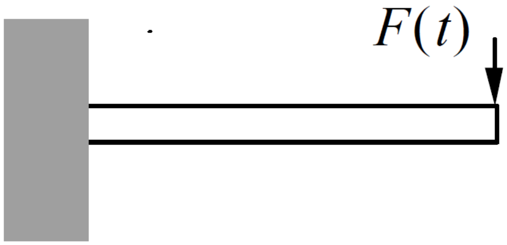
Structural Vibration and Dynamics

Structural Vibration and Dynamics

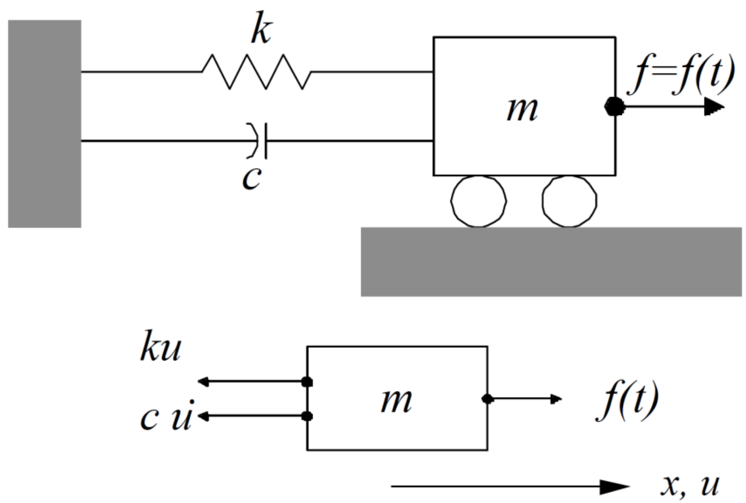
- Natural frequencies and modes
- Frequency response ($F(t)=F_o \sin \omega t$)
- Transient response ($F(t)$ arbitrary)



Natural frequencies and modes

I. Basic Equations

- Single DOF System



$$\begin{cases} m \text{ - mass} \\ k \text{ - stiffness} \\ c \text{ - damping} \\ f(t) \text{ - force} \end{cases}$$

From Newton's law of motion ($F = ma$), we have

$$m\ddot{u} = f(t) - ku - c\dot{u}, \qquad m\ddot{u} + c\dot{u} + ku = f(t),$$

where u is the displacement, $\dot{u} = du / dt$ and $\ddot{u} = d^2u / dt^2$.

Natural frequencies and modes

Free Vibration: $f(t) = 0$ and no damping ($c = 0$)

Eq. (1) becomes

$$m\ddot{u}+ku=0 \quad (2)$$

(meaning: inertia force + stiffness force = 0)

Assume:

$$u(t) = U \sin(\omega t),$$

where ω is the frequency of oscillation, U the amplitude.

Eq. (2) yields

$$-U\omega^2 m\sin(\omega t) + kU\sin(\omega t) = 0$$

Natural frequencies and modes

i.e.,

$$\left[-\omega^2 m + k\right] U = 0 \, .$$

For nontrivial solutions for U, we must have

$$\left|-\omega^2 m + k\right| = 0 \, ,$$

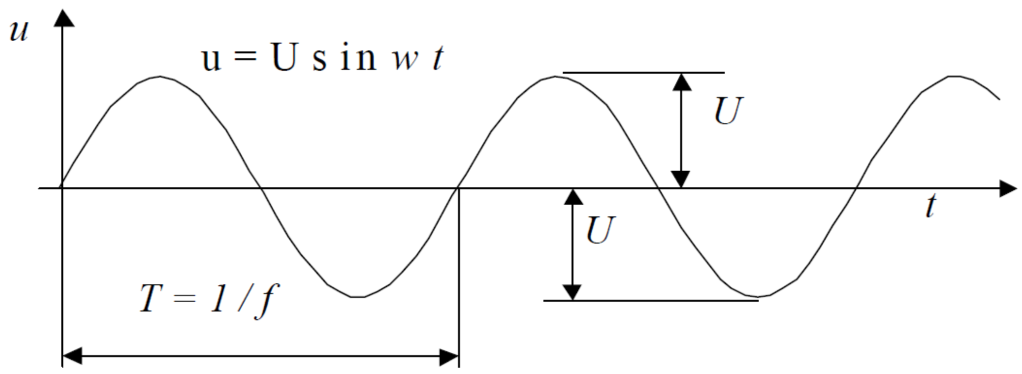
which yields

$$\omega = \sqrt{\frac{k}{m}} \, . \qquad (3)$$

This is the circular *natural frequency* of the single DOF system (rad/s). The cyclic frequency ($1/s$ = Hz) is

$$f = \frac{\omega}{2\pi} \, , \qquad (4)$$

Natural frequencies and modes



Undamped Free Vibration

With non-zero damping c, where

$$0 < c < c_c = 2m\omega = 2\sqrt{k\,m} \qquad (c_c = \text{critical damping}) \qquad (5)$$

we have the damped natural frequency:

$$\omega_d = \omega\sqrt{1-\xi^2}\,, \qquad (6)$$

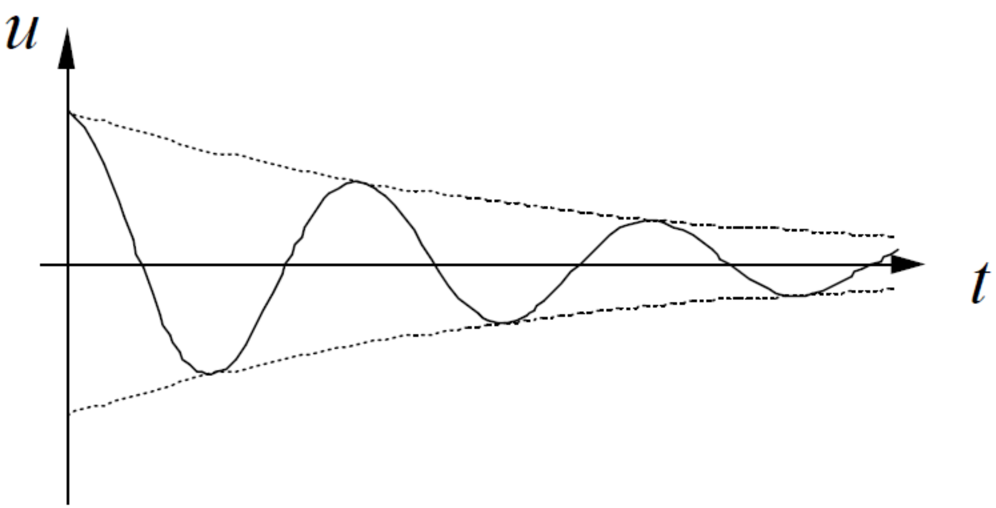
where $\xi = \dfrac{c}{c_c}$ (damping ratio).

Natural frequencies and modes

For structural damping: $0 \le \xi < 0.15$ (usually 1~5%)

$$\omega_d \approx \omega. \tag{7}$$

Thus, we can ignore damping in normal mode analysis.



Damped Free Vibration

Natural frequencies and modes

Multiple DOF System

Equation of Motion

Equation of motion for the whole structure is

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}(t)\,, \qquad (8)$$

in which:
- $\mathbf{u}$ — nodal displacement vector,
- $\mathbf{M}$ — mass matrix,
- $\mathbf{C}$ — damping matrix,
- $\mathbf{K}$ — stiffness matrix,
- $\mathbf{f}$ — forcing vector.

Physical meaning of Eq. (8):

Inertia forces + Damping forces + Elastic forces

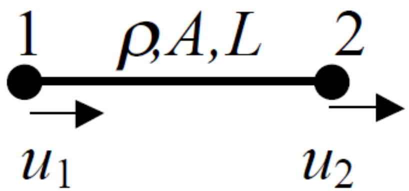
= Applied forces

Natural frequencies and modes

Mass Matrices:

Lumped mass matrix (1-D bar element):

$$m_1 = \frac{\rho AL}{2}$$



$$m_2 = \frac{\rho AL}{2}$$

Element mass matrix is found to be

$$\mathbf{m} = \underbrace{\begin{bmatrix} \dfrac{\rho AL}{2} & 0 \\ 0 & \dfrac{\rho AL}{2} \end{bmatrix}}_{\text{diagonal matrix}}$$

Natural frequencies and modes

In general, we have the *consistent mass matrix* given by

$$\mathbf{m} = \int_V \rho \mathbf{N}^T \mathbf{N} dV \qquad (9)$$

where **N** is the same shape function matrix as used for the displacement field.

This is obtained by considering the kinetic energy:

$$\begin{aligned} K &= \frac{1}{2} \dot{\mathbf{u}}^T \mathbf{m} \dot{\mathbf{u}} \qquad \left(\text{cf. } \frac{1}{2} m v^2\right) \\ &= \frac{1}{2} \int_V \rho \dot{u}^2 dV = \frac{1}{2} \int_V \rho (\dot{u})^T \dot{u} dV \\ &= \frac{1}{2} \int_V \rho (\mathbf{N} \dot{\mathbf{u}})^T (\mathbf{N} \dot{\mathbf{u}}) dV \\ &= \frac{1}{2} \dot{\mathbf{u}}^T \underbrace{\int_V \rho \mathbf{N}^T \mathbf{N} dV}_{\mathbf{m}} \dot{\mathbf{u}} \end{aligned}$$

Natural frequencies and modes

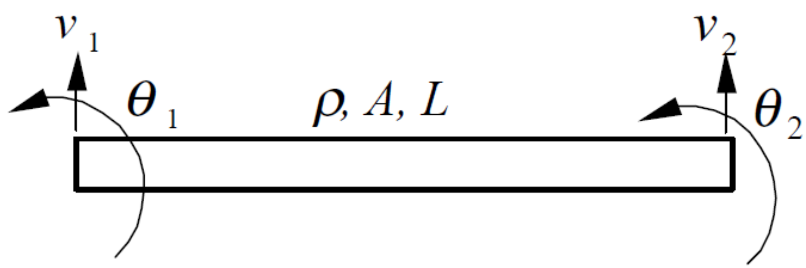
Bar Element (linear shape function):

$$\mathbf{m} = \int_V \rho \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \begin{bmatrix} 1-\xi & \xi \end{bmatrix} AL d\xi$$

$$= \rho AL \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \begin{matrix} \ddot{u}_1 \\ \ddot{u}_2 \end{matrix} \qquad (10)$$

Natural frequencies and modes

Simple Beam Element:



$$\mathbf{m} = \int_{\mathbf{V}} \rho \mathbf{N}^T \mathbf{N} dV$$

$$= \frac{\rho A L}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \begin{matrix} \ddot{v}_1 \\ \ddot{\theta}_1 \\ \ddot{v}_2 \\ \ddot{\theta}_2 \end{matrix} \qquad (11)$$

Natural frequencies and modes

II. Free Vibration

Study of the dynamic characteristics of a structure:

- natural frequencies
- normal modes (shapes)

Let $\mathbf{f}(t) = \mathbf{0}$ and $\mathbf{C} = \mathbf{0}$ (ignore damping) in the dynamic equation (8) and obtain

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0} \qquad (12)$$

Assume that displacements vary harmonically with time, that is,

$$\mathbf{u}(t) = \overline{\mathbf{u}} \sin(\omega t),$$
$$\dot{\mathbf{u}}(t) = \omega \overline{\mathbf{u}} \cos(\omega t),$$
$$\ddot{\mathbf{u}}(t) = -\omega^2 \overline{\mathbf{u}} \sin(\omega t),$$

where $\overline{\mathbf{u}}$ is the vector of nodal displacement amplitudes.

Natural frequencies and modes

Eq. (12) yields,

$$\left[\mathbf{K} - \omega^2 \mathbf{M}\right]\overline{\mathbf{u}} = \mathbf{0} \qquad (13)$$

This is a generalized eigenvalue problem (EVP).

Trivial solution: $\overline{\mathbf{u}} = \mathbf{0}$ for any values of ω (not interesting).

Nontrivial solutions: $\overline{\mathbf{u}} \neq \mathbf{0}$ only if

$$\left|\mathbf{K} - \omega^2 \mathbf{M}\right| = 0 \qquad (14)$$

This is an n-th order polynomial of ω^2, from which we can find n solutions (roots) or eigenvalues ω_i.

Natural frequencies and modes

- ω_i (i = 1, 2, …, n) are the natural frequencies (or characteristic frequencies) of the structure.
- ω_1 (the smallest one) is called the fundamental frequency.
- For each ω_i , Eq. (13) gives one solution (or eigen) vector

$$\left[\mathbf{K} - \omega_i^{\,2}\mathbf{M}\right]\bar{\mathbf{u}}_i = \mathbf{0}\,.$$

$\bar{\mathbf{u}}_i$ (i=1,2,…,n) are the *normal modes* (or *natural modes*, *mode shapes*, etc.).

Natural frequencies and modes

Properties of Normal Modes

$$\bar{\mathbf{u}}_i^T \mathbf{K} \bar{\mathbf{u}}_j = 0 \,,$$

$$\bar{\mathbf{u}}_i^T \mathbf{M} \bar{\mathbf{u}}_j = 0 \,, \qquad \text{for } i \neq j, \qquad (15)$$

if $\omega_i \neq \omega_j$. That is, modes are orthogonal (or independent) to each other with respect to **K** and **M** matrices.

Normalize the modes:

$$\bar{\mathbf{u}}_i^T \mathbf{M} \bar{\mathbf{u}}_i = 1,$$

$$\bar{\mathbf{u}}_i^T \mathbf{K} \bar{\mathbf{u}}_i = \omega_i^2 \,. \qquad (16)$$

Natural frequencies and modes

Note:

- Magnitudes of displacements (modes) or stresses in normal mode analysis have no physical meaning.
- For normal mode analysis, no support of the structure is necessary.

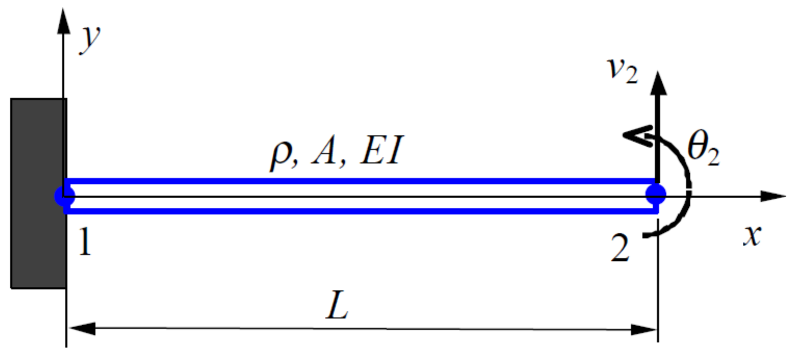
  $\omega_i = 0 \;\; \Leftrightarrow$ there are rigid body motions of the whole or a part of the structure.

  $\Rightarrow$ apply this to check the FEA model (check for mechanism or free elements in the models).

- Lower modes are more accurate than higher modes in the FE calculations (less spatial variations in the lower modes $\Rightarrow$ fewer elements/wave length are needed).

Natural frequencies and modes

Example:



$$\left[\mathbf{K}-\omega^2\mathbf{M}\right]\begin{Bmatrix}\bar{v}_2\\ \bar{\theta}_2\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix},$$

$$\mathbf{K}=\frac{EI}{L^3}\begin{bmatrix}12 & -6L\\ -6L & 4L^2\end{bmatrix},\qquad \mathbf{M}=\frac{\rho AL}{420}\begin{bmatrix}156 & -22L\\ -22L & 4L^2\end{bmatrix}.$$

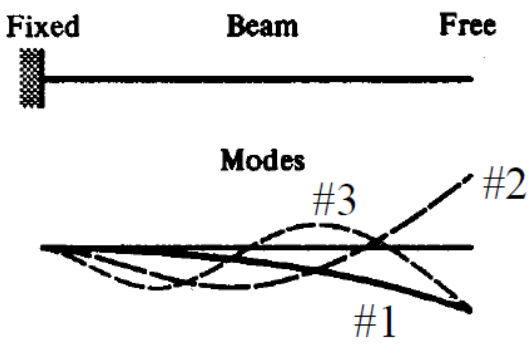
EVP:

$$\begin{vmatrix}12-156\lambda & -6L+22L\lambda\\ -6L+22L\lambda & 4L^2-4L^2\lambda\end{vmatrix}=0,$$

in which $\lambda=\omega^2\rho AL^4/420\,EI$.

Natural frequencies and modes

Solving the EVP, we obtain,



$$\omega_1 = 3.533\left(\frac{EI}{\rho AL^4}\right)^{1/2}, \quad \begin{Bmatrix} \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 1.38/L \end{Bmatrix},$$

$$\omega_2 = 34.81\left(\frac{EI}{\rho AL^4}\right)^{1/2}, \quad \begin{Bmatrix} \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ 7.62/L \end{Bmatrix}.$$

Exact solutions:

$$\omega_1 = 3.516\left(\frac{EI}{\rho AL^4}\right)^{1/2}, \quad \omega_2 = 22.03\left(\frac{EI}{\rho AL^4}\right)^{1/2}.$$

We can see that mode 1 is calculated much more accurately than mode 2, with one beam element.

III. Frequency Response Analysis

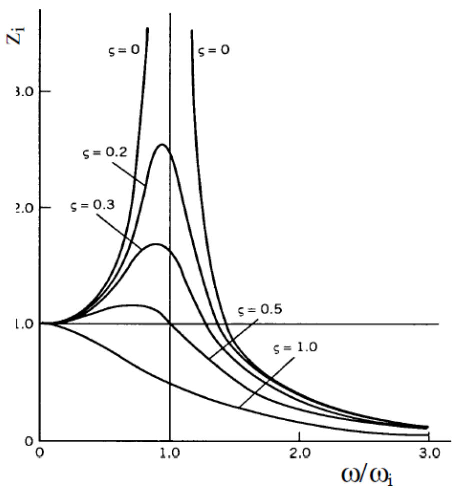
$$\mathbf{M\ddot{u}} + \mathbf{C\dot{u}} + \mathbf{Ku} = \underbrace{\mathbf{F}\sin\omega t}_{\text{Harmonic loading}} \qquad (25)$$

Modal method: Apply the modal equations,

$$\ddot{z}_i + 2\xi_i\omega_i\dot{z}_i + \omega_i^2 z_i = p_i \sin\omega t, \quad i=1,2,\ldots,m. \qquad (26)$$

These are 1-D equations. Solutions are

$$z_i(t) = \frac{p_i/\omega_i^2}{\sqrt{(1-\eta_i^2)^2 + (2\xi_i\eta_i)^2}}\sin(\omega t - \theta_i), \qquad (27)$$



where

$$\begin{cases} \theta_i = \arctan\dfrac{2\xi_i\eta_i}{1-\eta_i^2}, \text{ phase angle} \\ \eta_i = \omega/\omega_i, \\ \xi_i = \dfrac{c_i}{c_c} = \dfrac{c_i}{2m\omega_i}, \text{ damping ratio} \end{cases}$$

Frequency Response Analysis

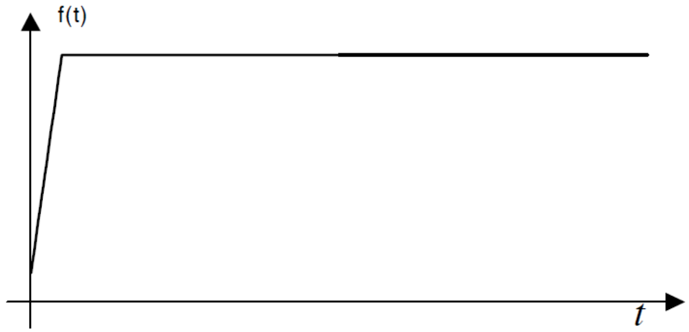
Direct Method: Solve Eq. (25) directly, that is, calculate the inverse. With $\mathbf{u} = \overline{\mathbf{u}} e^{i\omega t}$ (complex notation), Eq. (25) becomes

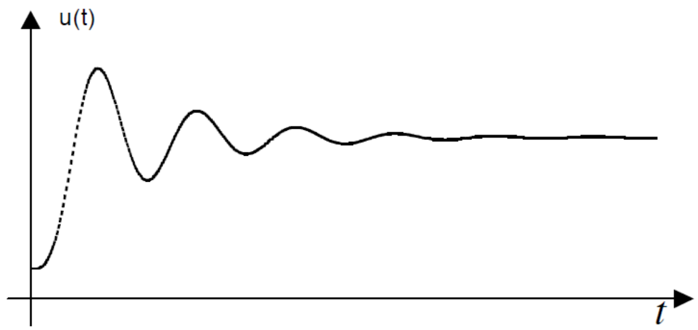
$$\left[\mathbf{K} + i\omega \mathbf{C} - \omega^2 \mathbf{M}\right]\overline{\mathbf{u}} = \overline{\mathbf{F}}.$$

This equation is expensive to solve and matrix is ill-conditioned if ω is close to any ω_i.

IV. Transient Response Analysis

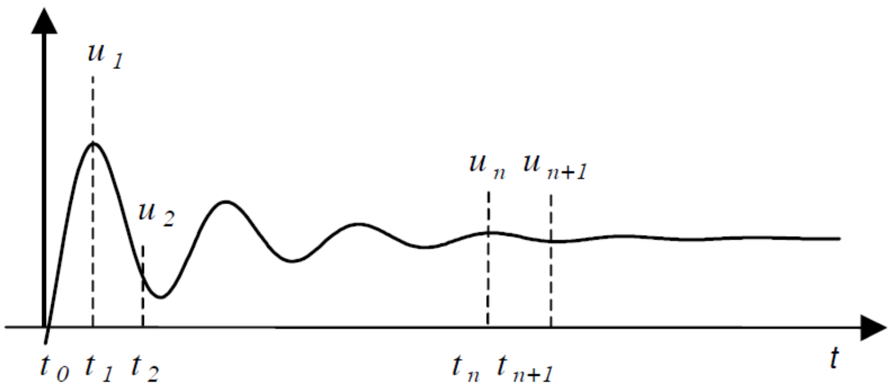
- Structure response to *arbitrary, time-dependent loading*.





Transient Response Analysis

Compute responses by integrating through time:



Equation of motion at instance t_n, $n = 0, 1, 2, 3, \cdots$:

$$\mathbf{M}\ddot{\mathbf{u}}_n + \mathbf{C}\dot{\mathbf{u}}_n + \mathbf{K}\mathbf{u}_n = \mathbf{f}_n.$$

Time increment: $\Delta t = t_{n+1} - t_n$, $n = 0, 1, 2, 3, \cdots$.

There are two categories of methods for transient analysis.

A. *Direct Methods* (Direct Integration Methods)

- ***Central Difference Method***

Approximate using finite difference:

$$\dot{\mathbf{u}}_n = \frac{1}{2\Delta t}(\mathbf{u}_{n+1} - \mathbf{u}_{n-1}),$$
$$\ddot{\mathbf{u}}_n = \frac{1}{(\Delta t)^2}(\mathbf{u}_{n+1} - 2\mathbf{u}_n + \mathbf{u}_{n-1})$$

Dynamic equation becomes,

$$\mathbf{M}\left[\frac{1}{(\Delta t)^2}(\mathbf{u}_{n+1} - 2\mathbf{u}_n + \mathbf{u}_{n-1})\right] + \mathbf{C}\left[\frac{1}{2\Delta t}(\mathbf{u}_{n+1} - \mathbf{u}_{n-1})\right] + \mathbf{K}\mathbf{u}_n = \mathbf{f}_n,$$

which yields,

$$\mathbf{A}\mathbf{u}_{n+1} = \mathbf{F}(t)$$

Transient Response Analysis

$$\mathbf{A}\mathbf{u}_{n+1} = \mathbf{F}(t)$$

$$\begin{cases} \mathbf{A} = \dfrac{1}{(\Delta t)^2}\mathbf{M} + \dfrac{1}{2\Delta t}\mathbf{C}, \\ \mathbf{F}(t) = \mathbf{f}_n - \left[\mathbf{K} - \dfrac{2}{(\Delta t)^2}\mathbf{M}\right]\mathbf{u}_n - \left[\dfrac{1}{(\Delta t)^2}\mathbf{M} - \dfrac{1}{2\Delta t}\mathbf{C}\right]\mathbf{u}_{n-1}. \end{cases}$$

$\mathbf{u}_{n+1}$ is calculated from $\mathbf{u}_n$ & $\mathbf{u}_{n-1}$, and solution is marching from $t_0, t_1, \cdots t_n, t_{n+1}, \cdots$, until convergent.

This method is *unstable* if Δt is too large.

Transient Response Analysis

- ***Newmark Method***:

Use approximations:

$$\mathbf{u}_{n+1} \approx \mathbf{u}_n + \Delta t \dot{\mathbf{u}}_n + \frac{(\Delta t)^2}{2}\left[(1-2\beta)\ddot{\mathbf{u}}_n + 2\beta \ddot{\mathbf{u}}_{n+1}\right], \rightarrow (\ddot{\mathbf{u}}_{n+1} = \cdots)$$

$$\dot{\mathbf{u}}_{n+1} \approx \dot{\mathbf{u}}_n + \Delta t\left[(1-\gamma)\ddot{\mathbf{u}}_n + \gamma \ddot{\mathbf{u}}_{n+1}\right],$$

where β & γ are chosen constants. These lead to

$$\mathbf{A}\mathbf{u}_{n+1} = \mathbf{F}(t)$$

where

$$\mathbf{A} = \mathbf{K} + \frac{\gamma}{\beta \Delta t}\mathbf{C} + \frac{1}{\beta (\Delta t)^2}\mathbf{M},$$

$$\mathbf{F}(t) = f(\mathbf{f}_{n+1}, \gamma, \beta, \Delta t, \mathbf{C}, \mathbf{M}, \mathbf{u}_n, \dot{\mathbf{u}}_n, \ddot{\mathbf{u}}_n).$$

Transient Response Analysis

This method is *unconditionally stable* if

$$2\beta \geq \gamma \geq \frac{1}{2}.$$

$$\text{e.g.,} \quad \gamma = \frac{1}{2}, \ \beta = \frac{1}{4}$$

which gives the constant average acceleration method.

Direct methods can be expensive! (the need to compute $\mathbf{A}^{-1}$, often repeatedly for each time step).

B. Modal Method

First, do the transformation of the dynamic equations using the modal matrix before the time marching:

$$\mathbf{u} = \sum_{i=1}^{m} \overline{\mathbf{u}}_i z_i(t) = \Phi \mathbf{z},$$

$$\ddot{z}_i + 2\xi_i \omega_i \dot{z}_i + \omega_i z_i = p_i(t), \qquad i = 1, 2, \cdots, m.$$

Then, solve the uncoupled equations using an integration method. Can use, e.g., 10%, of the total modes (m= n/10).

- Uncoupled system,
- Fewer equations,
- No inverse of matrices,
- More efficient for large problems.

Comparisons of the Methods

Direct Methods	*Modal Method*
• Small model	• Large model
• More accurate (with small Δt)	• Higher modes ignored
• Single loading	• Multiple loading
• Shock loading	• Periodic loading
• …	• …

Transient Response Analysis

Cautions in Dynamic Analysis

- *Symmetry*: It should not be used in the dynamic analysis (normal modes, etc.) because symmetric structures can have antisymmetric modes.

- Mechanism, rigid body motion means $\omega = 0$. Can use this to check FEA models to see if they are properly connected and/or supported.

- Input for FEA: loading $F(t)$ or $F(\omega)$ can be very complicated in real applications and often needs to be filtered first before used as input for FEA.

Examples

Impact, drop test, etc.

Reference:

1- Lecture Notes: INTRODUCTION TO THE FINITE ELEMENT METHOD, Yijun Liu, University of Cincinnati, 2003