دانشگاه صنعتی اصفهان

دانشکده مکانیک

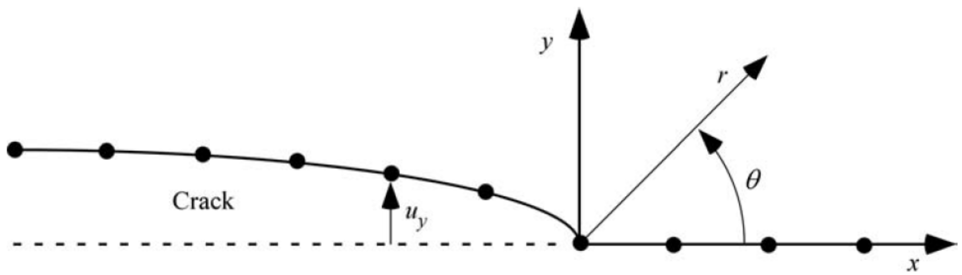
Computational Fracture Mechanics (2)

Computational fracture mechanics

- Introduction to Finite Element method
- Singular Stress Finite Elements
- Extraction of K (SIF), G
- J integral
- Finite Element mesh design for fracture mechanics
- Computational crack growth
- Traction Separation Relations

- K from local fields

1. Displacement

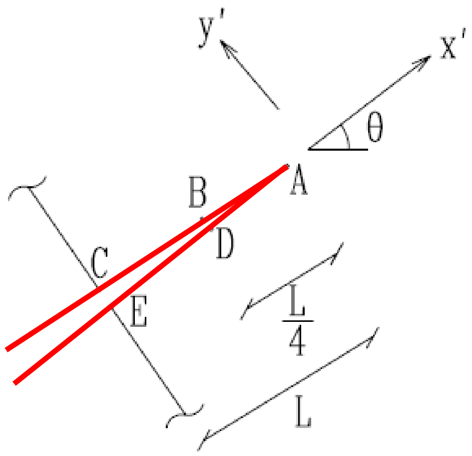


$$u_y(r,\theta=\pi)=\frac{4K_I\sqrt{r}}{\sqrt{2\pi}E'} \quad \Rightarrow \quad \boxed{K_I=\lim_{r\to 0}\left[\frac{E'u_y}{4}\sqrt{\frac{2\pi}{r}}\right] \quad (\theta=\pi)} \quad E'=\begin{cases} E & \text{plane stress} \\ \frac{E}{1-\nu^2} & \text{plane strain}\end{cases}$$

or alternatively from the first quarter point element:

Recall for 1D:

$$u=u_1+\frac{\sqrt{x}}{\sqrt{L}}(-3u_1-u_2+4u_3)+\frac{2x}{L}(u_1+u_2-2u_3)$$



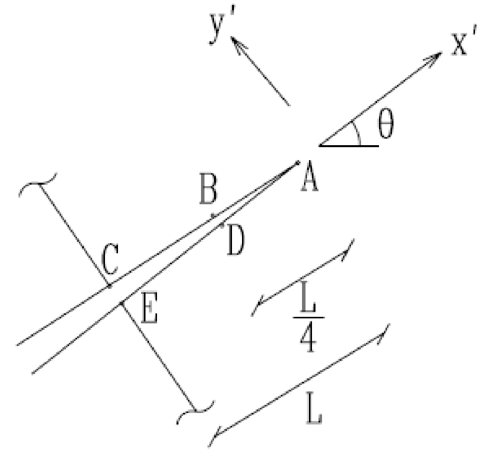
- K from local fields

1. Displacement

$$v = K_{\mathrm{I}} \frac{\kappa+1}{2G}\sqrt{\frac{r}{2\pi}}$$

$$u' = \overline{u}'_A + \left(-3\overline{u}'_A + 4\overline{u}'_B - \overline{u}'_C\right)\sqrt{\frac{r}{L}} + \left(2\overline{u}'_A + 2\overline{u}'_C - 4\overline{u}'_B\right)\frac{r}{L}$$

$$v' = \overline{v}'_A + \left(-3\overline{v}'_A + 4\overline{v}'_B - \overline{v}'_C\right)\sqrt{\frac{r}{L}} + \left(2\overline{v}'_A + 2\overline{v}'_C - 4\overline{v}'_B\right)\frac{r}{L}$$



$$\Rightarrow \quad K_{\mathrm{I}} = \frac{2G}{\kappa+1}\sqrt{\frac{2\pi}{L}}\left(-3\overline{v}'_A + 4\overline{v}'_B - \overline{v}'_C\right)$$

Mixed mode generalization:

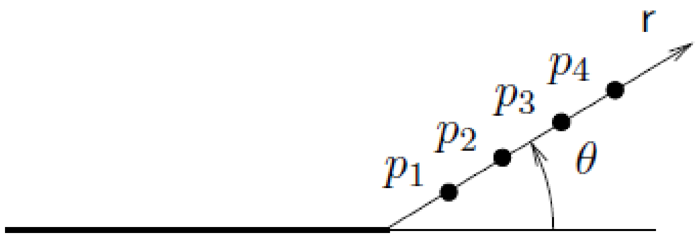
$$\left\{\begin{array}{c} K_{\mathrm{I}} \\ K_{\mathrm{II}} \end{array}\right\} = \frac{1}{2}\frac{2G}{\kappa+1}\sqrt{\frac{2\pi}{L}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -3\overline{u}'_A + 4\left(\overline{u}'_B - \overline{u}'_D\right) - \left(\overline{u}'_C - \overline{u}'_E\right) \\ -3\overline{v}'_A + 4\left(\overline{v}'_B - \overline{v}'_D\right) - \left(\overline{v}'_C - \overline{v}'_E\right) \end{bmatrix}$$

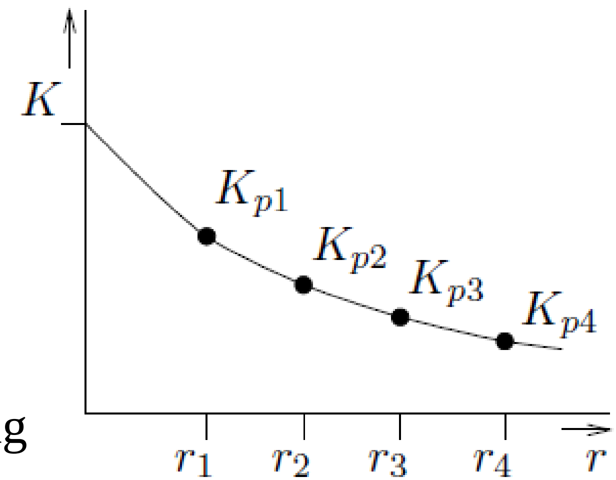
Extraction of K

- K from local fields

2. Stress

$$K_I = \lim_{r \to 0} \left(\sqrt{2\pi r}\ \sigma_{22}|_{\theta=0} \right) \quad ; \quad K_{II} = \lim_{r \to 0} \left(\sqrt{2\pi r}\ \sigma_{12}|_{\theta=0} \right)$$





or can be done for arbitrary angle (θ) taking σ angular dependence *f*(θ) into account

Stress based method is less accurate because:

- Stress is a derivative field and generally is one order less accurate than displacement
- Stress is singular as opposed to displacement
- Stress method is much more sensitive to where loads are applied(crack surface or far field)

Extraction of K

- *K* from energy approaches

1. Elementary crack advance (two FEM solutions for a and $a + \Delta a$)
2. Virtual Crack Extension: Stiffness derivative approach
3. J-integral based approaches (next section)

After obtaining G (or $J = G$ for LEFM) K can be obtained from:

$$K_I^2 = E'G$$

$$E' = \begin{cases} E & \text{plane stress} \\ \dfrac{E}{1-\nu^2} & \text{plane strain} \end{cases}$$

Extraction of K

- *K* from energy approaches

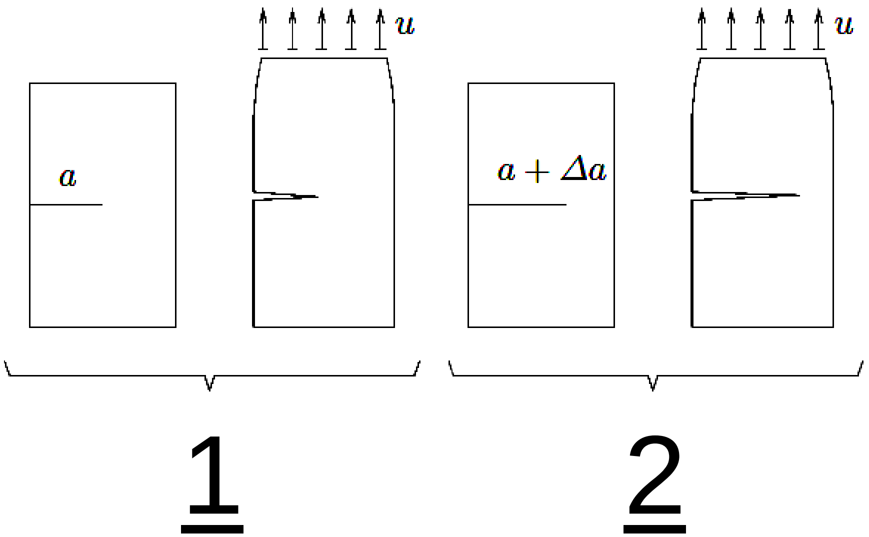
  1. Elementary crack advance

➤ For fixed grip boundary condition perform two simulations (1, *a*) and (2, *a*+∆*a*):

All FEM packages can compute strain (internal) energy U_i

Fixed grips: $\frac{dU_e}{da} = 0$

$$\Rightarrow G = -\frac{1}{B}\frac{dU_i}{da} \approx -\frac{1}{B}\frac{U_i(a+\Delta a) - U_i(a)}{\Delta a}$$



➤ Drawback:

Requires two solutions

Prone to Finite Difference (FD) errors

Extraction of K

- *K* from energy approaches

  2. Virtual crack extension

Potential energy is given by:

$$\Pi = \frac{1}{2}[u][K][u] - [u]\{p\}$$

$$-G = \frac{\partial \Pi}{\partial a} = \frac{\partial [u]}{\partial a}[K][u] + \frac{1}{2}[u]\frac{\partial [K]}{\partial a}[u] - \frac{\partial [u]}{\partial a}\{p\} - [u]\frac{\partial \{p\}}{\partial a}$$

$$= \frac{\partial [u]}{\partial a}\underbrace{\left([K][u] - \{p\}\right)}_{0} + \frac{1}{2}[u]\frac{\partial [K]}{\partial a}[u] - [u]\frac{\partial \{p\}}{\partial a}$$

$$\boxed{G = -\frac{1}{2}[u]\frac{\partial [K]}{\partial a}[u] + [u]\frac{\partial \{p\}}{\partial a}}$$

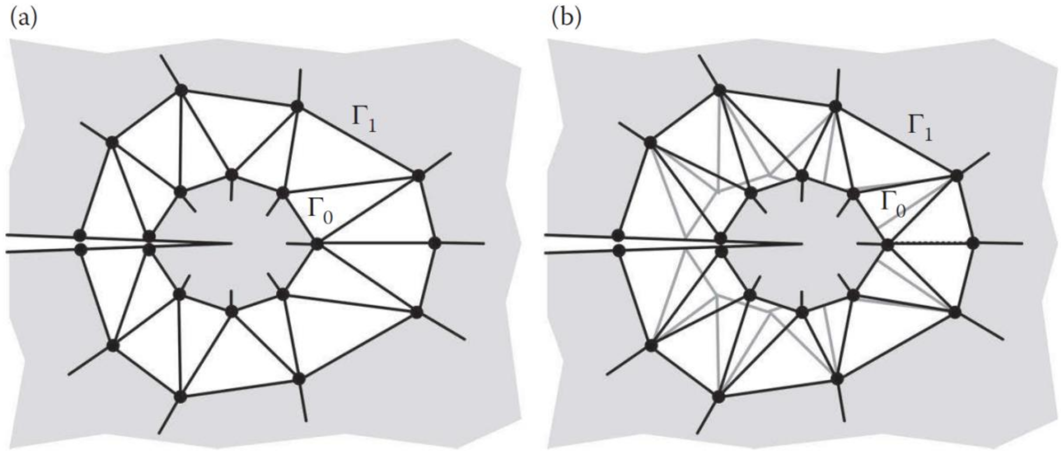
Furthermore when the loads are constant:

$$G = \frac{K_I^2}{E'} = -\frac{1}{2}[u]^T \frac{\partial [K]}{\partial a}[u]$$

- *K* from energy approaches

  2. Virtual crack extension

➢ Only the few elements that are distorted contribute to $\frac{\partial K}{\partial a}$.



➢ We may not even need to form elements and assemble **K** for a and a+∆a to obtain $\frac{\partial K}{\partial a}$. We can explicitly obtain $\frac{\partial K^e}{\partial a}$ for elements affected by crack growth by computing derivatives of actual geometry of the element to parent geometry.

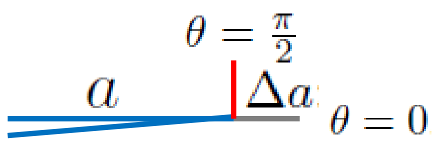
➢ This method is equivalent to J integral method (Park 1974)

- *K* from energy approaches
  2. Virtual crack extension: Mixed mode

➢ For LEFM energy release rates G_1 and G_2 are given by:

$$J_1 = G_1 = \frac{K_I^2 + K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \qquad J_2 = G_2 = \frac{-2K_I K_{II}}{E'}$$

Using Virtual crack extension (or elementary crack advance) compute G_1 and G_2 for crack lengths a, $a + \Delta a$ (along θ = 0, and determine G_1 and along θ = π determine G_2).



➢ Obtain K_I and K_{II} from:

$$K_I = \frac{s \mp \sqrt{s^2 + \frac{8G_1}{\alpha}}}{4}$$

$$K_{II} = \frac{s \mp \sqrt{s^2 + \frac{8G_2}{\alpha}}}{4}$$

that:

$$s = 2\sqrt{\frac{G_1 - G_2}{\alpha}},$$

$$\alpha = \frac{(1-\nu)(1+\kappa)}{E}$$

Note that there are two sets of solutions!

J integral

Uses of J integral:

1. LEFM: Can obtain K_I and K_{II} from J integrals (G = J for LEFM)

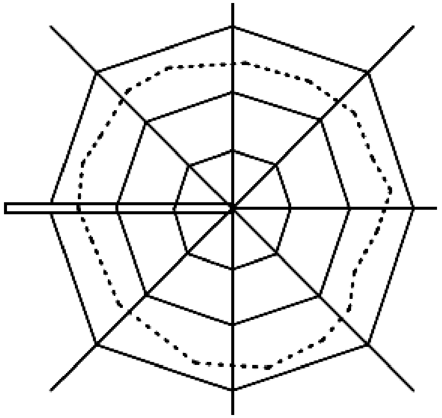
$$J_1 = G_1 = \frac{K_I^2 + K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \qquad J_2 = G_2 = \frac{-2K_I K_{II}}{E'}$$

2. Still valid for nonlinear (NLFM) and plastic (PFM) fracture mechanics.

- Methods to evaluate J integral:

  - Contour integral: $J = \int_\Gamma \left(w\,dy - \mathbf{t}\frac{\partial \mathbf{u}}{\partial x} d\Gamma \right)$

  - Equivalent (Energy) domain integral (EDI):

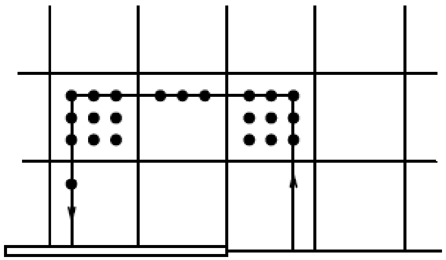


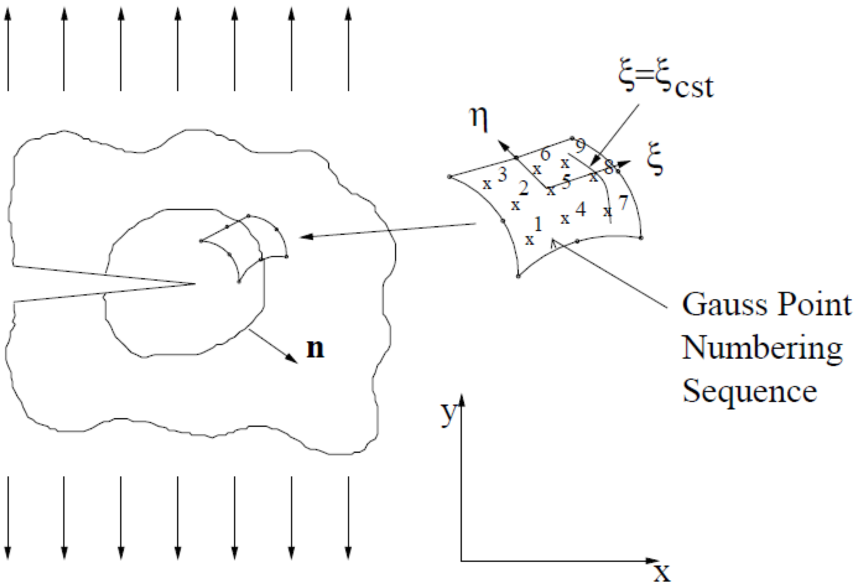
- Gauss theorem: line/surface (2D/3D) integral ➡ surface/volume integral
- Much simpler to evaluate computationally
- Easy to incorporate plasticity, crack surface tractions, thermal strains, *etc.*
- Prevalent method for computing J-integral

J integral

- Contour integral

➢ Stresses are available and also more accurate at Gauss points
➢ Integral path goes through Gauss points





$$J=\int_{\Gamma}\left(w\,dy-\mathbf{t}\frac{\partial \mathbf{u}}{\partial x}d\Gamma\right)$$

$$J=\int_{-1}^{1}\left\{\underbrace{\frac{1}{2}\left[\sigma_x\frac{\partial u}{\partial x}+\tau_{xy}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\sigma_y\frac{\partial v}{\partial y}\right]}_{w}\underbrace{\frac{\partial y}{\partial \eta}}_{dy}-\underbrace{\left[(\sigma_x n_1+\tau_{xy}n_2)\frac{\partial u}{\partial x}+(\tau_{xy}n_1+\sigma_y n_2)\frac{\partial v}{\partial x}\right]}_{\mathbf{t}\frac{\partial \mathbf{u}}{\partial x}}\underbrace{\sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2+\left(\frac{\partial y}{\partial \eta}\right)^2}}_{ds}\right\}d\eta$$

$$=\int_{-1}^{1}I\,d\eta$$

Cumbersome to formulate the integrand, evaluate normal vector, and integrate over lines (2D) and surfaces (3D) **Not commonly used**

J integral

- Equivalent Domain Integral

➢ General form of J integral

$$w = \int_0^{\varepsilon_{kl}^m} \sigma_{ij}\, d\varepsilon_{ij}^m$$

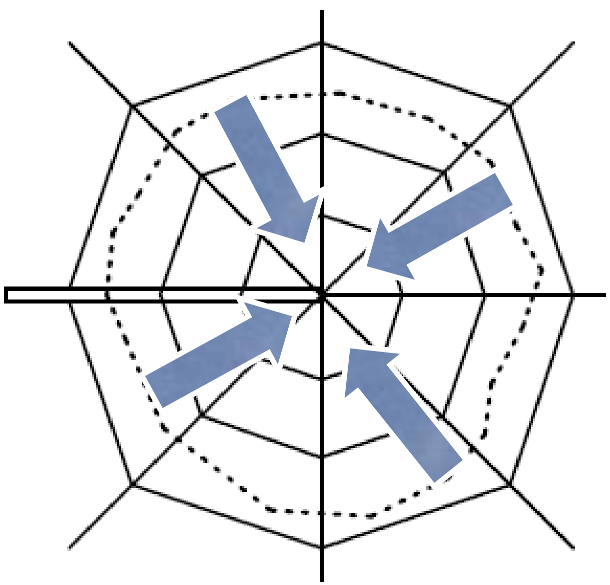
$$J = \lim_{\Gamma_0 \to 0} \int_{\Gamma_0} \left[(w + T)\delta_{1i} - \sigma_{ij} \frac{\partial u_j}{\partial x_1} \right] n_i \, d\Gamma$$

Inelastic stress

Kinetic energy density

$$T = \frac{1}{2}\rho \frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t}$$

Can include (visco-) plasticity, and thermal stresses



$$\varepsilon_{ij}^{total} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^{pl} + \alpha\Theta\delta_{ij} = \varepsilon_{ij}^{m} + \varepsilon_{kk}^{t}$$

Elastic

Plastic

Thermal (Θ temperature)

$\Gamma_0 \to 0$: J contour approaches Crack tip

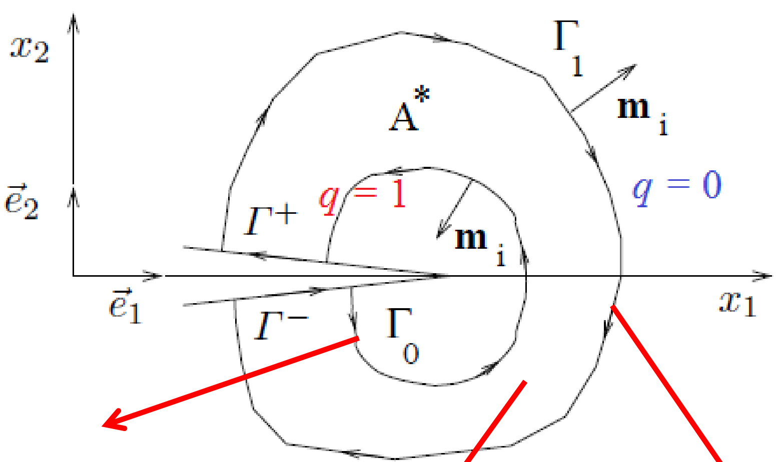
Accuracy of the solution deteriorates at Crack tip

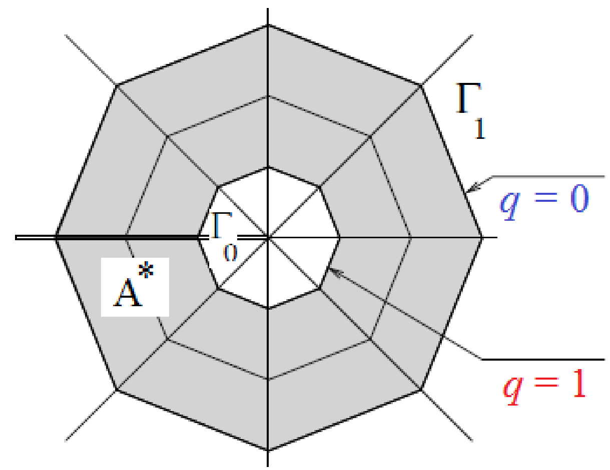
Inaccurate/Impractical evaluation of J using contour integral

- Equivalent Domain Integral

Divergence theorem: Line/Surface (2D/3D) integral → Surface/Volume Integral

Application in FEM meshes





Original J integral contour

Surface integral after using divergence theorem

$\Gamma_0 \to 0$ ⇒ 2D mesh covers crack tip

- Contour integral added to create closed surface
- By using $q = 0$ this integral in effect is zero