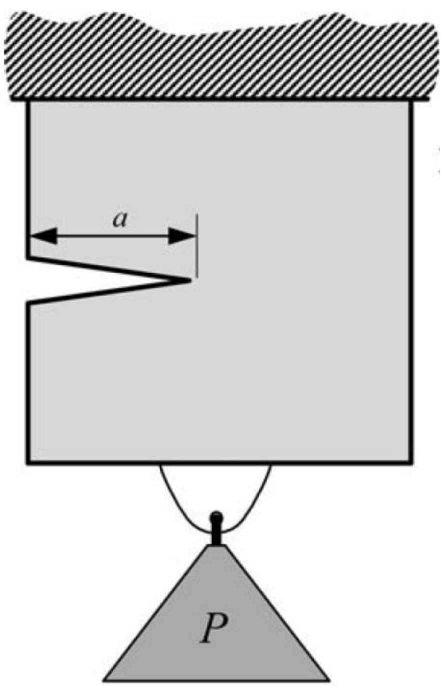
دانشگاه صنعتی اصفهان
دانشکده مکانیک

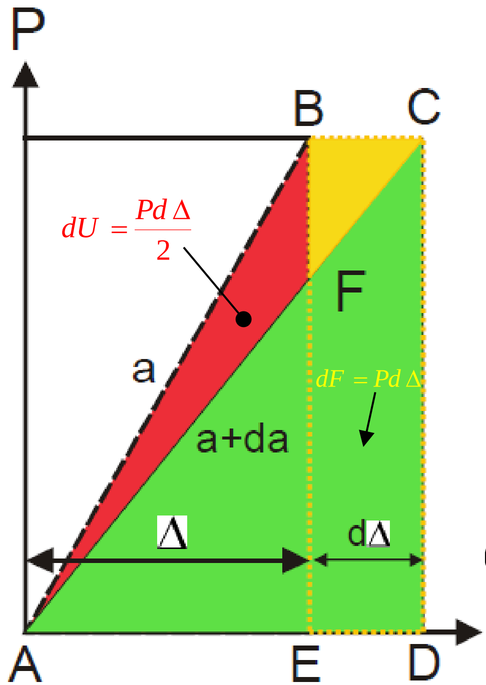
مکانیک شکست الاستیک خطی

load controlled *:* Consider a cracked plate that is dead loaded, since the load is fixed at *P*, the structure is said to be *load controlled*



(a)

Cracked plate at a fixed load *P*.



$$F = P\Delta$$

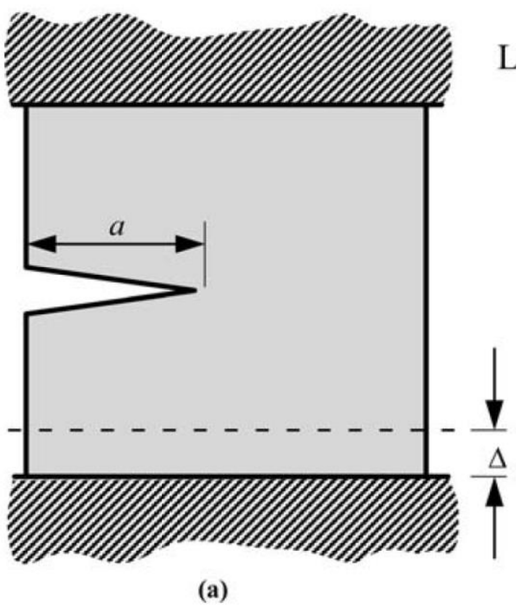
$$U = \frac{1}{2}\int_0^{\Delta} P d\Delta = \frac{P\Delta}{2}$$

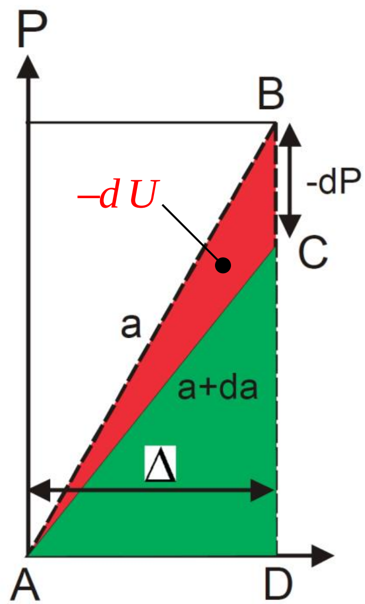
$$\Pi = U - F = -\frac{P\Delta}{2} = -U$$

$$G = -\frac{d\Pi}{dA} = \frac{1}{B}\left(\frac{dU}{da}\right)_P = \frac{P}{2B}\left(\frac{d\Delta}{da}\right)_P$$

نرخ آزادسازی انرژی

displacement controlled: when displacement is fixed





$$F = 0$$

$$\Pi = U - F = U$$

$$G = -\frac{d\Pi}{dA}$$

$$= -\frac{1}{B}\left(\frac{dU}{da}\right)_{\Delta} = -\frac{\Delta}{2B}\left(\frac{dP}{da}\right)_{\Delta}$$

Cracked plate at a fixed displacement Δ.

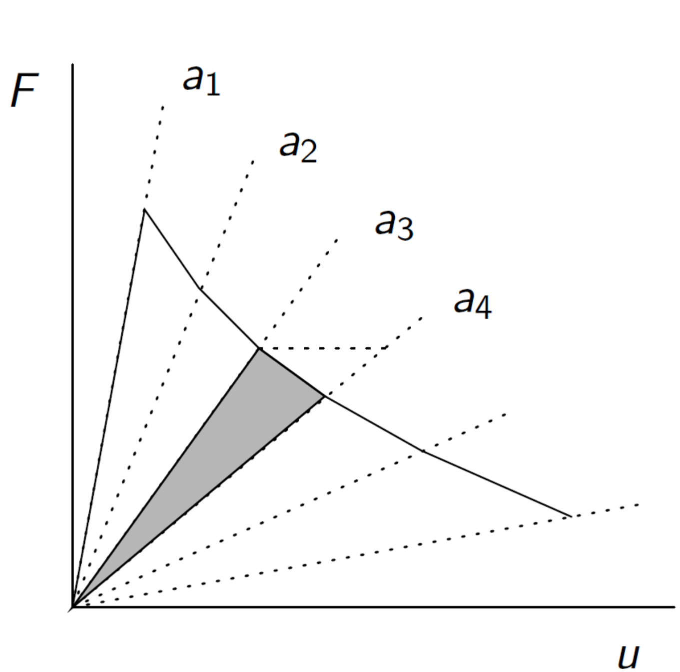
نرخ آزادسازی انرژی

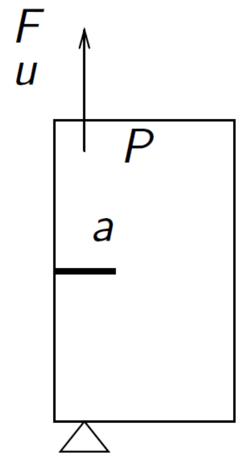
The compliance : the inverse of the plate stiffness $C \triangleq \dfrac{\Delta}{P}$

For both load control and displacement control: $G = \dfrac{P^2}{2B}\dfrac{dC}{da}$

$$\left(\frac{d\,\mathrm{U}}{da}\right)_P = -\left(\frac{d\,\mathrm{U}}{da}\right)_\Delta$$

In reality the loading of the plate may not be purely according to the extreme cases, as is shown in the figure. From such a real experiment the energy release rate can be determined from the force-displacement curve.

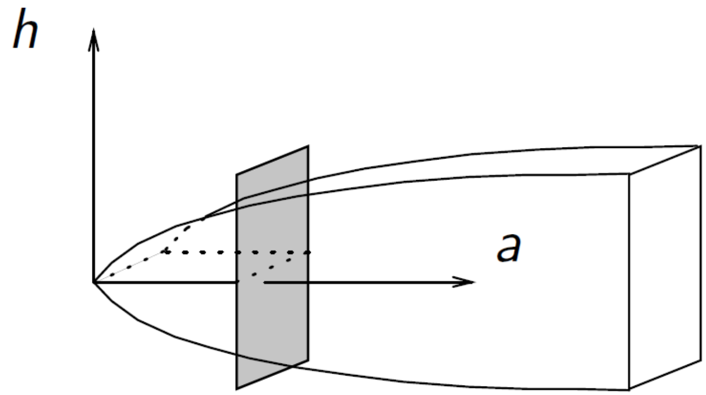




$$G = \frac{\text{shaded area}}{a_4 - a_3}\frac{1}{B}$$

Experimental force-displacement curve during crack growth.

نرخ آزادسازی انرژی



- **مثال**: در تیر زیر، تابع ارتفاع تیر $h(a)$ به نحوی تعیین کنید که $\frac{dC}{da}$ مستقل از a باشد.

$$C = \frac{\Delta u}{F} = \frac{2u}{F} = \frac{8a^3}{EBh^3} \quad \rightarrow \quad \frac{dC}{da} = \frac{24a^2}{EBh^3}$$

$$\text{choice} \quad : \quad h = h_0 a^n \quad \rightarrow$$

$$u = \frac{Fa^3}{3(1-n)EI} = \frac{4Fa^3}{(1-n)EBh^3} = \frac{4Fa^{3(1-n)}}{(1-n)EBh_0^3}$$

$$C = \frac{2u}{F} = \frac{8a^{3(1-n)}}{(1-n)EBh_0^3} \quad \rightarrow \quad \frac{dC}{da} = \frac{24a^{(2-3n)}}{EBh_0^3}$$

$$\frac{dC}{da} \quad \text{constant for} \quad n = \frac{2}{3} \quad \rightarrow \quad h = h_0 a^{\frac{2}{3}}$$

➢ ناپایداری و منحنی *R* :   *Instability and the R curve*

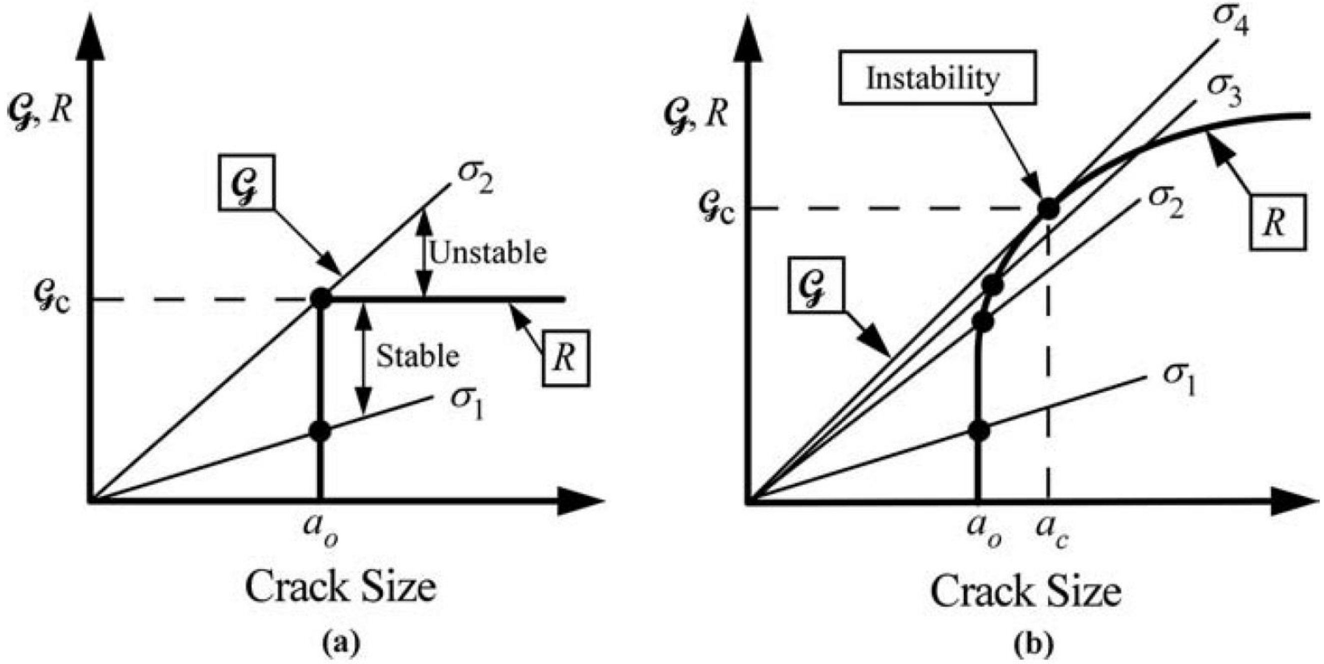
R : the material resistance to crack extension

Crack extension occurs when $G = 2w_f$; but crack growth may be stable or unstable, depending on how G and w_f vary with crack size.

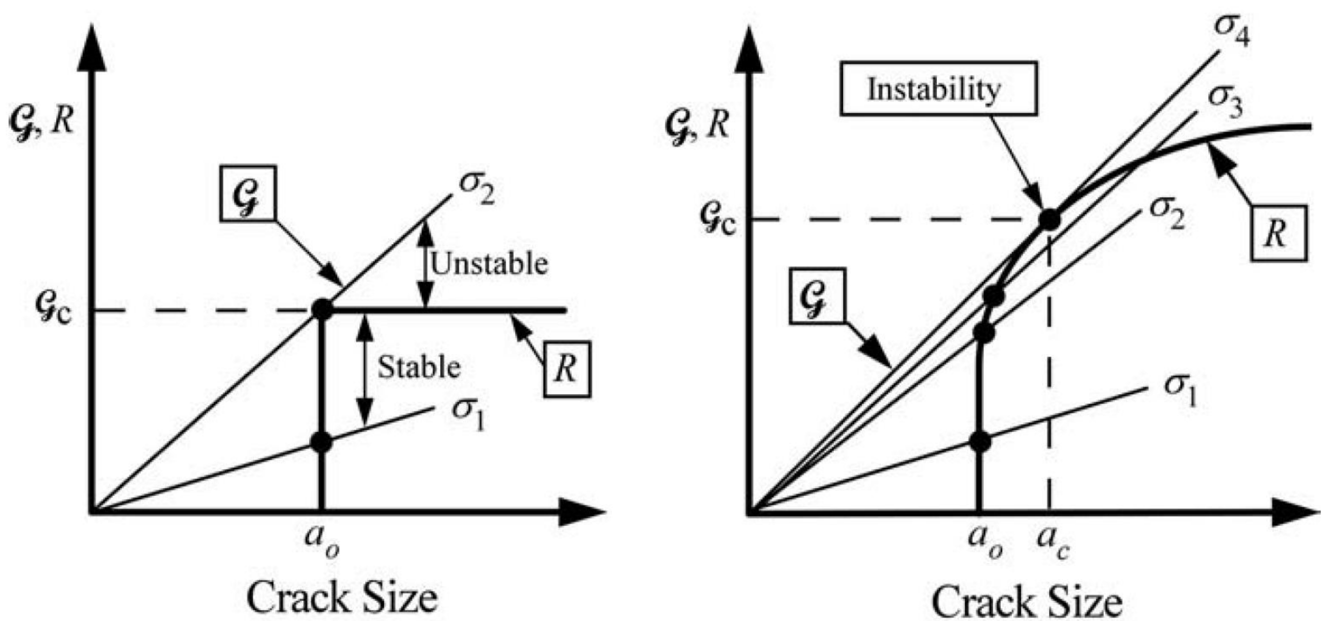
A plot of R vs. crack extension is called a *resistance curve* or R curve.

The corresponding plot of G vs. crack extension is the *driving force curve*.

ناپایداری و منحنی R



Schematic driving force vs. *R* curve diagrams (a) flat *R* curve and (b) rising *R* curve.



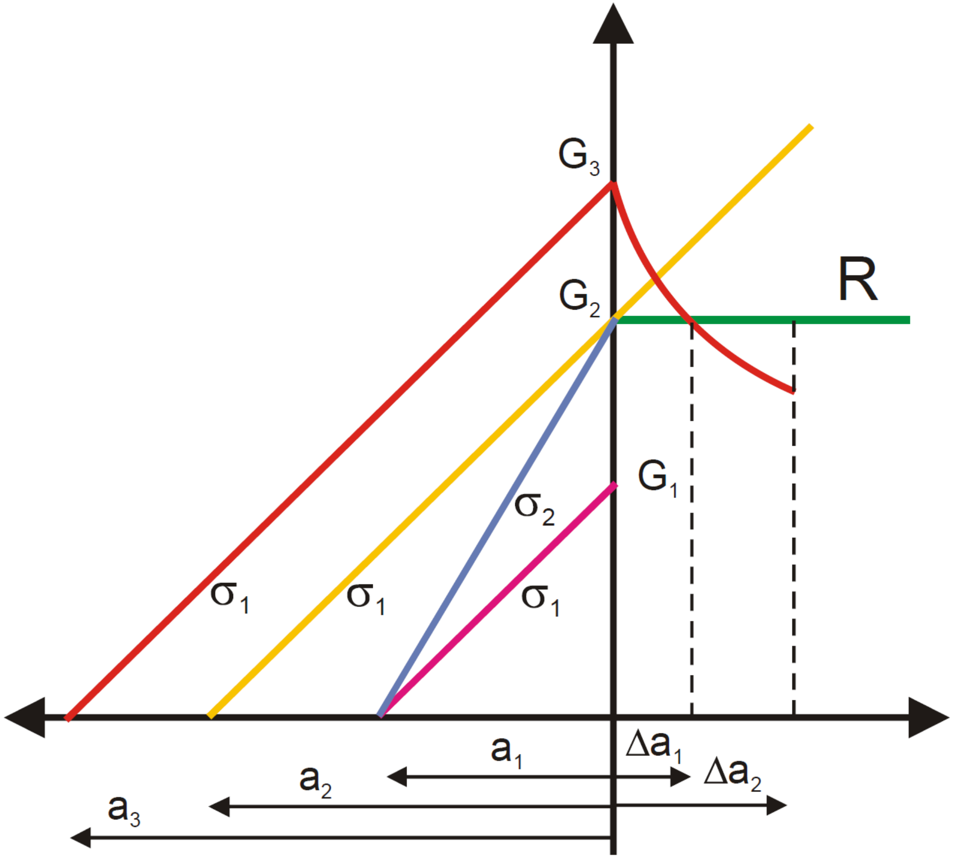
(a) (b)

The conditions for stable crack growth can be expressed as follows:

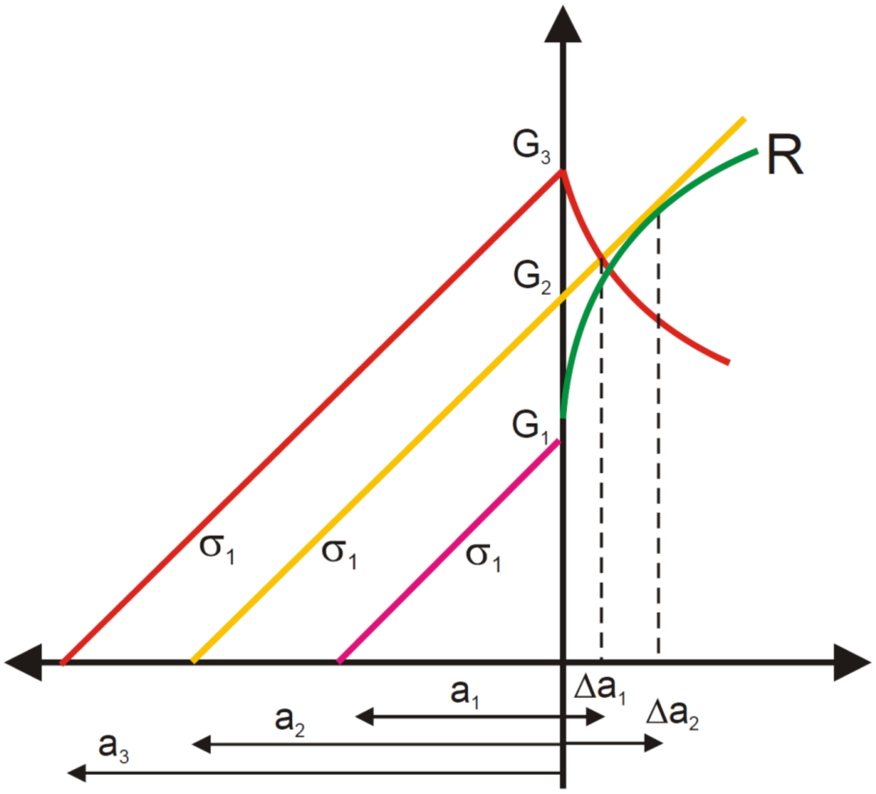
$$G = R \quad \text{and} \quad \frac{dG}{da} \leq \frac{dR}{da}$$

Unstable crack growth occurs when $\frac{dG}{da} > \frac{dR}{da}$

The energy approach can also be used for the analysis of different aspects of further crack growth such as *instability*, *dynamic crack growth*, and *crack arrest*.



Unstable crack growth occurs when $G > R$ and $\frac{dG}{da} > \frac{dR}{da}$



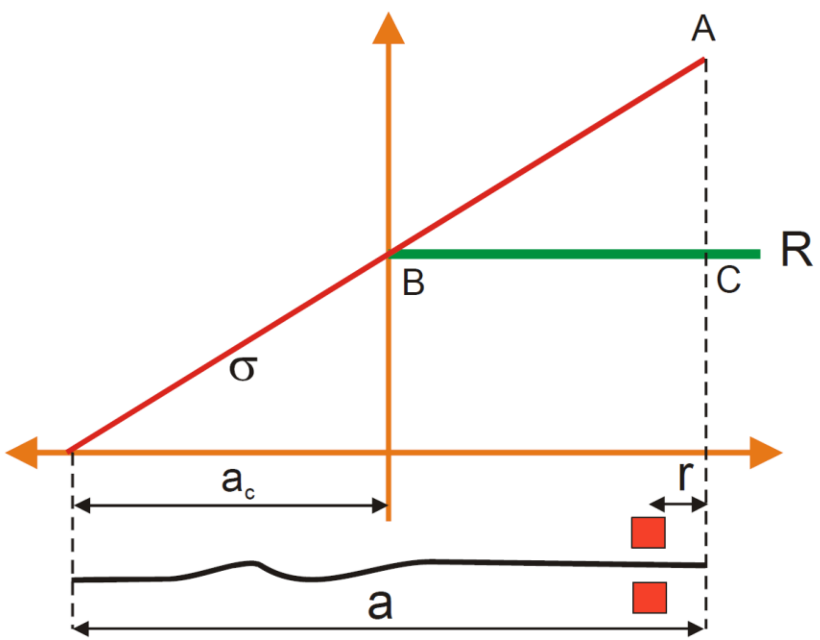
Crack Speed

an infinite sheet with a central crack of length *2a* under remote tensile stress σ , the horizontal and vertical displacements of these elements can be written as:

$$\begin{cases} u = \dfrac{2\sigma}{E}\sqrt{ar}\, f_u(\theta) \\ v = \dfrac{2\sigma}{E}\sqrt{ar}\, f_v(\theta) \end{cases}$$

$$r \propto a :$$

$$\begin{cases} u = C_1 \dfrac{\sigma a}{E} & \Rightarrow \dot{u} = C_1 \dfrac{\sigma \dot{a}}{E} \\ v = C_2 \dfrac{\sigma a}{E} & \Rightarrow \dot{v} = C_1 \dfrac{\sigma \dot{a}}{E} \end{cases}$$



Crack Speed

The kinetic energy for the plate can be defined and calculated as follows:

$$E_K = \frac{1}{2}\rho \iint \left(\dot{u}^2 + \dot{v}^2\right) dxdy$$

$$= \frac{1}{2}\rho \dot{a}^2 \frac{\sigma^2}{E^2} \iint \left(C_1^2 + C_2^2\right) dxdy$$

$$= \frac{1}{2} k \rho \dot{a}^2 a^2 \frac{\sigma^2}{E^2}$$

The kinetic energy for two crack tips, can be defined and calculated as follows:

$$E_S = 2\int_{a_c}^{a} (G - R)\, da$$

$$= -2R(a - a_c) + 2\int_{a_c}^{a} \frac{\pi\sigma^2 a}{E} da$$

$$= \frac{\pi\sigma^2}{E}(a - a_c)^2$$

The constant R is equal to G_{Ic} at the onset of instability($G_{IC} = \frac{\pi\sigma^2 a_c}{E}$).

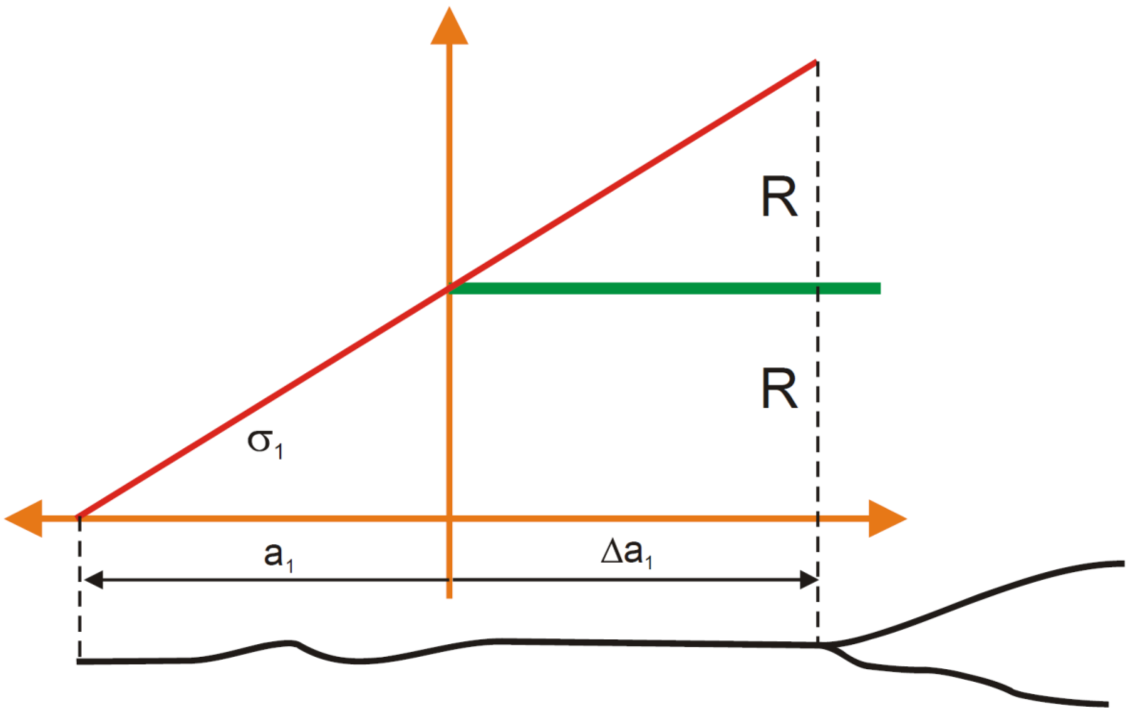
Crack Speed

Equating the two energies we may find the crack growth rate as:

$$\dot{a} = \sqrt{\frac{2\pi}{k}}\sqrt{\frac{E}{\rho}}\left(1-\frac{a_c}{a}\right)$$

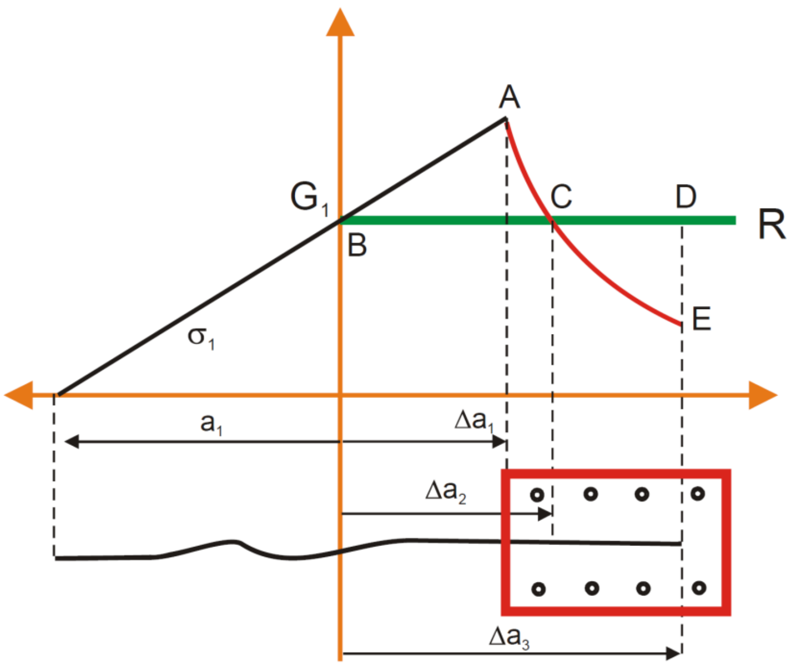
Crack Branching

under constant load where the energy release rate increases with further crack growth, there might be a point where the available energy becomes twice the energy required to grow a single crack.



Crack Arrest

One practical remedy is to use riveted patches or other types of stiffeners to simulate a fixed-displacement condition and arrest the crack. The location of the arrester must be chosen properly by considering the kinetic energy of the crack.The patch may decrease the energy down to the point C where the crack is expected to stop after a growth equal to Δa_2.



The area CDE equal to ABC

مبانی ریاضی مکانیک شکست الاستیک خطی

Mathematical Foundations of Linear Elastic Fracture Mechanics

The equations that follow are simplifications of more general relationships in elasticity and are subject to the following restrictions:

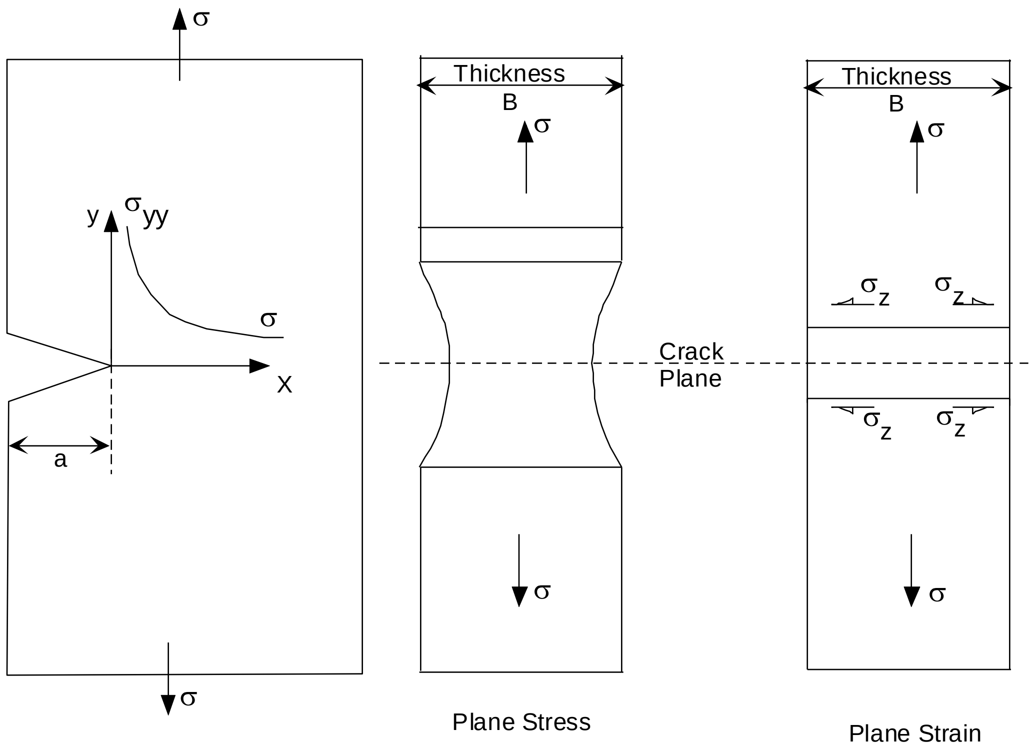
- Two-dimensional stress state (plane stress or plane strain)
- Isotropic material
- Quasistatic, isothermal deformation
- Absence of body forces from the problem

تنش صفحه‌ای: *plane stress*

➢ هنگامی که ضخامت جسم در برابر اندازه ناحیه پلاستیک قابل مقایسه است و یک تغییرشکل آزاد (انقباض) در سطوح جانبی جسم رخ می دهد.

کرنش صفحه‌ای: *plane strain*

➢ هنگامی که ضخامت جسم به اندازه کافی ضخیم است تا مانع تغییرشکل (انقباض) در ضخامت جسم (امتداد z) شود.



Cartesian Coordinates:

Strain-displacement relationships:

$$\varepsilon_{xx}=\frac{\partial u_x}{\partial x},\qquad \varepsilon_{yy}=\frac{\partial u_y}{\partial y},\qquad \varepsilon_{xy}=\frac{1}{2}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)$$

Stress-strain relationships:

1. Plane strain

$$\sigma_{xx}=\frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_{xx}+\nu\varepsilon_{yy}]$$

$$\sigma_{yy}=\frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_{yy}+\nu\varepsilon_{xx}]$$

$$\tau_{xy}=2\mu\varepsilon_{xy}=\frac{E}{1+\nu}\varepsilon_{xy}$$

$$\sigma_{zz}=\nu(\sigma_{xx}+\sigma_{yy})$$

$$\varepsilon_{zz}=\varepsilon_{xz}=\varepsilon_{yz}=\tau_{xz}=\tau_{yz}=0$$

Cartesian Coordinates:

Stress-strain relationships:

2. Plane stress

$$\sigma_{xx} = \frac{E}{1-\nu^2}[\varepsilon_{xx} + \nu\varepsilon_{yy}]$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}[\varepsilon_{yy} + \nu\varepsilon_{xx}]$$

$$\tau_{xy} = 2\mu\varepsilon_{xy} = \frac{E}{1+\nu}\varepsilon_{xy}$$

$$\varepsilon_{zz} = \frac{-\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy})$$

$$\sigma_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = \tau_{xz} = \tau_{yz} = 0$$

Cartesian Coordinates:

Equilibrium equations:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0$$

Compatibility equation:

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = 0 \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Cartesian Coordinates:

Airy stress function: For a two-dimensional continuous elastic medium, there exists a function Φ(x, y) from which the stresses can be derived:

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2}$$

$$\sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2}$$

$$\tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}$$

The equilibrium and compatibility equations are automatically satisfied if Φ has the following property:

$$\frac{\partial^4 \Phi}{\partial x^4} + 2\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = 0$$

$$\nabla^2 \nabla^2 \Phi = 0$$

Polar Coordinates:

Strain-displacement relationships:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

Stress-strain relationships:

1. Plane strain

$$\sigma_{rr} = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}]$$

Polar Coordinates:

Stress-strain relationships:

2. Plane stress

$$\sigma_{rr} = \frac{E}{1-\nu^2}[\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}]$$

Equilibrium equations:

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

$$\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} = 0$$

Polar Coordinates:

Compatibility equation:

$$\nabla^2(\sigma_{rr}+\sigma_{\theta\theta})=0$$

where:

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

Airy stress function(where Φ = Φ(r, θ)):

$$\nabla^2\nabla^2\Phi=0$$

Airy stress function:

$$\sigma_{rr}=\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}$$

$$\sigma_{\theta\theta}=\frac{\partial^2\Phi}{\partial r^2}$$

$$\tau_{r\theta}=-\frac{1}{r}\frac{\partial^2\Phi}{\partial r\partial\theta}+\frac{1}{r^2}\frac{\partial\Phi}{\partial\theta}$$