بسم الله الرحمن الرحیم

دانشگاه صنعتی اصفهان

دانشکده مکانیک

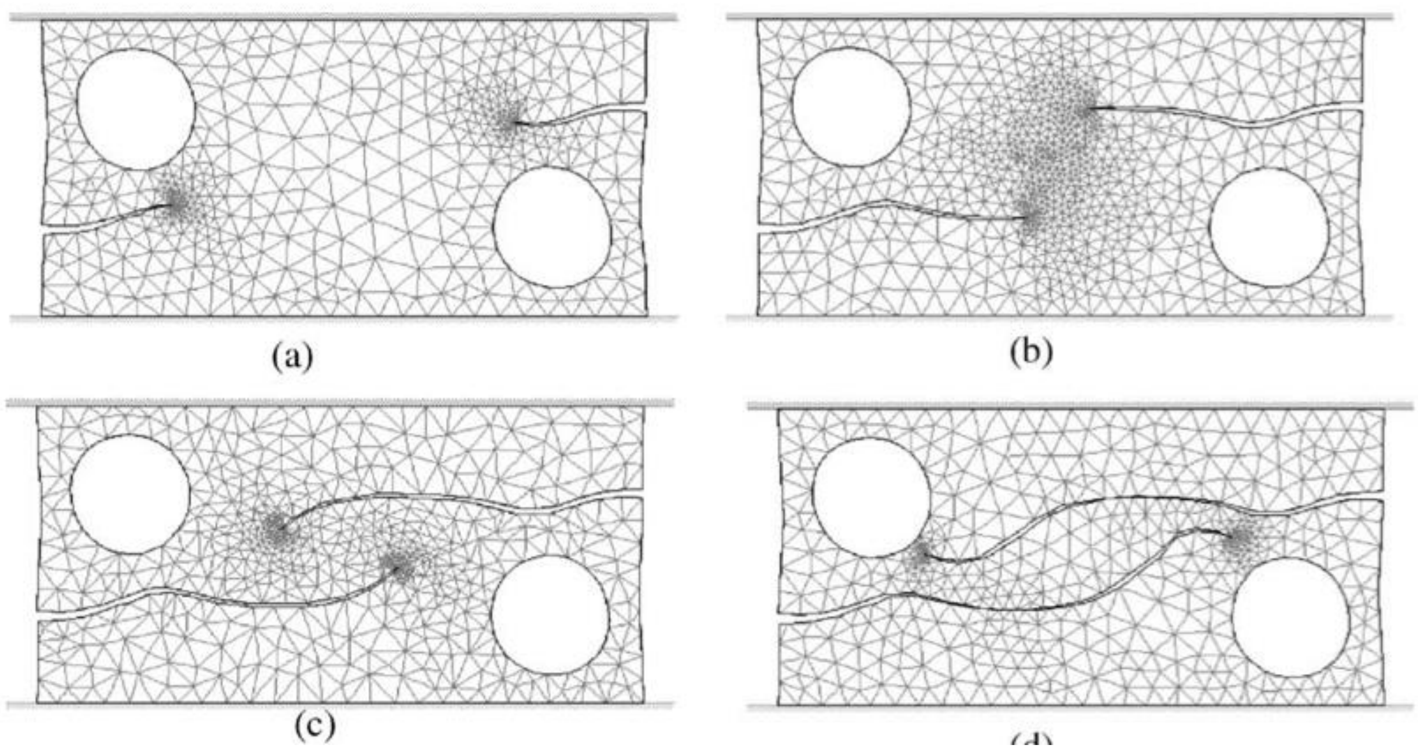
Computational Fracture Mechanics (4)

Computational fracture mechanics

- Introduction to Finite Element method
- Singular Stress Finite Elements
- Extraction of K (SIF), G
- J integral
- Finite Element mesh design for fracture mechanics
- Computational crack growth
- Traction Separation Relations

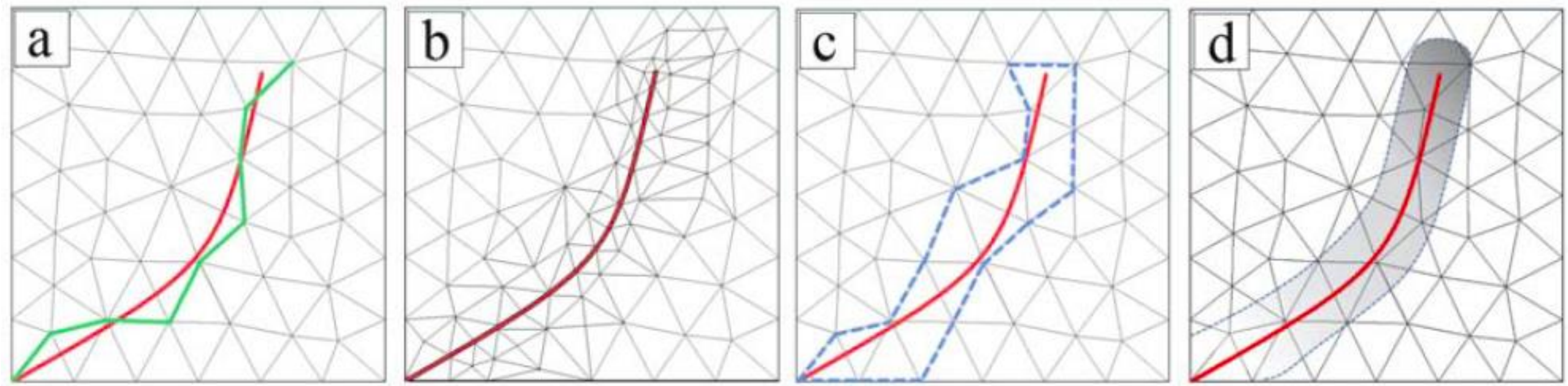
What's wrong with FEM for crack problems

- Element edges must conform to the crack geometry: make such a mesh is time-consuming, especially for 3D problems.
- Remeshing as crack advances: difficult. Example:



Mesh design for crack problems

Capturing/tracking cracks



Fixed mesh

Crack tracking

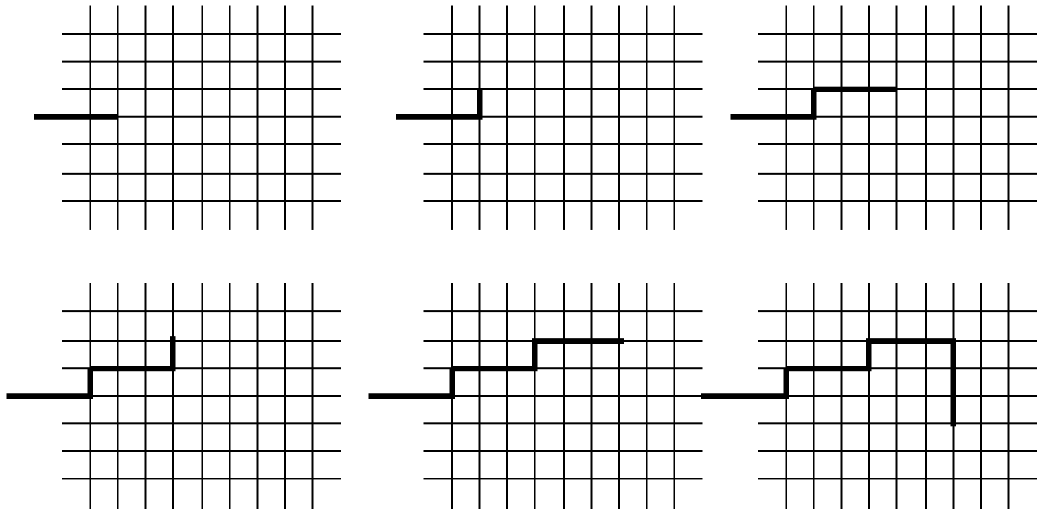
XFEM enriched elements

Crack/void capturing by bulk damage models

Mesh design for crack problems

Fixed meshes

- Nodal release method (typically done on fixed meshes)
  - Crack advances one element edge at a time by releasing FEM nodes
  - Crack path is restricted by discrete geometry

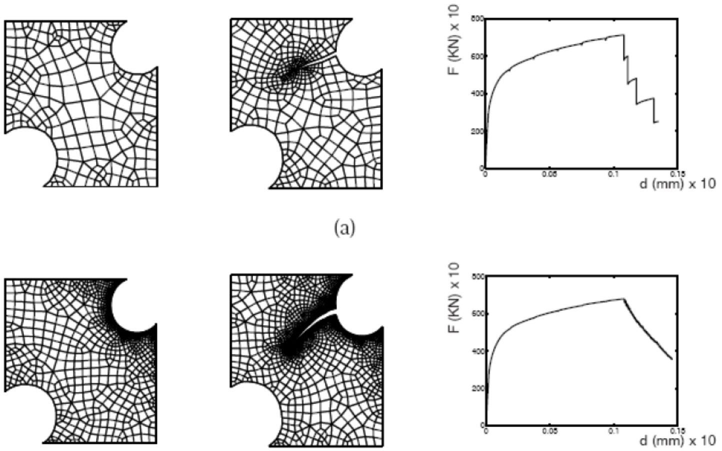


- Also for cohesive elements they can be used for both extrinsic and intrinsic schemes. For intrinsic ones, cohesive surfaces between all elements induces an artificial compliance (will be explained later)

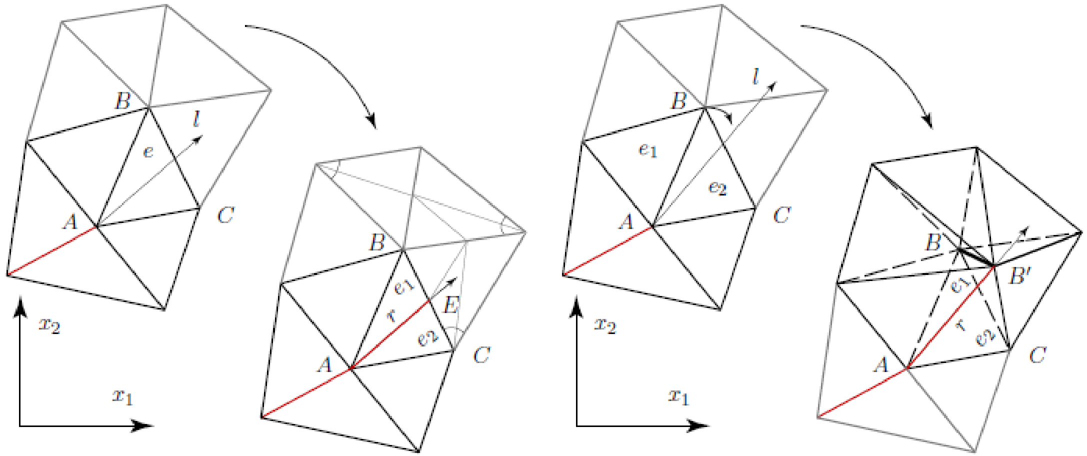
Mesh design for crack problems

Adaptive meshes

- Adaptive operations align element boundaries with crack direction



Element splitting:
Smoother crack path by element splitting: cracks split through and propagate between newly generated elements

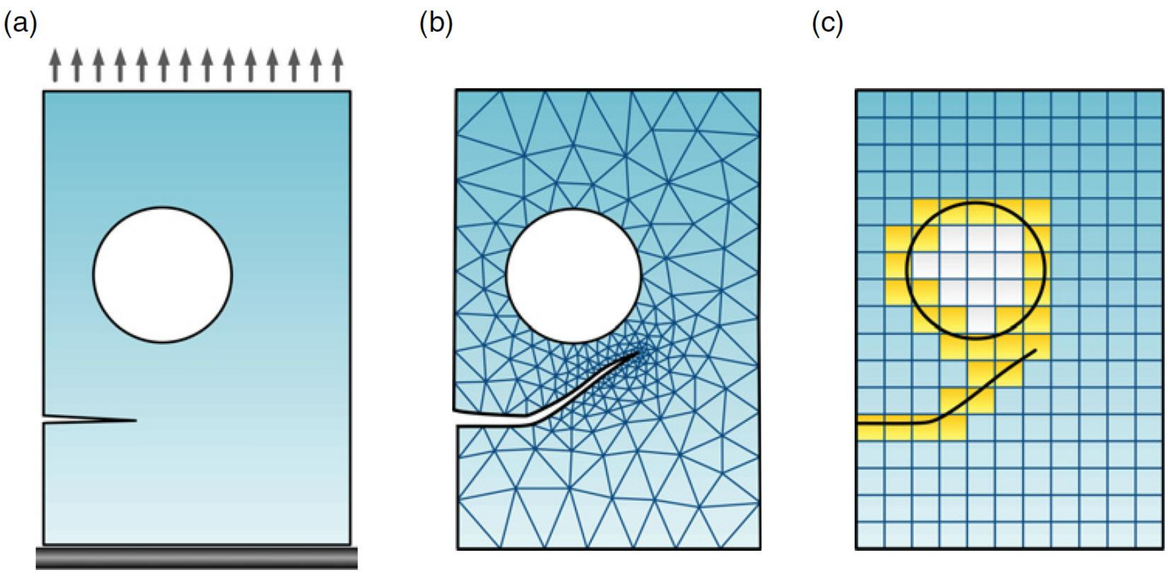


Cracks generated by refinement options

Element edges move to desired direction

Standard-FEM and Enriched-FEM:

Modeling of weak and strong discontinuities in the standard-FEM and enriched-FEM techniques:



(a) Crack propagation in a plate with a hole: (b) The standard-FEM using an adaptive mesh refinement in which the mesh conforms to the geometry of interfaces; (c) The enriched-FEM technique using a uniform mesh in which the elements cut by the interfaces are enriched.

Extended Finite Element Method (XFEM)

Belytschko and Black et al 1999

$$\mathbf{u}^h(\mathbf{x}) = \sum_{I \in \mathcal{S}} N_I(\mathbf{x})\mathbf{u}_I + \sum_{J \in \mathcal{S}^c} N_J(\mathbf{x})\Phi(\mathbf{x})\mathbf{a}_J$$

standard part enrichment part

S^C :set of enriched nodes

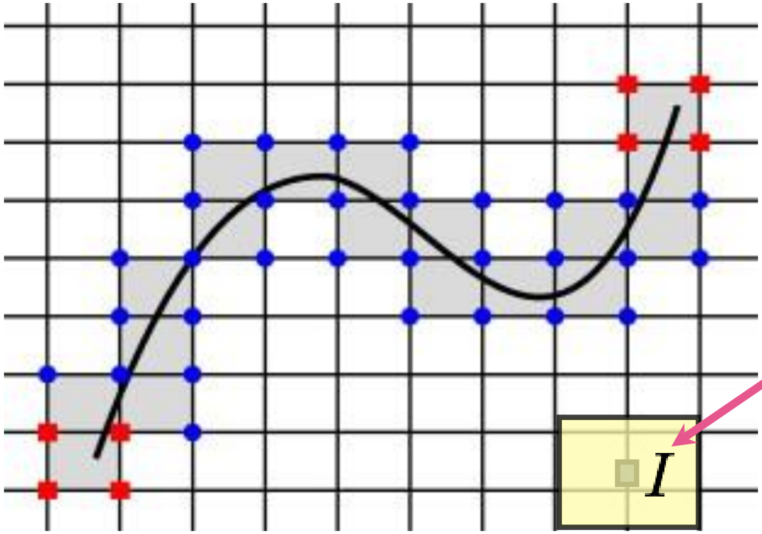
Partition of Unity (PUM):

enrichment function

$$\sum_J N_J(\mathbf{x}) = 1 \longrightarrow \sum_J N_J(\mathbf{x})\Phi(\mathbf{x}) = \Phi(\mathbf{x})$$

$\Phi(\mathbf{x})$ known characteristics of the problem (crack tip singularity, displacement jump etc.) into the approximate space.

Extended Finite Element Method (XFEM)



nodal support

$$N_I(\mathbf{x}) \neq 0$$

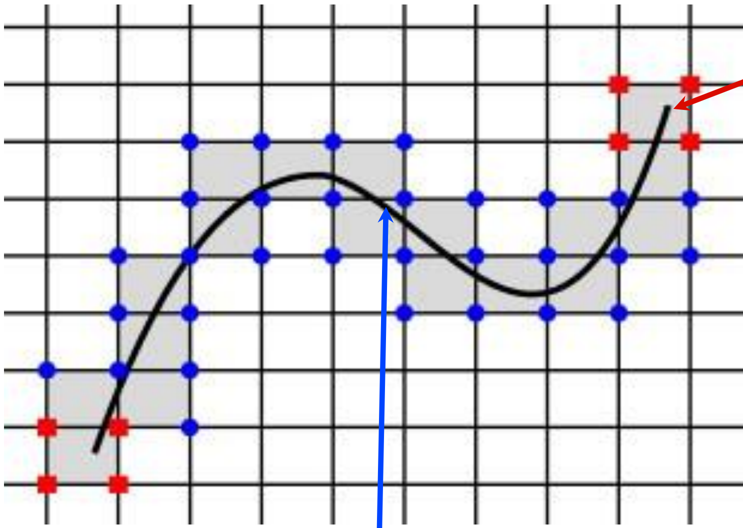
$$\sum_J N_J(\mathbf{x})\Phi(\mathbf{x}) = \Phi(\mathbf{x})$$

enriched nodes = nodes whose support is cut by the item to be enriched

enriched node I: standard degrees of freedoms (dofs) and additional dofs

XFEM for LEFM

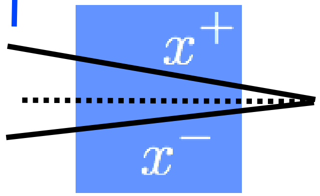
crack tip with known displacement



$$u = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left(\kappa - 1 + 2\sin^2\frac{\theta}{2}\right)$$
$$v = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left(\kappa + 1 - 2\cos^2\frac{\theta}{2}\right)$$

$$\Phi_1 = f(\sqrt{r}, \theta)$$

crack edge



displacement: discontinuous across crack edge

$$\Phi_2 : \Phi_2(x^+) \neq \Phi_2(x^-)$$

XFEM for LEFM

$$u = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left(\kappa - 1 + 2\sin^2\frac{\theta}{2}\right)$$
$$v = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left(\kappa + 1 - 2\cos^2\frac{\theta}{2}\right)$$

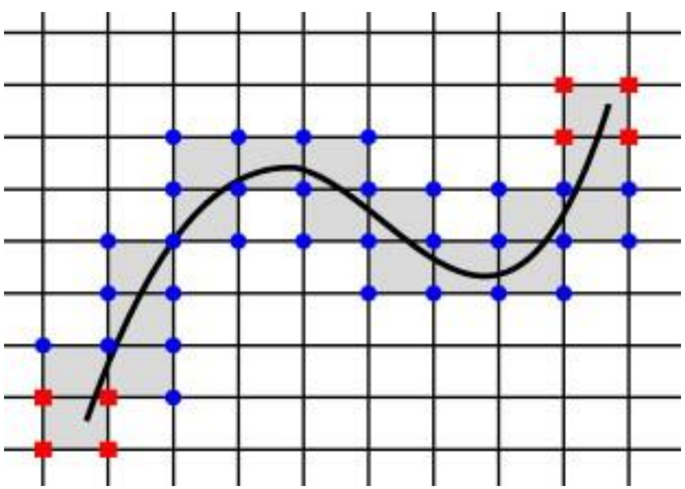
Crack tip enrichment functions:

$$[B_\alpha] = \left[\sqrt{r}\sin\frac{\theta}{2}, \sqrt{r}\cos\frac{\theta}{2}, \sqrt{r}\sin\frac{\theta}{2}\sin\theta, \sqrt{r}\cos\frac{\theta}{2}\sin\theta\right]$$

Crack edge enrichment functions:

$$H(\mathbf{x}) = \begin{cases} +1 & \text{if } (\mathbf{x} - \mathbf{x}^*)\cdot\mathbf{n} \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

$\mathcal{S}^c$	blue nodes
$\mathcal{S}^t$	red nodes

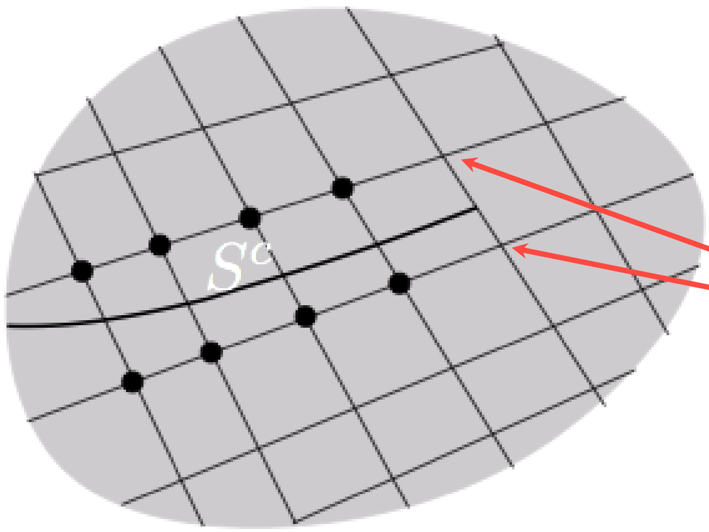


$$\mathbf{u}^h(\mathbf{x}) = \sum_{I\in\mathcal{S}} N_I(\mathbf{x})\mathbf{u}_I + \sum_{J\in\mathcal{S}^c} N_J(\mathbf{x})H(\mathbf{x})\mathbf{a}_J + \sum_{K\in\mathcal{S}^t} N_K(\mathbf{x})\left(\sum_{\alpha=1}^{4} B_\alpha \mathbf{b}_K^\alpha\right)$$

XFEM for LEFM

Wells, Sluys, 2001

$$\mathbf{u}^h(\mathbf{x}) = \sum_{I \in \mathcal{S}} N_I(\mathbf{x})\mathbf{u}_I + \sum_{J \in \mathcal{S}^c} N_J(\mathbf{x}) H(\mathbf{x}) \mathbf{a}_J$$

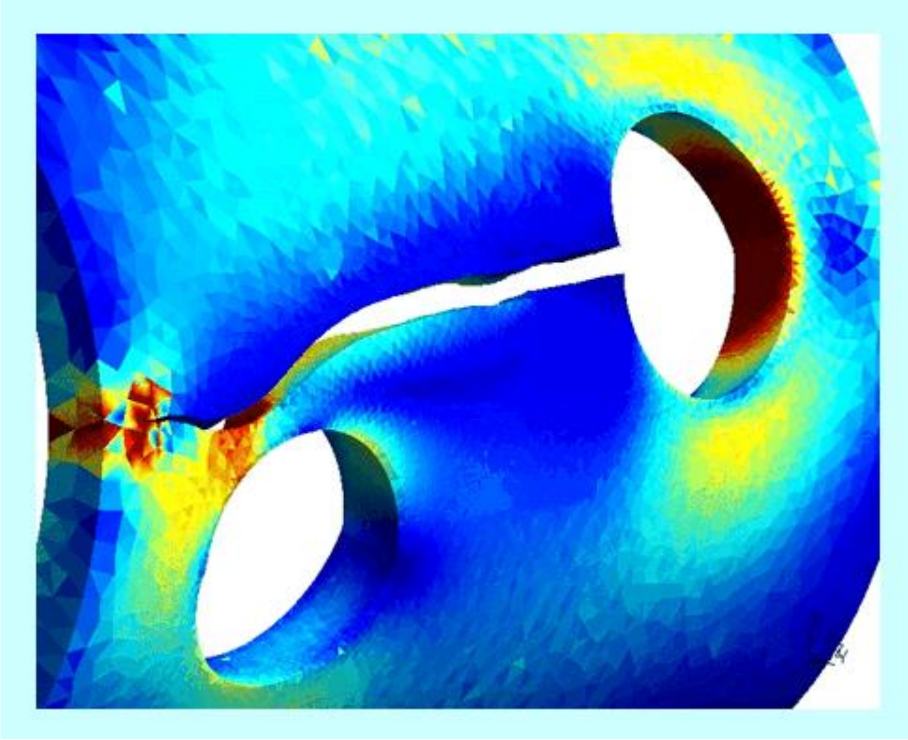


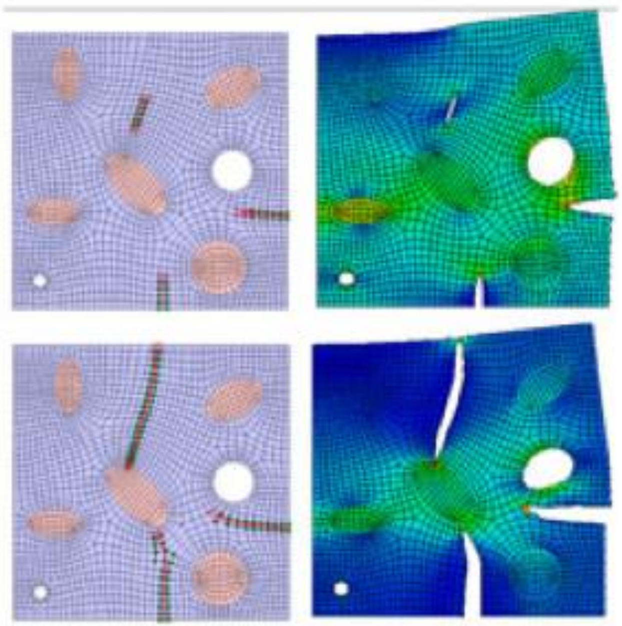
No crack tip solution is known, no tip enrichment!!!

not enriched to ensure zero crack tip opening!!!

$$H(\mathbf{x}) = \begin{cases} +1 & \text{if } (\mathbf{x} - \mathbf{x}^*) \cdot \mathbf{n} \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

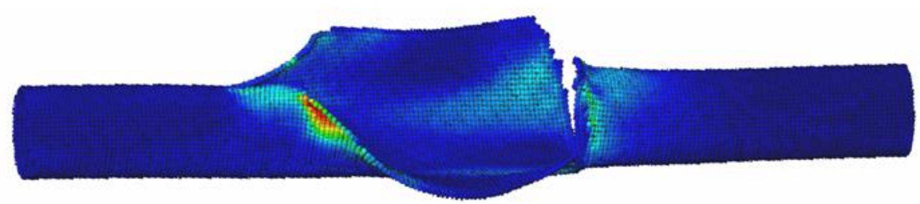
XFEM: examples





CENAERO, M. Duflot

Northwestern Univ.



A various X-FEM enrichment functions for different classes of solid mechanics

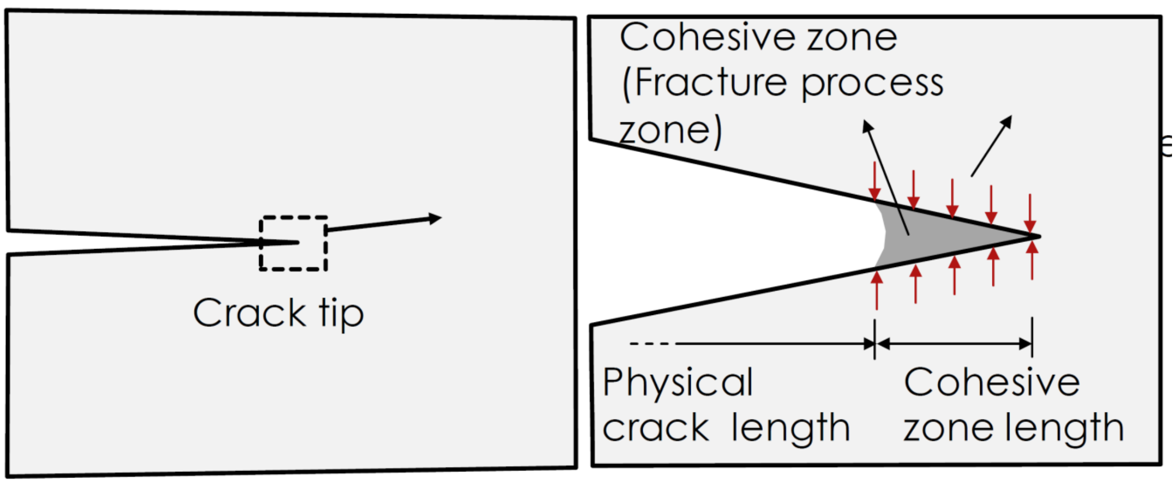
Kind of problem	Field variable (displacement)	Gradient of field variable (strain)	X-FEM enrichment functions
Bimaterial interfaces, voids, inclusions, grain boundaries	Continuous	Discontinuous	$\psi_{\text{ramp}}(\varphi(\boldsymbol{x})) = \lvert\varphi(\boldsymbol{x})\rvert$ $\psi_{\text{ridge}}(\varphi(\boldsymbol{x})) = \sum_{I\in\mathcal{N}} N_I(\boldsymbol{x})\,\lvert\varphi_I\rvert - \left\lvert\sum_{I\in\mathcal{N}} N_I(\boldsymbol{x})\,\varphi_I\right\rvert$
Strong discontinuity, crack interfaces	Discontinuous	—	$\psi_{\text{sign}}(\varphi(\boldsymbol{x})) = \text{sign}(\varphi(\boldsymbol{x}))$ $\psi_{\text{step}}(\varphi(\boldsymbol{x})) = H(\varphi(\boldsymbol{x}))$
Brittle crack tip (isotropic material)	Discontinuous	High gradient	$\psi_{\text{tip}}^{\text{brittle}}(r,\theta) = \left\{\sqrt{r}\sin\frac{\theta}{2},\ \sqrt{r}\cos\frac{\theta}{2},\ \sqrt{r}\sin\frac{\theta}{2}\sin\theta,\ \sqrt{r}\cos\frac{\theta}{2}\sin\theta\right\}$
Brittle crack tip (orthotropic material)	Discontinuous	High gradient	$\psi_{\text{tip}}^{\text{orthotropic}}(r,\theta) = \left\{\sqrt{r}\sin\frac{\theta_1}{2}\sqrt{g_1(\theta)},\ \sqrt{r}\cos\frac{\theta_1}{2}\sqrt{g_1(\theta)},\ \sqrt{r}\sin\frac{\theta_2}{2}\sqrt{g_2(\theta)},\ \sqrt{r}\cos\frac{\theta_2}{2}\sqrt{g_2(\theta)}\right\}$
Cohesive crack tip	Discontinuous	High gradient	$\psi_{\text{tip}}^{\text{cohesive}}(r,\theta) = r^k \sin\frac{\theta}{2} \quad (k=1\,,\,1.5\,,\,2)$
Plastic crack tip	Discontinuous	High gradient	$\psi_{\text{tip}}^{\text{plastic}}(r,\theta) = r^{\frac{1}{n+1}}\left\{\sin\frac{\theta}{2},\ \cos\frac{\theta}{2},\ \sin\frac{\theta}{2}\sin\theta,\ \cos\frac{\theta}{2}\sin\theta,\ \sin\frac{\theta}{2}\sin 3\theta,\ \cos\frac{\theta}{2}\sin 3\theta\right\}$
Multiple cracks (discontinuous *junction* function)	Discontinuous	High gradient	$\psi_{\text{junction}}^{\mathbb{H}}(\varphi(\boldsymbol{x})) = J^{\mathsf{H}}(\boldsymbol{x})$
Crack tip perpendicular to bimaterial interface	Discontinuous	High gradient	$\psi_{\text{tip}}^{\mathbb{N}\text{-bimaterial}}(r,\theta) = \left\{\psi_{\text{tip}}^1, \psi_{\text{tip}}^2, \psi_{\text{tip}}^3, \psi_{\text{tip}}^4\right\}$ $= \left\{r^\lambda\cos(\lambda+1)\theta\,,\ r^\lambda\sin(\lambda+1)\theta\,,\ r^\lambda\cos(\lambda-1)\theta\,,\ r^\lambda\sin(\lambda-1)\theta\right\}$
Crack tip terminating at a bimaterial interface	Discontinuous	High gradient	$\psi_{\text{tip}}^{\mathbb{O}\text{-bimaterial}}(r,\theta) = \left\{\psi_{\text{tip}}^1, \psi_{\text{tip}}^2, \ldots, \psi_{\text{tip}}^8\right\}$ $= \{r^{\lambda_1}\cos(\lambda_1+1)\theta\,,\ r^{\lambda_1}\sin(\lambda_1+1)\theta\,,\ r^{\lambda_1}\cos(\lambda_1-1)\theta\,,\ r^{\lambda_1}\sin(\lambda_1-1)\theta\,,$ $r^{\lambda_2}\cos(\lambda_2+1)\theta\,,\ r^{\lambda_2}\sin(\lambda_2+1)\theta\,,\ r^{\lambda_2}\cos(\lambda_2-1)\theta\,,\ r^{\lambda_2}\sin(\lambda_2-1)\theta\}$

A various X-FEM enrichment functions for different classes of solid mechanics

Kind of problem	Field variable (displacement)	Gradient of field variable (strain)	X-FEM enrichment functions
Bimaterial interfacial crack	Discontinuous	High gradient	$\psi_{\text{tip}}^{\text{I-bimaterial}}(r,\theta)=\{\psi_{\text{tip}}^{1},\psi_{\text{tip}}^{2},\ldots,\psi_{\text{tip}}^{12}\}$ $=\{\sqrt{r}\cos(\varepsilon\log r)e^{-\varepsilon\theta}\sin\frac{\theta}{2},\ \sqrt{r}\cos(\varepsilon\log r)e^{-\varepsilon\theta}\cos\frac{\theta}{2},$ $\sqrt{r}\cos(\varepsilon\log r)e^{+\varepsilon\theta}\sin\frac{\theta}{2},\ \sqrt{r}\cos(\varepsilon\log r)e^{+\varepsilon\theta}\cos\frac{\theta}{2},$ $\sqrt{r}\cos(\varepsilon\log r)e^{+\varepsilon\theta}\sin\frac{\theta}{2}\sin\theta,\ \sqrt{r}\cos(\varepsilon\log r)e^{+\varepsilon\theta}\cos\frac{\theta}{2}\sin\theta,$ $\sqrt{r}\sin(\varepsilon\log r)e^{-\varepsilon\theta}\sin\frac{\theta}{2},\ \sqrt{r}\sin(\varepsilon\log r)e^{-\varepsilon\theta}\cos\frac{\theta}{2},$ $\sqrt{r}\sin(\varepsilon\log r)e^{+\varepsilon\theta}\sin\frac{\theta}{2},\ \sqrt{r}\sin(\varepsilon\log r)e^{+\varepsilon\theta}\cos\frac{\theta}{2},$ $\sqrt{r}\sin(\varepsilon\log r)e^{+\varepsilon\theta}\sin\frac{\theta}{2}\sin\theta,\ \sqrt{r}\sin(\varepsilon\log r)e^{+\varepsilon\theta}\cos\frac{\theta}{2}\sin\theta\}$
Grain junctions in polycrystalline structures	Discontinuous	High gradient	$\psi_{\text{tip}}^{\text{notch}}(r,\theta)=r^{\lambda}\,\Psi(\theta)$
Multiple interfaces (*junction ramp* function)	Continuous	Discontinuous	$\psi_{\text{junction}}^{\mathbb{R}}(\varphi(\boldsymbol{x}))=J^{\mathbb{R}}(\boldsymbol{x})=\lvert\varphi_j(\boldsymbol{x})\rvert\lvert\varphi_k(\boldsymbol{x})\rvert$
Dislocation (tangential jump function)	Discontinuous	—	$\boldsymbol{\psi}_{\text{step}}^{\alpha}(\boldsymbol{x})=\mathbf{b}^{\alpha}\sum_{J\in\mathcal{N}_{\text{step}}^{\alpha}}N_J(\boldsymbol{x})\,[H(\varphi^{\alpha}(\boldsymbol{x}))-H(\varphi^{\alpha}(\boldsymbol{x}_J))]\,H(\vartheta^{\alpha}(\boldsymbol{x}))$
Dislocation (edge function)	Discontinuous	High gradient	$\boldsymbol{\psi}_{\text{core}}^{\alpha-\text{edge}}(\boldsymbol{x})=\sum_{J\in\mathcal{N}_{\text{core}}^{\alpha}}N_J(\boldsymbol{x})\frac{\mathbf{b}^{\alpha}.\mathbf{e}_t}{2\pi}\left[\left(\tan^{-1}\left(\frac{y}{x}\right)+\frac{xy}{2(1-\nu)(x^2+y^2)}\right)\mathbf{e}_t-\left(\frac{1-2\nu}{4(1-\nu)}\ln(x^2+y^2)+\frac{x^2-y^2}{4(1-\nu)(x^2+y^2)}\right)\mathbf{e}_n\right]$
Dislocation (screw function)	Discontinuous	High gradient	$\boldsymbol{\psi}_{\text{core}}^{\alpha-\text{screw}}(\boldsymbol{x})=\sum_{J\in\mathcal{N}_{\text{core}}^{\alpha}}N_J(\boldsymbol{x})\left[\frac{\mathbf{b}^{\alpha}.(\mathbf{e}_t\times\mathbf{e}_n)}{2\pi}\tan^{-1}\left(\frac{y}{x}\right)(\mathbf{e}_t\times\mathbf{e}_n)\right]$
Shear band localization	Discontinuous	Discontinuous	$\psi_{\tanh}(\varphi(\boldsymbol{x}))=\tanh(\ell\cdot\varphi(\boldsymbol{x}))$ $\psi_{\exp}(\varphi(\boldsymbol{x}))=\text{sign}(\varphi(\boldsymbol{x}))(1-\exp(-\ell\cdot\lvert\varphi(\boldsymbol{x})\rvert))$

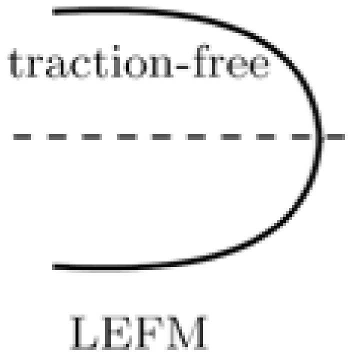
Cohesive models

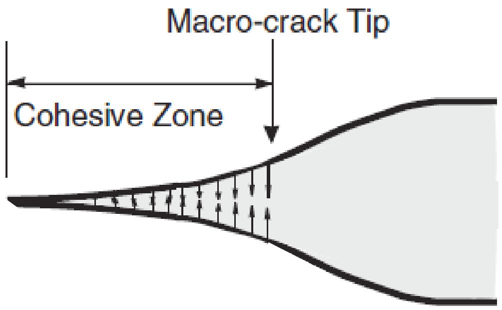
- Cohesive zone model has been widely used to solve crack propagation problems Explicit representation of cracks giving clear physical picture.
- Fracture formation is regarded as a gradual phenomenon in which the separation of the surfaces involving in the crack takes place across an extended crack tip, or cohesive zone, and is resisted by cohesive tractions (Dugdale 1960, Barenblatt 1962)



Cohesive models

- Cohesive models remove stress singularity predicted by Linear Elastic Fracture Mechanics (LEFM)





$$\sigma \propto \frac{1}{\sqrt{r}}$$

$$\sigma = \tilde{\sigma} f\ (\Delta u / \tilde{\delta})$$

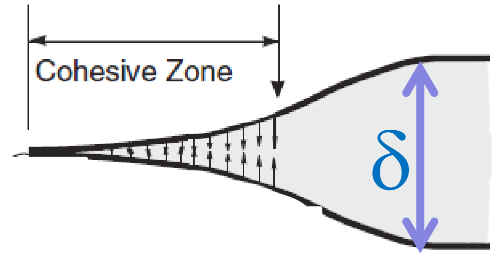
$\tilde{\sigma}$: Stress scale

$\tilde{\delta}$: Displacement scale

Traction is related to displacement jump across fracture surface

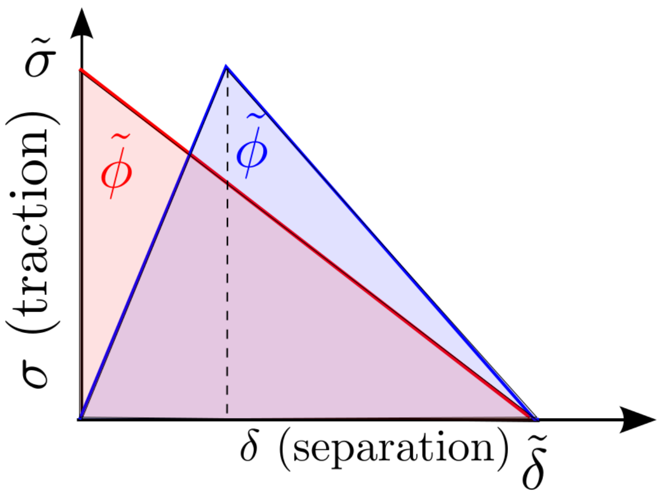
Cohesive models

- **Traction Separation Relation (TSR):** Relation between traction (stress) and displacement jump $\sigma = \tilde{\sigma} f\ (\delta / \tilde{\delta})$



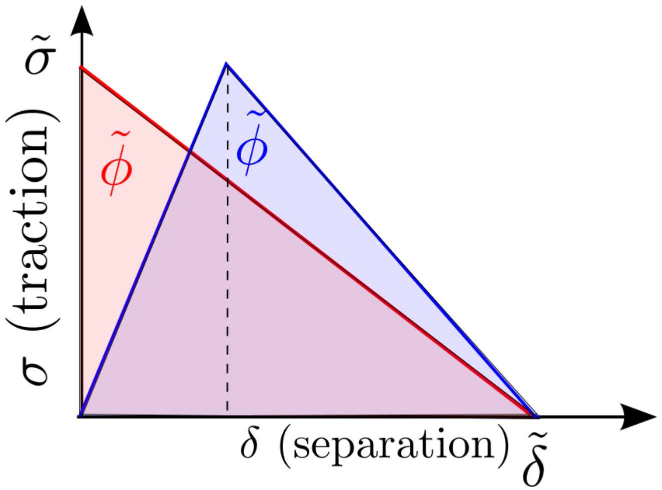
- **Parameters of a cohesive model** (Only 2 out of 3 are needed)

- Stress (traction) scale $\tilde{\sigma}$: Maximum traction in TSR
- Displacement scale $\tilde{\delta}$
- Work of Separation $\tilde{\phi}$:Area under σ–δ curve is the work needed to complete debond a unit surface area. This can he associated with G_c in LEFM theory.



Cohesive models

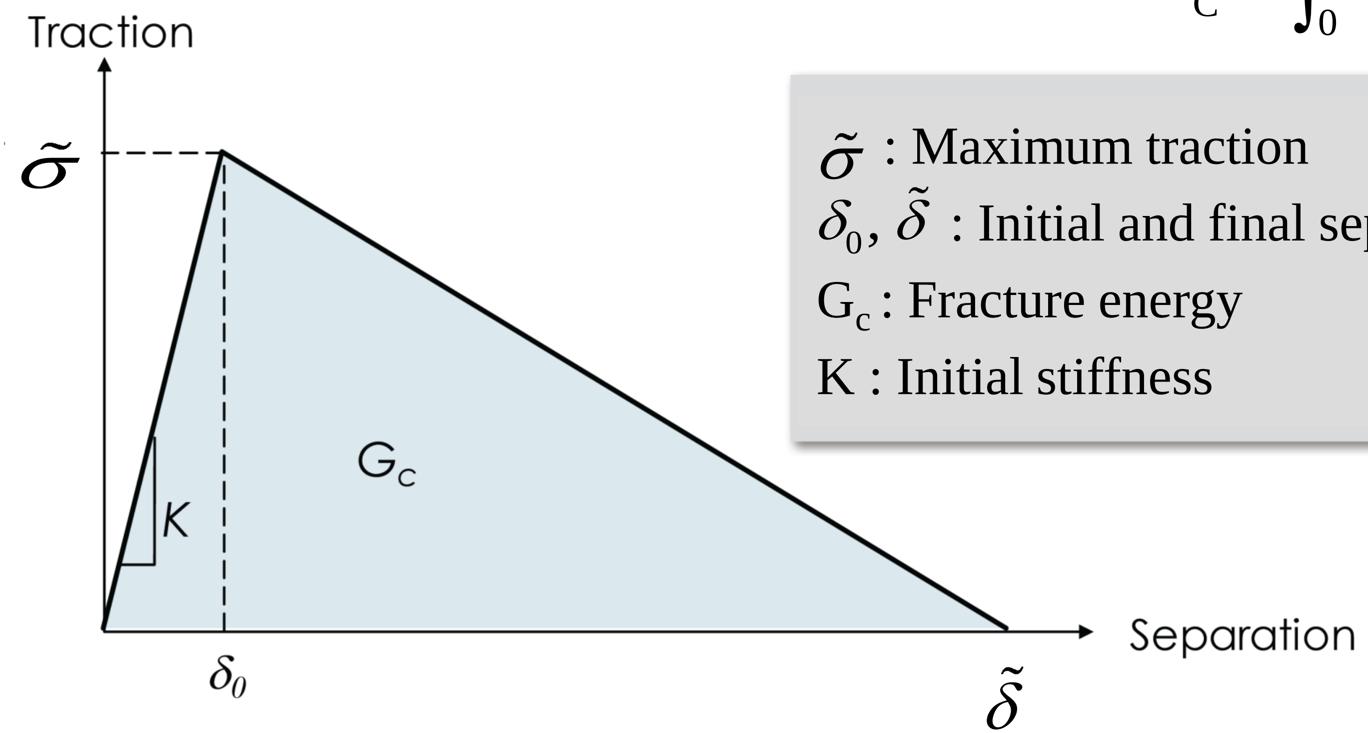
- Types of Cohesive models
  - Intrinsic cohesive models:
    - * It has an initial hardening σ–δ part
    - * σ starts from 0
    - * Can be inserted in FEM mesh from the start of the simulation (along certain lines or between all elements)
  - Extrinsic cohesive models:
    - * Generally has only softening σ–δ behavior.
    - * σ starts from maximum stress ($\tilde{\sigma}$)
    - * Should be adaptively inserted between elements when traction between elements approach $\tilde{\sigma}$



Cohesive models

- The behavior of CZMs is governed by traction-separation law
  - Exponential, bi-linear, trapezoidal, …

$$G_C = \int_0^{\tilde{\delta}} \sigma \, d\delta$$



$\tilde{\sigma}$: Maximum traction

δ_0, $\tilde{\delta}$: Initial and final separation

G_c : Fracture energy

K : Initial stiffness

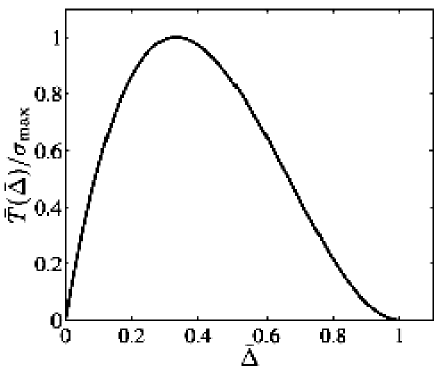
Cohesive models

- Cohesive model shape

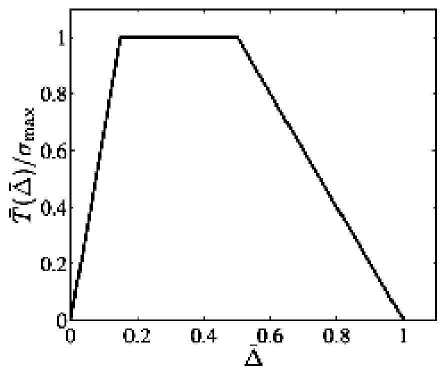
  - Can have important influence on the response of cohesive model

  -The shape can he based on ductile/brittle response of TSR and can make it intrinsic or extrinsic

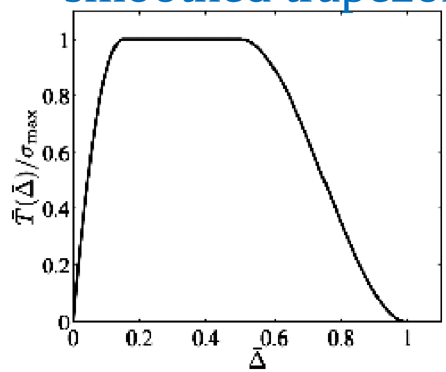
cubic polynomial



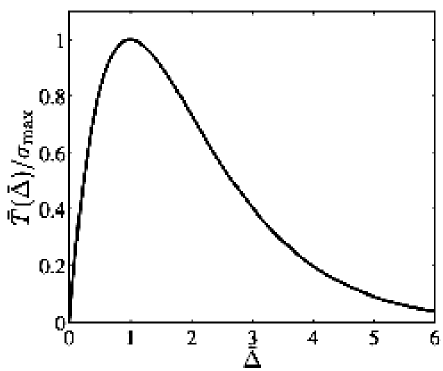
trapezoidal



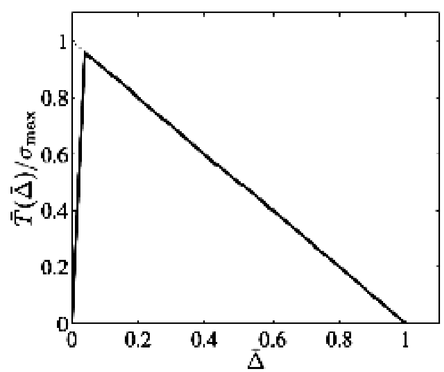
smoothed trapezoidal

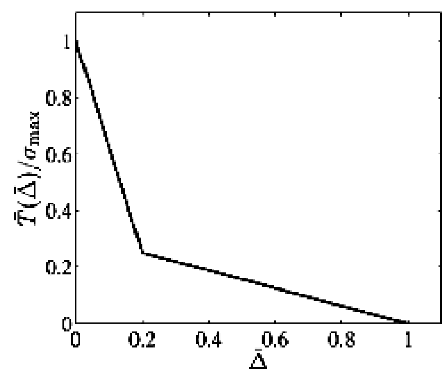


exponential



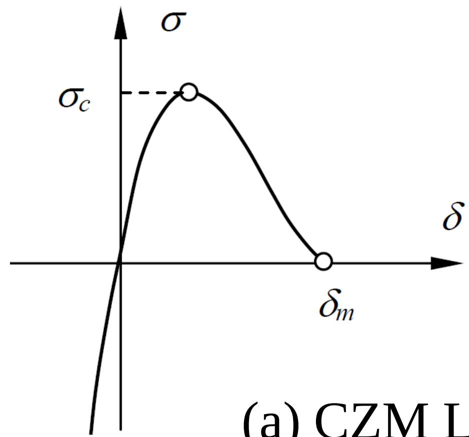
linear softening



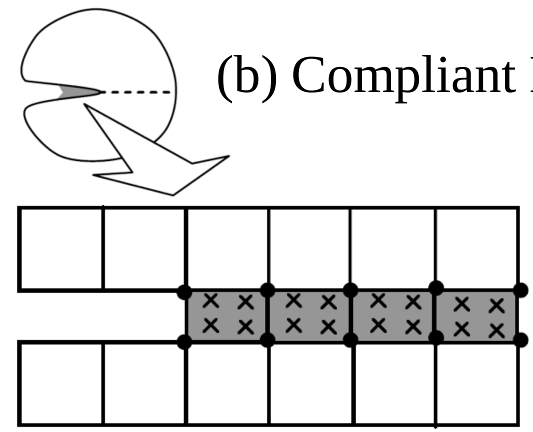


Cohesive models

- Continuum Cohesive Zone Model



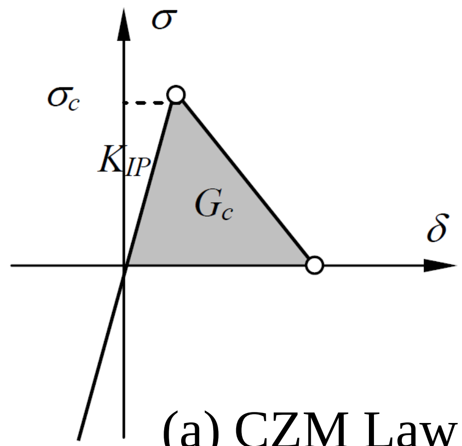
(a) CZM Law



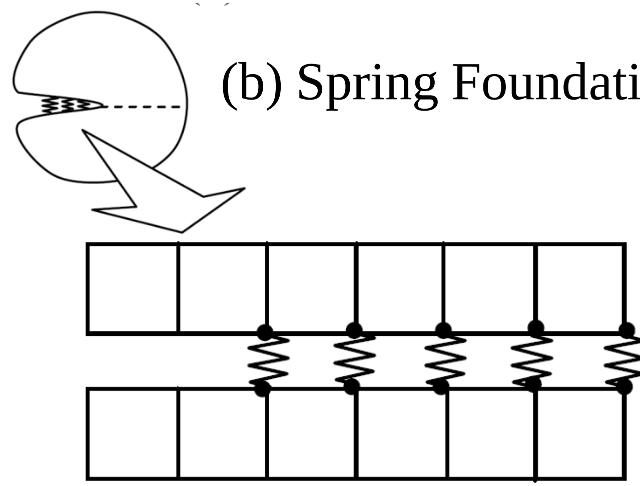
(b) Compliant Layer

(c) Continuum Element

- Discrete Cohesive Zone Model



(a) CZM Law

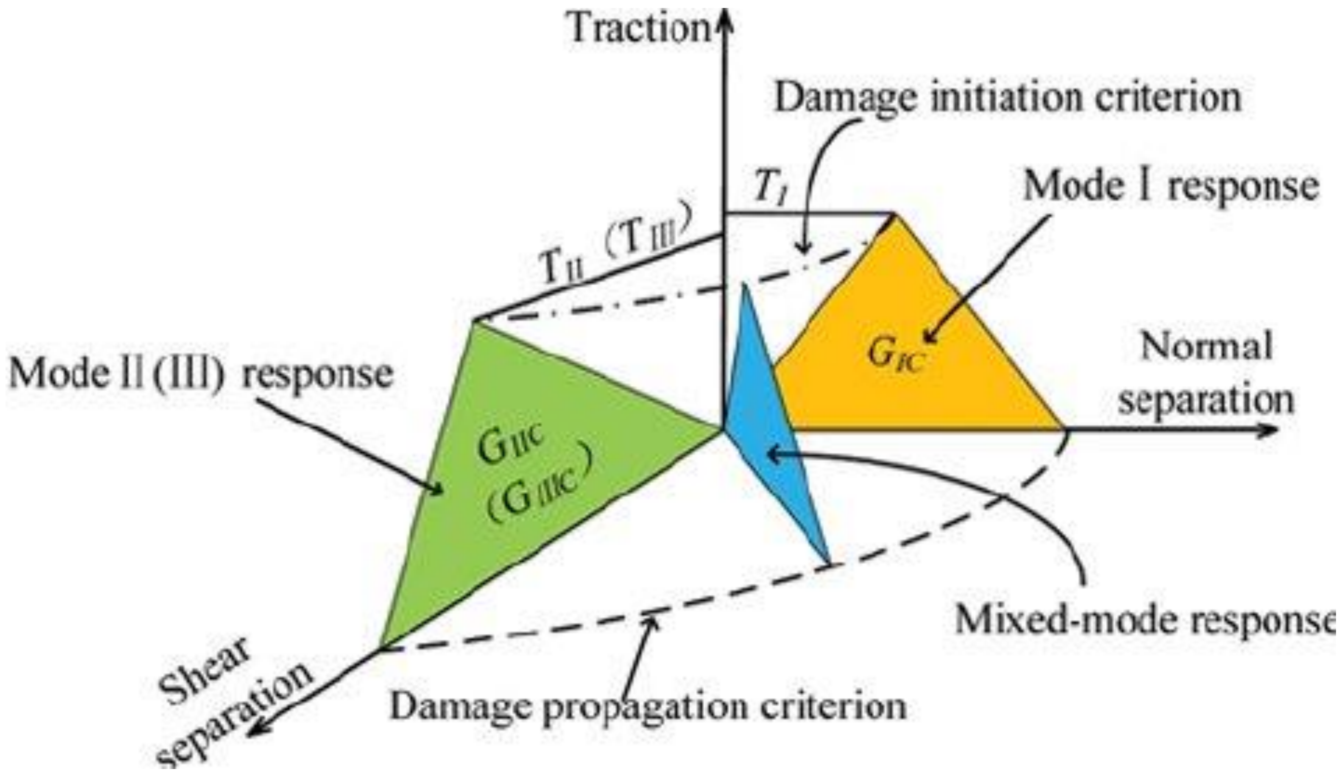


(b) Spring Foundation

(c) Discrete Element

Cohesive models

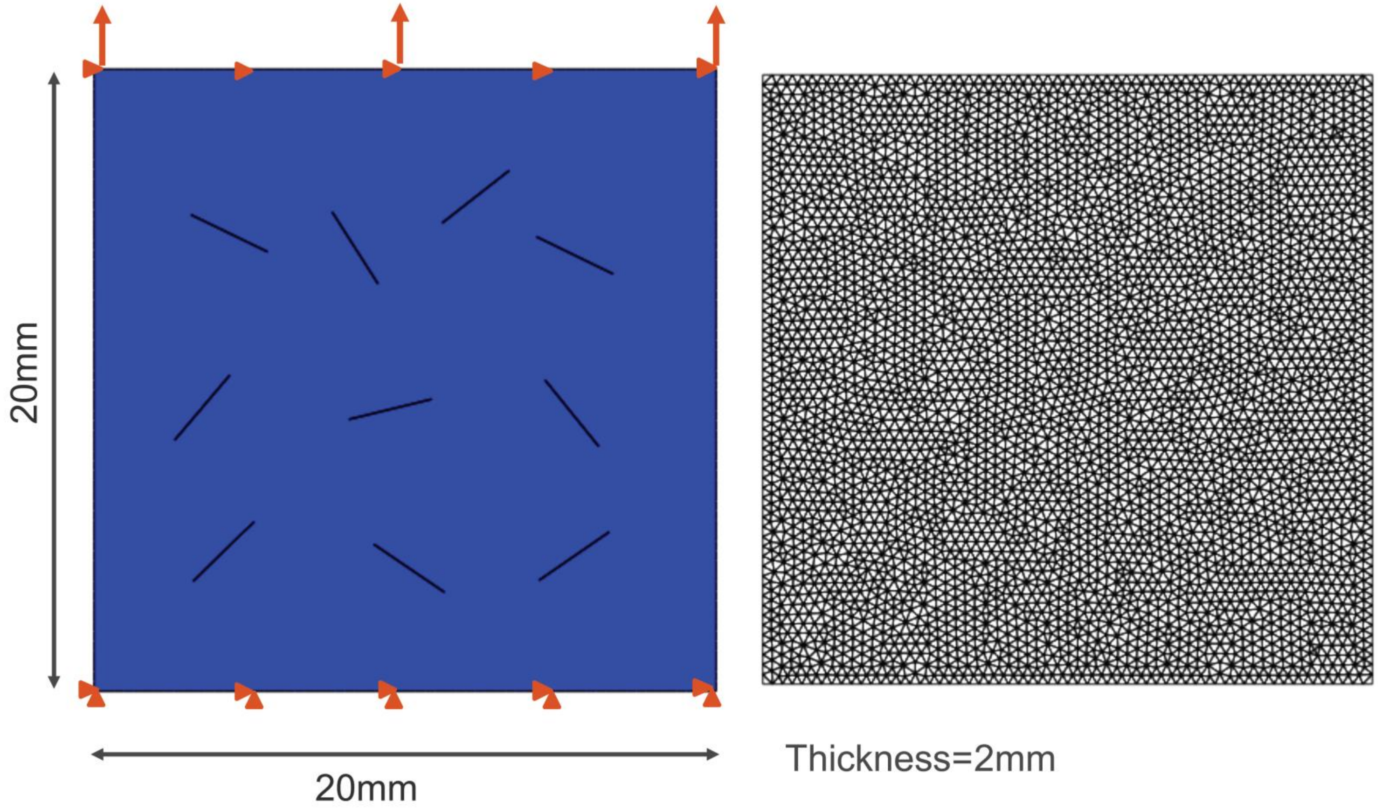
- Traction-Separation Law (Mixed Mode)



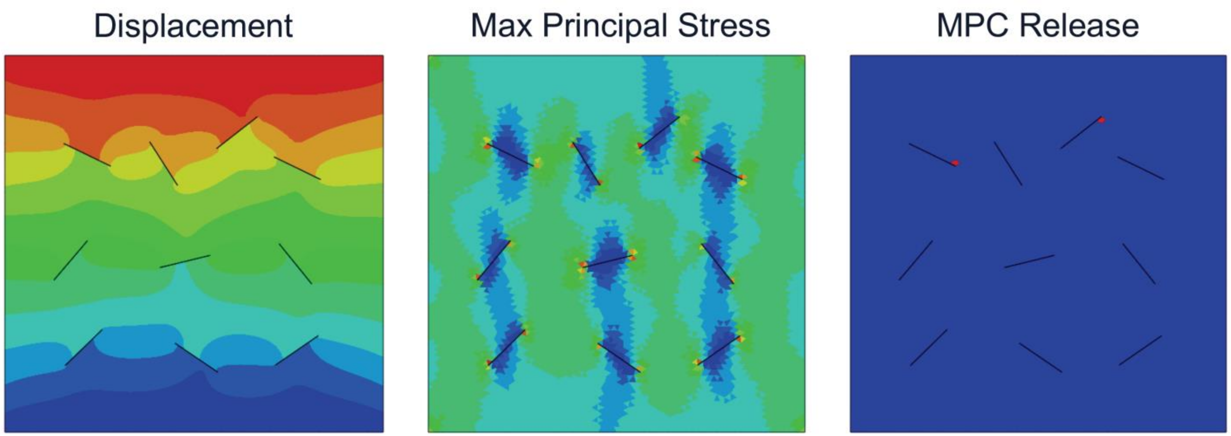
Mixed-mode triangle traction-separation law.

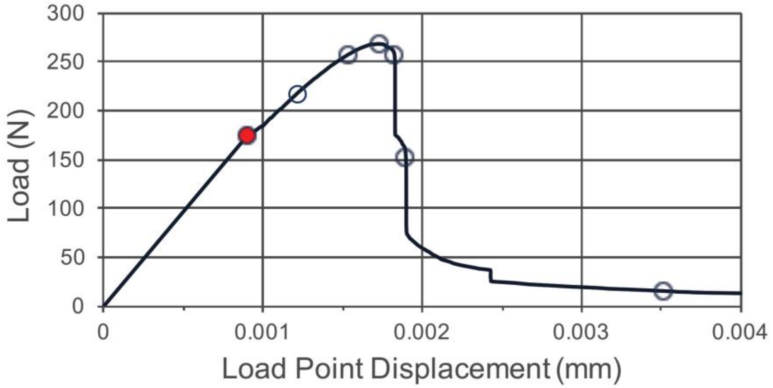
Multicrack Model

Geometry, BC, Mesh

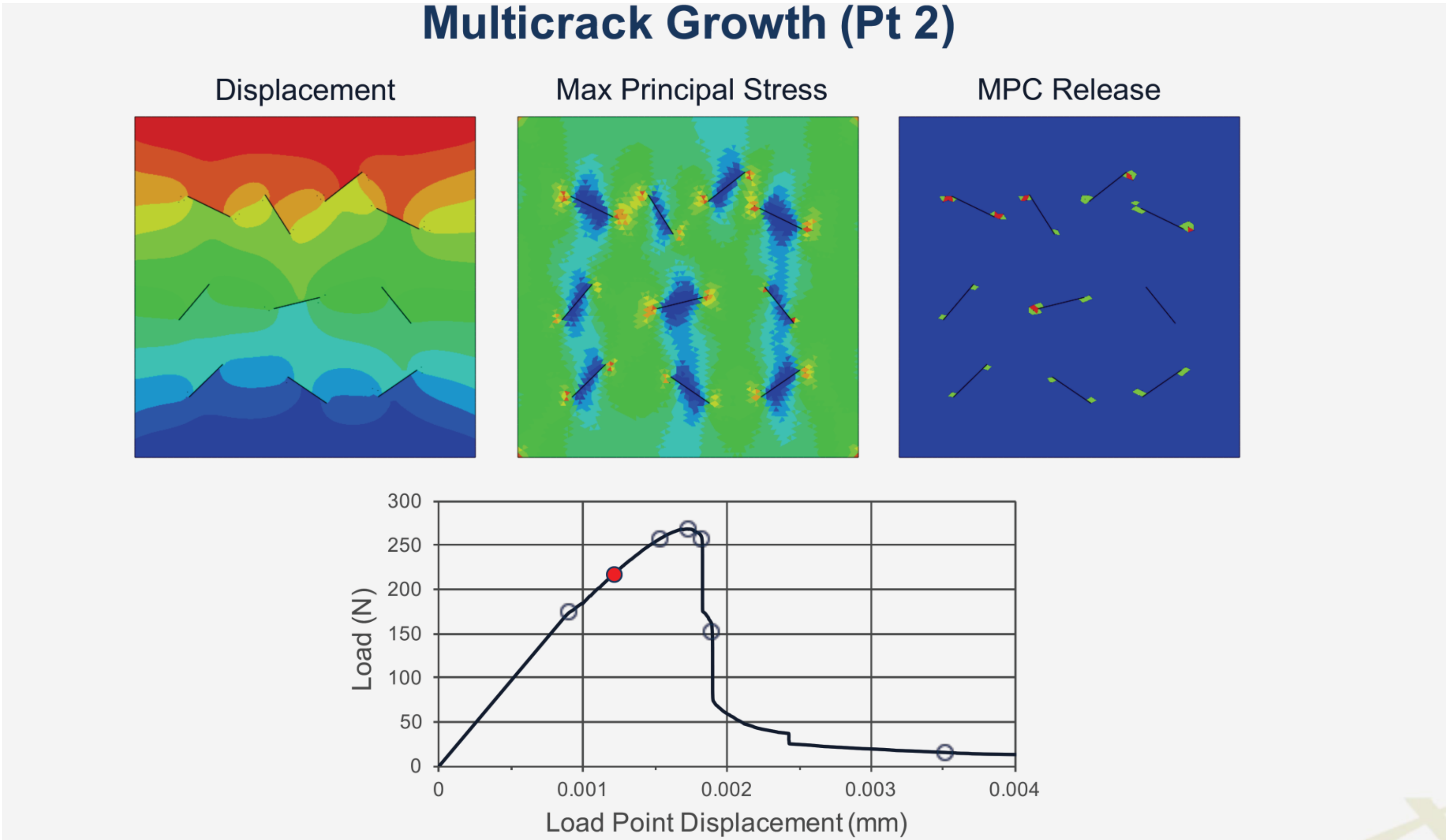


Multicrack Growth (Pt 1)

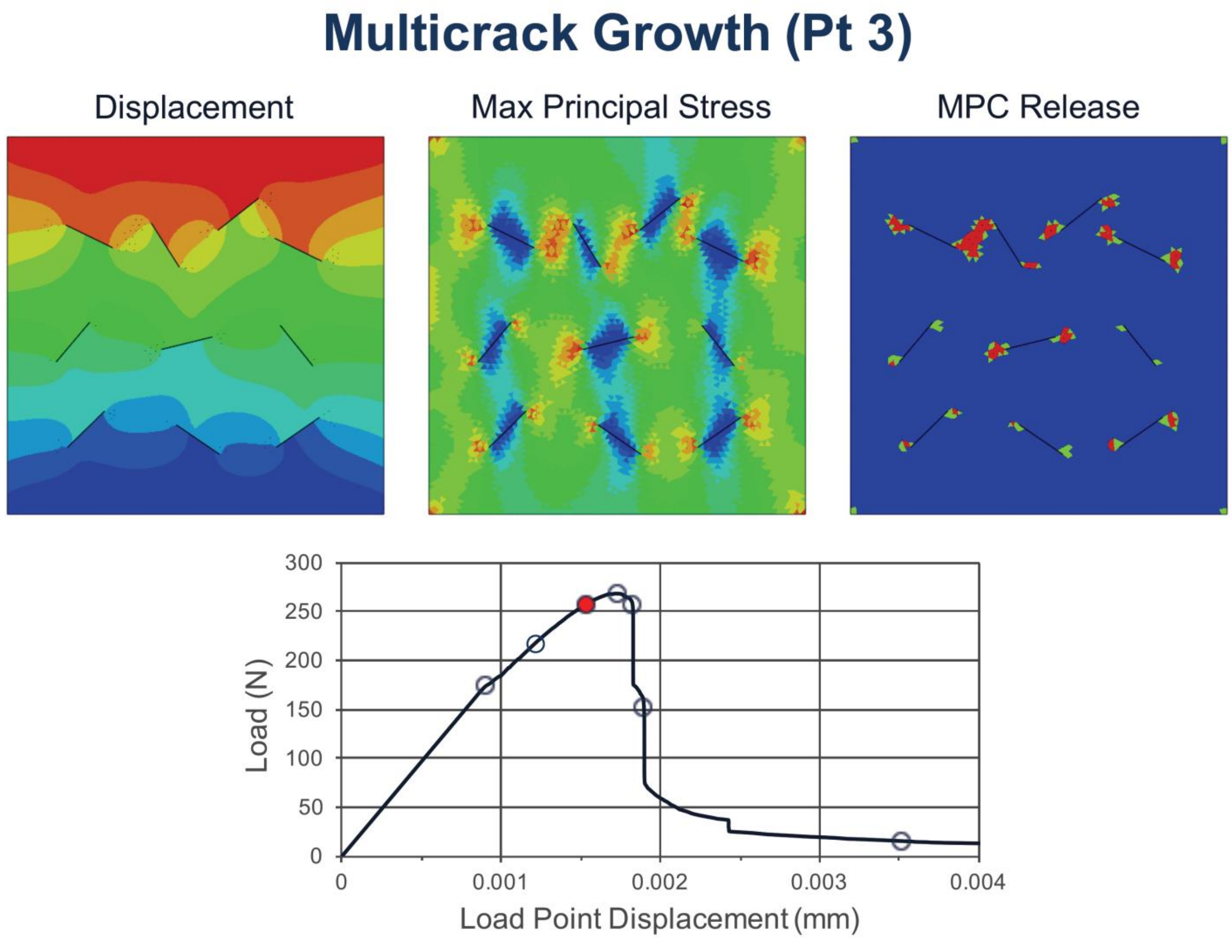




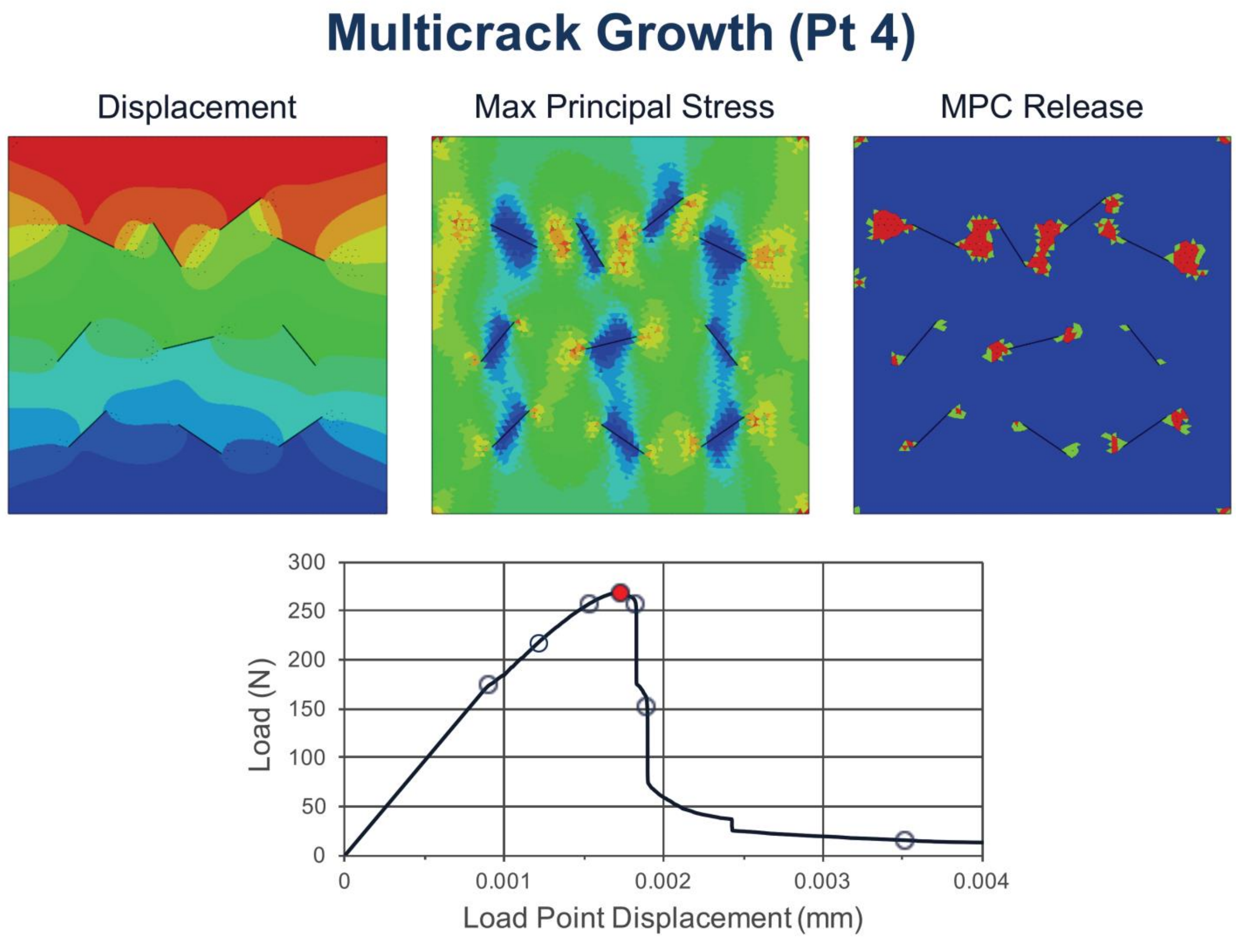
Cohesive models



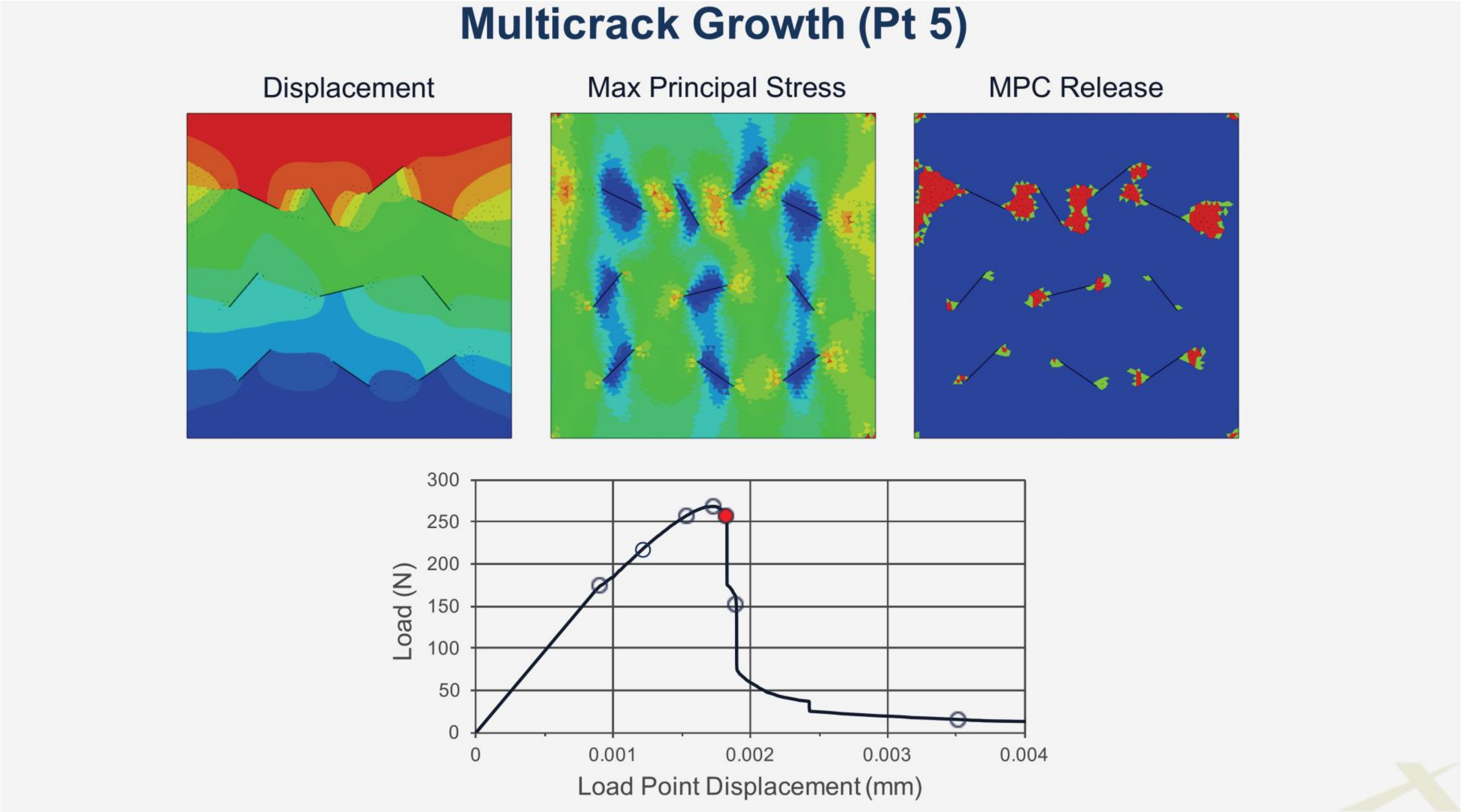
Cohesive models



Cohesive models

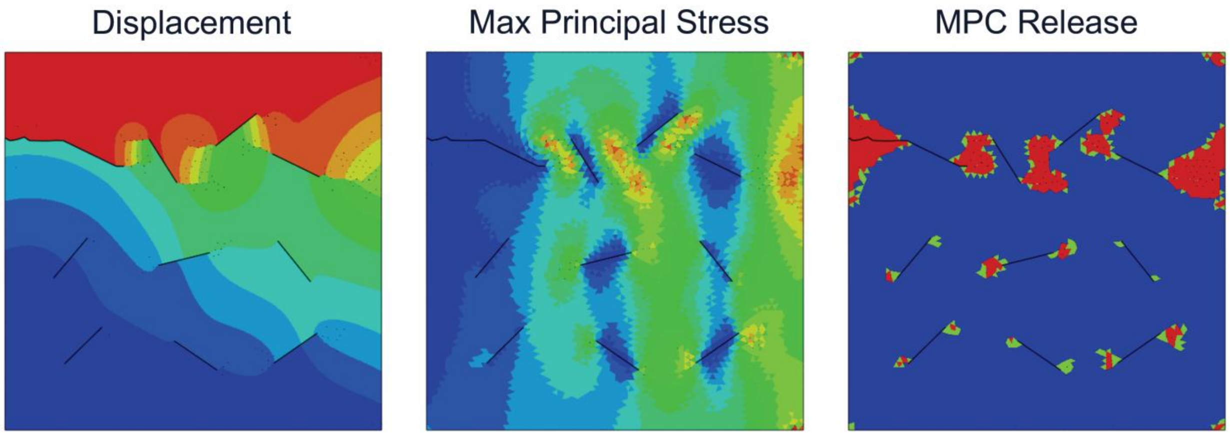


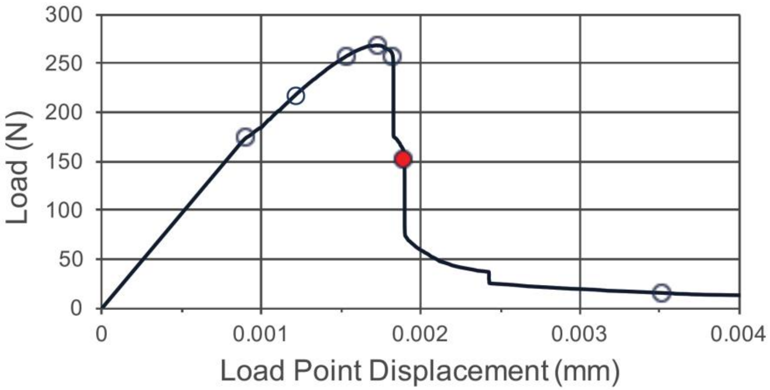
Cohesive models



Cohesive models

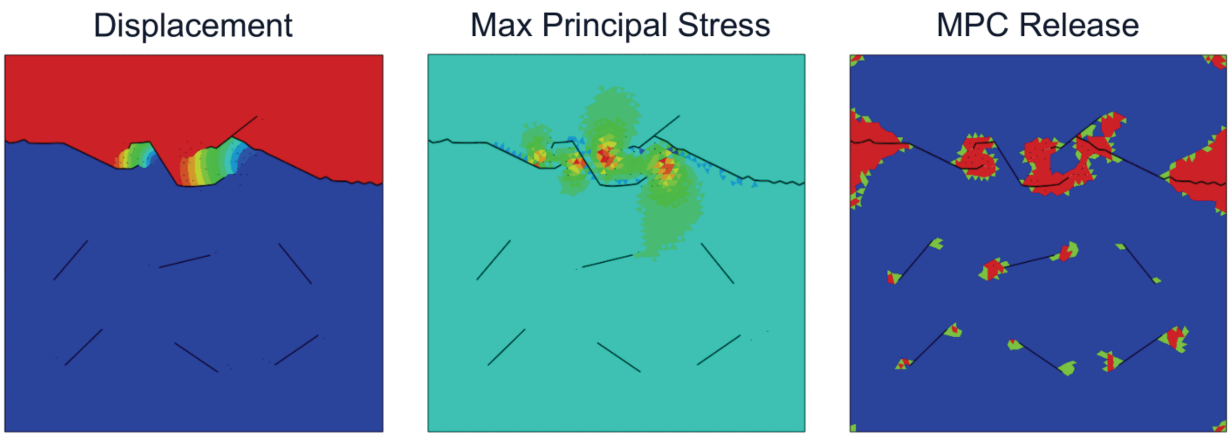
Multicrack Growth (Pt 6)

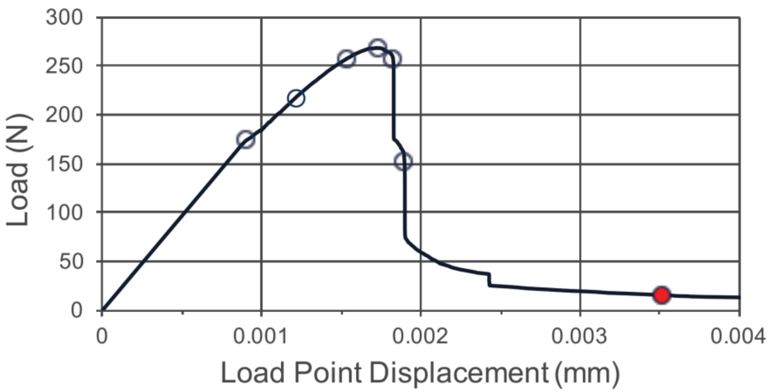




Cohesive models

Multicrack Growth (Pt 7)





Cohesive models

Multicrack Growth (Final)