دانشگاه صنعتی اصفهان

دانشکده مکانیک

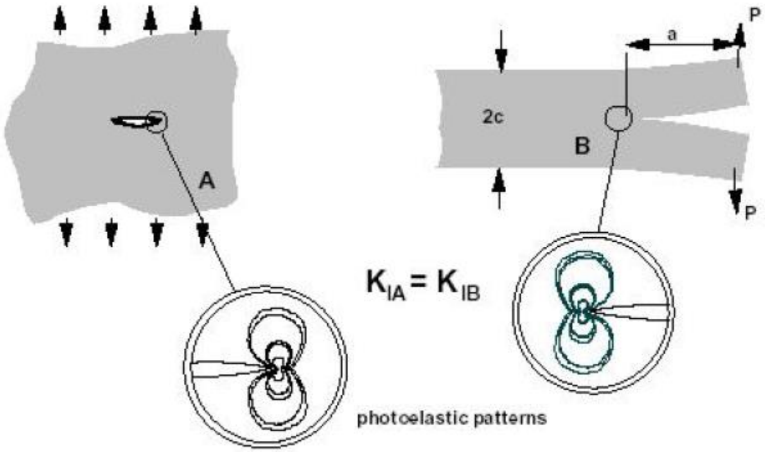
The stress intensity factor

Equal stress intensity factors for two different cracks with different lengths in different geometries under different loadings ensure similar crack tip stress fields. Hence, the critical stress intensity factor K_c, obtained at the onset of crack growth for a specific material and geometry, can be interpreted as a *mechanical property* named *fracture toughness*.

$$\sigma_{ij}^{(I)} = \frac{K_I}{\sqrt{2\pi r}} f_{ij}^{(I)}(\theta)$$

$$\sigma_{ij}^{(II)} = \frac{K_{II}}{\sqrt{2\pi r}} f_{ij}^{(II)}(\theta)$$

$$\sigma_{ij}^{(III)} = \frac{K_{III}}{\sqrt{2\pi r}} f_{ij}^{(III)}(\theta)$$



Relationship between K and Global Behavior

In order for the stress intensity factor to be useful, one must be able to determine *K* from remote loads and the geometry. Closed-form solutions for *K* have been derived for a number of simple configurations. For more complex situations, the stress intensity factor can be estimated by *experiment* or *numerical analysis*.

For an infinite plate subjected to remote tensile stress:

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} Cos\frac{\theta}{2}\left[1 - Sin\frac{\theta}{2} Sin\frac{3\theta}{2}\right] \rightarrow \qquad K_I = O(\sigma\sqrt{r})$$

K-based crack growth criteria:

$$K_I = K_{IC}, \qquad K_{II} = K_{IIC}, \qquad K_{III} = K_{IIIC}$$

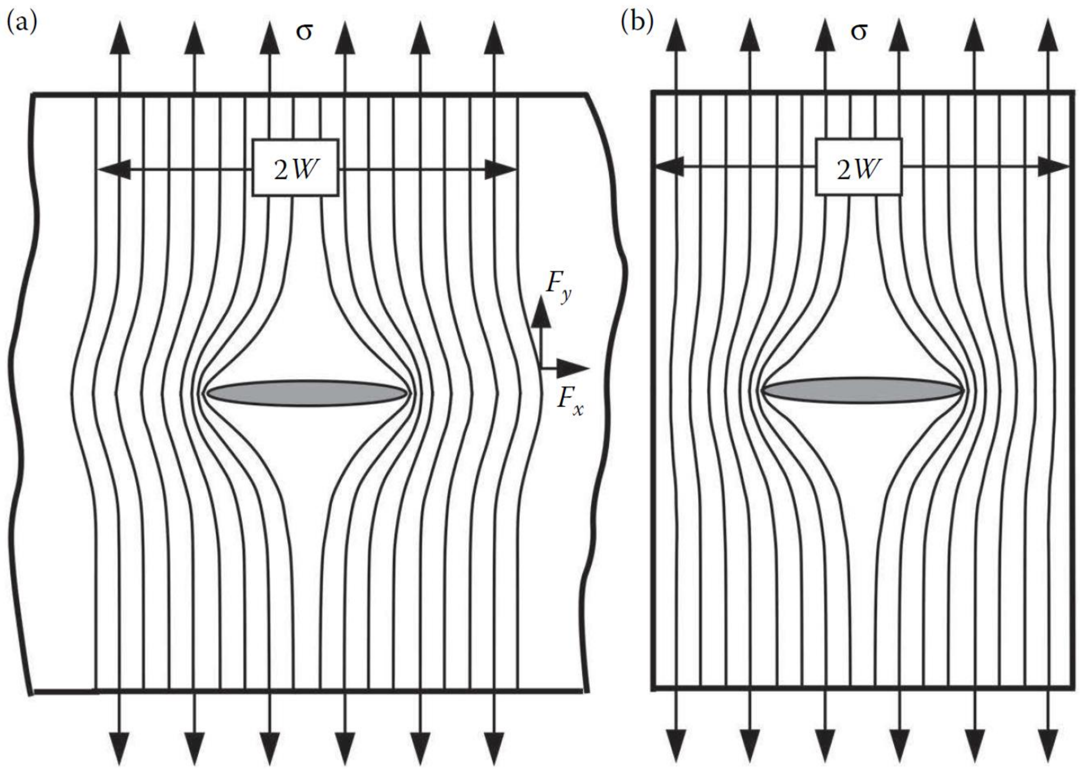
$$K_{IC} = Fracture\ \ Toughness$$

Effect of Finite Size

Most configurations for which there is a closed-form K solution consist of a crack with a simple shape (e.g., a rectangle or ellipse) in an infinite plate. Stated another way, the crack dimensions are small compared with the size of the plate; the crack tip conditions are not influenced by external boundaries. As the crack size increases, or as the plate dimensions decrease, the outer boundaries begin to exert an influence on the crack tip. In such cases, a closed-form stress intensity solution is usually not possible.

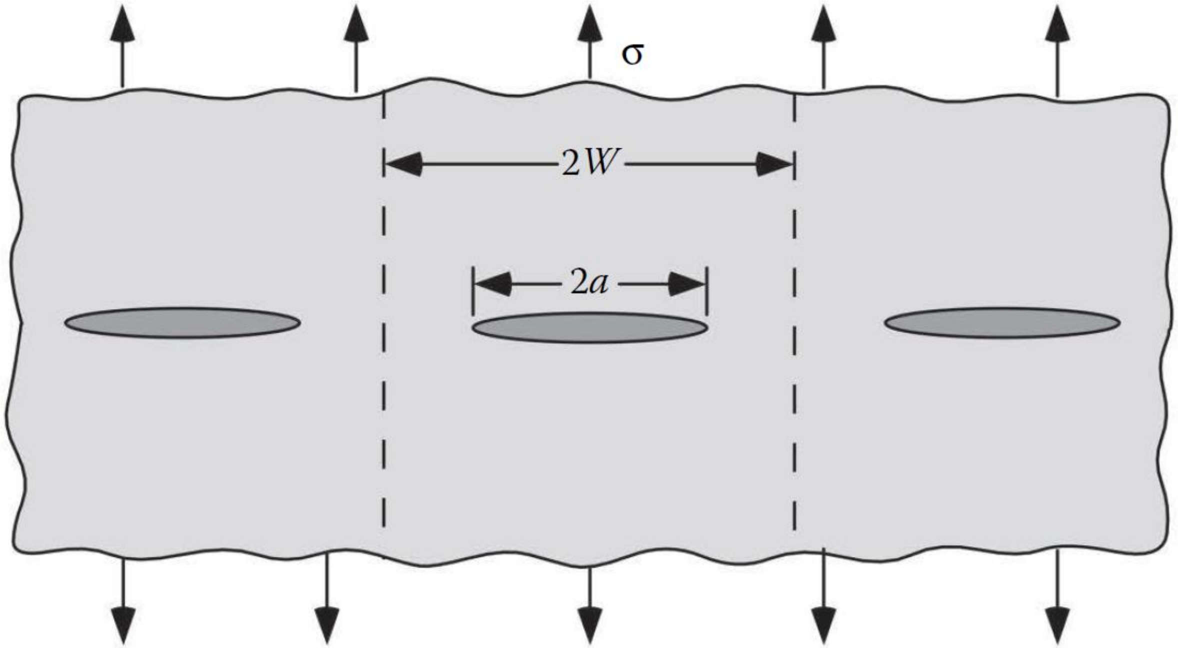
ضریب شدت تنش در نمونه‌های با ابعاد محدود

Effect of Finite Size



Stress concentration effects due to a through crack in finite and infinite width plates: (a) infinite plate and (b) finite plate

Effect of Finite Size



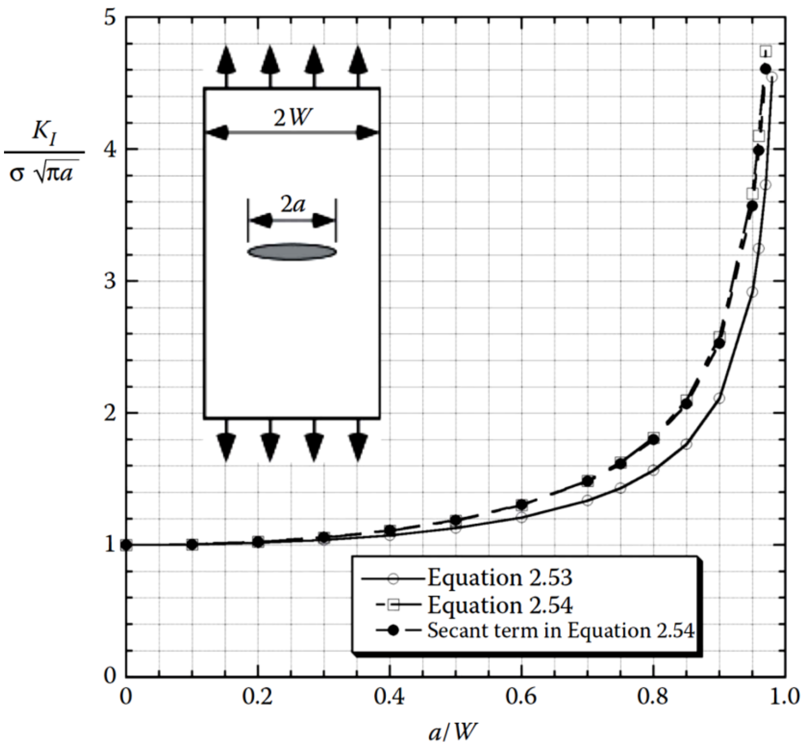
Collinear cracks in an infinite plate subjected to remote tension

$$K_I = \sigma\sqrt{\pi a}\left[\frac{2W}{\pi a}\tan\left(\frac{\pi a}{2W}\right)\right]^{1/2}$$

Effect of Finite Size

More accurate solutions for a through crack in a finite plate have been obtained from finite element analysis; solutions of this type are usually fit to a polynomial expression.

$$K_I = \sigma\sqrt{\pi a}\left[\sec\left(\frac{\pi a}{2W}\right)^{1/2}\right]\left[1-0.025\left(\frac{a}{W}\right)^2+0.06\left(\frac{a}{w}\right)^4\right]$$



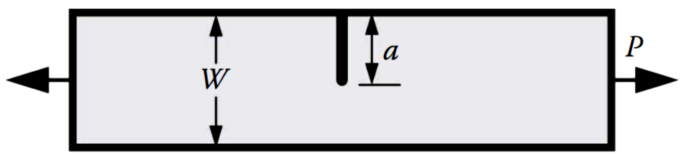
Although stress intensity solutions are given in a variety of forms, K can always be related to the through crack through an appropriate correction factor:

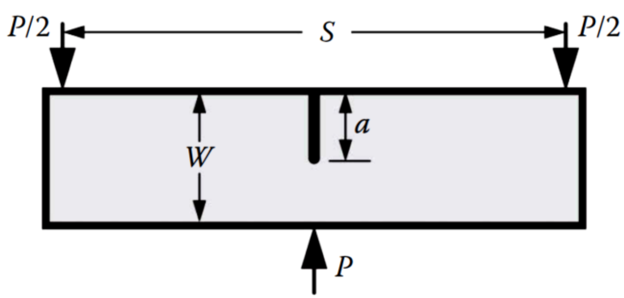
$$K_{(I,II,III)} = Y\sigma\sqrt{\pi a}$$

ضریب شدت تنش برای نمونه‌های خاص

Source: Tada, H., Paris, P.C., and Irwin, G.R., *The Stress Analysis of Cracks Handbook* (2nd Ed), Paris Productions, Inc., St. Louis, 1985.

Geometry	$f\left(\dfrac{a}{W}\right)$
Single-edge notched tension (SENT)	$\dfrac{\sqrt{2\tan(\pi a/2W)}}{\cos(\pi a/2W)}\left[0.752+2.02\left(\dfrac{a}{W}\right)+0.37\left(1-\sin\dfrac{\pi a}{2W}\right)^3\right]$
Single-edge notched bend (SE(B))	$\dfrac{3(S/W)\sqrt{a/W}}{2(1+2(a/W))(1-(a/W))^{3/2}}\left[1.99-\dfrac{a}{W}\left(1-\dfrac{a}{W}\right)\left\{2.15-3.93\left(\dfrac{a}{W}\right)+2.7\left(\dfrac{a}{W}\right)^2\right\}\right]$





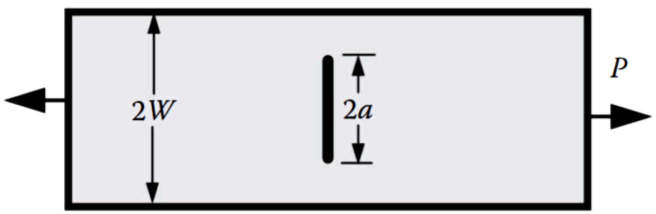
$$K_I=\frac{P}{B\sqrt{W}}f\left(\frac{a}{W}\right),$$ where B is the specimen thickness

ضریب شدت تنش برای نمونه‌های خاص

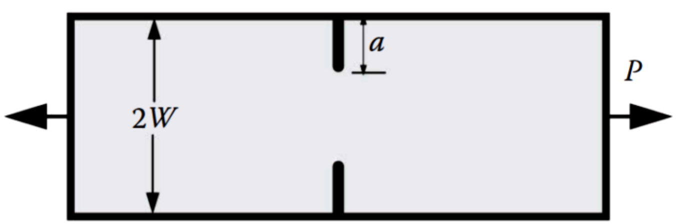
Source: Tada, H., Paris, P.C., and Irwin, G.R., *The Stress Analysis of Cracks Handbook* (2nd Ed), Paris Productions, Inc., St. Louis, 1985.

Geometry	$f\left(\frac{a}{W}\right)$
Center-cracked tension (CCT)	$\sqrt{\frac{\pi a}{4W}\sec\left(\frac{\pi a}{2W}\right)}\left[1-0.025\left(\frac{a}{W}\right)^2+0.06\left(\frac{a}{W}\right)^4\right]$
Double-edge notched tension (DENT)	$\frac{\sqrt{\pi a/2W}}{\sqrt{1-(a/W)}}\left[1.122-0.561\left(\frac{a}{W}\right)-0.205\left(\frac{a}{W}\right)^2+0.471\left(\frac{a}{W}\right)^3+0.190\left(\frac{a}{W}\right)^4\right]$

Center-cracked tension (CCT)



Double-edge notched tension (DENT)

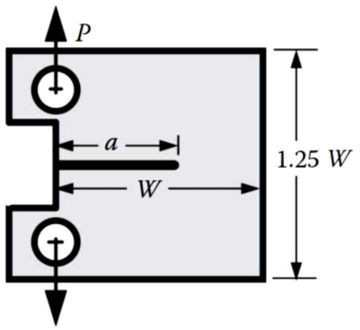


$$K_I = \frac{P}{B\sqrt{W}} f\left(\frac{a}{W}\right),$$ where B is the specimen thickness

ضریب شدت تنش برای نمونه‌های خاص

Source: Tada, H., Paris, P.C., and Irwin, G.R., *The Stress Analysis of Cracks Handbook* (2nd Ed), Paris Productions, Inc., St. Louis, 1985.

Geometry	$f\left(\frac{a}{W}\right)$
Compact specimen	$\frac{2+(a/W)}{(1-(a/W))^{3/2}}\left[0.886+4.64\left(\frac{a}{W}\right)-13.32\left(\frac{a}{W}\right)^2+14.72\left(\frac{a}{W}\right)^3-5.60\left(\frac{a}{W}\right)^4\right]$



$$K_I = \frac{P}{B\sqrt{W}} f\left(\frac{a}{W}\right),$$ where B is the specimen thickness

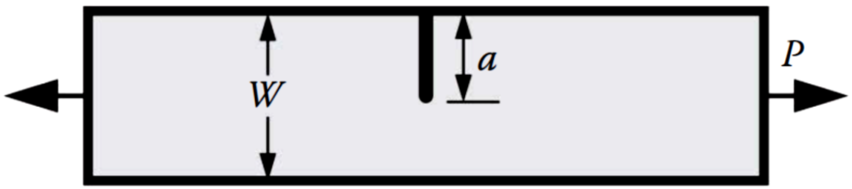
EXAMPLE

Show that the K_I solution for the single-edge notched tensile panel reduces to Equation $K_I = Y\sigma\sqrt{\pi a} = 1.12\sigma\sqrt{\pi a}$ when a<< W.

$$K_I = \frac{P}{B\sqrt{W}} f\left(\frac{a}{W}\right)$$

$$f\left(\frac{a}{W}\right) = \frac{\sqrt{2\tan(\pi a/2W)}}{\cos(\pi a/2W)}\left[0.752 + 2.02\left(\frac{a}{W}\right) + 0.37\left(1-\sin\frac{\pi a}{2W}\right)^3\right]$$

Single-edge notched tension (SENT)

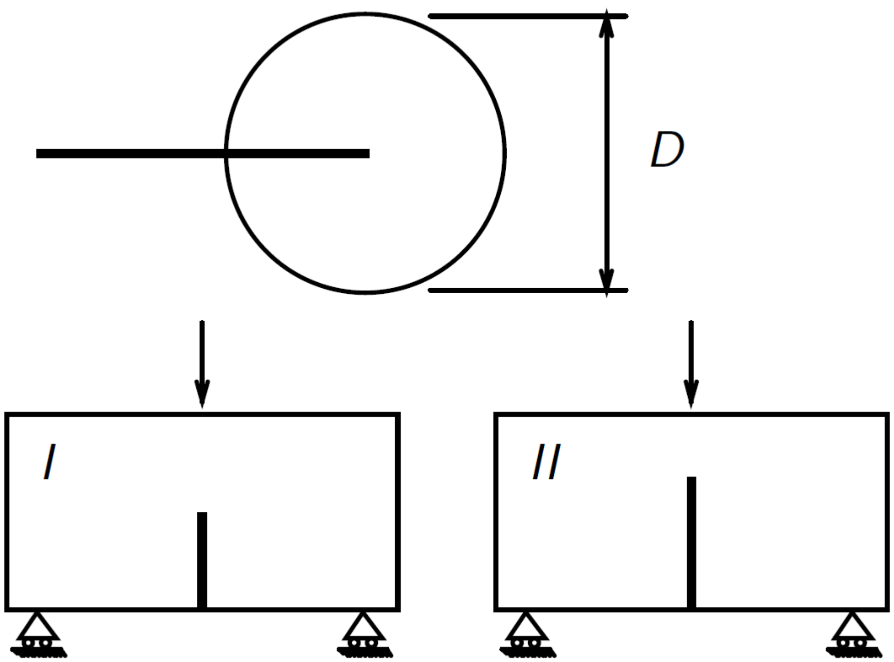


$$K_I = \frac{P}{B\sqrt{W}} f\left(\frac{a}{W}\right) = \frac{P}{BW} f\left(\frac{a}{w}\right)\sqrt{\frac{W}{\pi a}}\sqrt{\pi a} = Y\sigma\sqrt{\pi a} \longrightarrow Y = f\left(\frac{a}{W}\right)\sqrt{\frac{W}{\pi a}}$$

$$\lim_{a/W\to 0} Y = \lim_{a/W\to 0} f\left(\frac{a}{W}\right)\sqrt{\frac{W}{\pi a}} = \lim_{a/W\to 0}\sqrt{\frac{\pi a}{W}}[0.752 + 0.37]\sqrt{\frac{W}{\pi a}} \longrightarrow \lim_{a/W\to 0} Y = 1.122$$

The *K*-zone

what distance to the crack tip, displacement and stresses are still described accurately by the first term of the total solution?



$$\sigma_{ij}^{(I)} = \frac{K_I}{\sqrt{2\pi r}} f_{ij}^{(I)}(\theta) + \ldots$$

K-zone : D

$D_{II} << D_I$

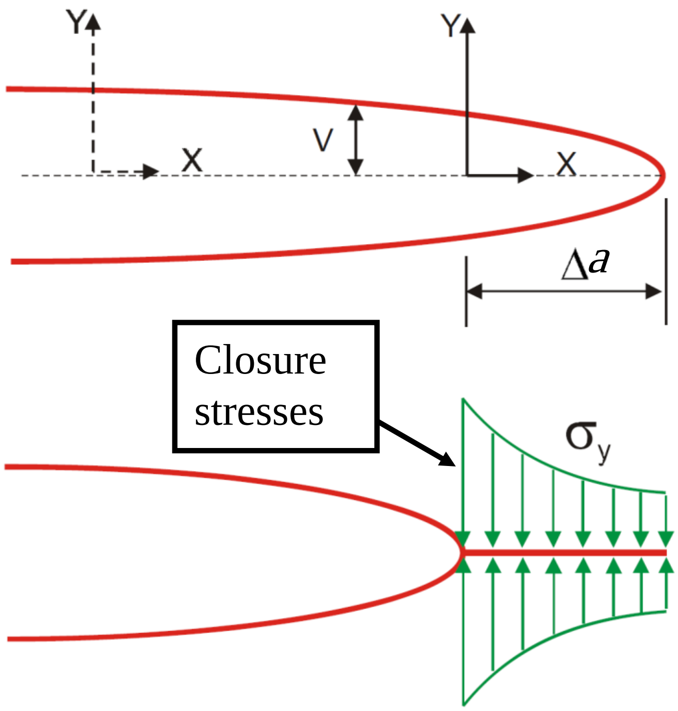
The *K*-zone depends on *geometry* and *loading*.

Relationship between *K* and *G*

There are two different criteria, based on energy considerations and crack tip stress field, for the onset of crack growth. So, there should be a relationship between methods.

Consider a crack of initial length $a + \Delta a$ subject to Mode *I*. It is convenient in this case to place the origin a distance Δa behind the crack tip (plate thickness=1).

Now assume that we may partially close the crack through application of a compressive stress field to the crack faces between x = 0 and x =Δa. The required work is:



$$W = 2\int_0^{\Delta a} \frac{\sigma_y(x)v(x)}{2}dx$$

Relationship between *K* and *G*

σ_y is the compressive stress distribution and v is the crack opening displacement. The factor of 2 on work is required because both crack faces are displaced.

As this work will be released as energy, the energy release rate *G* can be written (fixedload):

$$G = \lim_{\Delta a \to 0} \frac{W}{\Delta a} = \lim_{\Delta a \to 0} \frac{2}{\Delta a} \int_0^{\Delta a} \frac{\sigma_y v}{2} dx \qquad (*)$$

We may define the stresses and displacements in terms of the stress intensity factor (The crack opening displacement, *v*, for Mode *I* is obtained from Previous meeting by setting $\theta = \pi, r = \Delta a - x$):

$$v = \frac{1+\nu}{E} K_I \sqrt{\frac{r}{2\pi}} \sin\left(\frac{\theta}{2}\right)\left[\kappa + 1 - 2\cos^2\left(\frac{\theta}{2}\right)\right] \Longrightarrow v = \frac{4K_{I(a+\Delta a)}}{E'} \sqrt{\frac{\Delta a - x}{2\pi}}$$

Relationship between *K* and *G*

Noting that E'=E for plane stress and E'=E/(1-ν^2) for plane strain. Here $K_I(a+\Delta a)$ denotes the stress intensity factor at the original crack tip. The normal stress required to close the crack is related to K_I for the shortened crack:

$$\sigma_y = \frac{K_{I(a)}}{\sqrt{2\pi x}}$$

Substituting into Eq. (*):

$$G = \lim_{\Delta a \to 0} \frac{4K_{I(a+\Delta a)}K_{I(a)}}{2\pi E'\Delta a}\int_0^{\Delta a}\sqrt{\frac{\Delta a - x}{x}}\,dx = \lim_{\Delta a \to 0} \frac{4K_I^2}{2\pi E'\Delta a}\frac{\pi\Delta a}{2} = \frac{K_I^2}{E'}$$

For plane stress: $$G = \frac{K_I^2}{E}$$

For plane strain: $$G = (1-\nu^2)\frac{K_I^2}{E}$$

Relationship between *K* and *G*

Thus, Equation $G = \frac{K_I^2}{E'}$ is a general relationship for Mode I. The above analysis can be repeated for other modes of loading; the relevant closure stress and displacement for Mode II is τ_{yx} and u_x and the corresponding quantities for Mode *III* are τ_{yz} and u_z. When all three modes of loading are present, the energy release rate is given by:

$$G = \frac{K_I{}^2}{E'} + \frac{K_{II}{}^2}{E'} + \frac{K_{III}{}^2}{2G}$$

Crack Tip Plasticity

Linear elastic stress analysis of sharp cracks predicts infinite stresses at the crack tip. In real materials, however, stresses at the crack tip are finite because the crack tip radius must be finite. Inelastic material deformation, such as plasticity in metals leads to further relaxation of crack tip stresses.

Crack Tip Plasticity

The size of the crack tip yielding zone can be estimated by two methods: *the Irwin approach*, where the elastic stress analysis is used to estimate the elastic–plastic boundary, and the *strip yield model*. Both approaches lead to simple corrections for crack tip yielding. The term plastic zone usually applies to metals, but will be adopted here to describe inelastic crack tip behavior in a more general sense.

➢ The Irwin Approach

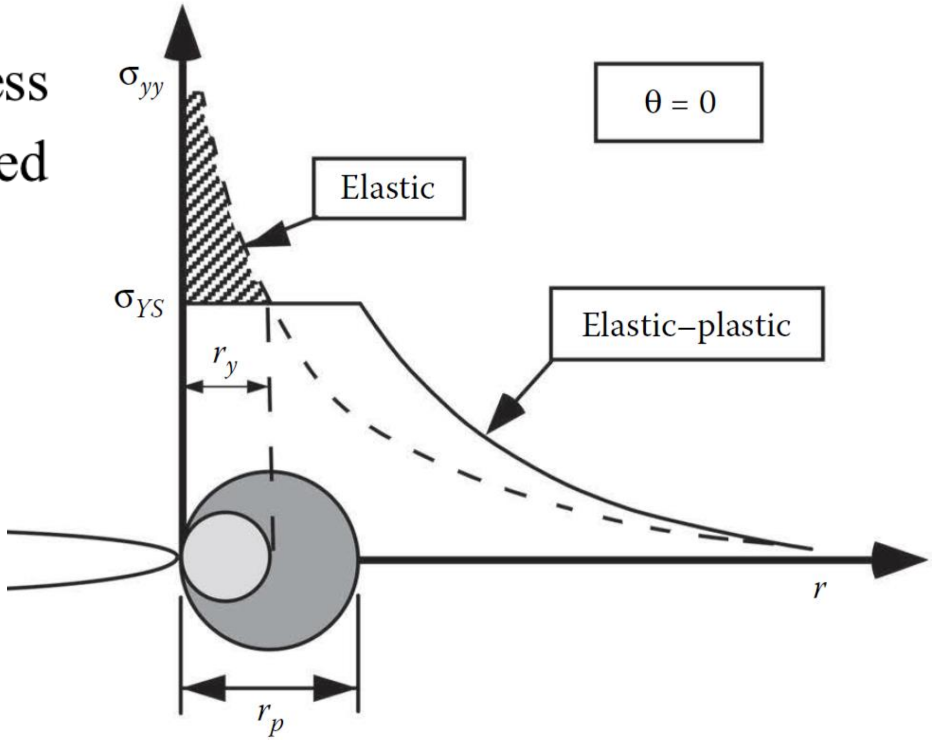
On the crack plane ($\theta = 0$) the normal stress, σ_{yy}, in a linear elastic material is given by $\sigma_{yy}=\frac{K_I}{\sqrt{2\pi r}}$. As a first approximation, we can assume that the boundary between elastic and plastic behavior occurs when the σ_{yy} satisfy a yield criterion.

Crack Tip Plasticity

- The Irwin Approach $\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} = \sigma_{YS}$ ⟹ $r_y = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2$

If we neglect strain hardening, the stress distribution for $r = r_y$ can be represented by a horizontal line at $\sigma_{yy} = \sigma_{YS}$.

The cross-hatched region represents forces that would be present in an elastic material but cannot be carried in the elastic–plastic material because the stress cannot exceed the yield.



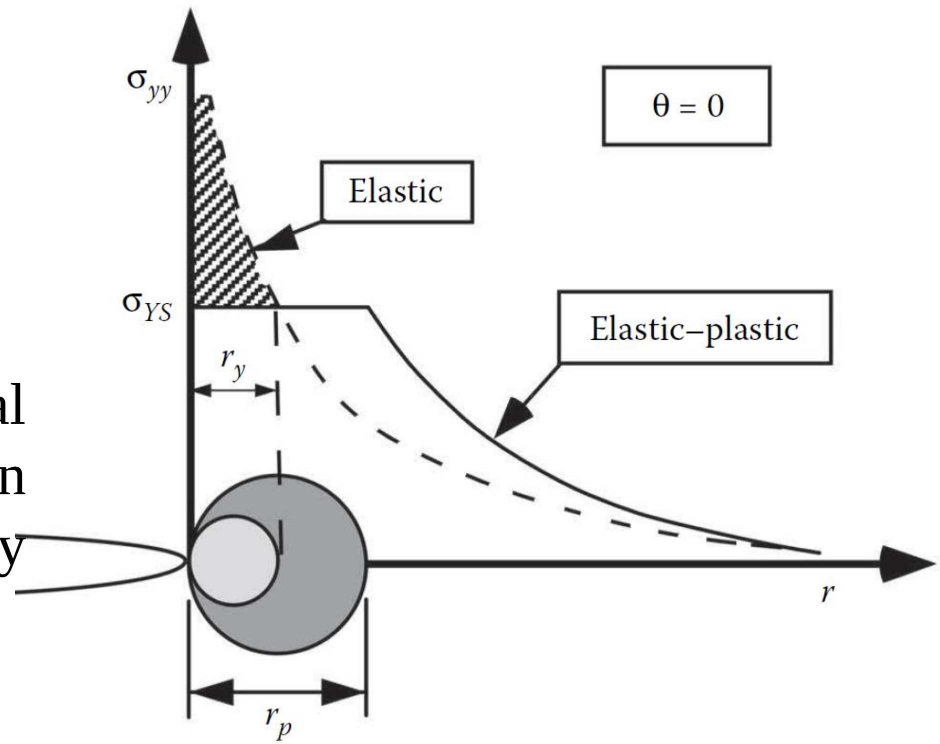
➢ The Irwin Approach

The cross-hatched area represents the load that must be redistributed, resulting in a larger plastic zone. A simple force balance leads to a second-order estimate of the plastic zone size, r_p

$$\sigma_{YS} r_p = \int_0^{r_y} \sigma_{yy} dr = \int_0^{r_y} \frac{K_I}{\sqrt{2\pi r}} dr$$

$$\Rightarrow \quad r_p = \frac{1}{\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2$$

Irwin accounted for the softer material in the plastic zone by defining an effective crack length that is slightly longer than the actual crack size.



➢ The Irwin Approach

The effective crack: $a_{eff} = a + r_y$

For plane stress: $r_y = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2$ *For plane strain:* $r_y = \frac{1}{6\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2$

The effective stress intensity is obtained by inserting a_{eff} into the K expression for the geometry of interest:

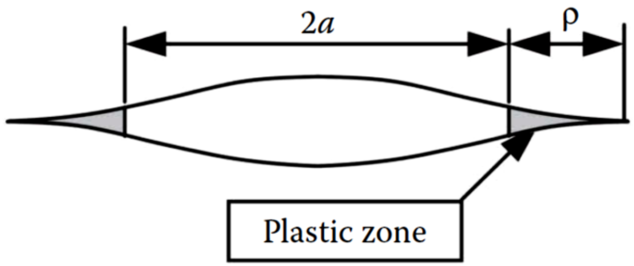
$$K_{eff} = Y(a_{eff})\sigma\sqrt{\pi a_{eff}}$$

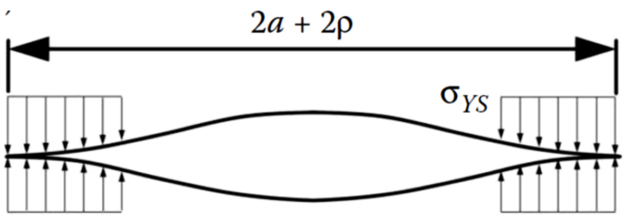
Since the effective crack size is taken into account in the geometry correction factor, Y, an iterative solution is usually required to solve for K_{eff}.

The Strip Yield Model

The strip yield model was first proposed by *Dugdale* and Barenblatt.

This model approximates the elastic–plastic behavior by superimposing two elastic solutions: a through crack under remote tension and a through crack with closure stresses at the tip. Thus, the strip yield model is a classical application of the principle of superposition.





➢ The Strip Yield Model

Load σ: $$K_I(\sigma) = \sigma\sqrt{\pi(a + r_p)}$$

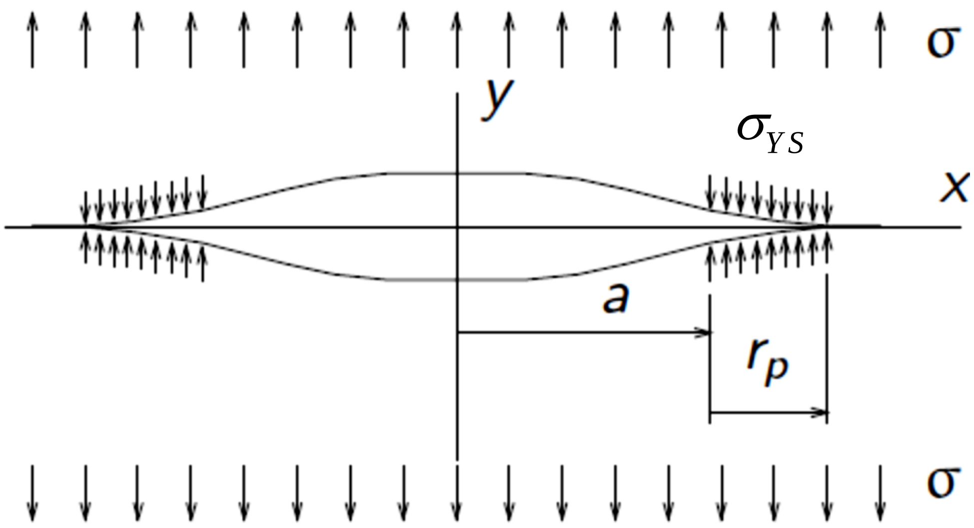
Load σ_{YS}: $$K_I(\sigma_{YS}) = K_{closure} = -2\sigma_{YS}\sqrt{\frac{a + r_p}{\pi}}\arccos\left(\frac{a}{a + r_p}\right)$$

singular term = 0

$$K_I(\sigma) = K_I(\sigma_{YS})$$

$$\frac{a}{a + r_p} = \cos\left(\frac{\pi\sigma}{2\sigma_{YS}}\right)$$

$$r_p = \frac{\pi}{8}\left(\frac{K_I}{\sigma_{YS}}\right)^2$$



Since $1/\pi = 0.318$ and $\pi/8 = 0.392$, the Irwin and strip yield approaches predict similar plastic zone sizes.

The effective crack: $a_{eff} = a + r_p$

$$K_{eff} = \sigma\sqrt{\pi a \sec\left(\frac{\pi\sigma}{2\sigma_{YS}}\right)}$$

Table: Fracture toughness of materials

Material	G_{Ic}(kJm^{-2})	K_{Ic}(MNm^{2})	E(GPa)	K^2_{Ic}/E'
Steel alloy	107	150	210	97.7
Aluminum alloy	20	37	69	18.1
Polyethylene	20 (J_{Ic})	—	0.15	
High-impact polystyrene	15.8(J_{Ic})	—	2.1	
Steel — mild	12	50	210	10.83
Rubber	13	—	0.001	
Glass-reinforced thermoset	7	7	7	6.37
Rubber-toughened epoxy	2	2.2	2.4	1.84
PMMA	0.5	1.1	2.5	0.44
Polystyrene	0.4	1.1	3	0.366
Wood	0.12	0.5	2.1	0.11
Glass	0.007	0.7	70	0.636