



دانشگاه صنعتی اصفهان
دانشکده مکانیک

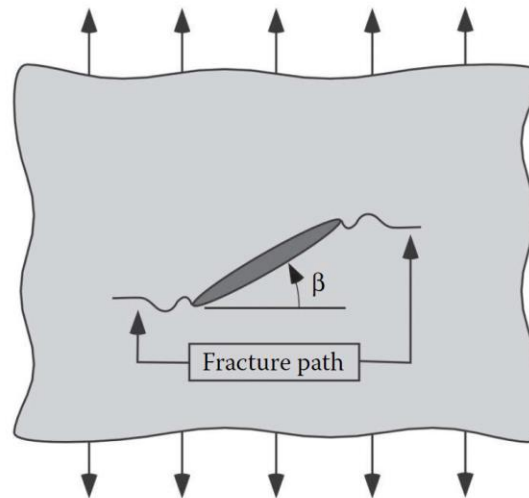
Mixed-Mode Fracture

Mixed-Mode Fracture

- When two or more modes of loading are present,
$$G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2G}$$

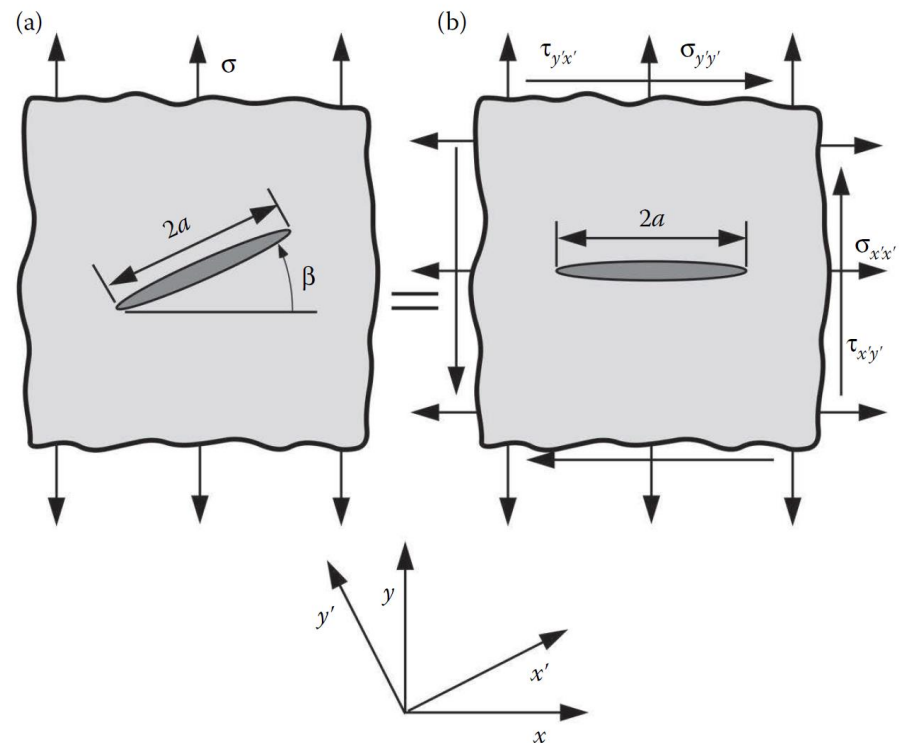
indicates that energy release rate contributions from each mode are additive. This equation, however, assumes self-similar crack growth.

- Typical propagation from an initial crack that is not orthogonal to the applied normal stress. The loading for the initial angled crack is a combination of Modes I and II, but the crack tends to propagate normal to the applied stress, resulting in pure Mode I loading.



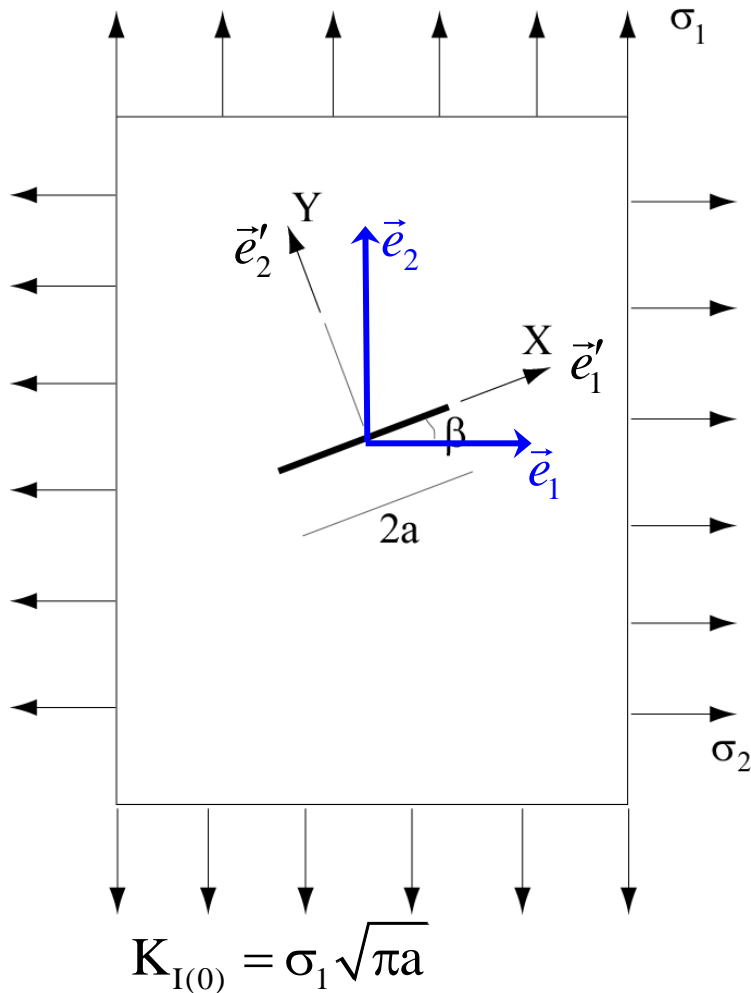
Mixed-Mode Fracture

- Through crack in an infinite plate for the general case where the principal stress is not perpendicular to the crack plane.



- A propagating crack seeks the path of least resistance (or the path of maximum driving force) and need not be confined to its initial plane. If the material is isotropic and homogeneous, the crack will propagate in such a way as to maximize the energy release rate.

Mixed-Mode Fracture biaxial loading



$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \text{ in global frame } (\vec{e}_1 \ \vec{e}_2)$$

→ expressed in local frame $(\vec{e}'_1 \ \vec{e}'_2)$

$$\vec{e}'_1 = \cos \beta \vec{e}_1 + \sin \beta \vec{e}_2$$

$$\vec{e}'_2 = -\sin \beta \vec{e}_1 + \cos \beta \vec{e}_2$$

$$(\mathbf{Q}) = \begin{pmatrix} \vec{e}_1 \cdot \vec{e}'_1 & \vec{e}_1 \cdot \vec{e}'_2 \\ \vec{e}_2 \cdot \vec{e}'_1 & \vec{e}_2 \cdot \vec{e}'_2 \end{pmatrix} \quad \mathbf{Q} = \text{Rotation tensor}$$

$$\text{Thus, } (\mathbf{Q}) = \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix}$$

Stress tensor components :

$$\begin{pmatrix} \sigma'_{11} & \sigma'_{12} \\ \sigma'_{21} & \sigma'_{22} \end{pmatrix} = (\mathbf{Q})^T \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} (\mathbf{Q})$$



Mixed-Mode Fracture

$$\begin{pmatrix} \sigma'_{11} & \sigma'_{12} \\ \sigma'_{21} & \sigma'_{22} \end{pmatrix} = \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix}$$

$$= \begin{bmatrix} c^2 \sigma_{11} + 2cs \sigma_{12} + s^2 \sigma_{22} & -cs \sigma_{11} + (c^2 - s^2) \sigma_{12} + cs \sigma_{22} \\ -cs \sigma_{11} + (c^2 - s^2) \sigma_{12} + cs \sigma_{22} & s^2 \sigma_{11} - 2cs \sigma_{12} + c^2 \sigma_{22} \end{bmatrix}$$

crack tip stresses

$$\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} f_{Iij}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} f_{IIij}(\theta)$$

with:

$$K_I = \varphi \sigma'_{22} \sqrt{\pi a}; \quad K_{II} = \gamma \sigma'_{12} \sqrt{\pi a}$$

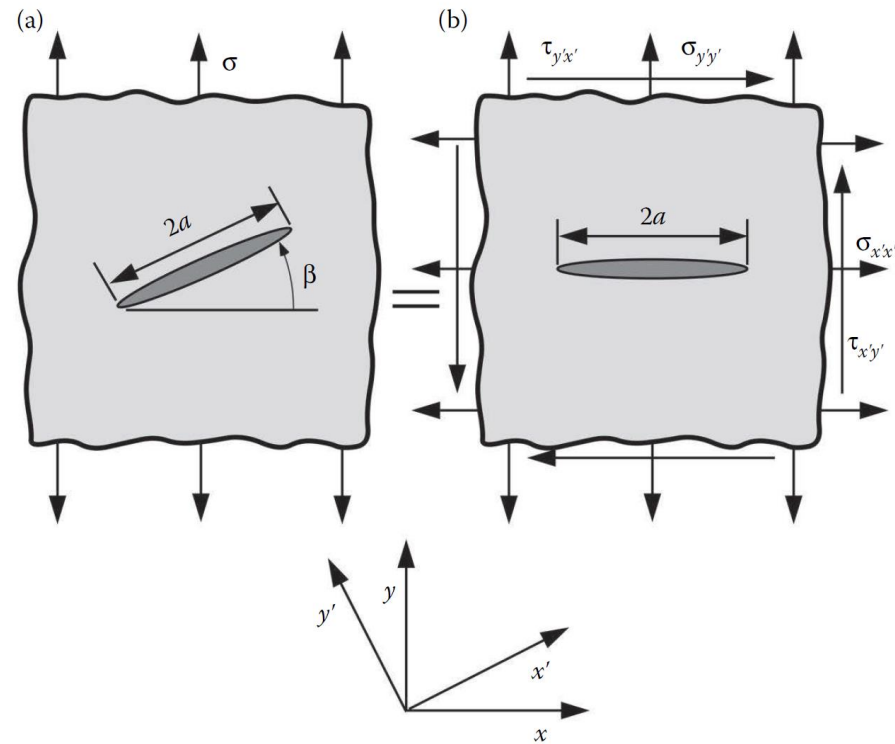
σ'_{11} “does not do anything”

Mixed-Mode Fracture

$$\sigma = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_{22} \end{bmatrix} \longrightarrow \sigma' = \begin{bmatrix} s^2 \sigma & cs \sigma \\ cs \sigma & c^2 \sigma \end{bmatrix}$$

$$K_I = K_{I(0)} \cos^2 \beta$$

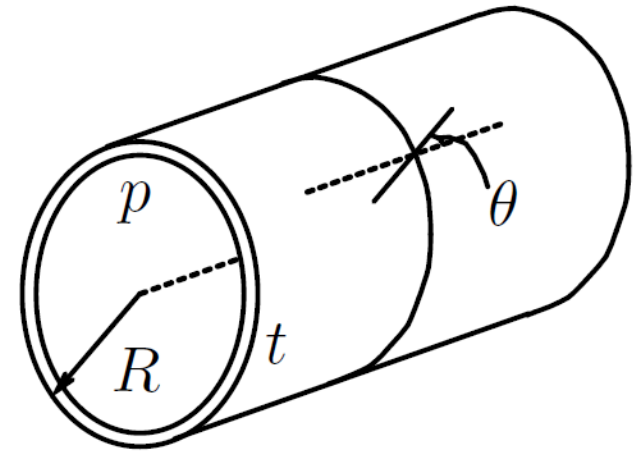
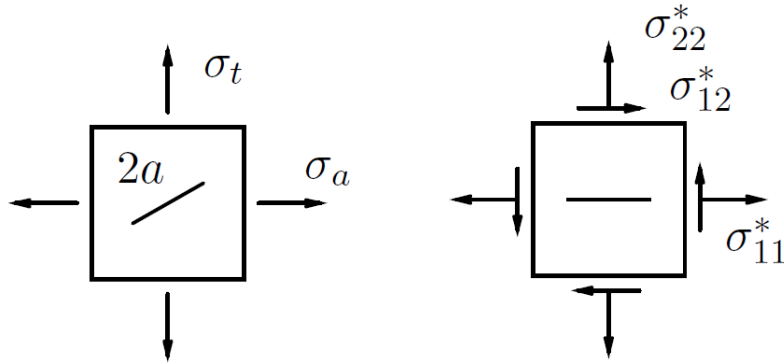
$$K_{II} = K_{I(0)} \cos \beta \sin \beta$$



$K_{I(0)}$ is the Mode I stress intensity when $\beta = 0$. $K_{I(0)} = \sigma \sqrt{\pi a}$

Example multi-mode load

Crack in tube wall at an angle θ with the tube axis.



$$\sigma_t = \frac{pR}{t} = \sigma \quad ; \quad \sigma_a = \frac{pR}{2t} = \frac{1}{2}\sigma$$

$$\sigma_{22}^* = s^2 \frac{1}{2} \sigma + c^2 \sigma \quad ; \quad \sigma_{12}^* = cs \left(1 - \frac{1}{2}\right) \sigma = \frac{1}{2} cs \sigma$$

$$K_I = \sigma_{22}^* \sqrt{\pi a} = \left(\frac{1}{2}s^2 + c^2\right) \sigma \sqrt{\pi a} = \left(\frac{1}{2}s^2 + c^2\right) \frac{pR}{t} \sqrt{\pi a}$$

$$K_{II} = \sigma_{12}^* \sqrt{\pi a} = \frac{1}{2} cs \sigma = \frac{1}{2} cs \frac{pR}{t} \sqrt{\pi a}$$



Crack Growth Direction

➤ criteria for crack growth direction :

- ❖ maximum tangential stress (MTS) criterion
- ❖ strain energy density (SED) criterion

requirement : crack tip stresses in cylindrical coordinates



Mixed-Mode Fracture

The crack tip stress fields (in polar coordinates) for the Mode I portion of the loading:

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}} \left[\frac{5}{4} \cos\left(\frac{\theta}{2}\right) - \frac{1}{4} \cos\left(\frac{3\theta}{2}\right) \right]$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}} \left[\frac{3}{4} \cos\left(\frac{\theta}{2}\right) + \frac{1}{4} \cos\left(\frac{3\theta}{2}\right) \right]$$

$$\tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}} \left[\frac{1}{4} \sin\left(\frac{\theta}{2}\right) + \frac{1}{4} \sin\left(\frac{3\theta}{2}\right) \right]$$

The singular stress fields for Mode II:

$$\sigma_{rr} = \frac{K_{II}}{\sqrt{2\pi r}} \left[-\frac{5}{4} \sin\left(\frac{\theta}{2}\right) + \frac{3}{4} \sin\left(\frac{3\theta}{2}\right) \right]$$

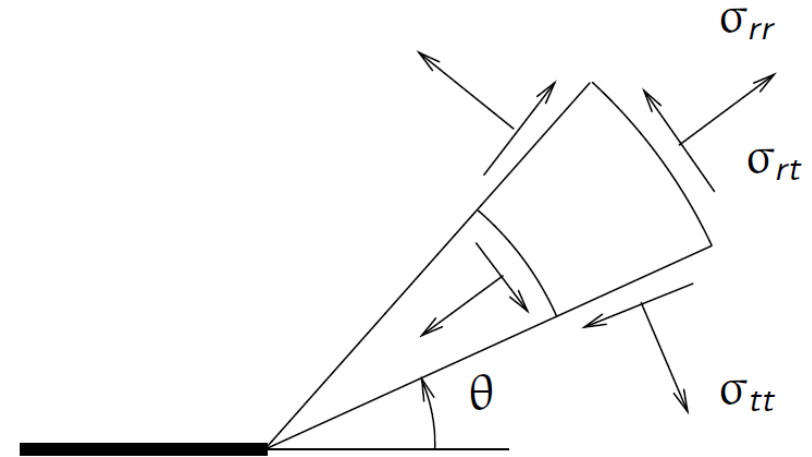
$$\sigma_{\theta\theta} = \frac{K_{II}}{\sqrt{2\pi r}} \left[-\frac{3}{4} \sin\left(\frac{\theta}{2}\right) - \frac{3}{4} \sin\left(\frac{3\theta}{2}\right) \right]$$

$$\tau_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}} \left[\frac{1}{4} \cos\left(\frac{\theta}{2}\right) + \frac{3}{4} \cos\left(\frac{3\theta}{2}\right) \right]$$

Crack Growth Direction

❖ Maximum tangential stress criterion

Erdogan & Sih (1963):



Hypothesis: *crack growth towards local maximum of σ_{tt}*

$$\frac{\partial \sigma_{tt}}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 \sigma_{tt}}{\partial \theta^2} < 0 \quad \rightarrow \quad \theta_c$$

$$\sigma_{tt}(\theta = \theta_c) = \sigma_{tt}(\theta = 0) = \frac{K_{Ic}}{\sqrt{2\pi r}} \quad \rightarrow \quad \text{crack growth}$$



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❖ Maximum tangential stress criterion

$$\frac{\partial \sigma_{tt}}{\partial \theta} = 0 \rightarrow$$

$$\frac{3}{2} \frac{K_I}{\sqrt{2\pi r}} \left[-\frac{1}{4} \sin\left(\frac{\theta}{2}\right) - \frac{1}{4} \sin\left(\frac{3\theta}{2}\right) \right] + \frac{3}{2} \frac{K_{II}}{\sqrt{2\pi r}} \left[-\frac{1}{4} \cos\left(\frac{\theta}{2}\right) - \frac{3}{4} \cos\left(\frac{3\theta}{2}\right) \right] = 0$$

$$K_I \sin \theta + K_{II} [3 \cos \theta - 1] = 0$$

$$\frac{\partial^2 \sigma_{tt}}{\partial \theta^2} < 0 \rightarrow$$

$$\frac{3}{4} \frac{K_I}{\sqrt{2\pi r}} \left[-\frac{1}{4} \cos\left(\frac{\theta}{2}\right) - \frac{3}{4} \cos\left(\frac{3\theta}{2}\right) \right] + \frac{3}{4} \frac{K_{II}}{\sqrt{2\pi r}} \left[\frac{1}{4} \sin\left(\frac{\theta}{2}\right) + \frac{9}{4} \sin\left(\frac{3\theta}{2}\right) \right] < 0$$

$$\sigma_{tt}(\theta = \theta_c) = \frac{K_{Ic}}{\sqrt{2\pi r}} \rightarrow$$

$$\frac{1}{4} \frac{K_I}{K_{Ic}} \left[3 \cos\left(\frac{\theta_c}{2}\right) + \cos\left(\frac{3\theta_c}{2}\right) \right] + \frac{1}{4} \frac{K_{II}}{K_{Ic}} \left[-3 \sin\left(\frac{\theta_c}{2}\right) - 3 \sin\left(\frac{3\theta_c}{2}\right) \right] = 1$$



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❖ Maximum tangential stress criterion

➤ Mode I: $K_{II} = 0$

$$\frac{\partial \sigma_{tt}}{\partial \theta} = K_I \sin(\theta) = 0 \quad \rightarrow \quad \theta_c = 0$$

$$\left. \frac{\partial^2 \sigma_{tt}}{\partial \theta^2} \right|_{\theta_c} < 0$$

$$\sigma_{tt}(\theta_c) = \frac{K_{Ic}}{\sqrt{2\pi r}} \quad \rightarrow \quad K_I = K_{Ic}$$

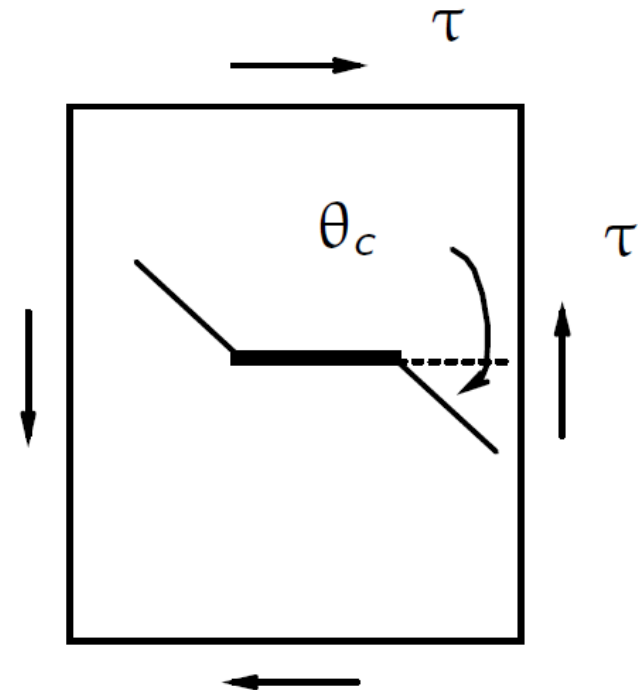
❖ Maximum tangential stress criterion

➤ Mode II: $K_I = 0$

$$\frac{\partial \sigma_{tt}}{\partial \theta} = K_{II} [3 \cos \theta_c - 1] = 0 \quad \rightarrow \quad \theta_c = \pm \arccos\left(\frac{1}{3}\right) = \pm 70.6^\circ$$

$$\left. \frac{\partial^2 \sigma_{tt}}{\partial \theta^2} \right|_{\theta_c = -70.6} < 0$$

$$\sigma_{tt}(\theta_c) = \frac{K_{Ic}}{\sqrt{2\pi r}} \quad \rightarrow \quad K_{IIc} = \sqrt{\frac{3}{4}} K_{Ic}$$





❖ Maximum tangential stress criterion

Multi-mode load

$$K_I \left[-\sin\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{2}\right) \right] + K_{II} \left[-\cos\left(\frac{\theta}{2}\right) - 3\cos\left(\frac{3\theta}{2}\right) \right] = 0$$

$$K_I \left[-\cos\left(\frac{\theta}{2}\right) - 3\cos\left(\frac{3\theta}{2}\right) \right] + K_{II} \left[\sin\left(\frac{\theta}{2}\right) + 9\sin\left(\frac{3\theta}{2}\right) \right] < 0$$

$$K_I \left[3\cos\left(\frac{\theta}{2}\right) + \cos\left(\frac{3\theta}{2}\right) \right] + K_{II} \left[-3\sin\left(\frac{\theta}{2}\right) - 3\sin\left(\frac{3\theta}{2}\right) \right] = 4K_{Ic}$$

$$\begin{cases} -K_I f_1 - K_{II} f_2 = 0 \\ -K_I f_2 + K_{II} f_3 < 0 \\ K_I f_4 - 3K_{II} f_1 = 4K_{Ic} \end{cases}$$

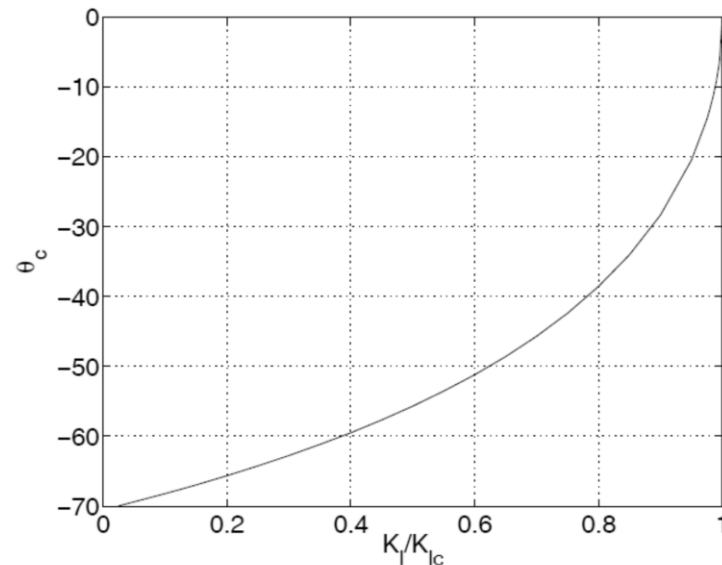
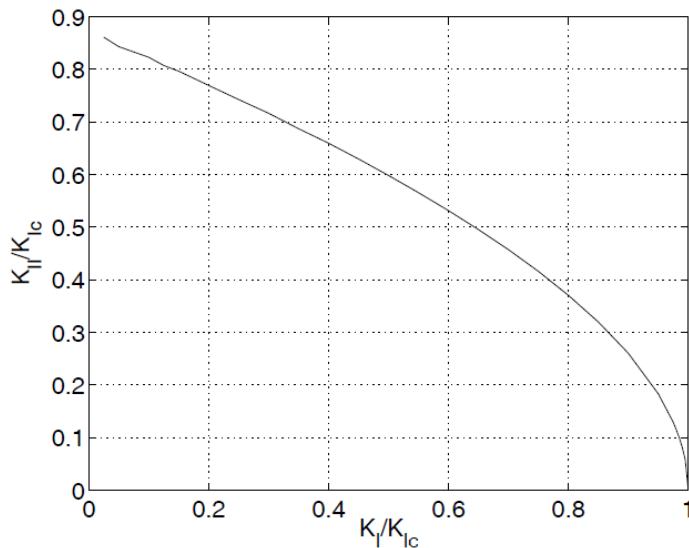


$$\begin{cases} -\left(\frac{K_I}{K_{Ic}}\right) f_1 - \left(\frac{K_{II}}{K_{Ic}}\right) f_2 = 0 \\ -\left(\frac{K_I}{K_{Ic}}\right) f_2 + \left(\frac{K_{II}}{K_{Ic}}\right) f_3 = 0 \\ \left(\frac{K_I}{K_{Ic}}\right) f_4 - 3\left(\frac{K_{II}}{K_{Ic}}\right) f_1 = 4 \end{cases}$$

❖ Maximum tangential stress criterion

Multi-mode load

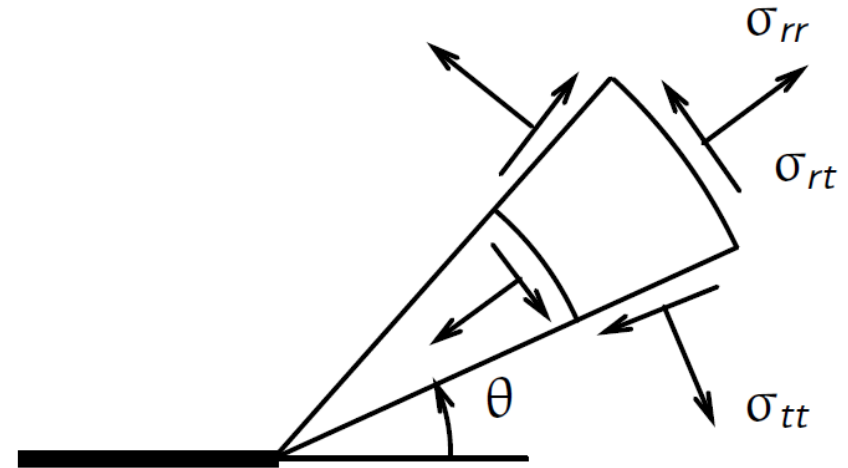
Determining the crack growth direction for multi-mode loading is only possible using numerical calculations. The relations which have to be satisfied for crack growth in a certain direction are written in terms of the parameter K_I/K_{Ic} .



In the first one K_{II}/K_{Ic} is plotted as a function of K_I/K_{Ic} and it can be used to determine which combination leads to crack growth. From the second plot the crack growth angle θ_c can then be determined.

❖ Strain energy density (SED) criterion

Sih (1973)



U_i = Strain Energy Density (Function)

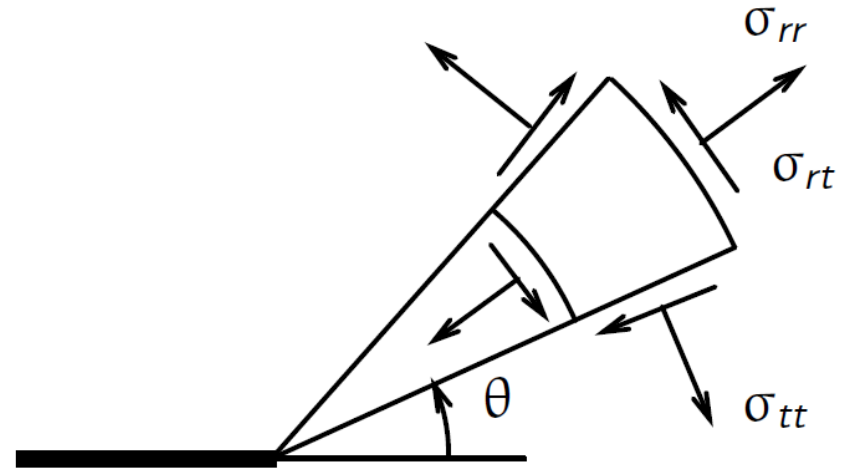
S = Strain Energy Density Factor = $rU_i = S(K_I, K_{II}, \theta)$

The strain energy density U_i is the stored elastic energy per unit of volume. For linear elastic material behavior this specific energy is easily calculated in the crack tip region and appears to be inverse proportional to the distance to the crack tip. The strain energy density factor S is now defined as the product rU_i , as such being independent of r .

❖ Strain energy density (SED) criterion

Sih (1973)

$$S = rU_i = r \int_0^{\varepsilon_{ij}} \sigma d\varepsilon_{ij}$$



Hypothesis: *crack growth towards local minimum of SED*

$$\frac{\partial S}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 S}{\partial \theta^2} > 0 \quad \rightarrow \quad \theta_c$$

$$S(\theta = \theta_c) = S(\theta = 0, \text{pl. strain}) = S_c \quad \rightarrow \quad \text{crack growth}$$

❖ Strain energy density (SED) criterion

$$U_i = \frac{1}{2E} (\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2) - \frac{\nu}{E} (\sigma_{xx} \sigma_{yy} + \sigma_{yy} \sigma_{zz} + \sigma_{zz} \sigma_{xx}) + \frac{1}{2G} (\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2)$$

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right] - \frac{K_{II}}{\sqrt{2\pi r}} \sin\left(\frac{\theta}{2}\right) \left[2 + \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right) \right]$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 + \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right] + \frac{K_{II}}{\sqrt{2\pi r}} \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right)$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right]$$

➔ $S = rU_i = S(K_I, K_{II}, \theta)$



❖ Strain energy density (SED) criterion

$$S = rU_i = S(K_I, K_{II}, \theta) = a_{11}k_I^2 + 2a_{12}k_I k_{II} + a_{22}k_{II}^2$$

$$a_{11} = \frac{1}{16G} (1 + \cos \theta) (\kappa - \cos \theta)$$

$$a_{12} = \frac{1}{16G} \sin \theta [2 \cos \theta - (\kappa - 1)]$$

$$a_{22} = \frac{1}{16G} [(\kappa + 1)(1 - \cos \theta) + (1 + \cos \theta)(3 \cos \theta - 1)]$$

$$k_i = K_I / \sqrt{\pi}$$



❖ Strain energy density (SED) criterion

Mode I: $K_{II} = 0, \quad K_I = \sigma\sqrt{\pi a}$

$$S = a_{11}k_I^2 = \frac{\sigma^2 a}{16G} (1 + \cos \theta) (\kappa - \cos \theta)$$

$$\frac{\partial S}{\partial \theta} = \sin \theta [2 \cos \theta - (\kappa - 1)] = 0 \quad \rightarrow \quad \theta_c = 0 \quad \text{or} \quad \theta_c = \arccos\left(\frac{\kappa - 1}{2}\right)$$

$$\frac{\partial^2 S}{\partial \theta^2} = 2 \cos 2\theta - (\kappa - 1) \cos \theta > 0 \quad \rightarrow \quad \theta_c = 0$$

$$S(\theta_c) = \frac{\sigma^2 a}{16G} (2) (\kappa - 1) = \frac{\sigma^2 a}{8G} (\kappa - 1)$$

$$S_c = S(\theta_c, Pl. Strain) = \frac{(1 + \nu)(1 - 2\nu)\sigma^2 a}{2\pi E} K_{Ic}^2$$



❖ Strain energy density (SED) criterion

Mode II: $K_I = 0, \quad K_{II} = \tau\sqrt{\pi a}$

$$S = a_{22}k_{II}^2 = \frac{\tau^2 a}{16G} [(\kappa + 1)(1 - \cos \theta) + (1 + \cos \theta)(3 \cos \theta - 1)]$$

$$\left. \begin{aligned} \frac{\partial S}{\partial \theta} &= \sin \theta [-6 \cos \theta + (\kappa - 1)] = 0 \\ \frac{\partial^2 S}{\partial \theta^2} &= 6 - \cos^2 \theta + (\kappa - 1) \cos \theta > 0 \end{aligned} \right\} \rightarrow \theta_c = \pm \arccos\left(\frac{\kappa - 1}{6}\right)$$

$$S(\theta_c) = \frac{\tau^2 a}{16G} \left[\frac{1}{12} (-\kappa^2 + 14\kappa - 1) \right]$$

$$S(\theta_c) = S_c \quad \tau_c = \frac{1}{\sqrt{a}} \sqrt{\frac{192GS_c}{-\kappa^2 + 14\kappa - 1}}$$



❖ Strain energy density (SED) criterion

Multi-mode load

It is again not possible to determine the crack growth direction for a general multi-mode loading. Numerically it can be done, however.

For this situation the scaled stress intensity factors k_I and k_{II} can be calculated, from which follows the expression for S as a function of β and θ . The requirements according to the SED criterion can be used to test a sequence of θ_c values for a sequence of β values and a plot can be made.

$$k_I = \sigma \sqrt{a} \sin^2 \beta ; \quad k_{II} = \sigma \sqrt{a} \sin \beta \cos \beta$$

$$S = S(K_I, K_{II}, \theta, \beta) = \sigma^2 a \sin^2 \beta [a_{11} \sin^2 \beta + 2a_{12} \sin \beta \cos \beta + a_{22} \cos^2 \beta]$$

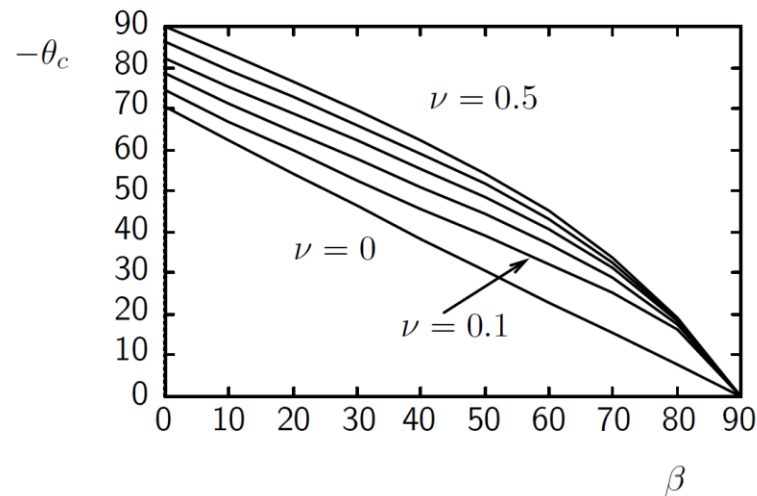
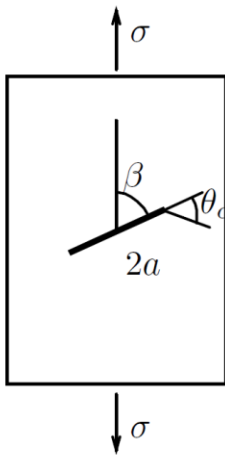
$$\frac{\partial S}{\partial \theta} = (\kappa - 1) \sin(\theta_c - 2\beta) - 2 \sin 2(\theta_c - \beta) - \sin 2\theta_c = 0$$

$$\frac{\partial^2 S}{\partial \theta^2} = (\kappa - 1) \cos(\theta_c - 2\beta) - 4 \cos 2(\theta_c - \beta) - 2 \cos 2\theta_c > 0$$

❖ Strain energy density (SED) criterion

Multi-mode load

The requirements according to the SED criterion can be used to test a sequence of θ_c values for a sequence of β values and a plot can be made. The figure shows the result for plane strain.





Propagation criteria

General criteria:

$$\Omega(K_I, K_{II}, K_{IC}, K_{IIC}, \beta_i, \dots) = 0 \quad \text{explicit form obtained experimentally}$$

Examples in Modes I and II

$$\left(\frac{K_I}{K_{IC}}\right)^m + C_0 \left(\frac{K_{II}}{K_{IIC}}\right)^n = 1 \quad m, n \text{ and } C_0 \text{ parameters determined experimentally}$$

Erdogan / Shih criterion (1963):

Crack growth occurs on directions normal to the maximum principal stress

$$K_I \left[\sin \frac{3\theta}{2} + \sin \frac{\theta}{2} \right] + K_{II} \left[\cos \frac{\theta}{2} + 3 \cos \frac{3\theta}{2} \right] = 0$$

Condition to obtain the crack direction