Finite Element Analysis of Boundary Value Problem

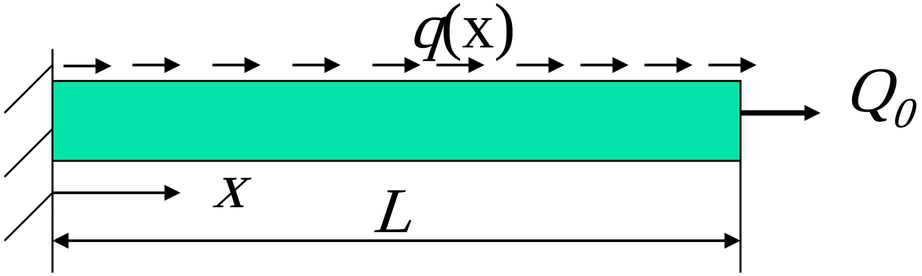
FE Analysis of 1D Bars

The DE is in the form of

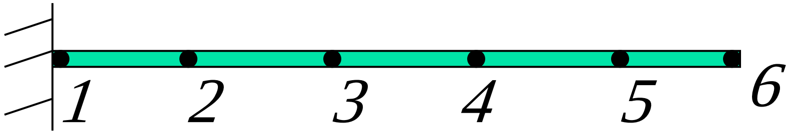
$$-\frac{d}{dx}(EA\frac{du}{dx})-q=0$$

q is the distributed load and Q_0 is the axial force.

$$u(0)=u_0, \quad (EA\frac{du}{dx})_{x=L}=Q_0$$



Physical Model

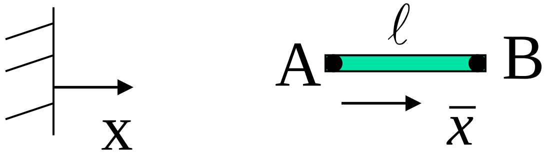


FE Model

FE Analysis of 1D Bars

Weak form

In FE analysis, we seek an approximation solution over each element.



$$Q_1 = -EA\left.\frac{du}{dx}\right|_{x=x_A}$$

$$Q_2 = EA\left.\frac{du}{dx}\right|_{x=x_B}$$

$$\int_{x_A}^{x_B}\left(EA\frac{dw}{dx}\frac{du}{dx}-wq\right)dx - w(x_A)Q_A - w(x_B)Q_B = 0$$

$$B(w,u)=\int_{x_A}^{x_B}\left(EA\frac{dw}{dx}\frac{du}{dx}\right)dx$$

$$\Rightarrow \quad B(w,u)=l(w)$$

$$l(w)=\int_{x_A}^{x_B} wqdx + w(x_A)Q_A + w(x_B)Q_B$$

FE Analysis of 1D Bars

Approximation of the solution

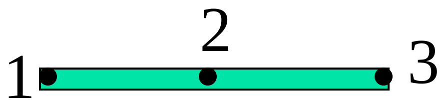
1- The approximation solution should be continuous and differentiable as required by the weak form. (nonzero coefficient matrix)

2- It should be a complete polynomial (capture all possible States, e.g. constant, linear, ….)

3- It should be an interpolant of variables at the nodes (satisfy EBCs)

First order

$$\begin{cases} U = a + bx, \quad U(x_1) = u_1, U(x_2) = u_2 \\ U = N_1 u_1 + N_2 u_2 \\ N_1 = 1 - \bar{x}/\ell, \quad N_2 = \bar{x}/\ell \end{cases}$$



Second Order

$$\begin{cases} U = a + bx + cx^2, \quad U(x_1) = u_1, U(x_2) = u_2, U(x_3) = u_3 \\ U = N_1 u_1 + N_2 u_2 + N_3 u_3 \\ N_1 = (1 - \bar{x}/\ell)(1 - 2\bar{x}/\ell), \quad N_2 = 4\bar{x}/\ell(1 - \bar{x}/\ell), \quad N_3 = -\bar{x}/\ell(1 - 2\bar{x}/\ell) \end{cases}$$

FE Analysis of 1D Bars

FE Model

$$u \approx U = \sum_{j=1}^{n} u_j N_j \qquad \text{and} \qquad \int_{x_A}^{x_B} \left(EA \frac{dw}{dx}\frac{du}{dx} - wq \right) dx - w(x_A)Q_A - w(x_B)Q_B = 0$$

$$w = N_j$$

If $n > 2$ then the above integral should modify to include interior nodal forces

$$\left\{ \begin{array}{l} \displaystyle\int_{x_A}^{x_B} \left(EA \frac{dN_1}{dx} \sum_{j=1}^{n} u_j \frac{dN_j}{dx} - N_1 q \right) dx - \sum_{j=1}^{n} N_1(x_j)Q_j = 0 \\ \displaystyle\int_{x_A}^{x_B} \left(EA \frac{dN_2}{dx} \sum_{j=1}^{n} u_j \frac{dN_j}{dx} - N_2 q \right) dx - \sum_{j=1}^{n} N_2(x_j)Q_j = 0 \\ \vdots \\ \displaystyle\int_{x_A}^{x_B} \left(EA \frac{dN_n}{dx} \sum_{j=1}^{n} u_j \frac{dN_j}{dx} - N_n q \right) dx - \sum_{j=1}^{n} N_n(x_j)Q_j = 0 \end{array} \right. \Longrightarrow \sum_{j=1}^{n} K_{ij} u_j - f_i - Q_i = 0$$

$$i = 1, 2, \ldots, n$$

Stiffness matrix

Force vector

Primary nodal DOF

Secondary nodal DOF

FE Analysis of 1D Bars

FE Model

where $$K_{ij} = \int_{x_A}^{x_B} \left(EA \frac{dN_i}{dx} \frac{dN_j}{dx} \right) dx = B(N_i, N_j)$$

$$f_i = \int_{x_A}^{x_B} qN_i dx = l(N_i)$$

Note ⟶ $$\sum_{j=1}^{n} N_j(x_i) Q_j = Q_i$$

Note that the problem has *2n* unknowns for each element, i.e. u_i and Q_i, so it cannot be solved without having another n conditions. Some of these will be provided by BCs and the remainder by balance of the secondary variables (forces) at node common to several element. (assembling process)

FE Analysis of 1D Bars

FE Model (Linear Element)

$$U = N_1 u_1 + N_2 u_2$$

$$N_1 = 1 - \bar{x}/\ell, \quad N_2 = \bar{x}/\ell$$

$$K_{11} = \int_0^\ell (EA)(-1/\ell)(-1/\ell)dx = AE/\ell$$

$$K_{12} = \int_0^\ell (EA)(-1/\ell)(1/\ell)dx = -AE/\ell$$

$$K_{22} = \int_0^\ell (EA)(1/\ell)(1/\ell)dx = AE/\ell$$

$$f_1 = \int_0^\ell q(1 - x/\ell)dx = 1/2q\ell$$

$$f_2 = \int_0^\ell q(x/\ell)dx = 1/2q\ell$$

Eventually for
Linear shape function

$$[K^e] = \frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \quad \{f\} = \frac{q\ell}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

FE Analysis of 1D Bars

FE Model (Quadratic Element)

$$U = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$N_1 = (1 - x/\ell)(1 - 2x/\ell), \quad N_2 = 4x/\ell(1 - x/\ell), \quad N_3 = -x/\ell(1 - 2x/\ell)$$

$$K_{11} = \int_0^\ell (EA)(-3/\ell + 4x/\ell^2)(-3/\ell + 4x/\ell^2)dx = 7AE/3\ell$$

$$K_{12} = K_{21} = \int_0^\ell (EA)(-3/\ell + 4x/\ell^2)(4/\ell - 8x/\ell)dx = -8AE/3\ell$$

. . . .

$$f_1 = \int_0^\ell q(1 - 3x/\ell + 2(x/\ell)^2)dx = 1/6q\ell$$

$$f_2 = \int_0^\ell q(4x/\ell)(1 - x/\ell)dx = 4/6q\ell$$

. . . .

FE Analysis of 1D Bars

FE Model (Quadratic Element)

For quadratic Shape function

$$[K^e] = \frac{AE}{3\ell}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}; \quad \{f\} = \frac{q\ell}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}$$

FE Analysis of 1D Bars

Assembly (or connectivity) of elements

In driving the element equation

-Isolate the element from mesh

-Formulate weak form (variational form)

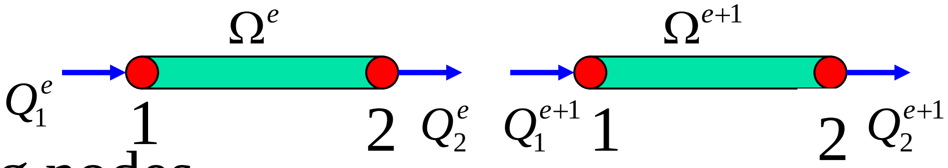
-Developed its finite element model

To solve the total problem

-put the element in its original position

-Impose continuity of PVs at nodal points

$$u_n^e = u_1^{e+1}$$



-Balance of SVs at connecting nodes

$$Q_n^e + Q_1^{e+1} = \begin{cases} 0 & \text{if no external point source is applied} \\ Q_0 & \text{if an external point source of } Q_0 \text{ is applied.} \end{cases} \quad (*)$$

FE Analysis of 1D Bars

Assembly (or connectivity) of elements (For linear element $n=2$)

The interelement continuity of the primary variables is imposed by renaming the two variable u_n^e and u_1^{e+1} at $x=x_N$ as one and same, namely the value of u at the global node N:

$$u_n^e = u_1^{e+1} = U_N$$

where $N=(n-1)e+1$

For a mesh of E linear finite elements ($n=2$):

$$\begin{aligned} u_1^1 &= U_1 \\ u_2^1 &= u_1^2 = U_2 \\ u_2^2 &= u_1^3 = U_3 \\ &\vdots \\ u_2^{E-1} &= u_1^E = U_E \\ u_2^E &= U_{E+1} \end{aligned}$$

FE Analysis of 1D Bars

Assembly (or connectivity) of elements (For linear element *n=2*)

To enforce balance of secondary variables Q_i^e , eq. (*), we must add *n*th equation of the element Ω^e to the first equation of the element Ω^{e+1} *:*

$$\sum_{j=1}^{n} K_{nj}^{e} u_j^e = f_n^e + Q_n^e$$

and

$$\sum_{j=1}^{n} K_{1j}^{e+1} u_j^{e+1} = f_1^{e+1} + Q_1^{e+1}$$

to give

$$\sum_{j=1}^{n} (K_{nj}^{e} u_j^e + K_{1j}^{e+1} u_j^{e+1}) = f_n^e + f_1^{e+1} + (Q_n^e + Q_1^{e+1})$$

$$= f_n^e + f_1^{e+1} + Q_0$$

This process reduces the number of equations from *2E* to *E+1*.

FE Analysis of 1D Bars

Assembly (or connectivity) of elements (For linear element *n=2*)

The first equation of the first element and the last equation of the last element will remain unchanged, except for renaming of the primary variables. The left-hand of the equation can be written in terms of the global nodal values as

$$
\begin{aligned}
&(K^e_{n1}u^e_1 + K^e_{n2}u^e_2 + \cdots + K^e_{nn}u^e_n) + (K^{e+1}_{11}u^{e+1}_1 + K^{e+1}_{12}u^{e+1}_2 + \cdots + K^{e+1}_{1n}u^{e+1}_n) \\
&\quad = (K^e_{n1}U_N + K^e_{n2}U_{N+1} + \cdots + K^e_{nn}U_{N+n-1}) + \\
&\quad\quad (K^{e+1}_{11}U_{N+n-1} + K^{e+1}_{12}U_{N+n} + \cdots + K^{e+1}_{1n}U_{N+2n-2}) \\
&\quad = K^e_{n1}U_N + K^e_{n2}U_{N+1} + \cdots + K^e_{n(n-1)}U_{N+n-2} + \\
&\quad\quad (K^e_{nn} + K^{e+1}_{11})U_{N+n-1} + K^{e+1}_{12}U_{N+n} + \cdots + K^{e+1}_{1n}U_{N+2n-2}
\end{aligned}
$$

where $N = (n-1)e + 1$

FE Analysis of 1D Bars

Assembly (or connectivity) of elements (For linear element *n=2*)

For a mesh of *E* linear finite elements (*n=2*):

$$K_{11}^{1}U_{1}+K_{12}^{1}U_{2}=f_{1}^{1}+Q_{1}^{1}\qquad (unchanged)$$

$$K_{21}^{1}U_{1}+(K_{22}^{1}+K_{11}^{2})U_{2}+K_{12}^{2}U_{3}=f_{2}^{1}+f_{1}^{2}+Q_{2}^{1}+Q_{1}^{2}$$

$$K_{21}^{2}U_{2}+(K_{22}^{2}+K_{11}^{3})U_{3}+K_{12}^{3}U_{4}=f_{2}^{2}+f_{1}^{3}+Q_{2}^{2}+Q_{1}^{3}$$

$$\vdots$$

$$K_{21}^{E-1}U_{E-1}+(K_{22}^{E-1}+K_{11}^{E})U_{E}+K_{12}^{E}U_{E+1}=f_{2}^{E-1}+f_{1}^{E}+Q_{2}^{E-1}+Q_{1}^{E}$$

$$K_{21}^{E}U_{E}+K_{22}^{E}U_{E+1}=f_{2}^{E}+Q_{2}^{E}\qquad (unchanged)$$

FE Analysis of 1D Bars

Assembly (or connectivity) of elements (For linear element *n=2*)

In matrix form

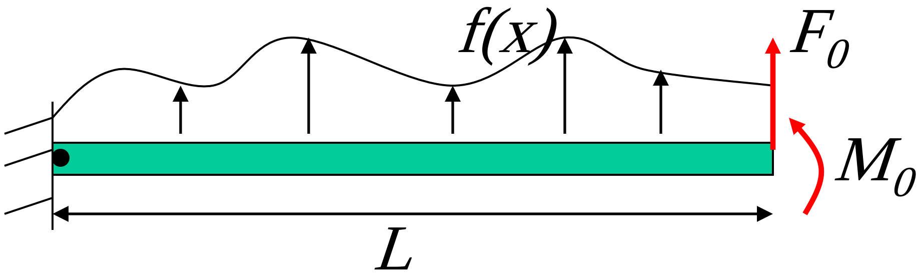
$$\begin{bmatrix} K_{11}^{1} & K_{12}^{1} & & & \\ K_{12}^{1} & K_{22}^{1}+K_{11}^{2} & K_{12}^{2} & & 0 \\ & K_{21}^{2} & K_{22}^{2}+K_{11}^{3} & & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ 0 & & & K_{22}^{E-1}+K_{11}^{E} & K_{12}^{E} \\ & & & & K_{22}^{E} \end{bmatrix} \begin{Bmatrix} U_{1} \\ U_{2} \\ U_{3} \\ \ldots \\ U_{E} \\ U_{E+1} \end{Bmatrix}$$

$$= \begin{Bmatrix} f_{1}^{1} \\ f_{2}^{1}+f_{1}^{2} \\ f_{2}^{2}+f_{1}^{3} \\ \ldots \\ f_{2}^{E-1}+f_{1}^{E} \\ f_{2}^{E} \end{Bmatrix} + \begin{Bmatrix} Q_{1}^{1} \\ Q_{2}^{1}+Q_{1}^{2} \\ Q_{2}^{2}+Q_{1}^{3} \\ \ldots \\ Q_{2}^{E-1}+Q_{1}^{E} \\ Q_{2}^{E} \end{Bmatrix}$$

FE Analysis of BEAM

The DE is in the form of

$$\frac{d^2}{dx^2}\left(b\frac{d^2w}{dx^2}\right) = f(x) \qquad 0 < x < L$$



FE Analysis of BEAM

Weak form

$$\int_{x_e}^{x_{e+1}} v\left(\frac{d^2}{dx^2}(b\frac{d^2w}{dx^2}) - f\right)dx = 0$$

$$Q_1^e = \left[\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)\right]_{x_e} ;Q_2^e = \left[b\frac{d^2w}{dx^2}\right]_{x_e}$$

$$Q_3^e = -\left[\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)\right]_{x_{e+1}} ;Q_4^e = -\left[b\frac{d^2w}{dx^2}\right]_{x_{e+1}}$$

or

BCs

$$\int_{x_e}^{x_{e+1}}\left(b\frac{d^2v}{dx^2}\frac{d^2w}{dx^2} - vf\right)dx - v(x_e)Q_1^e - \left(-\frac{dv}{dx}\right)_{x_e}Q_2^e - v(x_{e+1})Q_3^e - \left(-\frac{dv}{dx}\right)_{x_{e+1}}Q_4^e = 0$$

where

$$B(v,w) = \int_{x_e}^{x_{e+1}}\left(b\frac{d^2v}{dx^2}\frac{d^2w}{dx^2}\right)dx$$

$$l(v) = \int_{x_e}^{x_{e+1}} vf\,dx + v(x_e)Q_1^e + \left(-\frac{dv}{dx}\right)_{x_e}Q_2^e + v(x_{e+1})Q_3^e + \left(-\frac{dv}{dx}\right)_{x_{e+1}}Q_4^e$$

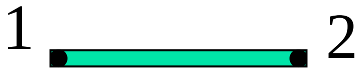
FE Analysis of BEAM

Approximation of the solution

1- The approximation solution should be continuous and differentiable as required by the weak form. (nonzero coefficient matrix)

2- It should be a complete polynomial (capture all possible States, e.g. constant, linear, ….)

3- It should be an interpolant of variables at the nodes (satisfy EBCs)



First order

$$w = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

$$w(x_e) = w_1, w(x_{e+1}) = w_2, \theta(x_e) = \theta_1, \theta(x_{e+1}) = \theta_2$$

or

$$w = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

$$u_1^e = w(x_e), u_2^e = -\left.\frac{dw}{dx}\right|_{x_e} ; u_3^e = w(x_{e+1}), u_4^e = -\left.\frac{dw}{dx}\right|_{x_{e+1}}$$

FE Analysis of BEAM

Shape Functions

Calculating Ci and substituting in the equation for w

$$w^e(x) = \sum_{j=1}^{4} u_j^e N_j$$

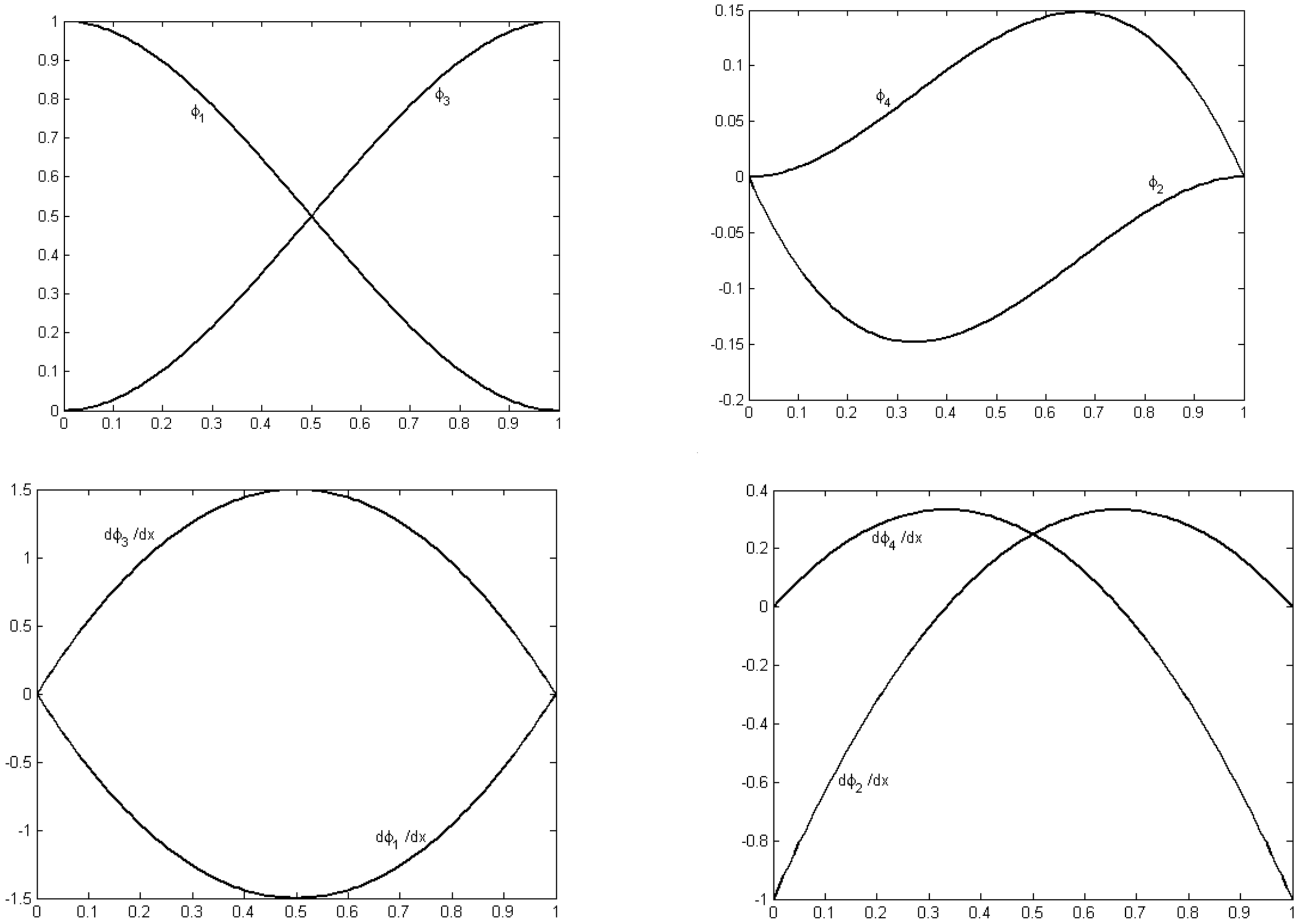
The interpolation functions in term of local coordinates are

$$N_1 = 1 - 3\left(\frac{x}{h}\right)^2 + 2\left(\frac{x}{h}\right)^3 ; N_2 = -x\left(1 - \frac{x}{h}\right)^2$$

$$N_3 = 3\left(\frac{x}{h}\right)^2 - 2\left(\frac{x}{h}\right)^3 ; N_4 = -x\left[\left(\frac{x}{h}\right)^2 - \frac{x}{h}\right]$$

FE Analysis of BEAM

Hermite cubic interpolation function



FE Analysis of BEAM

FE Model

$$\sum_{j=1}^{4}\left(\underbrace{\int_{x_e}^{x_{e+1}} b\,\frac{d^2N_i}{dx^2}\,\frac{d^2N_j}{dx^2}\,dx}\right)u_j-\left(\underbrace{\int_{x_e}^{x_{e+1}} N_i\,f dx}+Q_i^e\right)=0$$

or

$$\sum_{j=1}^{4}K_{ij}u_j-F_i=0$$

For b=EI constant and also a constant f over the element.

$$[K]=\frac{2EI}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h \\ -3h & 2h^2 & 3h & h^2 \\ -6 & 3h & 6 & 3h \\ -3h & h^2 & 3h & 2h^2 \end{bmatrix};\qquad \{F\}=\frac{fh}{12}\begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}+\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

FE Analysis of 1D FIN

Model Boundary Value Problem

The DE is in the form of

$$-\frac{d}{dx}(kA\frac{dT}{dx}) + P\beta T = Aq + P\beta T_{\infty}$$

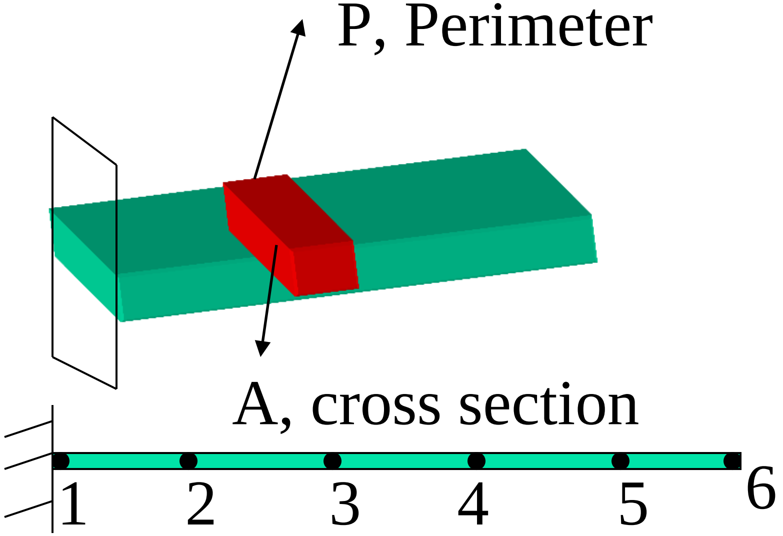
k is thermal conductivity

β is convection heat transfer coefficient

T_{∞} is the ambient temperature

q is the heat energy generated per unit volume

$$T(0) = T_0, \qquad Q = -kA\frac{\partial T}{\partial x} = Q_0$$



Physical Model

FE Model

FE Analysis of 1D FIN

Weak form

$$K_{ij} = \int_{x_A}^{x_B} \left(kA \frac{dN_i}{dx} \frac{dN_j}{dx} + P\beta N_i N_j \right) dx$$

$$f_i = \int_{x_A}^{x_B} N_i (qA + P\beta T_\infty) dx$$

$$Q_1^e = \left(-kA \frac{dT}{dx} \right)_{x_A} \quad ; \quad Q_2^e = \left(-kA \frac{dT}{dx} \right)_{x_B}$$

Assume the lateral surfaces of the bar are isolated and the BCs

$$-\frac{d}{dx}(kA\frac{dT}{dx}) = Aq$$

$$T(0) = T_1, \quad T(L) = T_2$$

FE Analysis of 1D FIN

Approximation of the solution

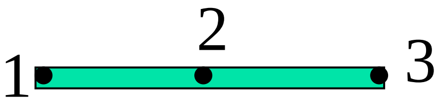
1- the approximation solution should be continuous and differentiable as required by the weak form. (nonzero coefficient matrix)

2- it should be a complete polynomial (capture all possible States, e.g. constant, linear, ….)

3- it should be an interpolant of variables at the nodes (satisfy EBCs)

First order

$$\left\{\begin{array}{l} T = a + bx, \quad T(x_1) = T_1, T(x_2) = T_2 \\ T = N_1 T_1 + N_2 T_2 \\ N_1 = 1 - \bar{x}/\ell, \quad N_2 = \bar{x}/\ell \end{array}\right.$$



Second Order

$$\left\{\begin{array}{l} T = a + bx + cx^2, \quad T(x_1) = T_1, T(x_2) = T_2, T(x_3) = T_3 \\ T = N_1 T_1 + N_2 T_2 + N_3 T_3 \\ N_1 = (1 - \bar{x}/\ell)(1 - 2\bar{x}/\ell), \quad N_2 = 4\bar{x}/\ell(1 - \bar{x}/\ell), \quad N_3 = -\bar{x}/\ell(1 - 2\bar{x}/\ell) \end{array}\right.$$

FE Analysis of 1D FIN

FE Model

Evaluating the integral using linear shape function

$$[K^e]\{T^e\} = \{f^e\} + \{Q^e\}$$

$$\frac{kA}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1^e \\ T_2^e \end{Bmatrix} = \frac{Aq\ell}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix}$$

For a uniform mesh $\ell = L/N$ and after assembling

$$\frac{kA}{\ell}\begin{bmatrix} 1 & -1 & 0 & \dots \\ -1 & 2 & -1 & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{N+1} \end{Bmatrix} = \frac{Aq\ell}{2}\begin{Bmatrix} 1 \\ 2 \\ \vdots \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ \vdots \\ Q_1^N \end{Bmatrix}$$

FE Analysis of 1D FIN

FE Model

Boundary conditions at nodes *1* and *N+1*

$$T_1 = T_1$$

$$T_{N+1} = T_{N+1}$$

Heat balance at global nodes *2,3,….,N*

$$Q_2^{e-1} + Q_1^{e} = 0 \quad for \quad e = 2,3,\ldots,N$$

After applying the above conditions:

$$\frac{kA}{\ell}\begin{bmatrix} 1 & -1 & 0 & \ldots \\ -1 & 2 & -1 & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{n+1} \end{Bmatrix} = \frac{Aq\ell}{2}\begin{Bmatrix} 1 \\ 2 \\ \vdots \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ 0 \\ \vdots \\ Q_1^N \end{Bmatrix}$$

Virtual work as the ‘weak form’ of equilibrium equations for analysis of solids

In a general three-dimensional continuum the equilibrium equations of an elementary volume can be written in terms of the components of the symmetric cartesian stress tensor as

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{xz}}{\partial z} + b_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial z} + b_y = 0 \\ \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} + b_z = 0 \end{Bmatrix} \Rightarrow \boldsymbol{L}(\mathbf{u}(\mathbf{x})) = \mathbf{0},$$

$\boldsymbol{b} = \begin{bmatrix} b_x & b_x & b_x \end{bmatrix}^T$ The body forces acting per unit volume

$\mathbf{u} = \begin{bmatrix} u & v & w \end{bmatrix}^T$ The displacement vector

Virtual work as the 'weak form' of equilibrium equations for analysis of solids

The weighting function vector defined as $\delta\mathbf{u} = [\delta u \quad \delta v \quad \delta w]^T$

We can now write the integral statement of equilibrium equations as

$$\int_V \delta\mathbf{u}^T \boldsymbol{L}(\mathbf{u})d\mathrm{v} = -\int_V \left[\delta u \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + b_x \right) + \delta v(L_2) + \delta w(L_3) \right] d\mathrm{v}$$

$$= 0$$

Integrating each term by parts and rearranging we can write this as

$$\int_V \left[\frac{\partial \delta u}{\partial x}\sigma_x + \left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} \right)\tau_{xy} + \cdots - \delta u\, b_x - \delta v\, b_y - \delta w\, b_z \right] d\mathrm{v}$$

(*)

$$+\int_\Gamma \left[\delta u(\sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z) + \delta v(..) + \delta w(..) \right] d\Gamma = 0$$

Virtual work as the ‘weak form’ of equilibrium equations for analysis of solids

where $$\mathbf{t}=\begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}=\begin{Bmatrix} \sigma_x n_x+\tau_{xy}n_y+\tau_{xz}n_z \\ \tau_{xy}n_x+\sigma_y n_y+\tau_{yz}n_z \\ \tau_{xz}n_x+\tau_{yz}n_y+\sigma_z n_z \end{Bmatrix}$$ are tractions acting per unit area of external boundary surface Γ of the solid

In the first set of bracketed terms in eq. (*) we can recognize immediately the small strain operators acting on $\delta\mathbf{u}$, which can be termed a virtual displacement.

We can therefore introduce a virtual strain defined as

$$\delta\boldsymbol{\varepsilon}^T=\left\{\frac{\partial\delta u}{\partial x},\frac{\partial\delta v}{\partial y},\frac{\partial\delta w}{\partial z},\frac{\partial\delta u}{\partial y}+\frac{\partial\delta v}{\partial x},\frac{\partial\delta v}{\partial z}+\frac{\partial\delta w}{\partial y},\frac{\partial\delta v}{\partial z}+\frac{\partial\delta w}{\partial y}\right\}^T=\mathbf{D}\delta\mathbf{u}^{\;T}$$

Arranging the six stress components in a vector $\boldsymbol{\sigma}$ in an order corresponding to that used for $\delta\boldsymbol{\varepsilon}$, we can write Eq. (*) simply as

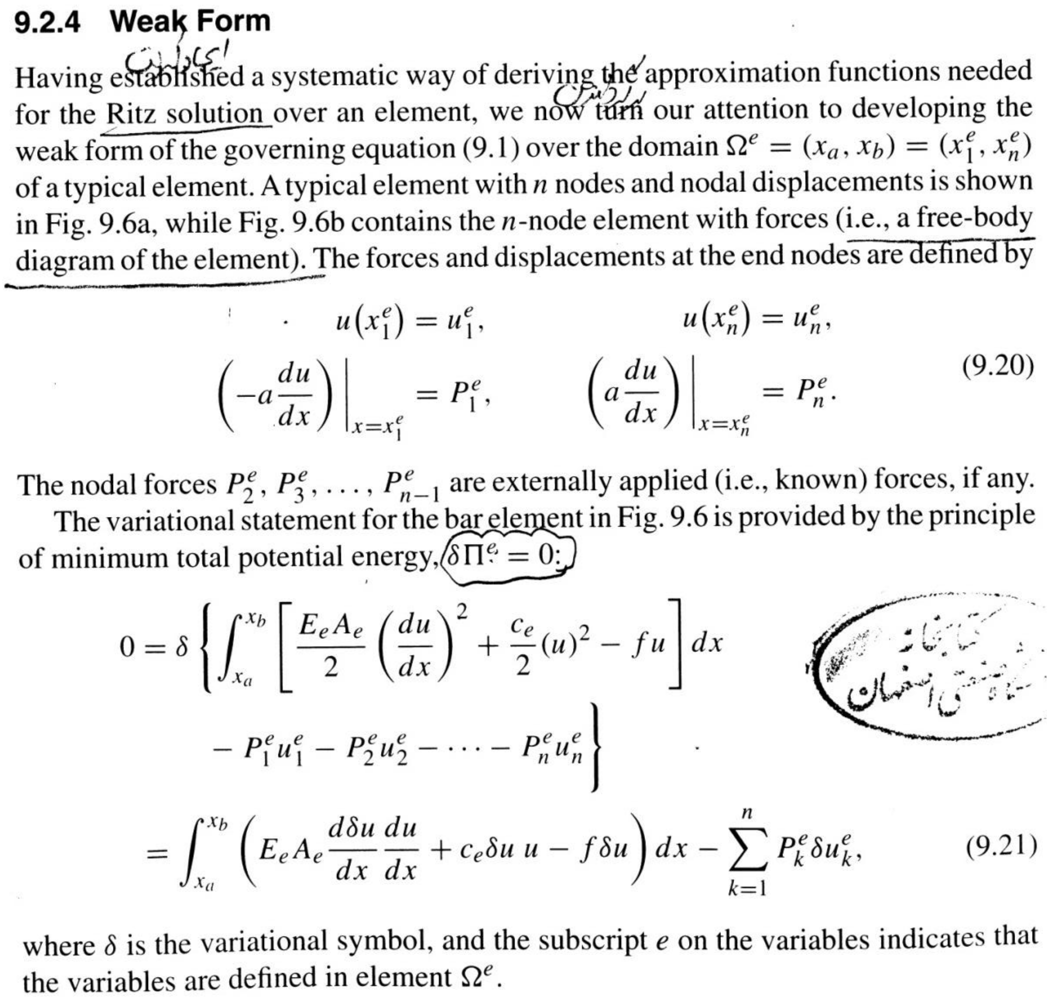
Virtual work as the ‘weak form’ of equilibrium equations for analysis of solids

$$\int_V \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma} d\mathrm{v} - \int_V \delta\mathbf{u}^T \mathbf{b} d\mathrm{v} - \int_\Gamma \delta\mathbf{u}^T \mathbf{t} d\Gamma = 0$$

we see from the above that the virtual work statement is precisely the weak form of equilibrium equations and is valid for non-linear as well as linear stress–strain (or stress–strain rate) relations.

Appendix

From “Energy Principles and Variational Methods in Applied Mechanics”, by: J. N. Reddy, 2nd ed., John Wiley (2002). (pp. 441--447)



9.2.4 Weak Form

Having established ایجادات a systematic way of deriving the approximation functions needed for the Ritz solution over an element, we now turn پرداختن our attention to developing the weak form of the governing equation (9.1) over the domain $\Omega^e = (x_a, x_b) = (x_1^e, x_n^e)$ of a typical element. A typical element with n nodes and nodal displacements is shown in Fig. 9.6a, while Fig. 9.6b contains the n-node element with forces (i.e., a free-body diagram of the element). The forces and displacements at the end nodes are defined by

$$\begin{aligned} u(x_1^e) &= u_1^e, & u(x_n^e) &= u_n^e, \\ \left(-a\frac{du}{dx}\right)\bigg|_{x=x_1^e} &= P_1^e, & \left(a\frac{du}{dx}\right)\bigg|_{x=x_n^e} &= P_n^e. \end{aligned} \tag{9.20}$$

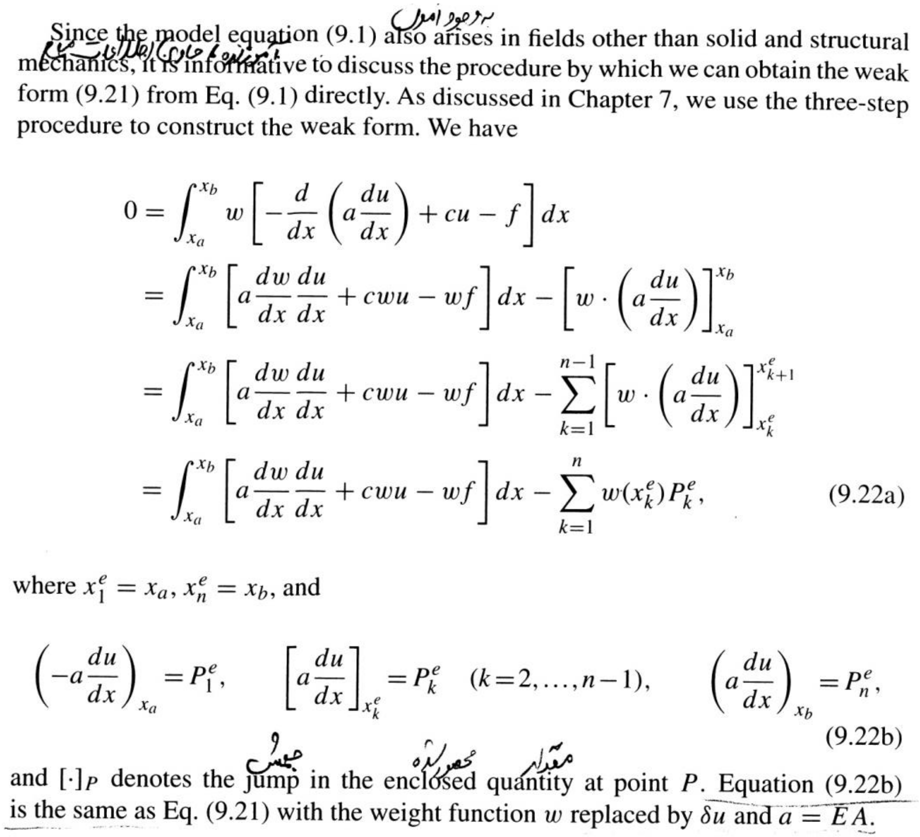
The nodal forces $P_2^e, P_3^e, \ldots, P_{n-1}^e$ are externally applied (i.e., known) forces, if any.

The variational statement for the bar element in Fig. 9.6 is provided by the principle of minimum total potential energy, $\boxed{\delta\Pi^e = 0}$:

$$\begin{aligned} 0 &= \delta\left\{\int_{x_a}^{x_b}\left[\frac{E_eA_e}{2}\left(\frac{du}{dx}\right)^2 + \frac{c_e}{2}(u)^2 - fu\right]dx \right. \\ &\qquad \left. - P_1^eu_1^e - P_2^eu_2^e - \cdots - P_n^eu_n^e\right\} \\ &= \int_{x_a}^{x_b}\left(E_eA_e\frac{d\delta u}{dx}\frac{du}{dx} + c_e\delta u\, u - f\delta u\right)dx - \sum_{k=1}^{n} P_k^e\delta u_k^e, \end{aligned} \tag{9.21}$$

where δ is the variational symbol, and the subscript e on the variables indicates that the variables are defined in element Ω^e.

Appendix



به وجود آمدن

آموزنده و حاوی اطلاعات مفید

Since the model equation (9.1) also arises in fields other than solid and structural mechanics, it is informative to discuss the procedure by which we can obtain the weak form (9.21) from Eq. (9.1) directly. As discussed in Chapter 7, we use the three-step procedure to construct the weak form. We have

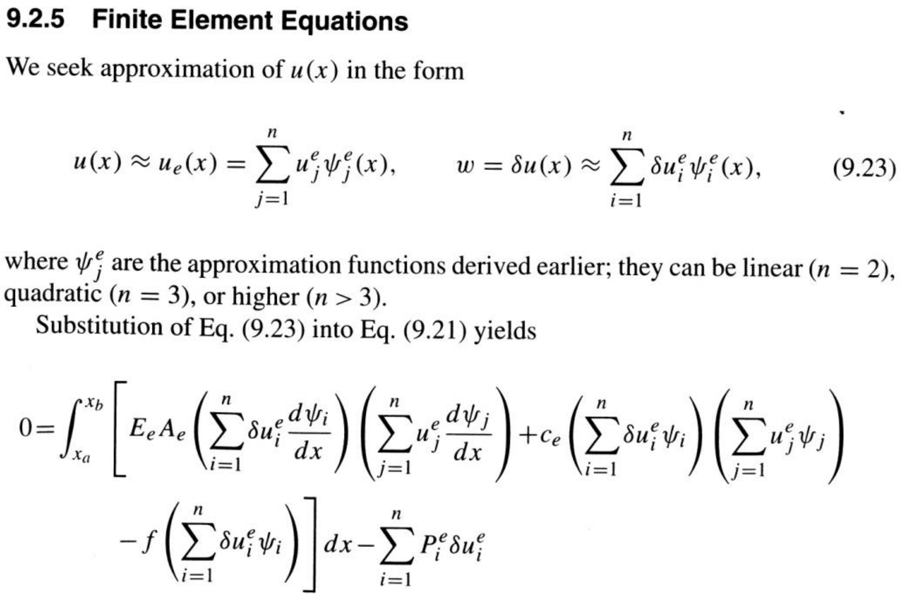
$$
\begin{aligned}
0 &= \int_{x_a}^{x_b} w\left[-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu-f\right]dx\\
&= \int_{x_a}^{x_b}\left[a\frac{dw}{dx}\frac{du}{dx}+cwu-wf\right]dx-\left[w\cdot\left(a\frac{du}{dx}\right)\right]_{x_a}^{x_b}\\
&= \int_{x_a}^{x_b}\left[a\frac{dw}{dx}\frac{du}{dx}+cwu-wf\right]dx-\sum_{k=1}^{n-1}\left[w\cdot\left(a\frac{du}{dx}\right)\right]_{x_k^e}^{x_{k+1}^e}\\
&= \int_{x_a}^{x_b}\left[a\frac{dw}{dx}\frac{du}{dx}+cwu-wf\right]dx-\sum_{k=1}^{n} w(x_k^e)P_k^e, \qquad (9.22a)
\end{aligned}
$$

where $x_1^e = x_a$, $x_n^e = x_b$, and

$$\left(-a\frac{du}{dx}\right)_{x_a}=P_1^e, \qquad \left[a\frac{du}{dx}\right]_{x_k^e}=P_k^e \quad (k=2,\ldots,n-1), \qquad \left(a\frac{du}{dx}\right)_{x_b}=P_n^e, \qquad (9.22b)$$

جهش محصور شده مقدار

and $[\cdot]_P$ denotes the jump in the enclosed quantity at point P. Equation (9.22b) is the same as Eq. (9.21) with the weight function w replaced by δu and $a = EA$.



9.2.5 Finite Element Equations

We seek approximation of $u(x)$ in the form

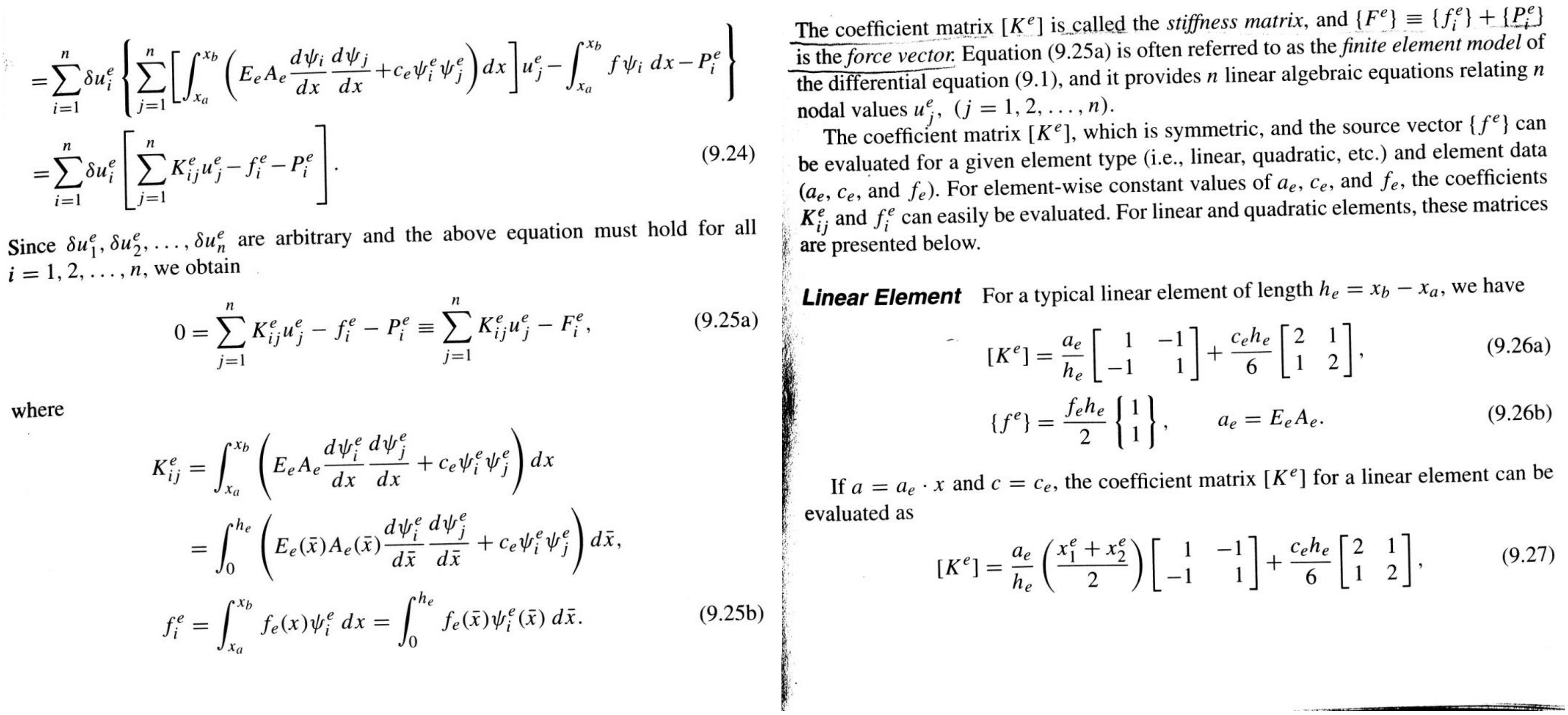
$$u(x) \approx u_e(x) = \sum_{j=1}^{n} u_j^e \psi_j^e(x), \qquad w = \delta u(x) \approx \sum_{i=1}^{n} \delta u_i^e \psi_i^e(x), \qquad (9.23)$$

where ψ_j^e are the approximation functions derived earlier; they can be linear ($n = 2$), quadratic ($n = 3$), or higher ($n > 3$).

Substitution of Eq. (9.23) into Eq. (9.21) yields

$$
\begin{aligned}
0=\int_{x_a}^{x_b}\Bigg[&E_eA_e\left(\sum_{i=1}^{n}\delta u_i^e\frac{d\psi_i}{dx}\right)\left(\sum_{j=1}^{n}u_j^e\frac{d\psi_j}{dx}\right)+c_e\left(\sum_{i=1}^{n}\delta u_i^e\psi_i\right)\left(\sum_{j=1}^{n}u_j^e\psi_j\right)\\
&-f\left(\sum_{i=1}^{n}\delta u_i^e\psi_i\right)\Bigg]dx-\sum_{i=1}^{n}P_i^e\delta u_i^e
\end{aligned}
$$

Appendix



$$= \sum_{i=1}^{n} \delta u_i^e \left\{ \sum_{j=1}^{n} \left[\int_{x_a}^{x_b} \left(E_e A_e \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} + c_e \psi_i^e \psi_j^e \right) dx \right] u_j^e - \int_{x_a}^{x_b} f \psi_i \, dx - P_i^e \right\}$$

$$= \sum_{i=1}^{n} \delta u_i^e \left[\sum_{j=1}^{n} K_{ij}^e u_j^e - f_i^e - P_i^e \right]. \tag{9.24}$$

Since $\delta u_1^e, \delta u_2^e, \ldots, \delta u_n^e$ are arbitrary and the above equation must hold for all $i = 1, 2, \ldots, n$, we obtain

$$0 = \sum_{j=1}^{n} K_{ij}^e u_j^e - f_i^e - P_i^e \equiv \sum_{j=1}^{n} K_{ij}^e u_j^e - F_i^e, \tag{9.25a}$$

where

$$K_{ij}^e = \int_{x_a}^{x_b} \left(E_e A_e \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx} + c_e \psi_i^e \psi_j^e \right) dx$$

$$= \int_{0}^{h_e} \left(E_e(\bar{x}) A_e(\bar{x}) \frac{d\psi_i^e}{d\bar{x}}\frac{d\psi_j^e}{d\bar{x}} + c_e \psi_i^e \psi_j^e \right) d\bar{x},$$

$$f_i^e = \int_{x_a}^{x_b} f_e(x) \psi_i^e \, dx = \int_0^{h_e} f_e(\bar{x}) \psi_i^e(\bar{x}) \, d\bar{x}. \tag{9.25b}$$

The coefficient matrix $[K^e]$ is called the *stiffness matrix*, and $\{F^e\} \equiv \{f_i^e\} + \{P_i^e\}$ is the *force vector*. Equation (9.25a) is often referred to as the *finite element model* of the differential equation (9.1), and it provides n linear algebraic equations relating n nodal values u_j^e, $(j = 1, 2, \ldots, n)$.

The coefficient matrix $[K^e]$, which is symmetric, and the source vector $\{f^e\}$ can be evaluated for a given element type (i.e., linear, quadratic, etc.) and element data (a_e, c_e, and f_e). For element-wise constant values of a_e, c_e, and f_e, the coefficients K_{ij}^e and f_i^e can easily be evaluated. For linear and quadratic elements, these matrices are presented below.

Linear Element For a typical linear element of length $h_e = x_b - x_a$, we have

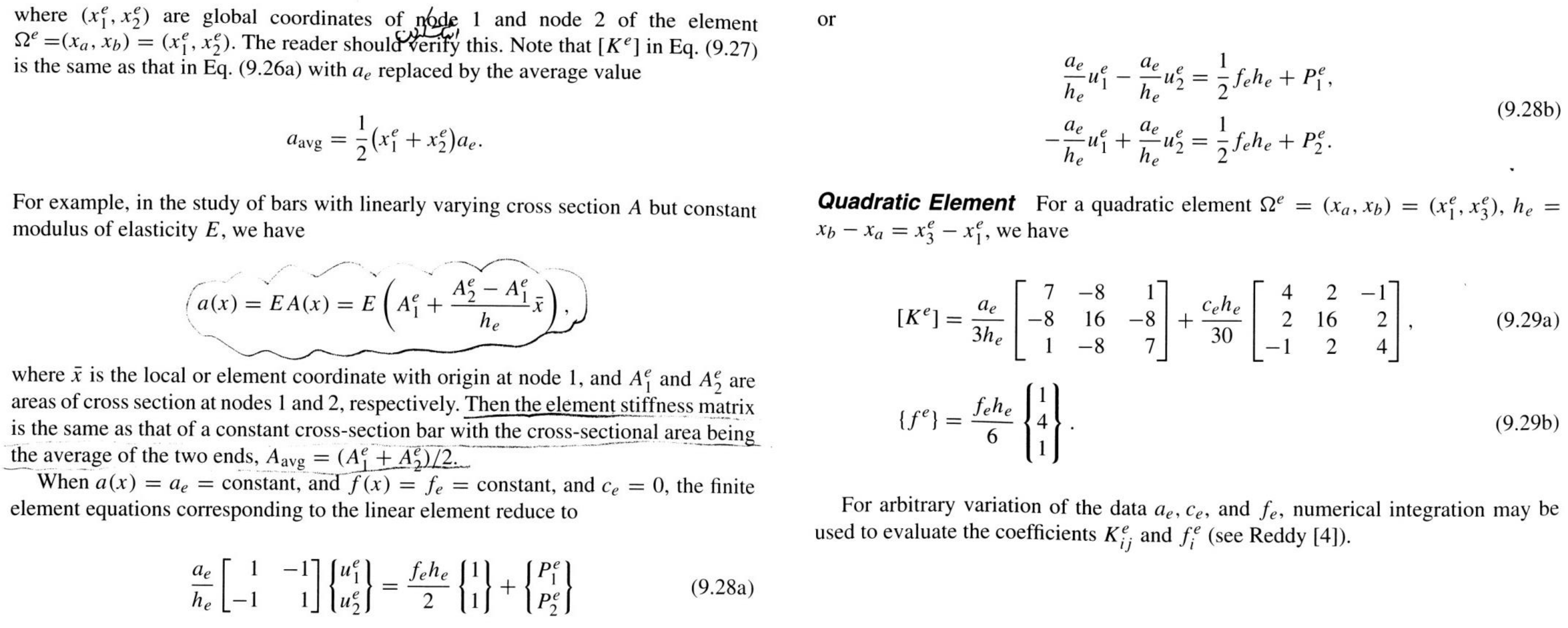
$$[K^e] = \frac{a_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \tag{9.26a}$$

$$\{f^e\} = \frac{f_e h_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \qquad a_e = E_e A_e. \tag{9.26b}$$

If $a = a_e \cdot x$ and $c = c_e$, the coefficient matrix $[K^e]$ for a linear element can be evaluated as

$$[K^e] = \frac{a_e}{h_e}\left(\frac{x_1^e + x_2^e}{2}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \tag{9.27}$$

Appendix



where (x_1^e, x_2^e) are global coordinates of node 1 and node 2 of the element $\Omega^e = (x_a, x_b) = (x_1^e, x_2^e)$. The reader should verify this. Note that $[K^e]$ in Eq. (9.27) is the same as that in Eq. (9.26a) with a_e replaced by the average value

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$$a_{\text{avg}} = \frac{1}{2}(x_1^e + x_2^e)a_e.$$

For example, in the study of bars with linearly varying cross section A but constant modulus of elasticity E, we have

$$a(x) = EA(x) = E\left(A_1^e + \frac{A_2^e - A_1^e}{h_e}\bar{x}\right),$$

where $\bar{x}$ is the local or element coordinate with origin at node 1, and A_1^e and A_2^e are areas of cross section at nodes 1 and 2, respectively. Then the element stiffness matrix is the same as that of a constant cross-section bar with the cross-sectional area being the average of the two ends, $A_{\text{avg}} = (A_1^e + A_2^e)/2$.

When $a(x) = a_e =$ constant, and $f(x) = f_e =$ constant, and $c_e = 0$, the finite element equations corresponding to the linear element reduce to

$$\frac{a_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \frac{f_e h_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} P_1^e \\ P_2^e \end{Bmatrix} \tag{9.28a}$$

or

$$\begin{aligned} \frac{a_e}{h_e}u_1^e - \frac{a_e}{h_e}u_2^e &= \frac{1}{2}f_e h_e + P_1^e, \\ -\frac{a_e}{h_e}u_1^e + \frac{a_e}{h_e}u_2^e &= \frac{1}{2}f_e h_e + P_2^e. \end{aligned} \tag{9.28b}$$

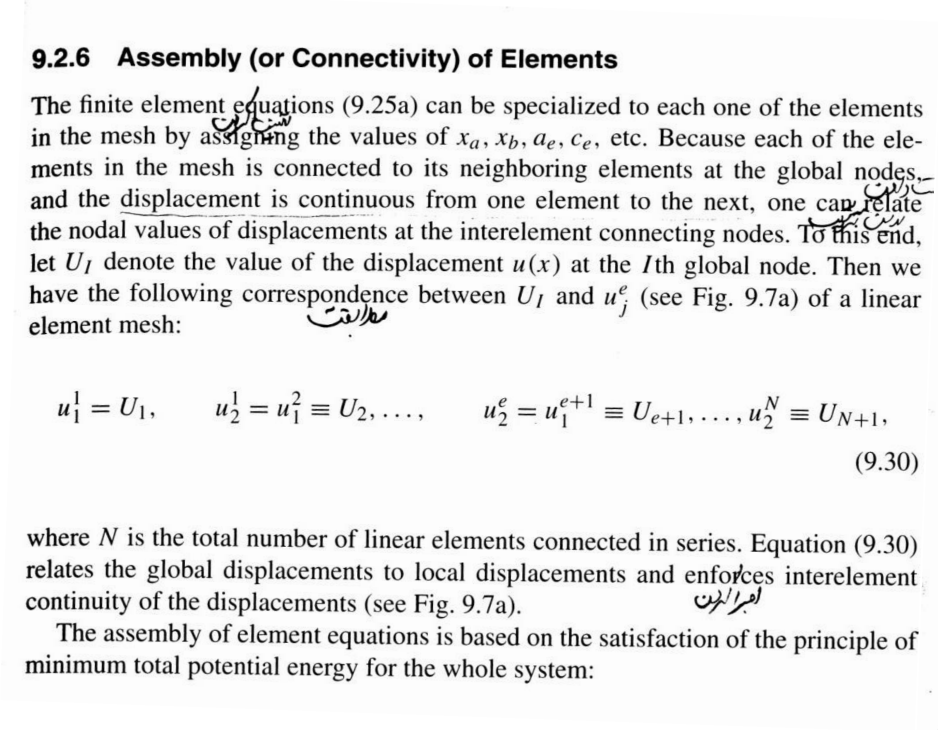
Quadratic Element For a quadratic element $\Omega^e = (x_a, x_b) = (x_1^e, x_3^e)$, $h_e = x_b - x_a = x_3^e - x_1^e$, we have

$$[K^e] = \frac{a_e}{3h_e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} + \frac{c_e h_e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}, \tag{9.29a}$$

$$\{f^e\} = \frac{f_e h_e}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}. \tag{9.29b}$$

For arbitrary variation of the data a_e, c_e, and f_e, numerical integration may be used to evaluate the coefficients K_{ij}^e and f_i^e (see Reddy [4]).

Appendix



9.2.6 Assembly (or Connectivity) of Elements

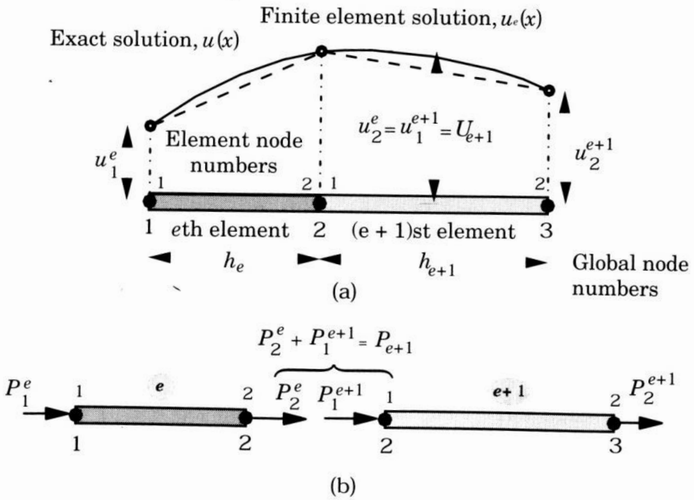
The finite element equations (9.25a) can be specialized [illegible] to each one of the elements in the mesh by assigning the values of x_a, x_b, a_e, c_e, etc. Because each of the elements in the mesh is connected to its neighboring elements at the global nodes, and the displacement is continuous from one element to the next, one can relate [illegible] the nodal values of displacements at the interelement connecting nodes. To this end, let U_I denote the value of the displacement $u(x)$ at the Ith global node. Then we have the following correspondence مطابقت between U_I and u_j^e (see Fig. 9.7a) of a linear element mesh:

$$u_1^1 = U_1, \qquad u_2^1 = u_1^2 \equiv U_2, \ldots, \qquad u_2^e = u_1^{e+1} \equiv U_{e+1}, \ldots, u_2^N \equiv U_{N+1}, \tag{9.30}$$

where N is the total number of linear elements connected in series. Equation (9.30) relates the global displacements to local displacements and enforces اجرا کردن interelement continuity of the displacements (see Fig. 9.7a).

The assembly of element equations is based on the satisfaction of the principle of minimum total potential energy for the whole system:

$$\delta\Pi = \sum_{I=1}^{N+1} \frac{\partial \Pi}{\partial U_I}\delta U_I = 0 \quad \text{or} \quad \frac{\partial \Pi}{\partial U_I} = 0, \qquad I = 1, 2, \ldots, N+1, \tag{9.31}$$

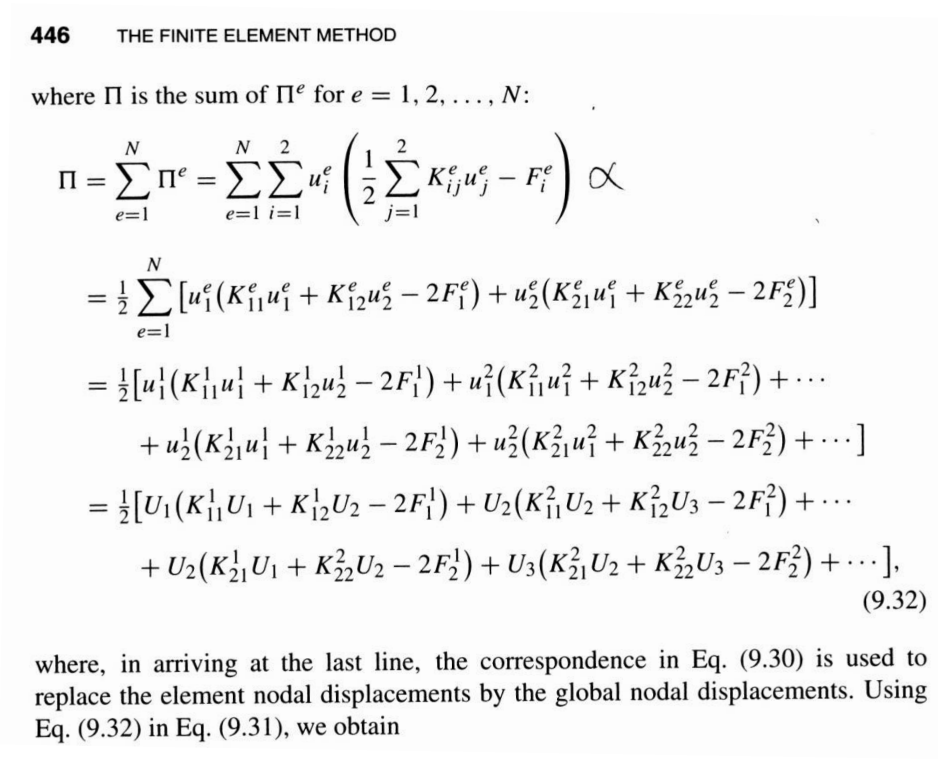


(a)

(b)

Figure 9.7 Connectivity of elements in one dimension. (a) Element-wise linear approximation of the displacement and interelement continuity of the displacements. (b) Balance of element forces at common nodes.

Appendix



where Π is the sum of Π^e for $e = 1, 2, \ldots, N$:

$$\Pi = \sum_{e=1}^{N} \Pi^e = \sum_{e=1}^{N} \sum_{i=1}^{2} u_i^e \left(\frac{1}{2} \sum_{j=1}^{2} K_{ij}^e u_j^e - F_i^e \right) \propto$$

$$= \tfrac{1}{2} \sum_{e=1}^{N} \left[u_1^e \left(K_{11}^e u_1^e + K_{12}^e u_2^e - 2F_1^e \right) + u_2^e \left(K_{21}^e u_1^e + K_{22}^e u_2^e - 2F_2^e \right) \right]$$

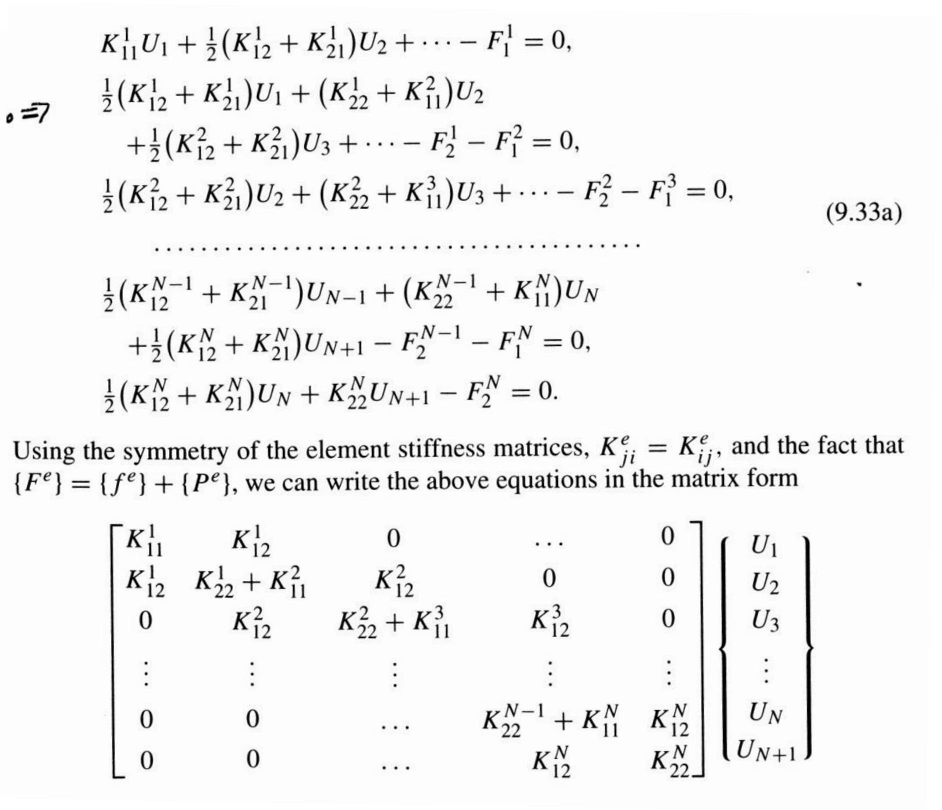
$$= \tfrac{1}{2} \big[u_1^1 \left(K_{11}^1 u_1^1 + K_{12}^1 u_2^1 - 2F_1^1 \right) + u_1^2 \left(K_{11}^2 u_1^2 + K_{12}^2 u_2^2 - 2F_1^2 \right) + \cdots$$

$$+ u_2^1 \left(K_{21}^1 u_1^1 + K_{22}^1 u_2^1 - 2F_2^1 \right) + u_2^2 \left(K_{21}^2 u_1^2 + K_{22}^2 u_2^2 - 2F_2^2 \right) + \cdots \big]$$

$$= \tfrac{1}{2} \big[U_1 \left(K_{11}^1 U_1 + K_{12}^1 U_2 - 2F_1^1 \right) + U_2 \left(K_{11}^2 U_2 + K_{12}^2 U_3 - 2F_1^2 \right) + \cdots$$

$$+ U_2 \left(K_{21}^1 U_1 + K_{22}^2 U_2 - 2F_2^1 \right) + U_3 \left(K_{21}^2 U_2 + K_{22}^2 U_3 - 2F_2^2 \right) + \cdots \big], \tag{9.32}$$

where, in arriving at the last line, the correspondence in Eq. (9.30) is used to replace the element nodal displacements by the global nodal displacements. Using Eq. (9.32) in Eq. (9.31), we obtain



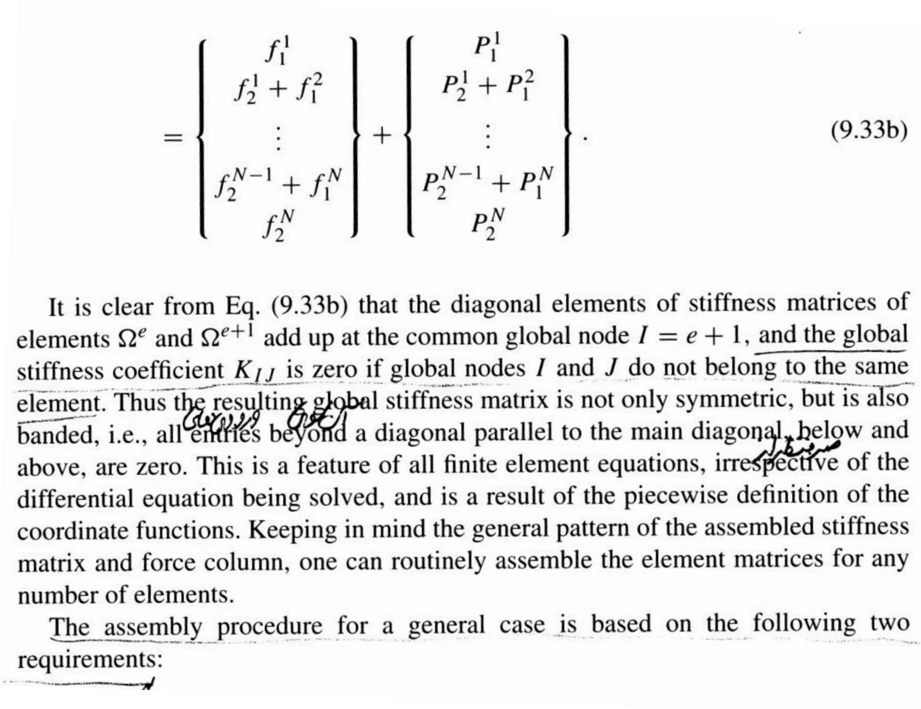
$\circ \Rightarrow$

$$\begin{aligned}
&K_{11}^1 U_1 + \tfrac{1}{2}\left(K_{12}^1 + K_{21}^1\right) U_2 + \cdots - F_1^1 = 0, \\
&\tfrac{1}{2}\left(K_{12}^1 + K_{21}^1\right) U_1 + \left(K_{22}^1 + K_{11}^2\right) U_2 \\
&\quad + \tfrac{1}{2}\left(K_{12}^2 + K_{21}^2\right) U_3 + \cdots - F_2^1 - F_1^2 = 0, \\
&\tfrac{1}{2}\left(K_{12}^2 + K_{21}^2\right) U_2 + \left(K_{22}^2 + K_{11}^3\right) U_3 + \cdots - F_2^2 - F_1^3 = 0, \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\tfrac{1}{2}\left(K_{12}^{N-1} + K_{21}^{N-1}\right) U_{N-1} + \left(K_{22}^{N-1} + K_{11}^N\right) U_N \\
&\quad + \tfrac{1}{2}\left(K_{12}^N + K_{21}^N\right) U_{N+1} - F_2^{N-1} - F_1^N = 0, \\
&\tfrac{1}{2}\left(K_{12}^N + K_{21}^N\right) U_N + K_{22}^N U_{N+1} - F_2^N = 0.
\end{aligned} \tag{9.33a}$$

Using the symmetry of the element stiffness matrices, $K_{ji}^e = K_{ij}^e$, and the fact that $\{F^e\} = \{f^e\} + \{P^e\}$, we can write the above equations in the matrix form

$$\begin{bmatrix}
K_{11}^1 & K_{12}^1 & 0 & \cdots & 0 \\
K_{12}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 & 0 & 0 \\
0 & K_{12}^2 & K_{22}^2 + K_{11}^3 & K_{12}^3 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & K_{22}^{N-1} + K_{11}^N & K_{12}^N \\
0 & 0 & \cdots & K_{12}^N & K_{22}^N
\end{bmatrix}
\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_N \\ U_{N+1} \end{Bmatrix}$$

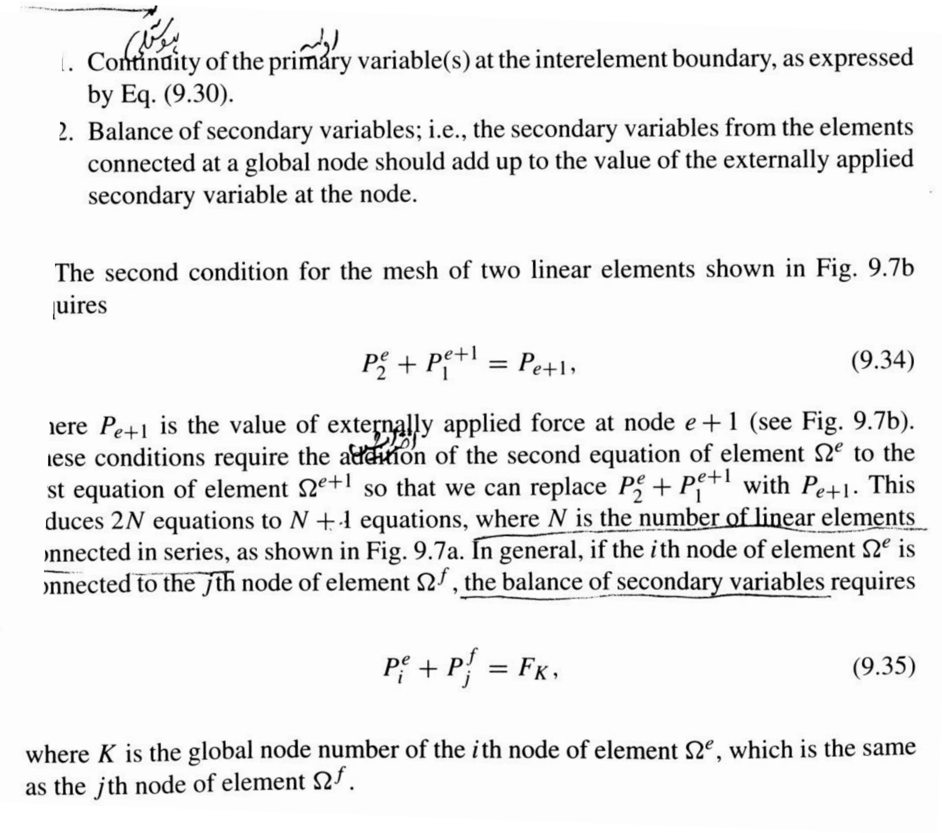
Appendix



$$= \begin{Bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ \vdots \\ f_2^{N-1} + f_1^N \\ f_2^N \end{Bmatrix} + \begin{Bmatrix} P_1^1 \\ P_2^1 + P_1^2 \\ \vdots \\ P_2^{N-1} + P_1^N \\ P_2^N \end{Bmatrix}. \tag{9.33b}$$

It is clear from Eq. (9.33b) that the diagonal elements of stiffness matrices of elements Ω^e and Ω^{e+1} add up at the common global node $I = e + 1$, and the global stiffness coefficient K_{IJ} is zero if global nodes I and J do not belong to the same element. Thus the resulting global stiffness matrix is not only symmetric, but is also banded, i.e., all entries ورودیها beyond آن سوی a diagonal parallel to the main diagonal, below and above, are zero. This is a feature of all finite element equations, irrespective صرفنظر of the differential equation being solved, and is a result of the piecewise definition of the coordinate functions. Keeping in mind the general pattern of the assembled stiffness matrix and force column, one can routinely assemble the element matrices for any number of elements.

The assembly procedure for a general case is based on the following two requirements:



1. Continuity (پیوستگی of the primary اولیه variable(s) at the interelement boundary, as expressed by Eq. (9.30).
2. Balance of secondary variables; i.e., the secondary variables from the elements connected at a global node should add up to the value of the externally applied secondary variable at the node.

The second condition for the mesh of two linear elements shown in Fig. 9.7b ıuires

$$P_2^e + P_1^{e+1} = P_{e+1}, \tag{9.34}$$

ıere P_{e+1} is the value of externally applied force at node $e + 1$ (see Fig. 9.7b). ıese conditions require the addition اضافه of the second equation of element Ω^e to the st equation of element Ω^{e+1} so that we can replace $P_2^e + P_1^{e+1}$ with P_{e+1}. This duces $2N$ equations to $N + 1$ equations, where N is the number of linear elements ınnected in series, as shown in Fig. 9.7a. In general, if the ith node of element Ω^e is ınnected to the jth node of element Ω^f, the balance of secondary variables requires

$$P_i^e + P_j^f = F_K, \tag{9.35}$$

where K is the global node number of the ith node of element Ω^e, which is the same as the jth node of element Ω^f.

Galerkin's Method in Elasticity

Governing equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0$$

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon}$$

$$\boldsymbol{\epsilon} = \left[\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}, \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]^{\mathrm{T}}$$

Interpolated Displ Field

$$u = \sum N_i(x, y, z) u_i$$

$$v = \sum N_j(x, y, z) u_j$$

$$w = \sum N_k(x, y, z) u_k$$

Interpolated Weighting Function

$$\phi_x = \sum N_i(x, y, z) \phi_{xi}$$

$$\phi_y = \sum N_j(x, y, z) \phi_{y\,j}$$

$$\phi_z = \sum N_k(x, y, z) \phi_{zk}$$

Galerkin's Method in Elasticity

$$\int_V \phi\left(L\tilde{u}-P\right)dV=0$$

$$\int_V \left[\left(\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}+f_x\right)\phi_x+\right.$$

$$\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}+f_y\right)\phi_y+$$

$$\left.\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{zy}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+f_z\right)\phi_z\right]dV=0$$

Integrate by part…

Galerkin's Method in Elasticity

Galerkin's Method in Elasticity Virtual Work

Virtual Total Potential Energy

$$\int_V \boldsymbol{\sigma}^T \boldsymbol{\epsilon}(\boldsymbol{\phi})\, dV - \int_V \boldsymbol{\phi}^T \mathbf{f}\, dV - \int_S \boldsymbol{\phi}^T \mathbf{T}\, dS - \sum_i \boldsymbol{\phi}^T \mathbf{P} = 0$$

Compare to Total Potential Energy

$$\Pi = \frac{1}{2}\int_V \boldsymbol{\sigma}^T \boldsymbol{\varepsilon} dV - \int_V \mathbf{u}^T \mathbf{f} dV - \int_S \mathbf{u}^T \mathbf{T} dS - \sum_i \mathbf{u}_i^T \mathbf{P}_i$$

Galerkin's Method in Elasticity

$$\left.\begin{aligned}
&\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}+b_x=0\\
&\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}+b_y=0\\
&\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+b_z=0
\end{aligned}\right\}\Rightarrow \mathbf{E}\left(\mathbf{u}\left(\mathbf{x}\right)\right)=\mathbf{0},$$

$$\left.\begin{aligned}
&\mathbf{u}-\mathbf{u}_0=\mathbf{0}\text{ on }A_u\\
&\left[\boldsymbol{\sigma}\right]\left(\hat{\mathbf{n}}\right)-\mathbf{t}=\mathbf{0}\text{ on }A_\sigma
\end{aligned}\right\}\Rightarrow \mathbf{B}\left(\mathbf{u}\left(\mathbf{x}\right)\right)=\mathbf{0}.$$