دانشگاه صنعتی اصفهان

دانشکده مکانیک

Computational Fracture Mechanics

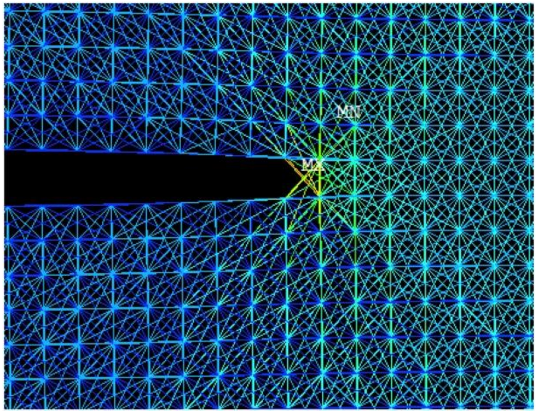
Introduction

Fracture models:

- Discrete crack models (discontinuous models): Cracks are explicitly modeled
  - LEFM
  - EPFM
  - Cohesive zone models

- Continuous models: Effect of (micro)cracks and voids are incorporated in bulk damage
  - Continuum damage models
  - Phase field models

- Peridynamic models: Material is modeled as a set of particles (www.peridynamics.org)



Computational fracture mechanics

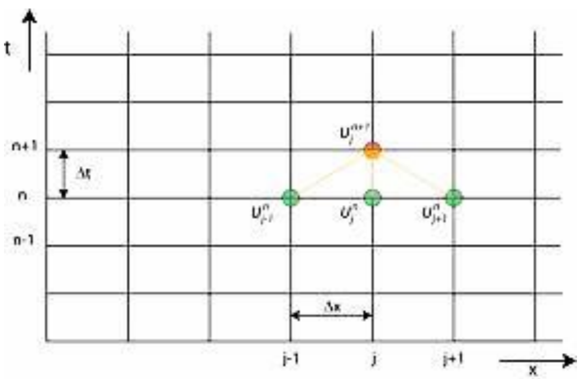
- Introduction to Finite Element method
- Singular Stress Finite Elements
- Extraction of K (SIF), G
- J integral
- Finite Element mesh design for fracture mechanics
- Computational crack growth
- Traction Separation Relations

Introduction to Finite Element method

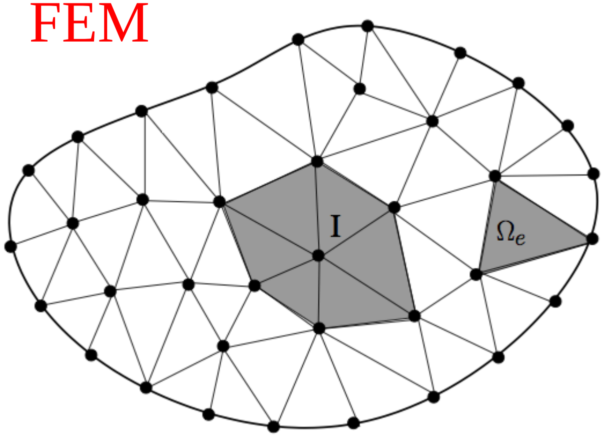
Numerical methods to solve PDEs:

- Finite Difference (FD) & Finite Volume (FV) methods
- FEM (Finite Element Method)
- BEM (Boundary Element Method)
- MMs (Meshless/Meshfree methods)

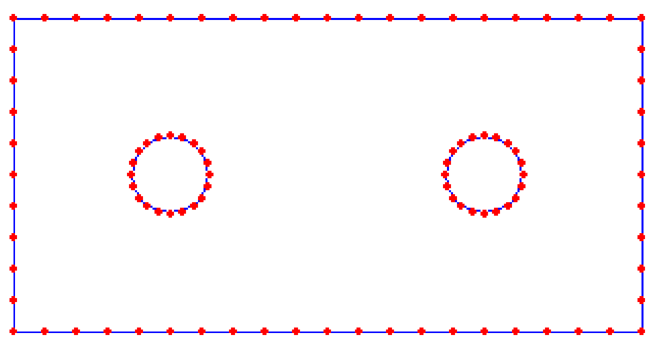
FD



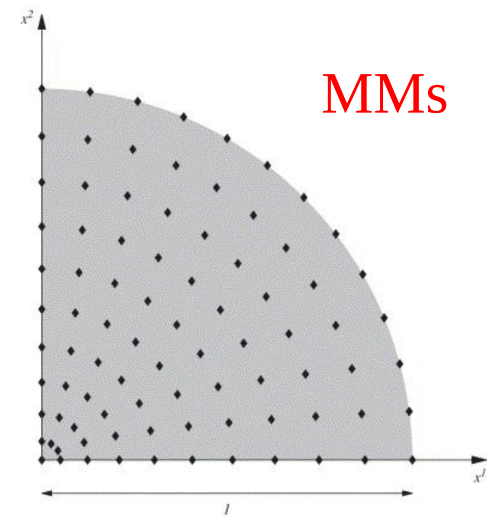
FEM



BEM



MMs



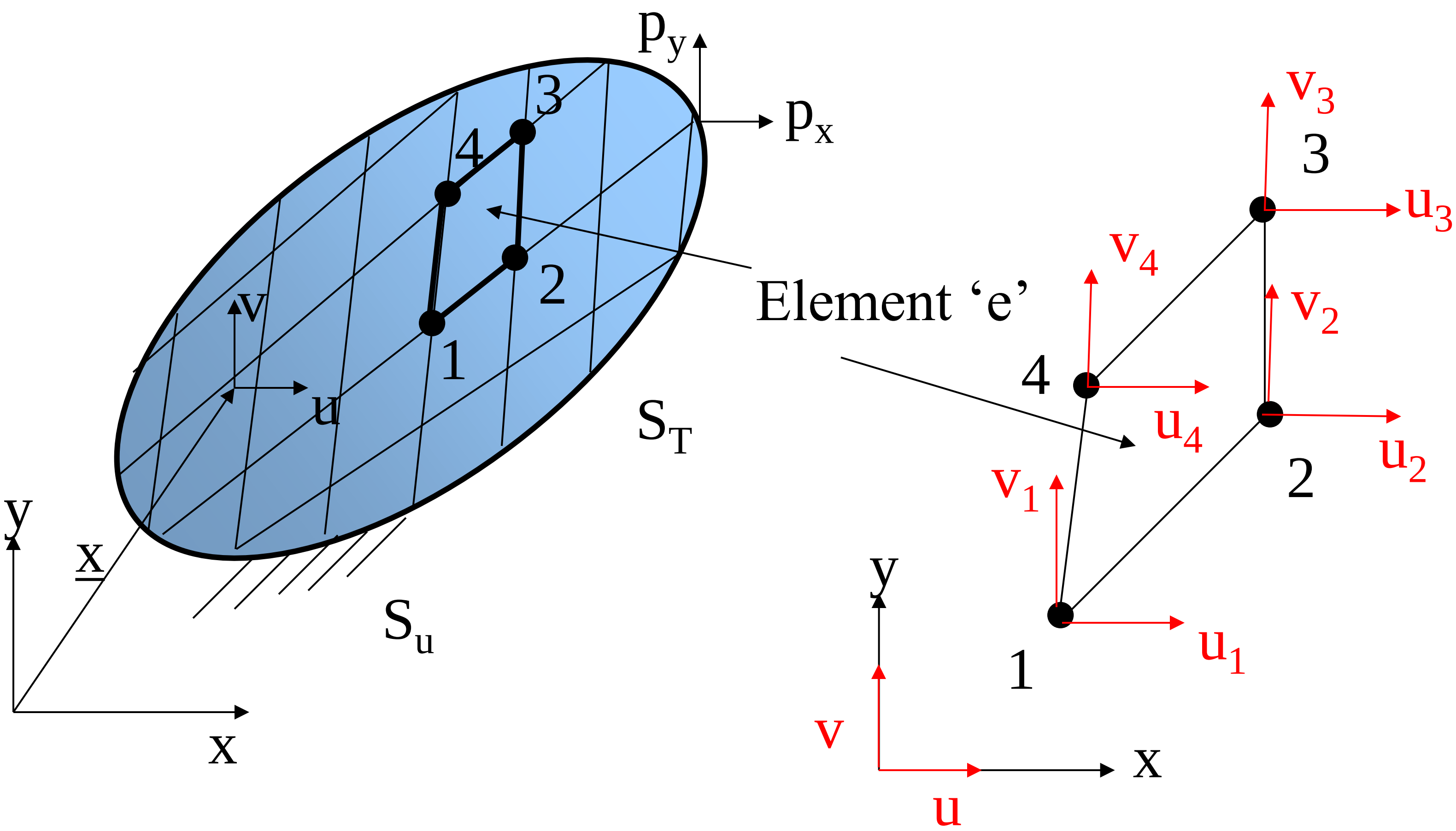
Finite Element Method:

- Approximating field is defined in a piecewise fashion by dividing the entire region over subregions
- Undetermined parameters u_i are the nodal values of the field
- The approximation functions can be generated systematically over these subregions
- FEM is the piecewise (or elementwise) application of the weighted residual method.
- We get different finite element approximations depending on the choice of the weighted residual method.

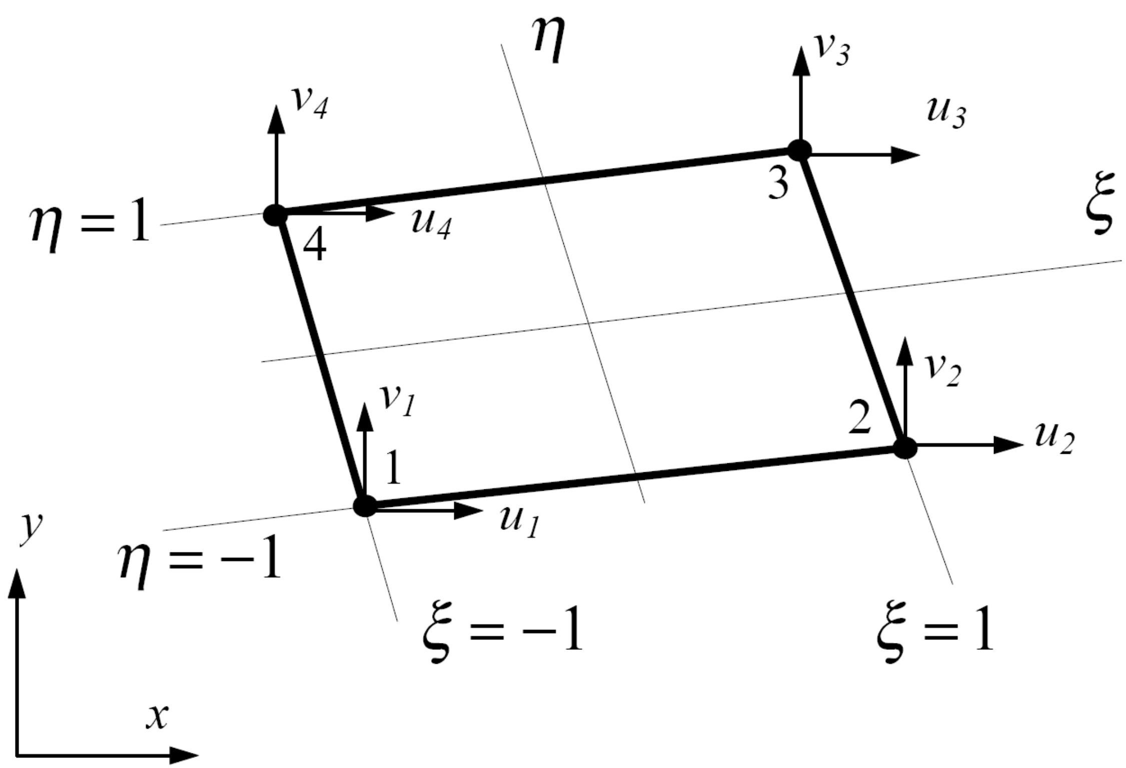
Steps in the finite element method:

- Discretization of the domain into a set of finite elements.
- Defining an approximate solution over the element.
- Weighted integral formulation of the differential equation.
- Substitute the approximate solution and get the algebraic equation

- Discretization of the domain into a set of finite elements.



Linear Quadrilateral Element (Q4)



Parent element

➢Defining an approximate solution over the element.

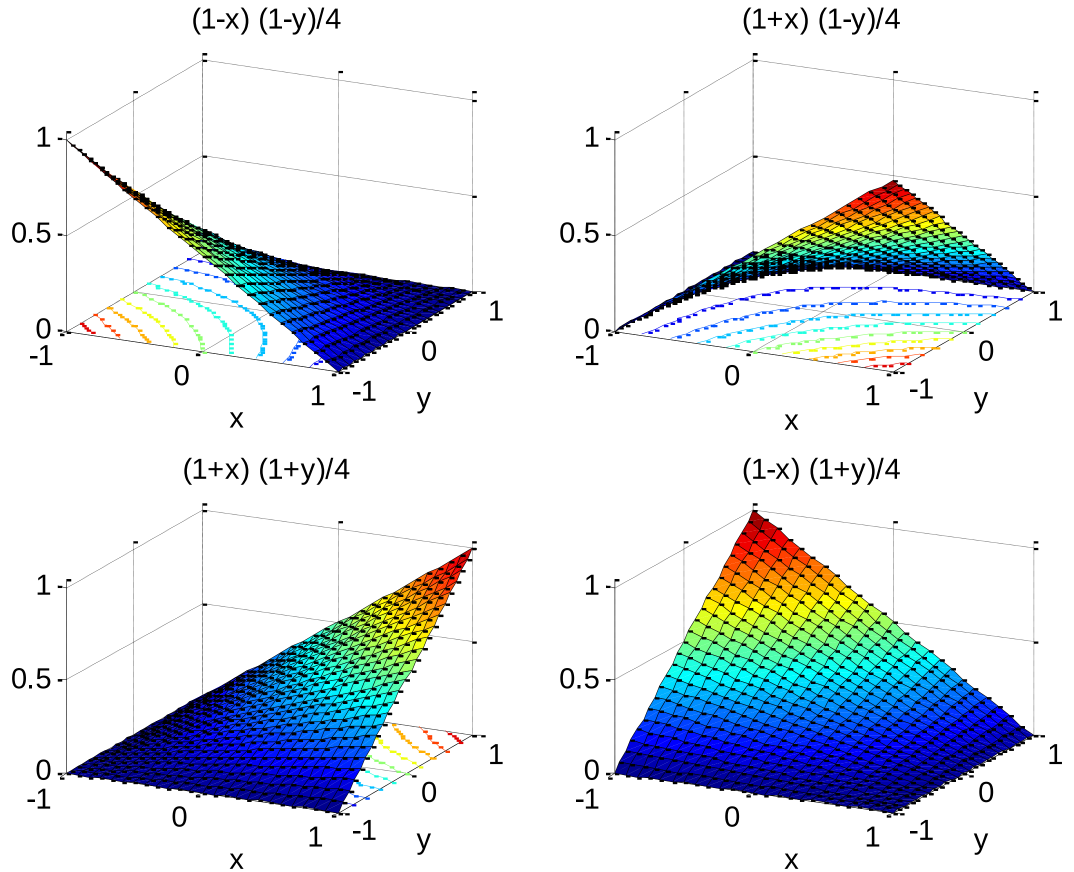
$$\mathbf{u} = \mathbf{N}\mathbf{d}$$

$$N_1 = \frac{(b-x)(h-y)}{4bh}$$

$$N_2 = \frac{(b+x)(h-y)}{4bh}$$

$$N_3 = \frac{(b+x)(h+y)}{4bh}$$

$$N_4 = \frac{(b-x)(h+y)}{4bh}$$



Interpolation functions (*b=h=1*)

- Defining an approximate solution over the element.

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}, \qquad \boldsymbol{\varepsilon} = \mathbf{Bd}$$

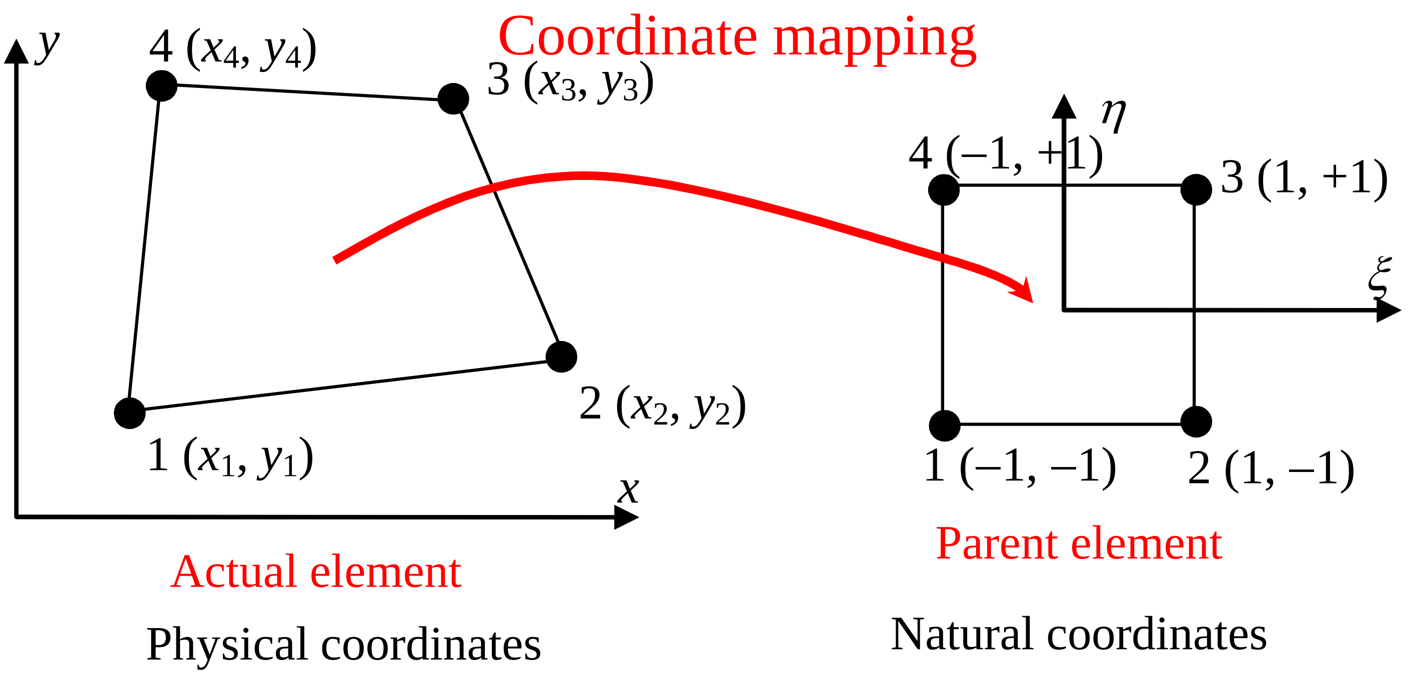
- Substitute the approximate solution and get the algebraic equation

$$\mathbf{B}=\frac{1}{4bh}\begin{bmatrix} -(h-y) & 0 & (h-y) & 0 & (h+y) & 0 & -(h+y) & 0 \\ 0 & -(b-x) & 0 & -(b+x) & 0 & (b+x) & 0 & (b-x) \\ -(b-x) & -(h-y) & -(b+x) & (h-y) & (b+x) & (h+y) & (b-x) & -(h+y) \end{bmatrix}$$

$$\mathbf{k}=\int_{-h}^{h}\int_{-b}^{b}\mathbf{B}^{T}\mathbf{E}\mathbf{B}t\,dx\,dy$$

$$\mathbf{f}=\iiint_{\mathbf{V}}\mathbf{N}^{T}\mathbf{X}\;dv+\iint_{\mathbf{S}}\mathbf{N}^{T}\mathbf{T}\,dS$$

Isoparametric Elements



$$\mathbf{u}(\xi,\eta) = \mathbf{N}(\xi,\eta)\mathbf{d}$$ (Interpolation of displacements)

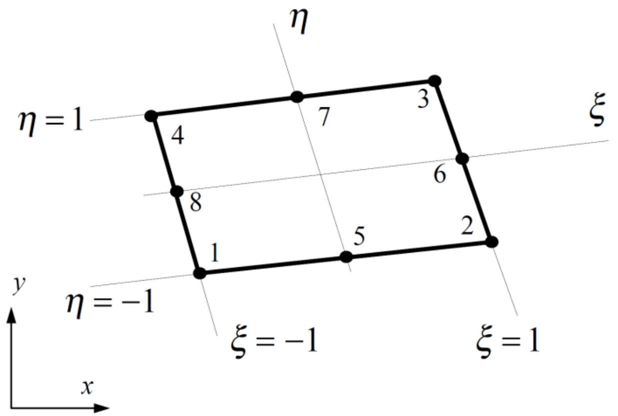
$$\mathbf{X}(\xi,\eta) = \mathbf{N}(\xi,\eta)\mathbf{x}_e$$ (Interpolation of coordinates)

Remarks

- Shape functions used for interpolating the coordinates are the same as the shape functions used for interpolation of the displacement field. Therefore, the element is called an *isoparametric element*.
- Note that the shape functions for coordinate interpolation and displacement interpolation do not have to be the same.
- Using the different shape functions for coordinate interpolation and displacement interpolation, respectively, will lead to the development of so-called *subparametric* or *superparametric* elements.

Quadratic Quadrilateral Element (Q4)

- In the natural coordinate system (ξ, η), the eight shape functions are,



$$N_1 = \frac{1}{4}(1-\xi)(\eta-1)(\xi+\eta+1)$$

$$N_2 = \frac{1}{4}(1+\xi)(\eta-1)(\eta-\xi+1)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1)$$

$$N_4 = \frac{1}{4}(\xi-1)(\eta+1)(\xi-\eta+1)$$

$$N_5 = \frac{1}{2}(1-\eta)(1-\xi^2)$$

$$N_6 = \frac{1}{2}(1+\xi)(1-\eta^2)$$

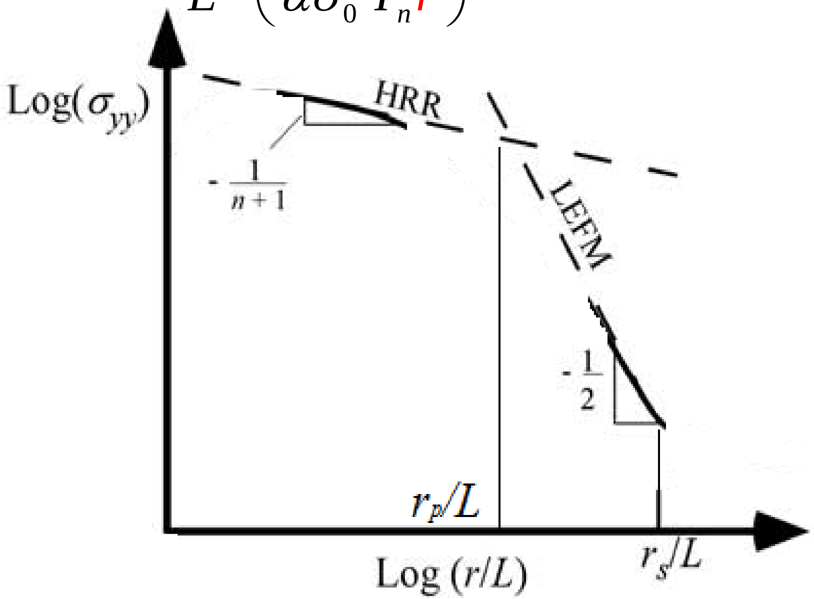
$$N_7 = \frac{1}{2}(1+\eta)(1-\xi^2)$$

$$N_8 = \frac{1}{2}(1-\xi)(1-\eta^2)$$

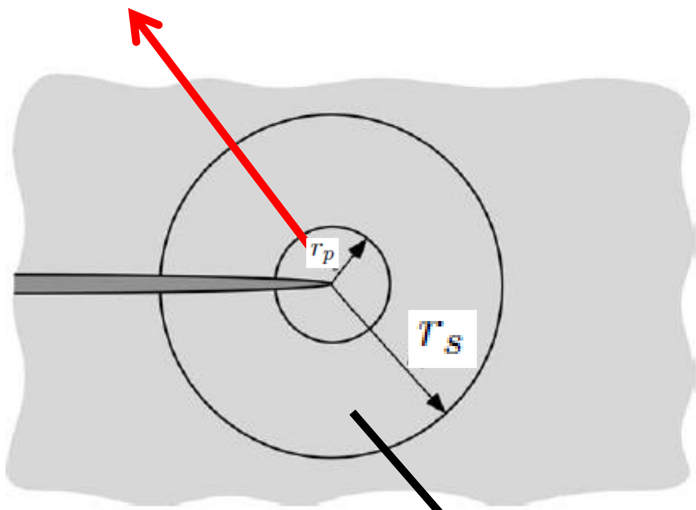
Singular crack tip solutions

$$\sigma_{ij} = \sigma_0 \left(\frac{EJ}{\alpha \sigma_0^2 I_n r} \right)^{\frac{1}{n+1}} \tilde{\sigma}_{ij}(n,\theta)$$

$$\varepsilon_{ij} = \frac{\alpha \sigma_0}{E} \left(\frac{EJ}{\alpha \sigma_0^2 I_n r} \right)^{\frac{n}{n+1}} \tilde{\varepsilon}_{ij}(n,\theta)$$



HRR solution



LEFM solution

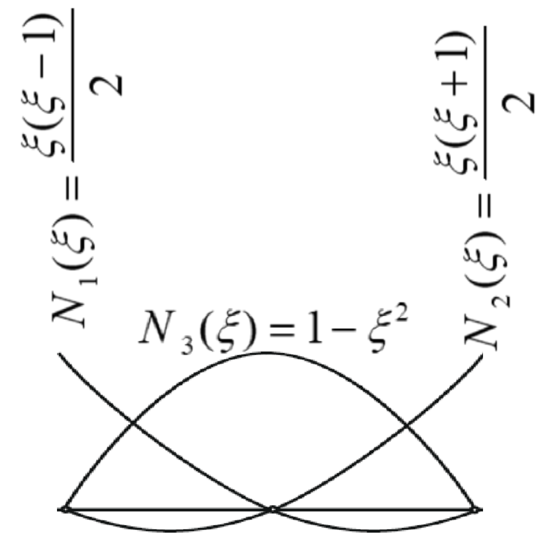
- NLFM (PFM): For HRR solution stress $\frac{1}{r^{\frac{1}{n+1}}}$ and strain $\frac{1}{r^{\frac{n}{n+1}}}$ are still singular

for Elastic-perfectly Plastic ($n \rightarrow \infty$) stress is bounded and strain is $\frac{1}{r}$ singular

Singular crack tip solutions

Motivation: 1D quadrature element

Find a that yields e singularity at x_1



Isoparametric element:

1. Geometry:

$$x = \Sigma_{i=1}^{3} N_i(\xi) x_i \quad \Rightarrow \quad \boxed{x = L\left\{\xi^2(\frac{1}{2}-\alpha)+\frac{1}{2}\xi+\alpha\right\}}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\xi} = L\left\{2\xi(\frac{1}{2}-\alpha)+\frac{1}{2}\right\}$$

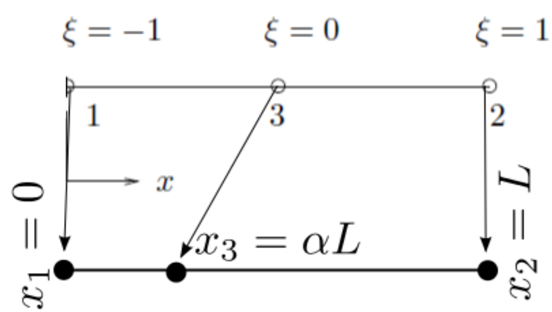
Singularity of $\epsilon(x) = \frac{\mathrm{d}u}{\mathrm{d}x} = \overbrace{\frac{\mathrm{d}u}{\mathrm{d}\xi}}^{\text{nonsingular}} / \frac{\mathrm{d}x}{\mathrm{d}\xi}$ at $x_1(\xi=-1) \Rightarrow$

$$\frac{\mathrm{d}x}{\mathrm{d}\xi}(\xi=-1)=0 \quad \Rightarrow -2(\frac{1}{2}-\alpha)+\frac{1}{2}=0 \Rightarrow \boxed{\alpha=\frac{1}{4}}$$

Hence,

$$x = \frac{L}{4}(\xi+1)^2 \quad \Rightarrow \quad \boxed{\xi = 2\sqrt{\frac{x}{L}}-1}$$

Parent element



Singular crack tip solutions

Motivation: 1D quadrature element

2. FEM solution

- Displacement

$$u = \Sigma_{i=1}^{3} N_i(\xi) u_i \quad \Rightarrow$$

$$u = u_1 \left[\frac{\xi(\xi-1)}{2}\right] + u_2 \left[\frac{\xi(\xi+1)}{2}\right] + u_3 \left[1-\xi^2\right] \Rightarrow$$

$$\boxed{u = u_1 + \frac{\sqrt{x}}{\sqrt{L}}(-3u_1 - u_2 + 4u_3) + \frac{2x}{L}(u_1 + u_2 - 2u_3)}$$

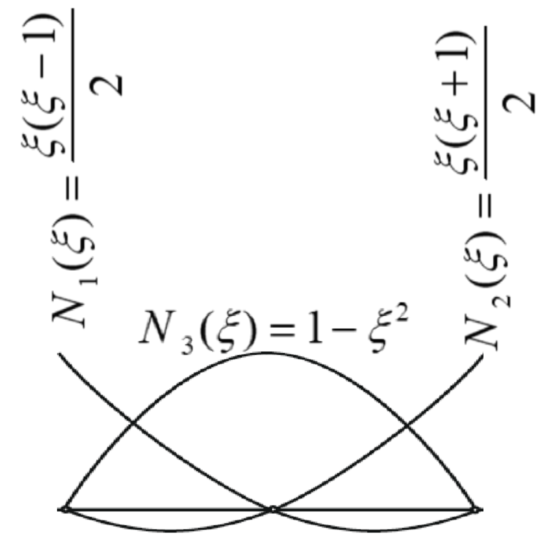
which matches $\sqrt{x}$ from asymptotic displacement solution.

- Strain

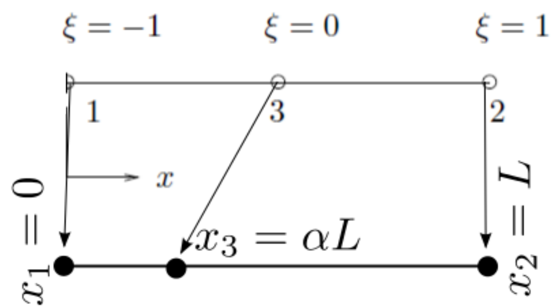
$$\left.\begin{aligned} \epsilon &= \frac{du}{dx} = \frac{du}{d\xi} / \frac{dx}{d\xi} \\ \frac{dx}{d\xi} &= L\frac{\xi+1}{2} = \sqrt{xL} \\ \frac{du}{d\xi} &= u_1\left[\frac{2\xi-1}{2}\right] + u_2\left[\frac{2\xi+1}{2}\right] - 2u_3\xi \end{aligned}\right\} \Rightarrow$$

$$\boxed{\epsilon = \frac{1}{\sqrt{xL}}\left(-\frac{3}{2}u_1 - \frac{1}{2}u_2 + 2u_3\right) + \frac{1}{L}(2u_1 + 2u_2 - 4u_3)}$$

Strain field too matches asymptotic term $\frac{1}{\sqrt{r}}$



Parent element



Moving singular ε position

Strain singularity at ξ means

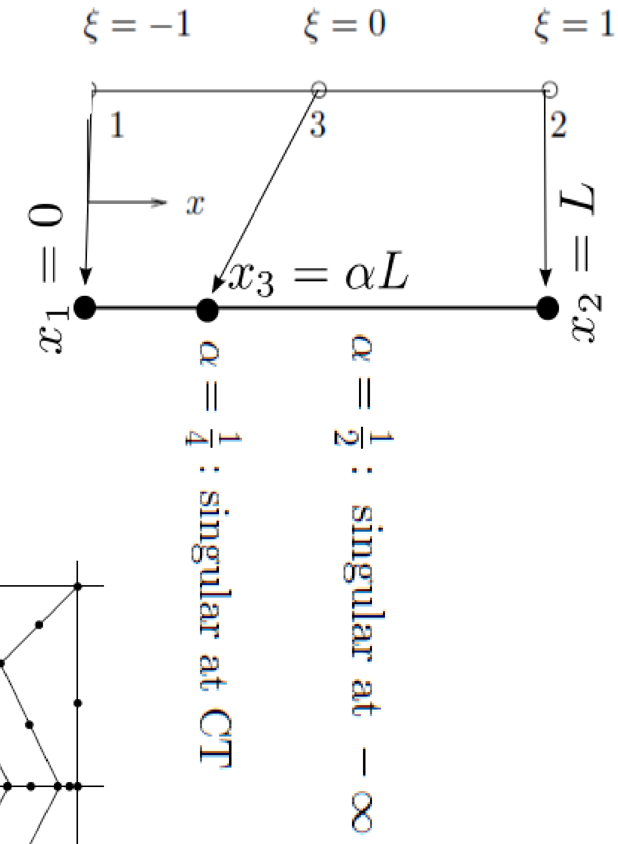
$$\frac{dx}{d\xi} = L\left\{2\xi\left(\frac{1}{2}-\alpha\right)+\frac{1}{2}\right\}$$

must be zero. Accordingly,

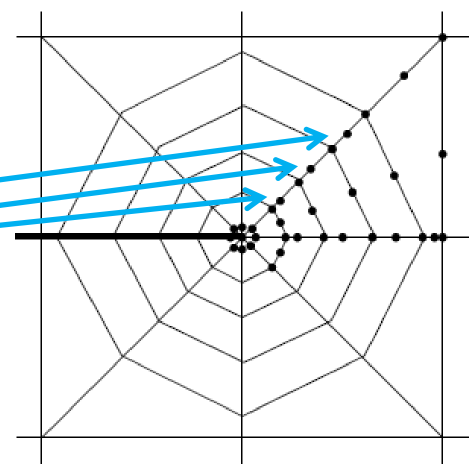
Singularity at infinity $(\xi \rightarrow -\infty)$ $\qquad \alpha -> \left(\frac{1}{2}\right)^{-}$

Singularity at crack tip $(\xi = -1)$ $\qquad \alpha = \frac{1}{4}$

Singularity inside element (not of interest) $(-1 < \xi < 0)$ $\qquad 0 < \alpha < \frac{1}{4}$



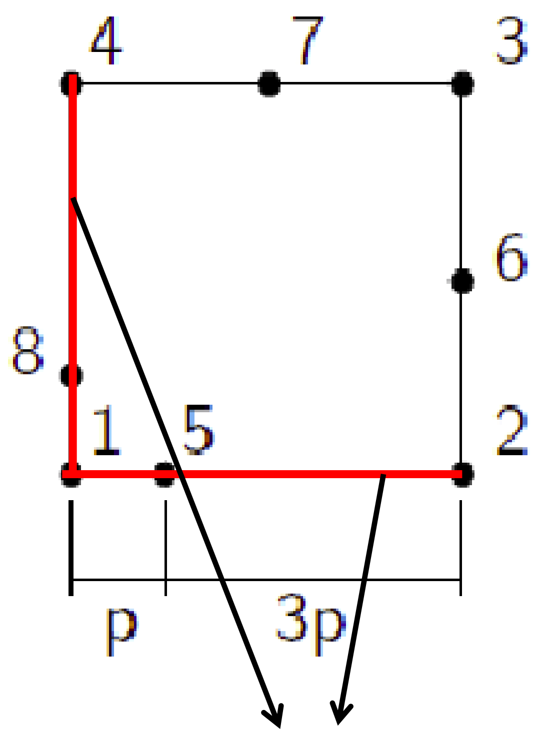
- Transition elements:
  According to this analysis
  mid nodes of next layers move to ½ point from ¼ point



Isoparametric singular elements

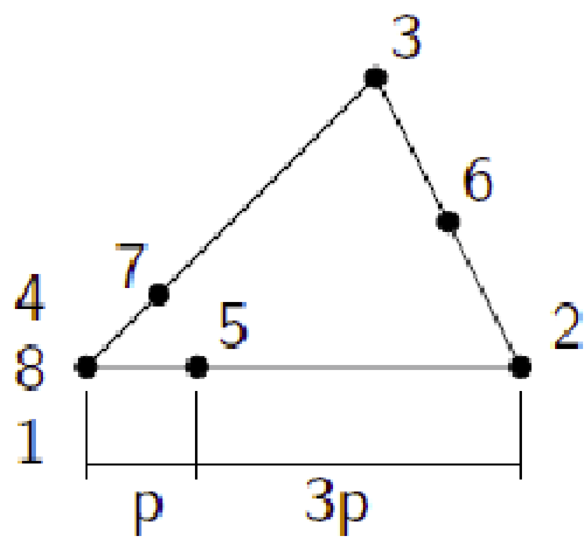
- **LEFM** $\varepsilon, \sigma : \frac{1}{\sqrt{r}}$

Quarter point
Quad element



singular form $\frac{1}{\sqrt{r}}$ only along these lines NOT recommended

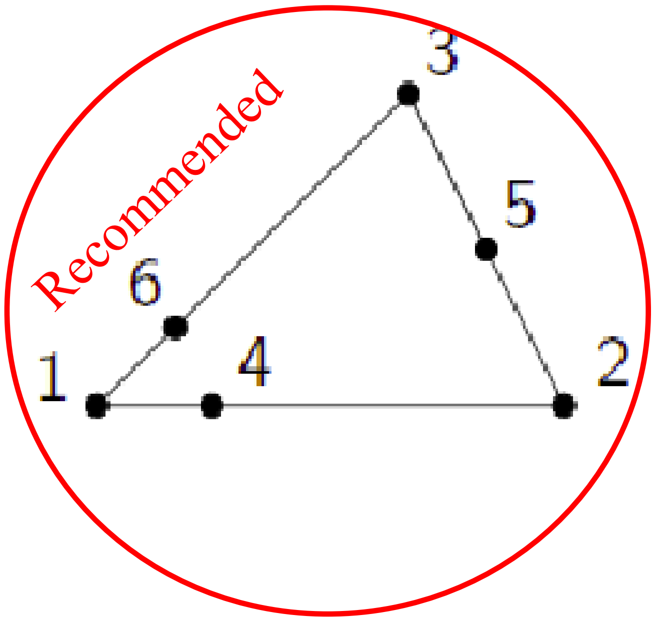
Quarter point **collapsed**
Quad element



Improvement:
- $\frac{1}{\sqrt{r}}$ from inside all element

Problem:
- Solution inaccuracy and sensitivity when opposite edge 3-6-2 is curved

Quarter point
Tri element

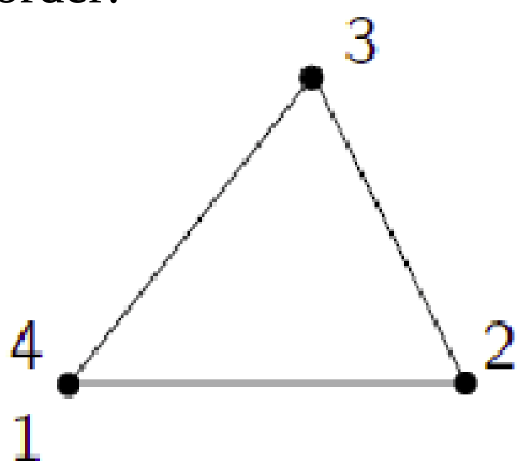


Improvement:
- Better accuracy and less mesh sensitivity

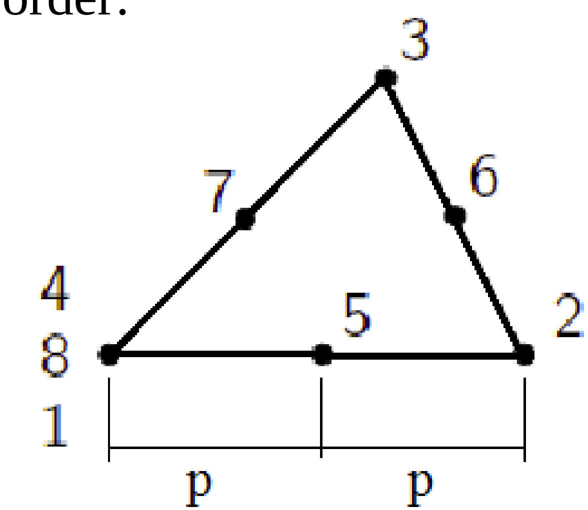
- Elastic-perfectly plastic $\varepsilon : \frac{1}{r}$

Collapsed Quad elements

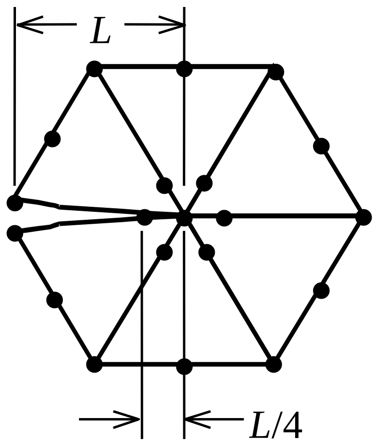
1st order:



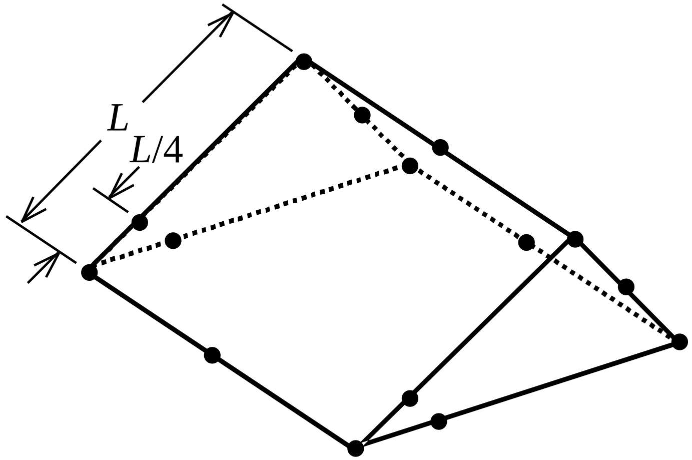
2nd order:



- Therefore, by shifting the nodes to quarter position, we approximating the stress and displacements more accurately.
- Other crack tip elements:



Triangular crack tip elements



A 3-D, wedge crack tip element