دانشگاه صنعتی اصفهان
دانشکده مکانیک

Crack-tip Stress Analysis

محاسبه میدان تنش نوک ترک (روش ویلیامز)

- ایروین (Irwin) و همکاران در دهه ۱۹۵۰ مفهوم *ضریب (فاکتور) شدت تنش* (stress intensity factor) را معرفی کردند که به وسیله آن می‌توان میدان تنش در اطراف نوک تیز ترک را محاسبه نمود و می‌توان آن را برحسب طول ترک، تنش اعمال شده و فاکتور شکل (Y) (برای اجسام با هندسه محدود) بیان کرد.

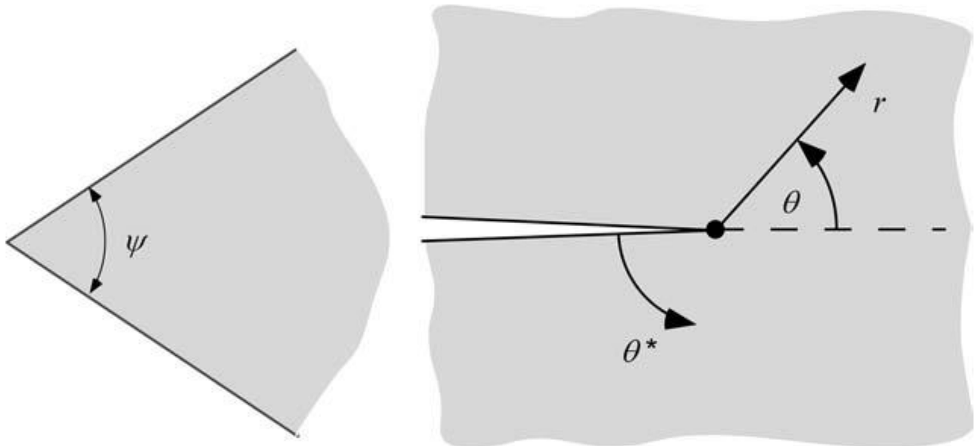
تابع تنش ایری **Airy stress function**

- در آنالیز تنش، هر نقطه (x,y,z) از یک جسم تحت تانسور تنش با مولفه‌های $(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz})$ قرار می گیرد؛ برای هنگامی که این جسم در صحفه‌ی خاصی (مثلا صفحه xy) بارگذاری گردد، ممکن است یکی از حالت‌های تنش صفحه‌ای و یا کرنش صفحه‌ای اتفاق افتد.

محاسبه میدان تنش نوک ترک (روش ویلیامز)

A variety of techniques are available for analyzing stresses in cracked bodies. We focus on Williams approach; the Williams approach considers the local crack-tip fields under generalized inplane loading.

Williams was the first to demonstrate the universal nature of the $1/\sqrt{r}$ singularity for elastic crack problems. Williams actually began by considering stresses at the corner of a plate with various boundary conditions and included angles; a crack is a special case where the included angle of the plate corner is 2π and the surfaces are traction free.



Williams postulated the following stress function:

$$\Phi = r^{\lambda+1}[c_1 \sin(\lambda+1)\theta^* + c_2 \cos(\lambda+1)\theta^* + c_3 \sin(\lambda-1)\theta^* + c_4 \cos(\lambda-1)\theta^*] \qquad (*)$$

$$= r^{\lambda+1}\Phi(\theta^*, \lambda)$$

where c_1, c_2, c_3, and c_4 are constants and $\theta^* = \theta + \pi$. Williams showed that the displacements vary with r^{λ}. In order for displacements to be finite in all regions of the body, λ must be > 0.

محاسبه میدان تنش نوک ترک (روش ویلیامز)

$$\sigma_{rr} = \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} = r^{\lambda-1}\left[F''\left(\theta^*\right) + (\lambda+1)F\left(\theta^*\right)\right]$$

$$\sigma_{\theta\theta} = \frac{\partial^2\Phi}{\partial r^2} = r^{\lambda-1}\left[\lambda(\lambda+1)F\left(\theta^*\right)\right]$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right) = r^{\lambda-1}\left[-\lambda F'\left(\theta^*\right)\right]$$

If the crack faces are traction free, $\sigma_{\theta\theta}(0) = \sigma_{\theta\theta}(2\pi) = \sigma_{r\theta}(0) = \sigma_{r\theta}(2\pi) = 0$ which implies the following boundary conditions:

$$F(0) = F(2\pi) = F'(0) = F'(2\pi) = 0 \qquad (\#)$$

Assuming the constants in Equation (*) are nonzero in the most general case, the boundary conditions can be satisfied only when $\sin(2\pi\lambda) = 0$. Thus,

$$\lambda = \frac{n}{2}, \qquad \text{where} \quad n = 1, 2, 3, \ldots$$

There are an infinite number of λ values that satisfy the boundary conditions; the most general solution to a crack problem, therefore, is a polynomial of the form:

$$\Phi = \sum_{n=1}^{N} \left(r^{\frac{n}{2}+1} F(\theta^*, \frac{n}{2}) \right)$$

and the stresses are:

$$\sigma_{ij} = \frac{\Gamma_{ij}(\theta^*, -\frac{1}{2})}{\sqrt{r}} + \sum_{m=0}^{N} \left(r^{\frac{m}{2}} \Gamma_{ij}(\theta^*, m) \right) \qquad (**)$$

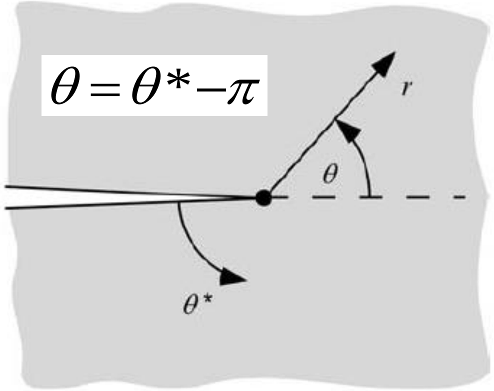
Γ is a function that depends on *F* and its derivatives. The order of the stress function polynomial, *N*, must be sufficient to model the stresses in all regions of the body. When , $r \to 0$ the first term in Equation (**) approaches infinity, while the higher-order terms remain finite (when m = 0) or approach zero (for m > 0).

$$\sigma_{ij} = \frac{\Gamma_{ij}(\theta^*, -\frac{1}{2})}{\sqrt{r}} + \sum_{m=0}^{N} \left(r^{\frac{m}{2}} \Gamma_{ij}(\theta^*, m) \right) \qquad (**)$$

Thus the higher-order terms are negligible close to the crack-tip, and stress exhibits an $1/\sqrt{r}$ singularity. Note that this result was obtained without assuming a specific configuration. It can be concluded that the inverse square-root singularity is universal for cracks in isotropic elastic media.

From Eqs. (*) and (#), it is possible to eliminate two of the constants:

$$\Phi(r,\theta) = r^{\frac{n}{2}+1} \left\{ c_3 \left[\sin\left(\frac{n}{2}-1\right)\theta^* - \frac{n-2}{n+2} \sin\left(\frac{n}{2}+1\right)\theta^* \right] + c_4 \left[\cos\left(\frac{n}{2}-1\right)\theta^* - \cos\left(\frac{n}{2}+1\right)\theta^* \right] \right\}$$



$$\Phi(r,\theta) = r^{3/2}\left[s_1\left(-\cos\frac{\theta}{2} - \frac{1}{3}\cos\frac{3\theta}{2}\right) + t_1\left(-\sin\frac{\theta}{2} - \sin\frac{3\theta}{2}\right)\right] + r^2\left[1-\cos 2\theta\right] + O(r^{5/2}) + \ldots$$

where s_i and t_i are constants to be defined.

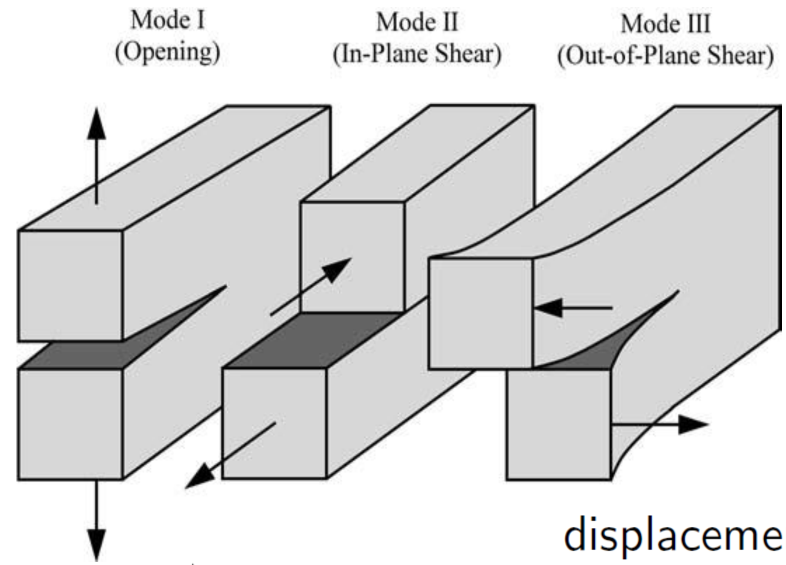
The stresses are:

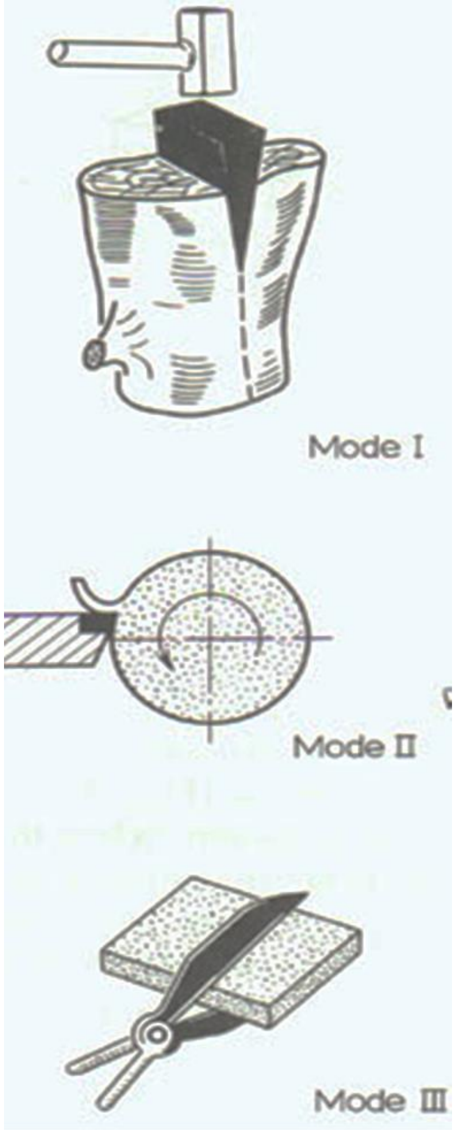
$$\sigma_{rr} = \frac{1}{4\sqrt{r}}\left\{s_1\left[-5\cos\frac{\theta}{2} + \cos\frac{3\theta}{2}\right] + t_1\left[-5\sin\frac{\theta}{2} + 3\sin\frac{3\theta}{2}\right]\right\} + 4s_2\cos^2\theta + O(r^{1/2}) + \ldots$$

$$\sigma_{\theta\theta} = \frac{1}{4\sqrt{r}}\left\{s_1\left[-3\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right] + t_1\left[-3\sin\frac{\theta}{2} - 3\sin\frac{3\theta}{2}\right]\right\} + 4s_2\sin^2\theta + O(r^{1/2}) + \ldots$$

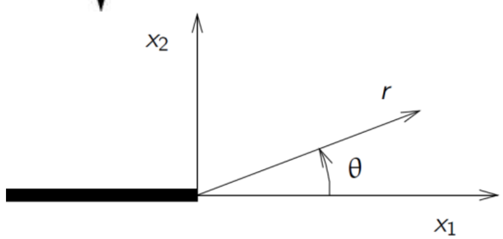
$$\sigma_{r\theta} = \frac{1}{4\sqrt{r}}\left\{s_1\left[-\sin\frac{\theta}{2} - \sin\frac{3\theta}{2}\right] + t_1\left[\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\right]\right\} - 2s_2\sin 2\theta + O(r^{1/2}) + \ldots$$

The constants s_i in the stress function are multiplied by cosine terms while the t_i are multiplied by sine terms. Thus, the stress function contains symmetric and antisymmetric components, with respect to θ = 0. When the loading is symmetric about θ = 0, $t_i = 0$, while $s_i = 0$ for the special case of pure antisymmetric loading.





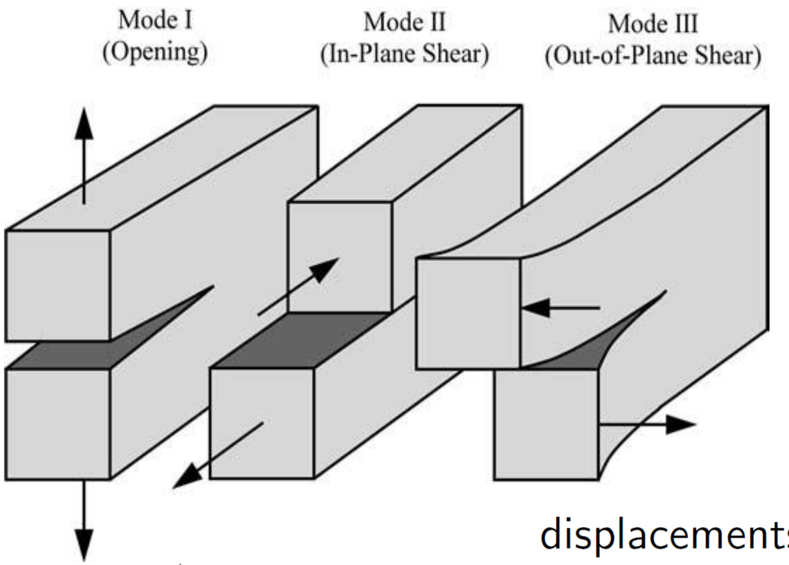
displacement for Mode I



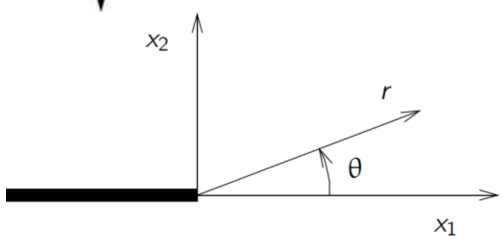
$$u_1(\theta > 0) = u_1(\theta < 0)$$
$$u_2(\theta > 0) = -u_2(\theta < 0)$$

Examples of symmetric loading include pure bending and pure tension; in both cases the principal stress is normal to the crack plane. Therefore, *symmetric* loading corresponds to *Mode I*.

محاسبه میدان تنش نوک ترک (روش ویلیامز)

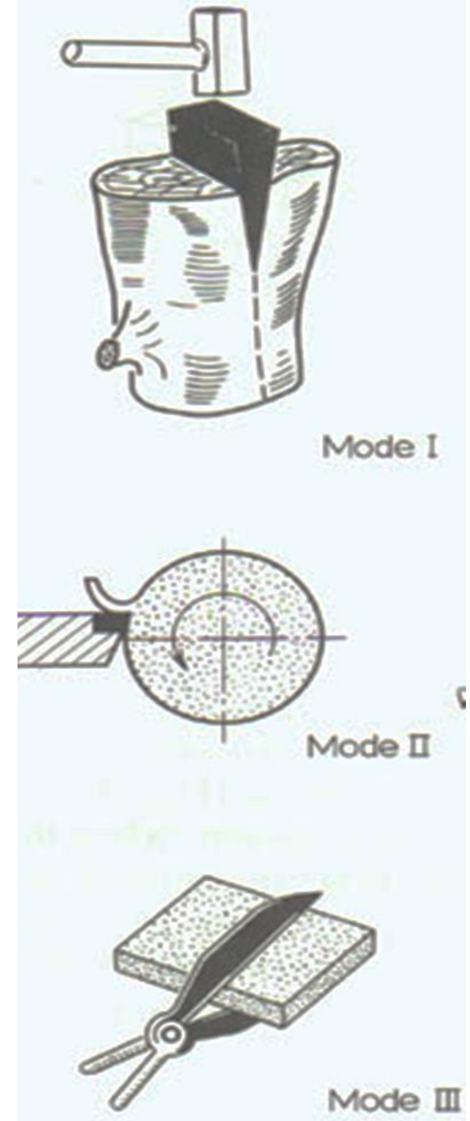


displacements for Mode II



$$u_1(\theta > 0) = -u_1(\theta < 0)$$
$$u_2(\theta > 0) = u_2(\theta < 0)$$

antisymmetric loading is produced by in-plane shear on the crack faces and corresponds to *Mode II*.



محاسبه میدان تنش نوک ترک (روش ویلیامز)

$$\sigma_{rr}=\frac{1}{4\sqrt{r}}\left\{s_1\left[-5\cos\frac{\theta}{2}+\cos\frac{3\theta}{2}\right]+t_1\left[-5\sin\frac{\theta}{2}+3\sin\frac{3\theta}{2}\right]\right\}+4s_2\cos^2\theta+O(r^{1/2})+\ldots$$

$$\sigma_{\theta\theta}=\frac{1}{4\sqrt{r}}\left\{s_1\left[-3\cos\frac{\theta}{2}-\cos\frac{3\theta}{2}\right]+t_1\left[-3\sin\frac{\theta}{2}-3\sin\frac{3\theta}{2}\right]\right\}+4s_2\sin^2\theta+O(r^{1/2})+\ldots$$

$$\sigma_{r\theta}=\frac{1}{4\sqrt{r}}\left\{s_1\left[-\sin\frac{\theta}{2}-\sin\frac{3\theta}{2}\right]+t_1\left[\cos\frac{\theta}{2}+3\cos\frac{3\theta}{2}\right]\right\}-2s_2\sin2\theta+O(r^{1/2})+\ldots$$

$$s_1=-\frac{K_I}{\sqrt{2\pi}}\ ,\qquad t_1=\frac{K_{II}}{\sqrt{2\pi}}$$

The crack-tip stress fields for symmetric (Mode I) loading (assuming the higher-order terms are negligible) are given by:

$$\sigma_{rr}=\frac{K_I}{\sqrt{2\pi r}}\left[\frac{5}{4}\cos\frac{\theta}{2}-\frac{1}{4}\cos\frac{3\theta}{2}\right]$$

$$\sigma_{\theta\theta}=\frac{K_I}{\sqrt{2\pi r}}\left[\frac{3}{4}\cos\frac{\theta}{2}+\frac{1}{4}\cos\frac{3\theta}{2}\right]$$

$$\sigma_{r\theta}=\frac{K_I}{\sqrt{2\pi r}}\left[\frac{1}{4}\sin\frac{\theta}{2}+\frac{1}{4}\sin\frac{3\theta}{2}\right]$$

محاسبه میدان تنش نوک ترک (روش ویلیامز)

The singular stress fields for Mode II are given by:

$$\sigma_{rr} = \frac{K_{II}}{\sqrt{2\pi r}}\left[-\frac{5}{4}\sin\frac{\theta}{2}+\frac{3}{4}\sin\frac{3\theta}{2}\right]$$

$$\sigma_{\theta\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left[-\frac{3}{4}\sin\frac{\theta}{2}-\frac{3}{4}\sin\frac{3\theta}{2}\right]$$

$$\sigma_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{1}{4}\cos\frac{\theta}{2}+\frac{3}{4}\cos\frac{3\theta}{2}\right]$$

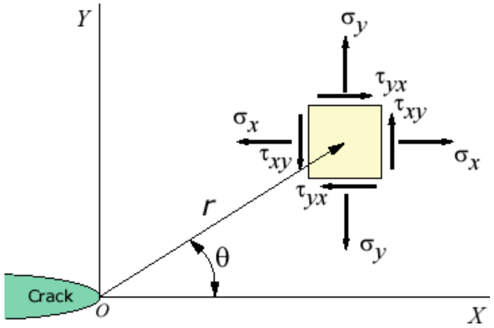
In the Cartesian coordinates

The crack-tip stress fields for Mode I:

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} Cos\frac{\theta}{2}\left[1 - Sin\frac{\theta}{2}Sin\frac{3\theta}{2}\right]$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} Cos\frac{\theta}{2}\left[1 + Sin\frac{\theta}{2}Sin\frac{3\theta}{2}\right]$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} \mathrm{Cos}\frac{\theta}{2}\mathrm{Sin}\frac{\theta}{2}\mathrm{Sin}\frac{3\theta}{2}$$



The crack-tip stress fields for Mode II:

$$\sigma_{xx} = \frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\left[2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right]$$

$$\sigma_{yy} = \frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$\sigma_{xy} = \frac{K_{II}}{\sqrt{2\pi r}} Cos\frac{\theta}{2}\left[1 - Sin\frac{\theta}{2}Sin\frac{3\theta}{2}\right]$$

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \qquad (Plane \;\; strain)$$

$$\sigma_{zz} = 0 \qquad (Plane \;\; stress)$$

The Stress Intensity Factor

Each mode of loading produces the $1/\sqrt{r}$ singularity at the crack tip, but the proportionality constants K (*The Stress Intensity Factor*) and f_{ij} depend on the mode. The stress intensity factor is usually given a subscript to denote the mode of loading, i.e., K_I, K_{II}, or K_{III}. Thus, the stress fields ahead of a crack tip in an isotropic linear elastic for Modes I, II, and III, respectively.

The stress fields ahead of a crack tip in an isotropic linear elastic material can be written as:

$$\sigma_{ij} \propto \frac{1}{\sqrt{r}}$$

$$\lim_{r \to 0} \sigma_{ij}^{(I)} = \frac{K_I}{\sqrt{2\pi r}} f_{ij}^{(I)}(\theta)$$

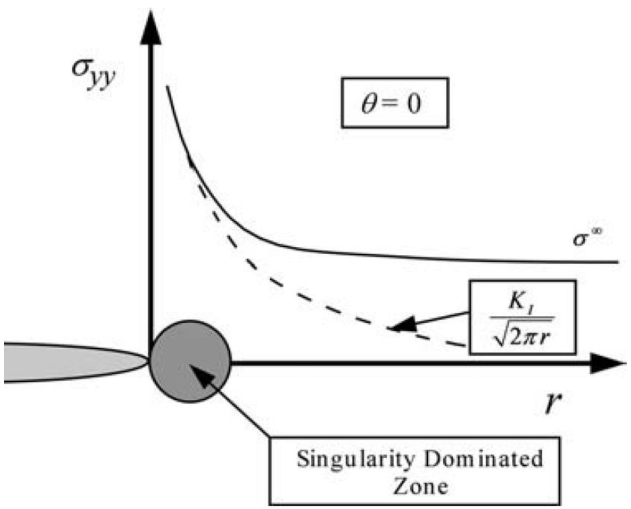
$$\lim_{r \to 0} \sigma_{ij}^{(II)} = \frac{K_{II}}{\sqrt{2\pi r}} f_{ij}^{(II)}(\theta)$$

$$\lim_{r \to 0} \sigma_{ij}^{(III)} = \frac{K_{III}}{\sqrt{2\pi r}} f_{ij}^{(III)}(\theta)$$

In a mixed-mode problem (i.e., when more than one loading mode is present):

$$\sigma_{ij}^{(\text{total})} = \sigma_{ij}^{(I)} + \sigma_{ij}^{(II)} + \sigma_{ij}^{(III)}$$

Consider the Mode I singular field on the crack plane, where $\theta = 0$:



$$\sigma_{xx} = \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}$$

The stress intensity factor defines the amplitude of the crack-tip singularity. That is, stresses near the crack tip increase in proportion to *K*. Moreover, the stress intensity factor completely defines the crack tip conditions; if *K* is known, it is possible to solve for all components of stress, strain, and displacement as a function of *r* and *θ*. This single-parameter description of crack tip conditions turns out to be one of the most important concepts in fracture mechanics.

Displacement Field

Finding the displacement field can be a more difficult problem than finding the stress field. One approach is to calculate the strains using the stress-strain laws, and then integrate the strain-displacement relations to determine the displacement fields.

Williams used the approach of starting from the solution of Coker and Filon in which it is shown that the displacement components in polar coordinates are related to the stress function by:

$$2\mu u_r = -\frac{\partial \Phi}{\partial r} + (1-\bar{\nu}) r \frac{\partial \Psi}{\partial \theta}$$

$$2\mu u_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} + (1-\bar{\nu}) r^2 \frac{\partial \Psi}{\partial r} \qquad (+)$$

where the displacement potential, Ψ is related to the stress function by

$$\nabla^2 \Phi = -\frac{\partial \left(r \frac{\partial \Psi}{\partial \theta}\right)}{\partial r} \qquad (\%)$$

Displacement Field

μ is the shear modulus, and $\bar{\nu} = \nu$ for plane strain and $\bar{\nu} = \nu/(1+\nu)$ for plane stress.

For Mode-I : $$\Phi = r^{\lambda+1}[c_2 \cos(\lambda+1)\theta + c_4 \cos(\lambda-1)\theta] \qquad (\%\%)$$

$$\Psi = r^m [a_1 \cos m\theta + a_2 \sin m\theta]$$

Evaluating the derivatives of Eq. (%%) and substituting into Eq. (%) yields:

$a_1 = 0, \quad a_2 = 4c_4/\lambda, \quad m = \lambda$

$$\Psi = r^{\lambda} \frac{4c_4}{\lambda} \sin\lambda\theta$$

Taking only the first term of the series (corresponding to λ=1/2):

$$\Phi = c_4 r^{3/2}[\cos\frac{\theta}{2} + \frac{1}{3}\cos\frac{3\theta}{2}]$$

$$\Psi = 8c_4 r^{-1/2} \sin\frac{\theta}{2}$$

Displacement Field

Substituting into Eq. (+) and replacing c_4 by $K_I/\sqrt{2\pi}$ yields:

$$\begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = K_I \frac{(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\begin{pmatrix} (\frac{5}{2}-4\bar{\nu})\cos\frac{\theta}{3}-\frac{1}{2}\cos\frac{3\theta}{2} \\ -(\frac{7}{2}-4\bar{\nu})\sin\frac{\theta}{2}+\frac{1}{2}\sin\frac{3\theta}{2} \end{pmatrix}$$

In the Cartesian coordinates:

$$u_i \propto \sqrt{r}$$

$$u = \frac{1+\nu}{E}K_I\sqrt{\frac{r}{2\pi}}\cos\left(\frac{\theta}{2}\right)\left[\kappa - 1 + 2\sin^2\left(\frac{\theta}{2}\right)\right]$$

$$v = \frac{1+\nu}{E}K_I\sqrt{\frac{r}{2\pi}}\sin\left(\frac{\theta}{2}\right)\left[\kappa + 1 - 2\cos^2\left(\frac{\theta}{2}\right)\right]$$

where u, v = displacements in x, y directions

$\kappa = (3-4\nu)$ for plane strain problems

$\kappa = \left(\frac{3-\nu}{1+\nu}\right)$ for plane stress problems