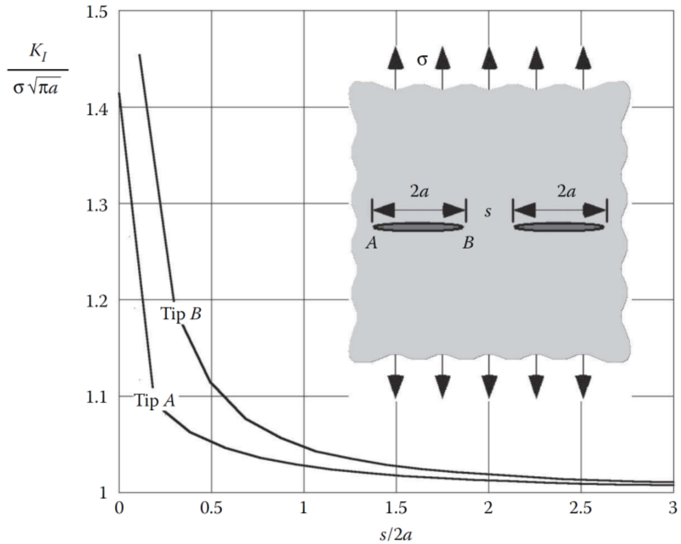
- The local stress field and crack driving force for a given flaw can be significantly affected by the presence of one or more neighboring cracks. Depending on the relative orientation of the neighboring cracks, the interaction can either magnify or diminish the stress intensity factor.

## Coplanar Cracks

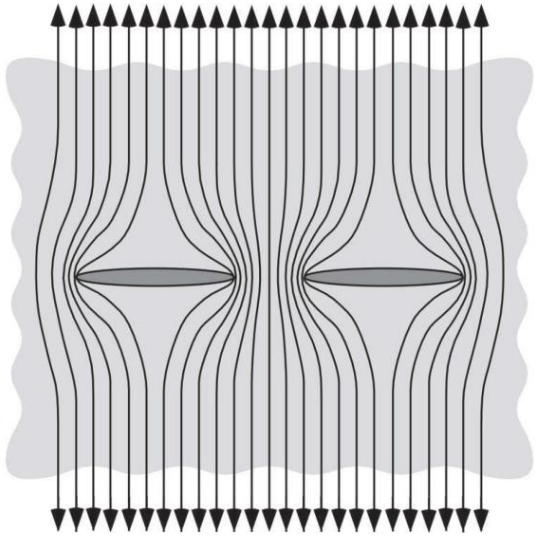
- Typical propagation from an initial crack that is not orthogonal to the applied normal stress. The loading for the initial angled crack is a combination of Modes I and II, but the crack tends to propagate normal to the applied stress, resulting in pure Mode I loading.

## Coplanar Cracks

- The figure illustrates two identical coplanar cracks in an infinite plate. The lines of force represent the relative stress concentrating effect of the cracks. As the ligament between the cracks shrinks in size, the area through which the force must be transmitted decreases. Consequently, $K_I$ is magnified for each crack as the two cracks approach one another.



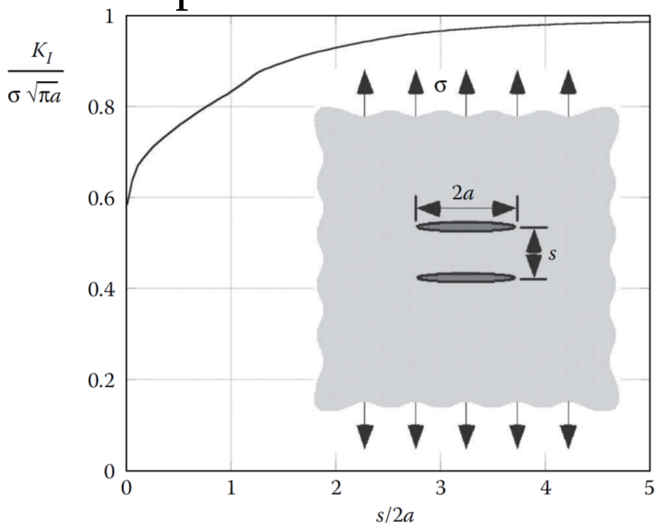

Interaction of two identical coplanar through-wall cracks in an infinite plate



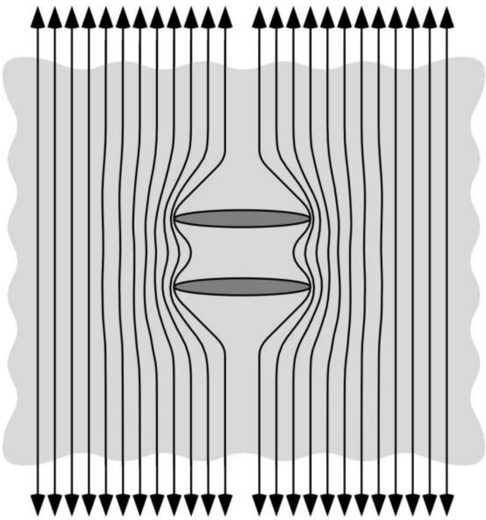
Coplanar cracks. Interaction between cracks results in a magnification of $K_I$

## Parallel Cracks

- The figure illustrates two parallel cracks. In this case, the cracks tend to shield one another, which results in a decrease in $K_I$ relative to the case of the single crack. This is indicative of the general case where two or more parallel cracks have a mutual shielding interaction when subject to Mode I loading. Consequently, multiple cracks that are parallel to one another are of less concern than multiple cracks in the same plane.




Interaction between two identical parallel through-wall cracks in an infinite plate



Parallel cracks. A mutual shielding effect reduces $K_I$ in each crack.